COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 303

Note: Cowles Foundation Discussion Papers are preliminary
materials circulated to stimulate discussion and
critical comment. Requests for single copies of a
Paper will be filled by the Cowles Foundation within
the limits of the supply. References in publications
to Discussion Papers (other than mere acknowledgment
by a writer that he has access to such unpublished
material) should be cleared with the author to protect
the tentative character of these papers.

THE BADLY BEHAVED ECONOMY WITH THE WELL BEHAVED PRODUCTION FUNCTION

Joseph E. Stiglitz

October 15, 1970

P. 3, 1. 21   should read: "more interesting anomaly ...."

P. 5, 1. 16   should read: "model, as told, for instance,"

P. 7, 1. 1   should read: "(5) In the same models, it was established that all paths along which expec-"

P. 10, 1. 6   should read: "(2.4') $a_{11}b_{21}b_{02}(1+r)^2 + (b_{01}a_{11} - b_{11}a_{01} - b_{21}b_{02})(\ldots)$

P. 11, 1. 11   should read: "Since the cost...."

       1. 20   should read: "switching requires the presence of at least two capital goods.) This theorem"

p. 12, 1. 20-1 should read: "creasing with r , it is monotonic in r . The relative amount of "capital" in two processes will be monotonic in r near an interest rate at which they are equally profitable if p_2/p_1 is a monotonic function of r .[9]"

P. 16, 1. 4   delete " ' "

       1. 5   delete " ' "

P. 17, 1. 5   should read: "capital goods industry uses its output as an input, ..."

       1. 11   should read: "In the previous section, we have seen how ..."

p. 21, 1. 16   should read: value of a machine which has   T-v   years to live is"

p. 22, 1. 1   should read: $\dfrac{(1-w)}{r} (1 - e^{-r(T-v)})$.

       1. 16   denominator of RHS should be: $(T^{-v} + T^{-1})\ldots$

P. 24, 1. 13   numerator should be: $be^{rb}(e^r - 1) + e^r$

P. 26, 1. 5   numerator should be: $N_1 + N_2 - N_1'T$

       1. 7   should read: $N_1 + \alpha N_2 > N_1'T$

       1. 8   should read: "If $\alpha > 1$, then (3.20) can be satisfied and $N_1 + N_2 < N_1'T$, so $dC/dT < 0$."

P. 33, l. 15   should read: $(4.9)\quad \sum\limits_{1}^{\infty} C^*_A (\frac{1}{1+\delta})^t = C^*_A/\delta = \sum\limits_{1}^{T} \hat{C}^t (\frac{1}{1+\delta})^t + \sum\limits_{T+1}^{\infty} C^*_B (\frac{1}{1+\delta})^t$

P. 37, l. 6   should read: "capital goods; a machine of type s of age u
has a price p_{su} in terms"

     l. 14   should read: "and the value of intermediate commodities per man
is, as usual, simply"

     l. 20   should read: " $V^k_D + V^k_I$, we plot for each process its total capital
requirements per man, [39] "

P. 39a ***

P. 40, l. 5   should read: $r + \mu = \max\limits_{\imath} \frac{b(\imath) - \imath w}{w} = ...$ (7.3)

     l. 11   should read: $-\frac{dw}{dr} = -\frac{dw/d\imath}{dr/d\imath} = \frac{w^2}{b} = \frac{-w}{r+\imath+\mu}$ (7.4)

     l. 14   should read: "where n is the rate of growth of population, except
if n = r , i.e. at"

P. 41, l. 4   should read: $Y(\imath) = ...$ (7.7)

     l. 10   should read: "described in the..."

     l. 20   should read: "canonical model, where even though there is..."

P. 43, l. 5   should read: $V = C + \lambda(C-\underline{C}) + \sum p(\imath, t)(N_I(\imath) - \mu K(\imath)) + ...$ (7.9)

P. 45, l. 10   should read: $p^*(\hat{\imath}^*) = ...$ (7.17)

     l. 12   should read: $p^*(\hat{\imath}^*) = w^*$ (7.19)

P. 46, l. 2   should read: $-\left(\imath \left(\frac{b(\imath)}{\imath}\right)\right) e^{-(\mu+\delta)\tau} d\tau$.

P. 46a, Fig.7.4a   The schedule relating $\frac{b-b'\imath}{b'}$ and $\imath$ should be <u>upward</u> sloping
rather than <u>downward</u> sloping.

P. 55, l. 13   should read: " $\kappa(\mathcal{T}, t)$ is ..."

P. 57, l. 15   should read: $\frac{d\Pi}{dt} = \frac{dwN_I}{dt} = -\mu wN_I + (b(\hat{\imath}) - w\hat{\imath})N_I - \dot{w}N_C$ (9.10)

P. 58, l. 3   should read: "depreciating machines plus the increase in profits
from the new machines, plus the change in production
costs from the wage change.

***P. 39a, Figure 7(a): w is the slope of the straight line tangent to b($\imath$)

P. 58, l. 7   should read: "The value of investment, wN_I, is constant along the curve (when $\dot{w} = 0$) "

l. 11   should read: "of investment is decreasing when $\dot{w} \geq 0$, to the left increasing, when $\dot{w} \leq 0$. w is constant along"

P. 59, l. 12   should read: $d\kappa(\ell, t'') = d\kappa(\ell, t')e^{-u(t''-t')} \qquad \ell > \hat{\ell}(t')$

P. 66, l. 10   numerator should read: $b(\ell(t+1)) - b'(\ell(t+1))\ell(t+1)$

P. 67, l. 23   equation number should be: (9.34)

P. 67b, l. 1   should read: " ℓ "

l. 2   should read: $N_C(t+1) = \dfrac{(1 - N_C(t))}{1+n} g^{-1}(w(\ell_2)s^{-1}(1 - N_C(t)))$

l. 4   should read: " ℓ_1 "

l. 5   should read: $N_C(t+1) = \dfrac{\ell_1(1 - N_C(t))}{1+n}$

l. 6   should read: $N_C(t+1) = \dfrac{1 - N_C(t)}{1+n} g^{-1}(w(\ell_1)s^{-1}N_C(t+1))$

l. 7   should read: $N_C(t+1) = \dfrac{\ell_2(1 - N_C(t))}{1+n}$

P. 74, l. 18   should read: "[23]   Levhari, D. "A Nonsubstitution...""

P. F5, l. 30   should read: "ployment in the consumption goods industry is C . Under our normalization, total"

P. F6, l. 3   should read: "Since at b = 1 , the numerator is positive and the deriva-"

l. 4   should read: "tive with respect to b of the numerator is positive for r > 0 ."

l. 5   delete.

P. F8, l. 16   should read: $\sum\limits_{0}^{\infty} C(t)(\dfrac{1}{1+\delta})^t$

P. F10, l. 6   should read: $\dfrac{dw}{dr} = -V_K - (r-n)\dfrac{dV_K}{dr}$

l. 33   should read: $V = C + \lambda(C-\underline{C}) + \int p(\ell, t)(d\tilde{N}_I(\ell) - u d\kappa(\ell)) + \ldots$

P. F11, l. 17 should read: "true economic explanation of China's policy of back-
yard furnaces. In con-"

P. F12, l. 22 should read: " $N_C(t)$ as a function of $N_C(t-1)$. $g(\ell) \equiv b - b'\ell$."

P. A2, l. 26 should read: "Assuming $-\mu + (b(\hat{\ell})\bar{\ell}/b(\bar{\ell}) - \hat{\ell})$ is ..."

P. A2a, l. 7 should read: " $\bar{\dot{\ell}} > 0$ "

P. A4, l. 3 should read: "Second, if in the first stage, $\dot{N}_I$ is ..."

l. 4 should read: "negative. To see this, observe that the derivative
of $\dot{N}_I$ with respect"

l. 5 should read: "to $\bar{\ell}$ has the same sign as $-\dot{\ell}$, and the derivative
of $\dot{N}_I$ with respect"

THE BADLY BEHAVED ECONOMY WITH THE WELL BEHAVED PRODUCTION FUNCTION*

by

J.E. Stiglitz

The story of economic growth formalized some fifteen years ago by
Solow (1956) and Swan (1956) is a very appealing one; it is a simple model
and yet a very rich one, as attested to by the multitude of variations
around the central theme to be found in the literature of the last decade
and a half. Yet, at least since Professor Robinson's attack against the
use of capital aggregates (1953), there have been doubts about the usefulness
of the model in describing "real" economies with heterogeneous capital goods.
These doubts have found some substance in the recent discussions of the re-
switching of techniques. These discussions have, however, been solely con-
cerned with comparisons of steady states, and not with the growth path of
an economy inheriting a particular capital stock, with particular consump-
tion patterns, reproduction rates, etc. It is the latter with which growth
theory must ultimately be concerned.

I have been asked to address myself to the implications of the recent
discussion of reswitching to truly dynamic economies, i.e. economies out of

steady state. I shall attempt to show that (a) although the presence of heterogeneous capital goods does present severe difficulties for the simpler neoclassical stories of capital accumulation, difficulties which are by no means completely resolved at the present time, these difficulties are not those associated with the reswitching phenomena, and (b) neither the reswitching phenomena (and the associated valuation perversities) nor the more important difficulties with the conventional growth models presented by heterogeneous capital goods which we shall discuss below (including those associated with the uncertainty of future prices in economies without a complete set of futures markets) have any bearing on the validity of the more fundamental aspects of neoclassical analysis. What must be altered are our simpler views of the process of accumulation as one of steadily increasing consumption, wage rates, and capital intensity of newly constructed machines.

The argument of the paper proceeds as follows: Part I establishes that even in the limited domain of steady state analysis, the reswitching phenomena and the associated valuation perversities are not of great interest:

First, we show that reswitching can be ruled out under fairly weak conditions. In section 2 five theorems giving sufficient conditions for no reswitching are presented:

(a) A process can be used at two interest rates and not at an intervening one only if the process used at the intervening one requires more of some capital goods and less of others (implying, of course, that there must be at least two capital goods).[1]

(b) If there are only two capital goods, reswitching can only occur under those conditions in which the factor price equalization theorem does not

obtain, i.e., the relative prices of the two capital goods cannot be a monotonic function of the interest rate.

(c) It requires only a limited amount of smooth substitutability to make reswitching impossible:

> (i) If <u>any</u> capital good (as an input) is smoothly substitutable (either directly or indirectly) for itself (as an output), then reswitching is impossible.

> (ii) If any capital good is smoothly substitutable for every other capital good (either as input or as output), reswitching is impossible.

> (iii) If labor is smoothly substitutable for any capital good (either as an output or as an input) which requires a capital good in its production, reswitching is impossible.

These smooth substitutability relationships need only exist in one industry.

Many of the participants on both sides of the reswitching controversy have suggested that more significant than reswitching itself are the valuation perversities, which always occur when there is reswitching but can occur even without reswitching: a lower level of consumption and value of capital is associated with a lower interest rate (below the golden rule).

In section 3 we show that these perversities--and indeed, the far more interesting anomoly of a lower level of consumption being associated with a higher value of capital--are not inconsistent with neoclassical economies; indeed, they may occur in a slight modification of the very model Wicksell used to argue against such perversities.

Finally, in section 4, we argue that these perversities have no fun-

damental significance for neoclassical doctrine; the impression that they
do arises from a confusion of the analysis of steady states with true dynamics.
For instance, the fact that the value of capital is larger on one steady
state path than on another has nothing to do with the question of whether
savings are required if one is to go from the first path to the second,
i.e. whether consumption must be foregone, and it is wrong--as some have
done--to measure the "cost" of the transition by the change in the value
of capital. Even when reswitching can occur, it is still true that the
rate of interest correctly reflects the marginal rates of transformation
available to the economy. (In the case of discrete technologies, as usual
careful note must be taken of left and right handed derivatives, but we
should be used to this by now.)

Having argued in Part I that if reswitching is to have any relevance
to capital theory, it must be in the analysis of truly dynamic economies,
in Part II we show that there are important problems associated with hetero-
geneous capital goods, but they are not those of reswitching.

After pointing out (section 5) that there is no obvious way of ex-
tending the notion of reswitching to dynamic economies, the concept of
recurrences is introduced: technique α is used, then replaced by β
but eventually, the economy returns to using technique α . It is shown
that recurrences may occur in economies in which reswitching does not occur
and need not occur in economies in which reswitching does occur: indeed
the reswitching phenomena appears to be irrelevant to the analysis of truly
dynamic economies. Using a slight modification of the Wicksell model, we
inquire, in the remaining sections of the paper, into the validity of the

conventional neoclassical stories. In section 7, we establish that if there are ex post fixed coefficients,

(a) Along an optimal path of accumulation, there may be recurrences.

(b) There may be <u>discontinuities</u> in the choices of technique. (The economy goes from technique γ to technique α, skipping technique β.)

(c) Consumption may not be a monotonic function of time. (Of course, even in the conventional models, the savings <u>rate</u> may not be a monotonic function of time.)

(d) The wage rate need not be a monotonic function of time.

(e) The consumption rate of interest may not be a monotonic function of time.

Thus, we have shown that the optimal plan of development may look very different from that suggested by the traditional Ramsey analysis for economies with a single malleable capital good, and that which Wicksell seems to have envisioned. That the corresponding story of development in a descriptive model when there are heterogeneous capital goods, as told, for instance, by Solow, runs into difficulties when there are heterogeneous capital goods has been pointed out and extensively discussed by Hahn [1966], Shell-Stiglitz [1967], and Samuelson [1967b], among others. Attention has focused on the following sources of difficulties: (1) In the absence of futures markets extending infinitely far into the future, individuals must form expectations of what prices and rentals will be in the future in order to make their investment decision. These expectations may or may not turn out to be correct. Research has focused on two classes of paths: those in which expectations of prices in the immediate future are fulfilled (short run perfect foresight)

and those in which expectations of prices are based on past experience (adaptive expectations), including, in particular, the case where individuals expect prices next period to be the same as those of this period (static expectations). (2) With given expectations, there are a variety of paths which are consistent with short run perfect foresight (i.e. monetary equilibrium may very well not be uniquely determined). (3) The short-run perfect foresight paths may not converge to balanced growth; in other words there is no way of ensuring that the initial prices will be those which lead to convergence.[2] (4) Although in the two models which have been perhaps most fully investigated thus far (Shell-Stiglitz [1967], Cass-Stiglitz [1969]) (i.e. global as well as local behavior completely characterized) it has been established that static expectations would ensure that the economy converged to balanced growth, the growth path of the economy, as it converged, might look very different from that described by the Solow-Swan-Meade-Uzawa malleable capital goods models.

The result that static expectations might ensure stability was somewhat disturbing; for static expectations implied that investment decisions were being made (resources allocated) on incorrect expectations, and hence the resulting allocations were likely to be inefficient. This result suggested that the economy might have to sacrifice efficiency for stability, and that it was in fact the "frictions" and "imperfections" in the economy which provided the most important forces for stability.

(5) In the same models, it was established that all paths along with expec-
tations were fulfilled forever must converge to balanced growth; i.e. along
those paths which did not converge eventually the perfect foresight assump-
tion would be violated (in finite time).

Unfortunately, it now appears that perfect foresight paths which are
consistent with perfect foresight forever need not converge to balanced growth
and that static expectations need not ensure stability.

Section 9 uses our modified Wicksellian model to investigate these
and other questions raised by the existence of heterogeneous capital goods.
We first consider an economy in which all of profits are saved but none
of wages (the Marxian savings assumption) and in which individuals have
static expectations. The growth path of the economy is likely to be oscil-
latory, i.e. there are likely to be recurrences in the type of machine
constructed. If the wage is high, this is likely to mean that profits are
low and hence savings are low, and that capital intensive machines will be
built. Demand for labor in the consumption goods sector will decline (as
of any given wage). But the excess labor cannot be hired in the investment
goods sector unless savings rise, i.e. unless profits rise. Thus wages
fall, enabling more workers to be hired in both the consumption and invest-
ment goods sector, restoring full employment. But at the lower wage, just
the reverse occurs. Nonetheless, it can be established that these capital
intensity-distribution cycles are all damped, i.e. the economy converges
to balanced growth.

The stability of the balanced growth path turns out to depend, however,
on the assumption that capital decays exponentially. If capital has a finite

life, the balanced growth path may be unstable, and the economy may converge
instead to a limit cycle. This result holds whether there are static expec-
tations or perfect foresight and under a variety of savings assumptions.
Indeed, when we attempt to make the model more "reasonable" by introducing
life cycle savings with overlapping generations, further difficulties arise.
The savings (thriftiness) conditions are not sufficient, by themselves, to
determine the rate of profit (the dynamic path of the economy); evidently,
at least in part, "the rate of profit is what it is because individuals
expect it to be what it is," and had they expected it to be different, it
indeed would be different. Moreover, the economy may neither converge to
balanced growth, diverge, nor converge to a limit cycle. It simply "wobbles"
along.

These models serve to illustrate the crucial role which expectations
play in the determination not only of the distribution of income today, but
also of the entire growth path of the economy. This raises important doubts
about the validity of the conventional growth models which are so structured
that expectations play no role at all.

PART I. Steady States

2.1. Definition of Reswitching

The reswitching phenomenon may be simply described as follows.[3]
Consider a competitive economy with a single primary factor, labor, no joint
production, and constant returns to scale which is in balanced growth. (Hence
the dynamic nonsubstitution theorem obtains (see, e.g., Mirrlees [1969],
Stiglitz [1970]).) Then there may exist three rates of interest, such that

at the highest and lowest, the same technology is used, and in between, an alternative technology is employed.[4]

2.2. Example of Reswitching

The possibility of reswitching is illustrated by the following simple example. An economy has a single final good X_1 which can be produced by two alternative technologies, given in the following input-output table:

Requirement of input per unit of output of	Technology A	Technology B	
	X_1	X_1	X_2
Labor	a_{01}	b_{01}	b_{02}
X_1	a_{11}	b_{11}	0
X_2	0	b_{21}	0

Technology A requires a_{01} units of labor and a_{11} units of X_1 to produce one unit of X_1. Technology B requires b_{01} units of labor, b_{11} units of X_1 and b_{21} units of X_2 to produce one unit of X_1, while to produce one unit of X_2, b_{02} units of labor are required (and nothing else). Then if the interest rate is r and we let labor be our numeraire, the cost of production of X_1 using technology A is[5] (in long run equilibrium)

$$(2.1) \qquad P_1 = a_{01} + (1+r)a_{11}P_1 = \frac{a_{01}}{1 - (1+r)a_{11}}$$

while that using Technology B is

$$(2.2) \qquad P_1 = b_{01} + (1+r)(P_1 b_{11} + P_2 b_{21}) = \frac{b_{01} + b_{21}b_{02}(1+r)}{1 - (1+r)b_{11}} ,$$

since the cost of producing X_2 is

(2.3)
$$P_2 = b_{02} .$$

The two are equal when

(2.4)
$$\frac{a_{01}}{1 - (1+r)a_{11}} = \frac{b_{01} + b_{21}b_{02}(1+r)}{1 - (1+r)b_{11}}$$

or

(2.4')
$$a_{11}b_{21}b_{02}(1+r)^2 + (b_{01}a_{11} - b_{11}a_{01})(1+r) + (a_{01} - b_{01}) = 0 .$$

It is clear that if

(2.5)
$$a_{11} > b_{11} \quad \text{and} \quad a_{01} > b_{01}$$

then, provided that the B technology does not dominate the A technology at all values of r, there will be, in general, two interest rates at which the two technologies have the same cost of production.[6,7] (See Figure 2.1.) For very low and very high r, the B technology dominates the A, and for intervening interest rates, the A technology dominates the B technology.
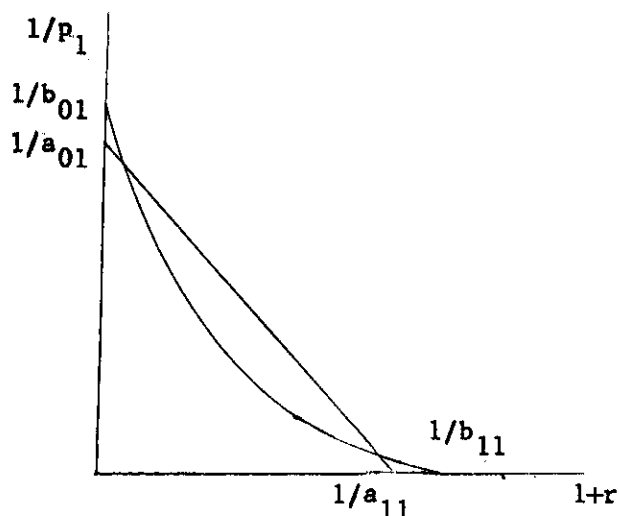

FIGURE 2.1

Factor Price Frontier for Simple Two Process Economy

2.3. <u>Dated Labor and Multiple Internal Rates Return</u>

This example also serves to illustrate the following point:

Whenever the non-substitution theorem obtains, we can reduce the costs

of production to a (possibly infinite) series of dated labor. That the dif-

ference between two such series is zero at a number of rates of interest

has been well known for a long time.

2.4. <u>Capital Intensities and Reswitching</u>

The example also suggests two general theorems providing sufficient

conditions for the impossibility of reswitching. If we let labor be the

numeraire, it is clear that as the rate of interest rises, all prices rise.

Since the costs of capital is rising, at higher interest rates, the economy

should choose the technique with the lower capital costs. Reswitching can

only occur then if one process has a higher capital cost than another at

two different interest rates, while the second process has a higher capital

cost at an intervening interest rate. It immediately follows[8] that if <u>one</u>

<u>process has a higher requirement of every capital good than another,</u> then

since the capital cost of one will be increasing monotonically relative to

the other, <u>it is impossible for the first process to be used at two different</u>

<u>interests rates and the second process at an intervening one.</u> (Thus, re-

switching requires the presence of at least two capital goods. This theorem

says, in other words, that two processes cannot be involved in reswitching

if one of them is more "capital intensive" than the other regardless of the

weights used to form the (additive) capital aggregate. This theorem is helpful

in clarifying that it is just those situations where one process requires

more of one capital good than another, but less of some other capital good, with which reswitching is concerned.

One might conjecture that if the relative prices of the capital goods move in a systematic way as the interest rate changes it might be possible to rule out the possibility of reswitching. This is the substance of the theorem presented in the next section.

2.5. Reswitching and the Factor Price Equalization Theorem

If there are only two capital goods, reswitching can occur only under those circumstances in which the factor price equalization theorem does not obtain.

To see this, write the costs of production of the jth commodity with the kth technique as

$$a_{0j}^k + (1+r)(p_1 + p_2)\left\{\frac{p_1}{p_1 + p_2} a_{1j}^k + \frac{p_2}{p_1 + p_2} a_{2j}^k\right\}$$

where a_{1j}^k is the requirement of the ith capital good in the production of the jth commodity by the kth technique (a_{0j}^k is the requirement of labor). The term in the bracket we can think of as the "real" capital, and is a weighted average of the capital requirements of the two different kinds of capital goods, with the weights being the relative prices. $(1+r)(p_1 + p_2)$ is the "cost" of "capital," and since p_1 and p_2 are monotonically increasing with r, it is monotonic in r. The amount of "capital" will be monotonic in r if and only if p_2/p_1 is a monotonic function of r.[9] But monotonicity of p_2/p_1 as a function of r is exactly what is required for the factor price equalization theorem to obtain.

In our two capital goods models, we can show that a necessary and sufficient condition for relative prices to be monotonic in r is that, if we choose our units so that a unit of (gross) output requires a unit of labor, the total (unweighted) capital requirements of one sector be un-ambiguously larger (at every interest rate) than that of the other sector.[10]

In the more general model, the connection between the factor price equalization theorem and the reswitching phenomena does not obtain; even when all relative prices move monotonically with the rate of interest,[11] the weighted average of capital requirements of one process may be greater than that of another at two interest rates, while the opposite obtains at an intervening interest rate. On the other hand, in a world with free international trade and all economies in balanced growth, we will never observe the reswitching phenomenon even in a single industry: a technique which is used by a low interest rate economy, but discarded by an economy with a higher interest rate will never be brought back into use at still a third, higher, interest rate. The reason for this is that if there is a single set of international commodity prices, there is a single set of weights by which we can aggregate the various kinds of capital to form the aggregate capital requirements. A process then is either more or less capital inten-sive than another (at that particular set of international prices).

2.6. Substitutability and Reswitching

It was recognized early in the reswitching controversy that if all production functions were differentiable, reswitching could not occur. Stiglitz [1966] argued, however, that only a limited amount of substitutability

was required to rule out the possibility of reswitching. (All the examples of reswitching presented in the literature involve discrete technologies allowing no smooth substitutability in any sector of the economy.) The following three theorems represent extensions and refinements of the argument presented there and subsequently by Solow [1967] and Starrett [1969].

Consider any homogeneous of degree zero transformation schedule

$$(2.6) \quad \phi(C_1, \ldots\ldots C_n, X_1, \ldots\ldots X_m, X_{m+1}, \ldots\ldots X_{2m}, L) = 0$$

where $(C_1, \ldots\ldots C_n)$ is the vector of consumption goods (not used in production), $(X_1, \ldots\ldots X_m)$ is the vector of capital inputs, $(X_{m+1}, \ldots\ldots X_{2m})$ is the corresponding vector of capital outputs, and L is the labor input. A "steady state technology" is defined by the vector (C, X) (or any scalar multiple of that vector, where $\phi(C, X, (n+1)X, L) = 0$, where n is the growth rate of labor.[12]

(1) <u>If there is any capital good which is used directly or indirectly in its own production, and it is possible to increase its output by increasing the input of the capital good (leaving all other inputs and outputs unchanged) then there can be no reswitching.</u> More formally, a technology can be used at only one interest rate if there exists a set of subscripts $i_1 \ldots i_j \ldots i_n$, such that $i_1 \leq m$, and

$$(2.7) \qquad \frac{\partial X_{i_{j+1}}}{\partial X_{i_j}} \quad \text{and} \quad \frac{\partial X_{i_1+m}}{\partial X_{i_n}} \quad \text{exist.}$$

Indeed, since

$$(2.8) \qquad \frac{\partial X_j}{\partial X_k} = \begin{cases} p_j/p_k & \text{if} \quad (j-m)(j-m) > 0 \\ p_j(1+r)/p_k & j < m, \quad k > m \quad , \\ p_j/p_k(1+r) & j > m, \quad k < m \end{cases}$$

$$(2.9) \qquad \frac{\partial X_{i_2}}{\partial X_{i_1}} \times \frac{\partial X_{i_3}}{\partial x_{i_2}} \cdots \frac{\partial X_{i_1+m}}{\partial X_{i_n}} = (1+r)$$

(2) <u>If there is any capital good which is smoothly substitutable for every</u> <u>other capital good, either as an input or as an output, then reswitching</u> <u>cannot occur</u>; more formally, a technology cannot be involved in reswitching if for some i and every j, $1 \leq i \leq m$, $1 \leq j \leq m$

$$(2.10)(a) \qquad \text{Either} \quad \frac{\partial X_j}{\partial X_i} \quad \text{or} \quad \frac{\partial X_{j+m}}{\partial X_{i+m}} \quad \text{exists}$$

or

$$(2.10)(b) \qquad \frac{\partial X_i}{\partial X_{j+m}} \quad \text{exists}$$

or

$$(2.10)(c) \qquad \frac{\partial X_{i+m}}{\partial X_j} \quad \text{exists}$$

The proof follows immediately upon observing that, for instance in the case of (a),

$$(2.11) \qquad \frac{p_j}{p_i} = \frac{\partial X_j}{\partial X_i} \quad \left(\text{or} \quad \frac{\partial X_{j+m}}{\partial X_{i+m}} \right)$$

independent of r . Let A denote the input-output matrix for the capital

goods corresponding to the given technology. Thus, whenever the given tech-

nology is used, the price of the capital goods must be given by

$$P_k = a_0 + P_i(1+r)\left(\frac{P_i}{P_i} A'\right)$$

where $\left(\frac{P_i}{P_i} A'\right)$ is independent of r . Thus P_k is a linear function of

$P_i(1+r)$. Consider any other technology, with its input-output matrix for

capital goods B . If we compare the cost of production using the B

technology when the prices are those corresponding to the A technology,

$$b_0 + P_i(1+r)\left(\frac{P_i}{P_i} B\right)$$

to the cost of production using the A technology at the same prices, we ob-

serve that there is at most one value of r at which they are equal.[13]

The desired result follows immediately.[14]

(3) If there is any capital good which is smoothly substitutable for labor,

either as an input or as an output (i.e., if for any i , $\partial X_i/\partial L$ or

$\partial X_{i+m}/\partial L$ exists) and it requires some commodity other than labor in its pro-

duction, then there can be no reswitching.

Assume wages are paid at the beginning of the period of production

and labor is smoothly substitutable for the ith capital good as a factor

of production.[15] Then $\partial X_i/\partial L = 1/P_i$. But P_i is a monotonic function of

r if X_{i+m} requires capital in its production; hence the given technology

can be used at most at one r . The case of labor being substitutable for

capital as an output may be handled similarly.

It is important to observe that in each of the theorems all that we require for the existence of the indicated derivatives of the transformation schedule is that these derivatives be defined for at least one of the sectoral production functions of the economy.[16] Thus, for example, if one capital goods industry uses its output as an output, and there is a smooth substitutability relationship between the two, reswitching will not be possible, even if all other sectoral production functions have fixed coefficients and that sector itself has a non-differentiable relationship between other of its inputs and output.

3.1. Wicksell and Wicksell Effects

In the previous section, we have seen that how the changes in the relative prices of the different capital goods (as we change the rate of interest) may lead to the relative "capital" intensities of two processes changing with the rate of interest; this in turn results in the possibility of the reswitching phenomenon. Wicksell, on the other hand, discussed at great length the consequences of the change in the price of capital goods relative to consumption goods as the interest rate changes. Somewhat heuristically, it is now a familiar proposition from the literature on two sector growth models that if the capital goods sector is less labor intensive than the consumption goods sector, as the wage rate rises and interest rate falls the price of capital goods will accordingly fall. Hence, the value of capital (in consumption numeraire) may increase less rapidly than the "real capital" stock.[17] This raises the possibility that an increase in the capital stock (number of machines) may be accompanied by a

fall in the total value of capital in consumption numeraire (as we compare

steady states), which in turn would imply that an economy with a lower value

of capital would have a higher output.

The major thrust of Wicksell's brilliant analysis of Akerman's problem[18]

was to argue that

> "A growth of capital, as long as it is such as to be pro-
> fitable, is always accompanied by an increase in the total
> product."

In establishing his argument, he also shows that the length of life of a

machine, the number of machines, the value of capital and the output of

consumption goods increase monotonically as the rate of interest falls.

The reswitching phenomenon provides immediate counterexamples to

the generality of the last two propositions; for if one technique has a higher

consumption (net output) per man than another, at a switch point it must

also have a higher value of capital and profits (since the wages are identical

for the two technologies at a switch point). That is (assuming there is a

single consumption good C) in the stationary state, if w is the wage,

V_K the value of capital, and r the interest rate, $C = w + rV_K$ or

$V_K = (C - w)/r$. If two technologies are competitive at the same w and

r , $V_{K1} = (C_1 - w)/r$, $V_{K2} = (C_2 - w)/r$, so $V_{K1} \gtrless V_{K2}$ as $C_1 \gtrless C_2$.

Thus the reswitching phenomenon implies that the technology used at an inter-

mediate interest rate has a lower (higher) consumption per man and a lower

(higher) value of capital than that used at lower or higher interest rates.[19]

Nonetheless, these models do not contradict the fundamental Wicksellian

argument that a "larger value of capital be associated with an increase in

total product." In the following subsections, we shall show (a) that the

Wicksell argument does not hold even for a slight modification of his system and (b) that further slight modifications of the Wicksell model show the same kind of "perversities" as the discrete heterogeneous capital good model, but (c) none of these perversities are of crucial importance to neoclassical economics.

3.2. A Wicksellian Capital Model

We follow Wicksell in assuming that there are two sectors in the economy, one producing machines by means of labor alone, and the other producing consumption goods, with labor and machines. All machines are identical except in their durability, a machine lasting T years requiring N(T) man-years to construct. Wicksell assumed that the production function in the consumption goods sector was Cobb-Douglas; we depart from him here in assuming that it is fixed coefficients--a machine with one labor produces one unit of consumption goods per year.[20] We wish to contrast stationary economies differing in their rate of interest. We shall express all variables in per capita terms.

If the wage is w (in consumption good numeraire) and the interest rate is r , the present discounted value of the quasi-rents from a machine of durability T is

$$(3.1) \qquad \frac{(1-w)(1 - e^{-rT})}{r} = (1-w) \int_0^T e^{-rt} dt$$

The competitive firm will choose that technique which maximizes returns per dollar invested:

(3.2)
$$\max_{T} \frac{(1-w)(1 - e^{-rT})}{w\bar{N}r}$$

i.e. if $v = N'T/N$, the elasticity of N, then

(3.3)
$$v = \frac{rTe^{-rT}}{1-e^{-rT}}$$

A sufficient condition to ensure that a value of T satisfying (3.3) is indeed the global maximum is that $v' \geq 0$.

It is immediate that as r falls, durability increases:[21]

(3.4)
$$\frac{dT}{dr} = - \frac{\{1 - rT - e^{-rT}\}Te^{-rT}}{(1 - e^{-rT})^2} \Big/ \frac{\{1 - rT - e^{-rT}\}re^{-rT}}{(1 - e^{-rT})^2} - v' < 0 .$$

Since consumption C equals the total number of machines[22]

(3.5)
$$C = \frac{T}{N + T}$$

and is monotonically increasing with T,

(3.6)
$$\frac{dC}{dT} = \frac{N(1 - v)}{(N + T)^2} > 0$$

(using (3.3) and the fact[23] that $rTe^{-rT}/(1 - e^{-rT}) < 1$), C is inversely related to r,

(3.7)
$$\frac{dC}{dr} < 0 .$$

We are interested in determining the value of capital, V_K. The value of a machine which has $T-V$ years to live is

$$\frac{(1-w)}{r} (1 - e^{-r(T-V)}) .$$

Since there are $\frac{1}{N+T}$ machines of each age, the total value of capital, V_K, is

$$(3.8) \qquad V_K = \frac{(1-w)(rT - 1 + e^{-rT})}{(N+T)r^2}$$

The competitive wage is determined to make the present discounted value of the "optimal" durability machine equal its cost:

$$\frac{(1-w)}{r} (1 - e^{-rT}) = wN(T)$$

or

$$(3.9) \qquad w = \frac{(1 - e^{-rT})/r}{((1 - e^{-rT})/r) + N}$$

Substituting (3.9) into (3.8) and simplifying, we obtain

$$(3.10) \qquad V_K = \frac{N}{(N+T)r} \frac{rT - 1 + e^{-rT}}{1 - e^{-rT} + rN}$$

That dV_K/dr will be negative if ν is constant (so, for appropriately chosen units, $N = T^{\nu}$) may be seen by rewriting (3.10), letting $rT = d$ (since ν is constant (3.3) implies that d is a constant):

$$(3.10') \qquad V_K = \frac{T^{\nu+1}(d - 1 + e^{-d})}{(T^{\nu} + T)[1 - e^{-d} + dT^{\nu-1}]d}$$

$$\frac{dV_K}{dT} = \frac{V_K\{(1 - e^{-d})(T^{-2} + \nu T^{-(1+\nu)}) + d[(2-\nu)T^{\nu-3} + T^{-2}]\}}{(T^{\nu} + T^{-1})(1 - e^{-d} + dT^{\nu-1})} > 0$$

On the other hand, observing that by making ν' arbitrarily large, we can

make dT/dr arbitrarily small at a point, we can make the sign of dV_K/dr

depend on that of $\partial V_K/\partial r_{\underline{T}}$; but consider the case where rT is small

(the value of rT depends only on the value of ν , not its derivative).

Then (3.10) may be approximated by

$$(3.10") \qquad\qquad V_K \approx \frac{NT^2}{2(N+T)[N + T - rT^2/2]}$$

which is an increasing function of r .[24]

Thus, we have the possibility that even in a Wicksellian model, a

higher value of capital will be associated with a lower level of net output.

The "paradox" which Wicksell was so concerned to show could not occur in

an economy with "perfectly free competition"--"a fall in the national dividend

resulting from continued saving and capital accumulation"--may indeed occur.

3.3. Reswitching in a Wicksellian Model

I think Wicksell would have been equally surprised--and disturbed--

to find out that in a slight variant of his model, consumption per man and

T^* , the optimal durability, need not be monotonic in r --although I

think he needn't have been.

To see this, we depart from Wicksell's formulation, and that presented

above in Section 3.2, by assuming (as seems reasonable) that there are gesta-

tion periods for the production of machines. Thus, to produce a machine

which will last T years requires an input of labor described by the function

$N(u,T)$, the total cost of which is, if r is the interest rate and w

the wage,

(3.11)
$$w \int N(u,T) e^{-ru} du$$

For simplicity, we shall assume that $N(u,T)$ takes on the simple form

$$N(u,T) = \begin{cases} N_1(T) & 0 \geq u \geq -1 \\ N_2/b = \text{a constant} & -1 \geq u \geq -b \\ 0 & \text{otherwise} \end{cases}$$

(where $u = 0$ is the date of completion of the machine).

Then a profit maximizing firm will choose the technique for which the present discounted value of quasi-rents per dollar invested,

(3.13)
$$\frac{(1-w)(1 - e^{-rT})}{\frac{w}{r}\left[\frac{N_2}{b}(e^{rb} - e^r) + N_1(e^r - 1)\right]}$$

is largest. Defining

(3.14)
$$\alpha(r) = \frac{e^{rb} - e^r}{(e^r - 1)b}$$

so[25]

(3.15)
$$\alpha'(r) = \frac{be^{rb}(e^r - 1) - e^r(e^{rb} - e^1)}{(e^r - 1)^2 b} > 0$$

we see that a profit maximizing firm will choose T so that

(3.16)
$$\frac{rTe^{-rT}}{1-e^{-rT}} = \frac{-rTh'(rT)}{h(rT)} = \left(\frac{N_1'T}{N_1}\right)\frac{N_1}{N_1 + \alpha N_2}$$

where $h(x) = 1 - e^{-x}$. The elasticity of h must equal the elasticity of

N_1 times the ratio of N_1 to total "weighted" labor input, where the weight on N_2 is given by (3.14). In the simple Wicksell model, we saw that lower rates of interest are always associated with longer lived machines. Now, because as the interest rate decreases, the capital costs also decrease, the opposite may also be true, and indeed, it is even possible for there to be "reswitching" of techniques, i.e. a given durability will be optimal for two different interest rates with another durability optimal for an intervening interest rate. Thus, we observe that

$$(3.17) \qquad \frac{dT}{dr} \sim \frac{1 - rT - e^{-rT}}{1 - e^{-rT}} + \frac{rN_2\alpha'}{(N_1 + \alpha N_2)}$$

The first term is always negative,[26] the second always positive, and there is an apparent ambiguity in the sign of dT/dr .

Consider, for instance, the following example

$$N(u, T) = \begin{cases} T & 0 \geq u \geq -1 \\ .093 & -1 \geq u \geq -5 \\ 0 & \text{otherwise} \end{cases}$$

Then, when $r = 1$, $T = 1$, since $\alpha = 7.6$,[27]

$$\frac{r}{e^{rT} - 1} = \frac{1}{1.71} = \frac{1}{N_1 + \alpha N_2} = \frac{1}{1 + .71} \cdot$$

(3.17) is positive, since, when $rT = 1$

$$\frac{1 - e^{-rT} - rT}{1 - e^{-rT}} = \frac{-1}{e^{rT} - 1} = -\frac{1}{1.71} = -.59$$

while, since $\alpha' = 19.8$

$$\frac{rN_2\alpha'}{N_1 + \alpha N_2} = \frac{19.8 \times .093}{1.71} = 1.087 > .59$$

It is also apparent that consumption need not be monotonic in T :

(3.18)
$$C = \frac{T}{N_1 + N_2 + T}$$

(3.19)
$$\frac{dC}{dT} = \frac{N_1 + N_2 - N_2'T}{(N_1 + N_2 + T)^2} .$$

The first order condition for optimal durability only guarantees that

(3.20)
$$N_2 + \alpha N_1 > N_2'T$$

If $\alpha > 1$, then (3.20) can be satisfied and $N_1 + N_2 < N_2'T$, so $dC/dT < 0$. Conversely, a sufficient condition for C to be a monotonically increasing function of T is that[28] $\alpha \leq 1$.

This model makes clear why the consumption per man "perversity" is no paradox at all. Consumption per man depends simply on the total labor requirements per machine year:

(3.21)
$$\frac{N_1 + N_2}{T}$$

while the choice of technique depends on weighted labor input (weights given by the intertemporal prices) and the weighted returns (i.e., not T, but $\frac{1 - e^{-rT}}{r}$).

The important point to observe is that the first and second order conditions imply constraints on the weighted labor input per unit of value output and its changes, but not necessarily on the unweighted expression (3.21).

4.1. Wicksell, Reswitching and Neoclassical Doctrine

Should Wicksell have been upset if he had discovered that these "perversities" could in fact occur? Does it have any serious implications for neoclassical doctrine? I think the answer to both of these questions is no. Wicksell made two related errors in attempting to interpret the "paradox": (1) He confused comparisons of steady states with truly dynamic paths (Mrs. Robinson, in spite of her constant warnings to others not to fall into this sinister trap, seems to have fallen into it herself). (2) He confused savings in value terms with savings in real terms. "True" neoclassical doctrine asserts the following two propositions: (a) Foregoing consumption today will allow the economy to take increased consumption some time in the future, and the marginal rate at which consumption today may be transformed into consumption tomorrow is given, in a competitive economy, by the rate of interest. More formally, consider an economy with an initial endowment vector of capital goods, X ; then the consumption possibilities of that economy may be represented by[29] (for simplicity we assume there is a single consumption good)

$$(4.1) \qquad c^0 = \emptyset(c^1, \ldots, c^t, \ldots; X_1, \ldots, X_n)$$

where c^t is consumption at time t . Neoclassical doctrine asserts that

if r_t is the rate of interest between the t and $t+1$ period, and p_{X_i} is the competitive price of the ith capital good, then

$$(4.2) \qquad \frac{\partial c^1}{\partial c^0} = \frac{\partial c^1/\partial X_i}{\partial c^0/\partial X_i} = \frac{\partial c^1/\partial X_i}{p_{X_i}} = (1 + r_0) > 0$$

$$\frac{\partial c^t}{\partial c^0} = \prod_{i=0}^{t-1} (1 + r_i) = \frac{\partial c^t/\partial X_i}{p_{X_i}} > 0$$

The competitive rate of interest is equal to the marginal rate of transformation between consumption (in the relevant periods) and is equal to the marginal physical product of the ith capital good divided by the price of the ith capital good. (b) There is a diminishing marginal rate of transformation:

$$(4.3) \qquad \frac{\partial(\partial c^1/\partial c^0)}{\partial c^0} \le 0 , \qquad \frac{\partial(\partial c^0/\partial X_i)}{\partial X_i} \le 0$$

These fundamental neoclassical propositions are very different from the propositions with which the reswitching, capital valuation and consumption per man perversities are concerned. The latter consider steady states, i.e. situations where $C_0 = C_1 = C_2 = \ldots\ldots\ldots$ and let $\underset{\sim}{X}$ be that endowment of capital goods which will sustain the given steady state. (For a discrete technology with one consumption good, there are only a finite number of such steady states.) If the transformation function is differentiable, each steady state will be characterized by an interest rate, so that we can write unambiguously,

(4.4) $$C(r) = \emptyset(C(r), \ldots, C(r); \underset{\sim}{X}(r))$$

Consider a problem such as that discussed by Wicksell in which V_K and C are continuous functions of r . Wicksell drew attention to the fact that, if we define $V_K(r) = \sum_i p_{X_i}(r)X_i(r)$ where $X_i(r)$ is the equilibrium price of the ith capital good in consumption good numeraire when the interest rate is r , then

(4.5) $$\frac{dC(r)/dr}{dV_K(r)/dr} = \frac{dC}{dV_K} \neq r \; .$$

But (4.5) is very different from (4.2), even though it has certain super-ficial similarities. Indeed, the valuation perversity is just the extreme case of (4.5) where

$$\frac{dC}{dV_K} < 0$$

It would be foolish to suggest that because the value of capital is lower in one steady state than another, no consumption need be foregone in going from one steady state to another; yet, remarkably enough, Pasinetti [1969] (among others) have actually suggested that the "cost" of going from one steady state to another be measured by the change in the value of capital.[30]

The Wicksell effects discussed above are simply a reflection of the fact that in different steady states, the price vector and the capital vector are different. To isolate the two effects we might compare the value of capital in the two steady states at the same price system; a natural price system to use is that at which the two corresponding technologies

are equally profitable (i.e. at the switch point). But at a switch point, as we have already noted, in a stationary economy,

$$(4.6) \qquad \frac{\partial C}{\partial V_K} = (1+r)$$

But (4.6) conveys little if any more information than (4.5); it does not tell us whether the transition from one steady state to another is even feasible (as it will not be if the second technology requires a capital good which cannot be produced by the capital goods of the first technology); when it is feasible, it is clear that making the transition may well entail increases in consumption in some periods and decreases in others. Indeed, it would appear that comparing consumptions, values of capital, etc. across steady states conveys little if any information about the true consumption opportunities available to an economy.

It should be noted that in the one sector Solow-Swan model, there can be no Wicksell effects (the price of capital goods in terms of consumption goods is fixed at unity) and hence these valuation perversities cannot occur. But clearly neoclassical analysis does not depend on this assumption; to suggest that is to suggest that Wichsell, Uzawa, Meade, and indeed Solow and Samuelson in their earlier articles are not neoclassical.

Similarly, the consumption per man perversity represents a confusion between steady state analysis and true dynamics, and between special properties of the simple Solow-Swan model and the properties of more general neoclassical models. Observe that (4.3) may be rewritten as (using (4.2)), $\partial r_0 / \partial c^0 < 0$, or, somewhat informally,

(4.7)
$$\frac{\partial c^0}{\partial r_0} < 0$$

higher interest rates between this period and next correspond to lower values

of consumption this period (keeping consumption in periods 2 and after an

endowment vector of capital goods constant; other interest rates and prices

will, of course, also be changing as r_0 changes). The consumption per

man perversity is that, along steady states

(4.8)
$$\frac{dC(r)}{dr} > 0$$

But again (4.8) is very different from (4.7). In particular, as we change

r in one case, we are changing all the capital endowments. Thus (4.8)

does not constitute a violation of the law of diminishing returns, as it

would appear at first sight.[31]

4.2. Differentiability and Reswitching

In the original examples of reswitching, there were always a finite

number of technologies. This led to the conjecture that if there were "enough"

processes in each (or in any) industry, then reswitching could not in fact

occur. These results were formalized in the theorems presented in Section

2.6. Far weaker conditions than differentiability of the transformation

surface were required to rule out reswitching. On the other hand, the example

of Section 3.3 showed reswitching could occur even if there were an infinite

number of techniques; "differentiability" by itself is not sufficient to

rule out reswitching.

It should be clear that there is in fact no contradiction between our theorems and our example. Our theorems required, for instance, that when one capital good was substitutable for another, it was substitutable in both directions (i.e. we could increase the first and decrease the second, or decrease the first and increase the second); we required, in other words, that the relevant capital goods be used in the given technology in strictly positive amounts. In our example, labor alone produces machines, and machines produce consumption goods with labor. The application of different amounts of labor results in the production of different kinds of machines (for each durability of machine is really a different kind of machine) in a smooth, differentiable manner. But in any technology, only one type of machine is employed, and it is not employed in its own construction.

4.3. Reswitching and Neoclassical Distribution Theory

Our example also serves to clarify the relationship between the re-switching phenomenon and neoclassical-marginalist distribution theory. When, as in the case of the propositions just presented, marginal products are well defined, competitive prices will be equal to those marginal products. When there are a discrete number of technologies, or, as in our example, there are an infinite number of technologies, but in each not every (capital) good is produced, there may be several sets of prices which correspond to (i.e. will support) any particular path of the economy. Which price system will be chosen will depend on the preferences of consumers (for instance, on their time preferences).

The lack of smooth substitutability in the economy--even in our example,

with an infinite number of technologies, there is only limited substituta-
bility--does have some implications for the concept of the rate of return;
for in the absence of such substitutability, it may not be possible to reduce
consumption this period, increase it next period, keeping consumption in all
future periods unchanged and the economy at full employment. Consider, for
instance, an economy in steady state with technology A , and consumption
C_A^* , and consider an efficient path which begins with the steady state endow-
ment corresponding to technology A and eventually reaches steady state using
technology B , with steady state consumption C_B^* . Let $\hat{C}^t$ be the consump-
tion in the transition periods $t = 1, \ldots, T$. In general, it will require
more than one period to make the transition $(T \geq 2)$, and $\hat{C}^t - C_A^*$ will
be negative some periods, positive others.

We can define the internal rate of return as that rate of discount
for which

$$(4.9) \qquad \sum_0^\infty C_A^* \left(\frac{1}{1-\delta}\right)^t = C_A^*/\delta = \sum_0^T \hat{C}^t \left(\frac{1}{1-\delta}\right)^t + \sum_{\pi+1}^\infty C_B^* \left(\frac{1}{1-\delta}\right)^t$$

We can show[32] that if $\hat{C}^t$ is an efficient path along which all capital
goods and labor are fully employed, if δ_i is the ith solution to (4.9)
either (a) the rate of return is equal to the rate of interest at a switch
point, $\delta_i = r^*$, or (b) at $r^* = \delta_i$, there is some other technology
whose steady state prices are less than those of A or B (i.e. if A
and B were the only technologies, then it would be a switch point, but
there is some other technology D which at the given interest rate, dominates
A and B). Thus, the fact that there may be many switch points is simply

a reflection of the fact that (4.9) may have several solutions (i.e. that $\hat{c}^t - c_A^*$ will be negative some periods, positive others).[33]

Out of steady state, the (internal) rate of return has no natural economic interpretation, and we must return to the use of intertemporal prices (interest rates) to evaluate the consequences of any change.[34]

4.4. Other Issues in the Analysis of Steady States

Much of recent growth literature has focused on issues other than those discussed in the preceding sections. In this sub-section, we comment briefly on the relationship between the reswitching phenomonon and two of the more widely discussed issues in growth theory:

(a) Existence of balanced growth with full employment (equality of warranted and natural rates of growth). The existence of many capital goods (and consumption goods) means that even if there were no substitution in techniques, the ratio of the value of capital to the value of output would change as the rate of interest changes, so that the natural and warranted rates of growth could be brought into equality by price adjustments. The reswitching phenomenon has, of course, nothing to say on this issue, but it does have implications for

(b) Uniqueness of balanced growth, since the pseudo production possibilities schedule of steady states will not be concave and monotone. But neither is it even in the two sector neoclassical growth model or in the Wicksellian model. Uniqueness of balanced growth equilibrium when a constant fraction of income is saved depends on the value of net output per unit of value capital being a monotone concave function of capital per man (in steady state), and as we have argued, above, there is no particular reason such value constructs will be "well behaved."

This means, of course, that the simple parables of accumulation, such as those discussed by Solow and Swan, are not likely to be valid in more general models, particularly when there are heterogeneous capital goods. And indeed, this has already been recognized in the works of Hahn, Cass, Shell, Stiglitz, and others.

PART II. Dynamics

5. The Meaning of Reswitching in a Dynamic Economy

5.1. It is apparent that the definition of reswitching given above in Section 2.1 will not be immediately applicable to truly dynamic situations. This may be viewed in several different ways:

(a) The preceding definition required that the choice of technique be independent of preferences among alternative commodities, i.e. that the non-substitution theorem obtain. But out of steady state, there is in general no non-substitution theorem.[35] Indeed, each of the inherited capital goods and labor in each of the periods may be considered separate primary factors (since they cannot be produced); since there is more than one primary factor, the non-substitution theorem will definitely not obtain. (See Mirrlees [1969], Stiglitz [1970c].)

(b) There is no longer any such animal as "the interest rate"; since relative prices of different capital goods are changing, there is an own rate of return for each capital good.

5.2. What is to be meant then by the reswitching of techniques in a dynamic context? There are two approaches that we could take:

(a) We could consider an economy which is "poor" (as measured by for instance the maximum level of sustainable consumption), and ask whether on its optimal development trajectory, it will use (or construct) machines of a particular type over one interval of time, another type over an intervening interval of time, and then return to the earlier technique. Thus, we can address ourselves to the fundamental question--the question which I take it both sides of the Cambridge dispute are really interested in (as opposed to the imaginary question of comparing islands in steady state equilibrium)--of what sense can we make of the Wicksell-Solow-Robinson neoclassical story of capital accumulation.[36] It need hardly be pointed out, of course, that this reswitching in time, which perhaps we should call recurrence, is a very different phenomenon than the reswitching of techniques along the factor price frontier. As will be clear, recurrence of techniques may occur in technologies which do not allow reswitching, and in technologies in which there is reswitching, there may be no recurrences. These questions are investigated in Section 7.

(b) Instead of characterizing the evolution of the economy along the optimal path, i.e. a path where savings are chosen optimally and where there is perfect foresight of future prices (of commodities and factors), we could characterize the path for a "descriptive" model of the competitive economy, e.g. where a constant fraction of income is saved. Again, we can ask whether it is possible for there to be recurrences in the choice of technique. This is discussed in Section 9.[37]

6. Accumulation in a Small Country Facing Fixed International Prices[38]

There is one situation where the neoclassical parable seems to have some general validity. Consider a small economy facing internationally given prices of all commodities, which are assumed to be constant and independent of its behavior. There is a perfect international market for used capital goods; a machine of type s of age u has a price p_{su}, in terms of our numeraire (say the first commodity). Consider a technique of production which requires, to produce one unit of commodity j, a_{0j}^k units of labor, a_{ij}^k units of commodity i, b_{suj}^k units of machines of type s of age u. The value of the output per man of the process is

$$(6.1) \qquad q^k = \frac{p_j + \sum_s \sum_u p_{s(u+1)} b_{suj}^k}{a_{0j}} ;$$

the value of durable capital per man of the process is

$$(6.2) \qquad V_D^k = \sum_s \sum_u p_{su} b_{suj}^k / a_{0j}^k$$

and the value of intermediate commodities per man is, as usual, simple

$$(6.3) \qquad V_I^k = \sum a_{ij}^k p_i / a_{0j}^k .$$

We can now easily derive the factor price frontier as the outer envelope of the straight lines defined by (6.4):

$$(6.4) \qquad w = q^k - (1+r)(V_D^k + V_I^k)$$

The "production function" may be similarly derived. Defining $V_K^k = V_D^k + V_I^k$, we plot for each process its total capital requirements per man,
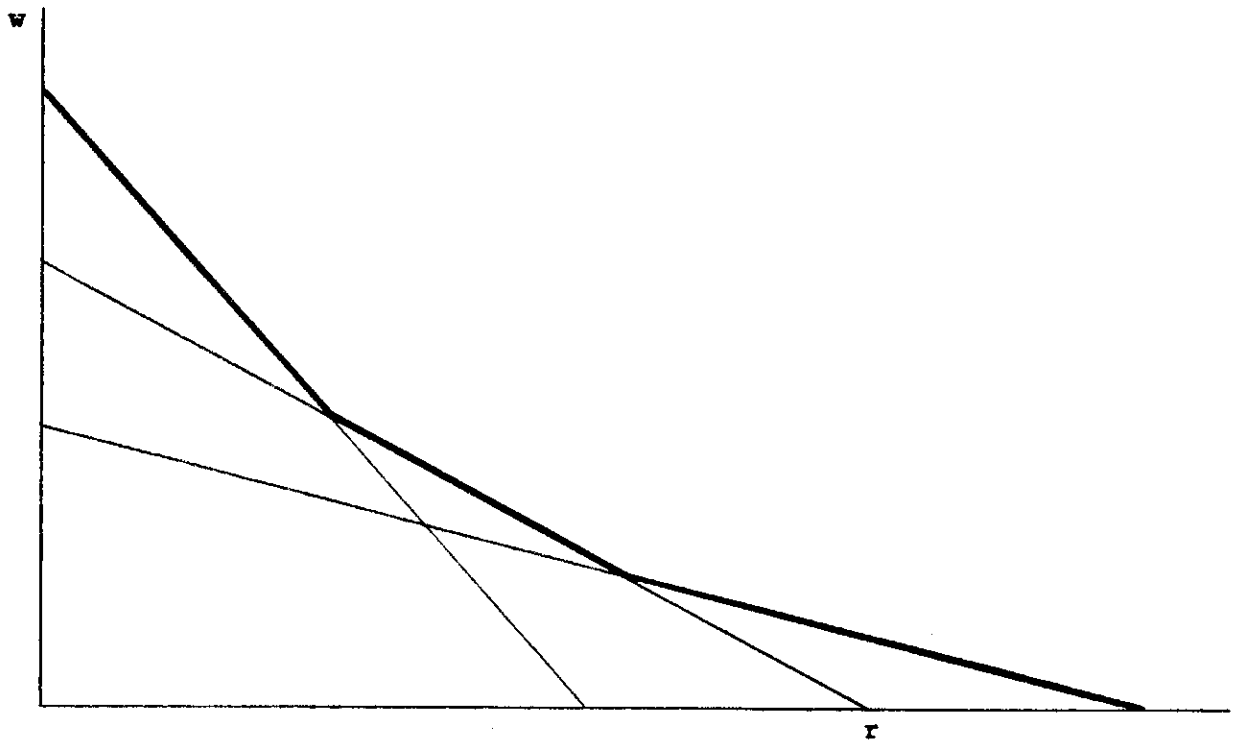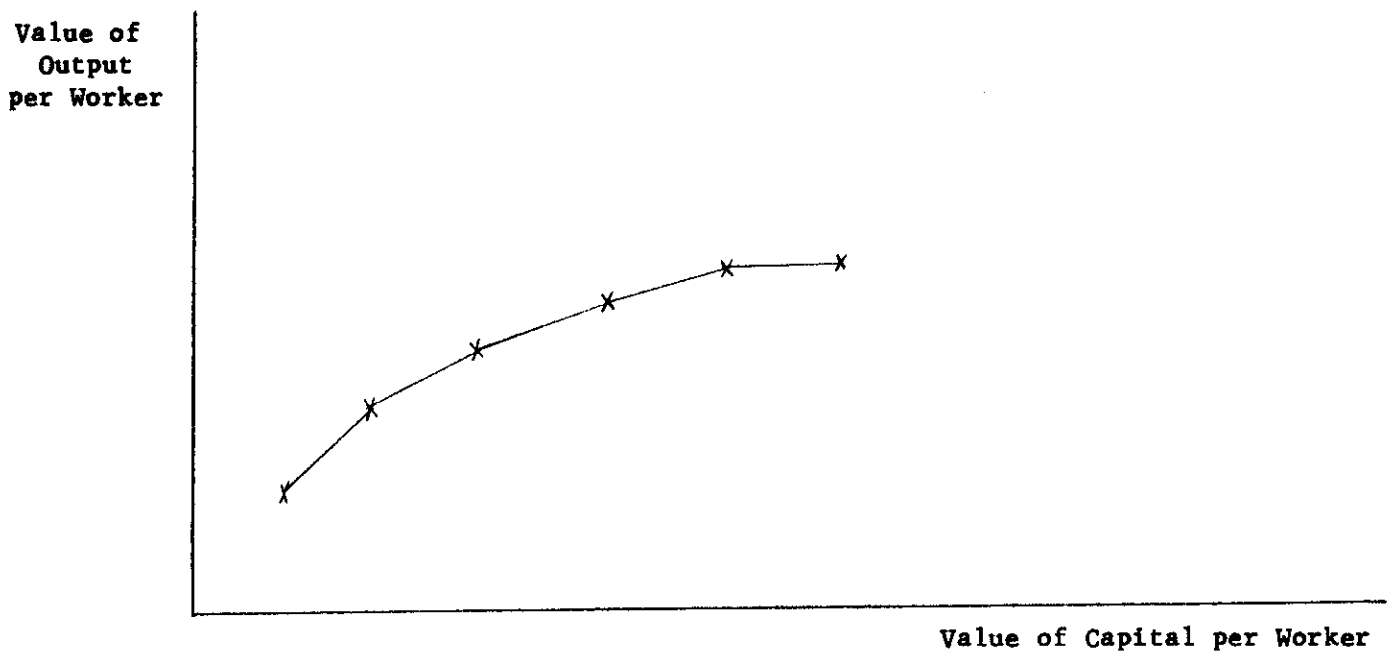
FIGURE 6.1

Value of Capital per Worker

FIGURE 6.2

v_K^k and its output per man, q^k, as in Figure 6.2. The production func-
tion displays the properties of a well-behaved neoclassical production func-
tion; at each corner point, only one industry is operated (with one process),
along straight line segments two industries are operated. For such an economy,
the process of accumulation consists of moving along the production possi-
bilities schedule, from less capital intensive processes and industries to
more capital intensive processes and industries.

7. The Wicksell Neoclassical Model of Capital Accumulation

The major salient features of the neoclassical model of capital ac-
cumulation may be summarized as follows:

(a) There is a monotonic increase in consumption per capita accompanied by

(b) A monotonic increase in the wage rate and fall in the consumption rate
of interest.

(c) The increase in capital may take one of several forms:

    (i) Machines may be made more durable.

    (ii) More productive machines may be constructed (i.e. machines which
        have a higher output per worker).

    (iii) Workers may move to a more capital intensive sector.[40]

The first two may be referred to as capital deepening; the third as capital
widening. The process of capital accumulation is characterized by steady
capital deepening, although not necessarily by capital widening.[41]

The reswitching possibilities alert us to the fact that the intro-
duction of machines having a higher output per worker may not be the appro-
priate indicator of capital deepening, that indeed no meaning can really be

attached to one technique being more capital intensive than another. But as we have suggested already the problems with the neoclassical story of capital accumulation are deeper--and quite apart from--those of reswitching.

7.1. The Two Sector Putty-Clay Model[42]

The simplest model in which to examine most of these issues is the two sector model in which capital is produced by labor alone, and consumption goods are produced by means of machines and labor. We shall focus only on the second kind of capital deepening; i.e. on the question of whether, on an optimal path of accumulation, the output per man on new machines is steadily increased.

We shall assume that a machine of type ℓ, when fully manned, which requires ℓ laborers, produces an output of $b(\ell)$. All machines require one man-year to be constructed and depreciate exponentially at the rate μ. ℓ may either be a continuous or discrete variable; in any case, $b(\ell)$ will be assumed to be a monotonically increasing concave function of ℓ. (If b is twice differentiable, $b' \geq 0$, $b'' \leq 0$.) See Figure 7.1a and 7.1b. This is equivalent to assuming that output per man is a declining function of ℓ.

7.2. Steady States[43]

If the wage rate were w (forever), then the technique which would minimize costs would be that for which
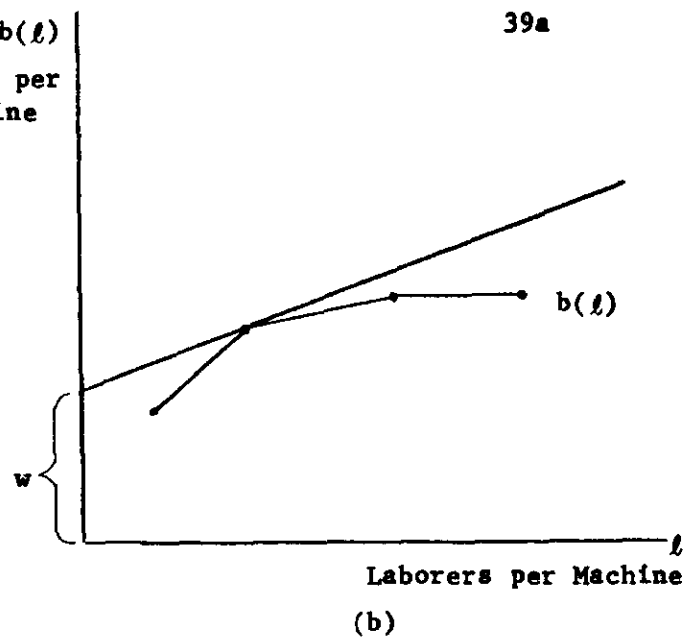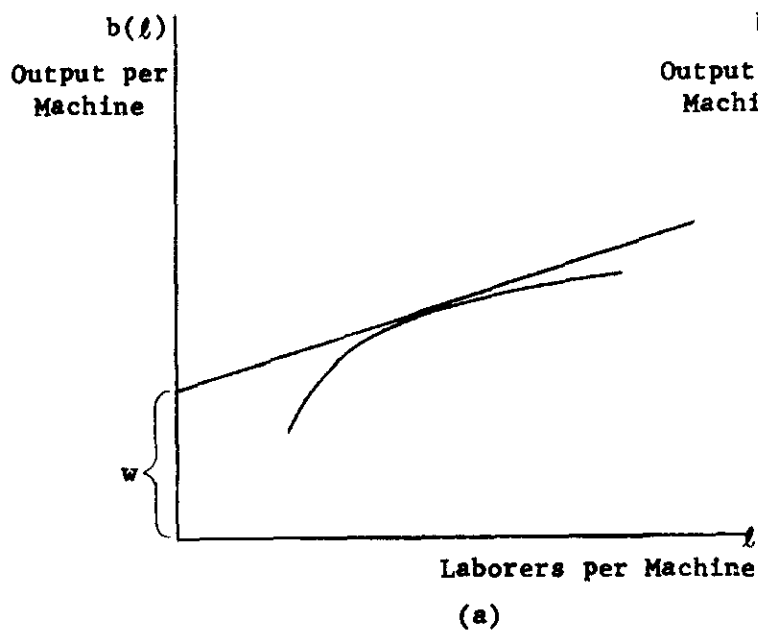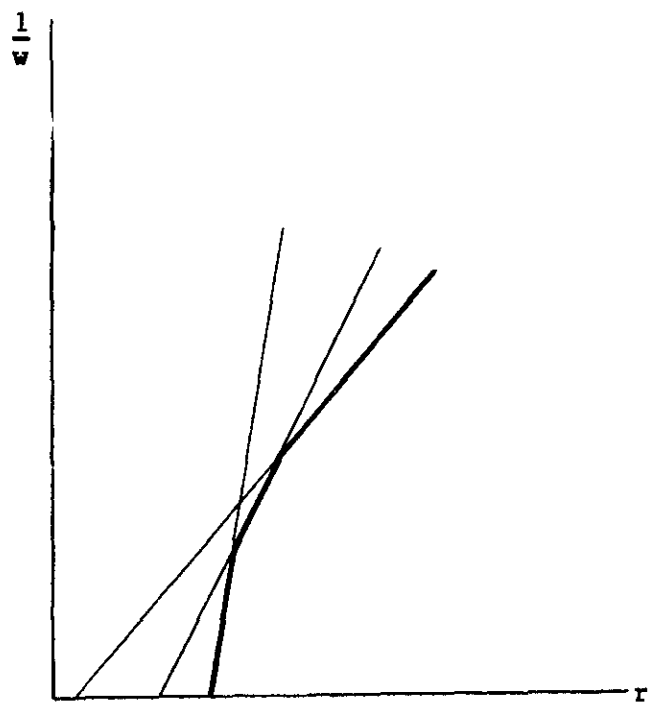
$$(7.1) \qquad b(\ell) - w\ell$$

b(ℓ)
Output per
Machine

w

Laborers per Machine

(a)

b(ℓ)
Output per
Machine

39a

b(ℓ)

w

Laborers per Machine

(b)

FIGURE 7.1

$\frac{1}{w}$

r

FIGURE 7.2

Factor Price Frontier

y

V_K

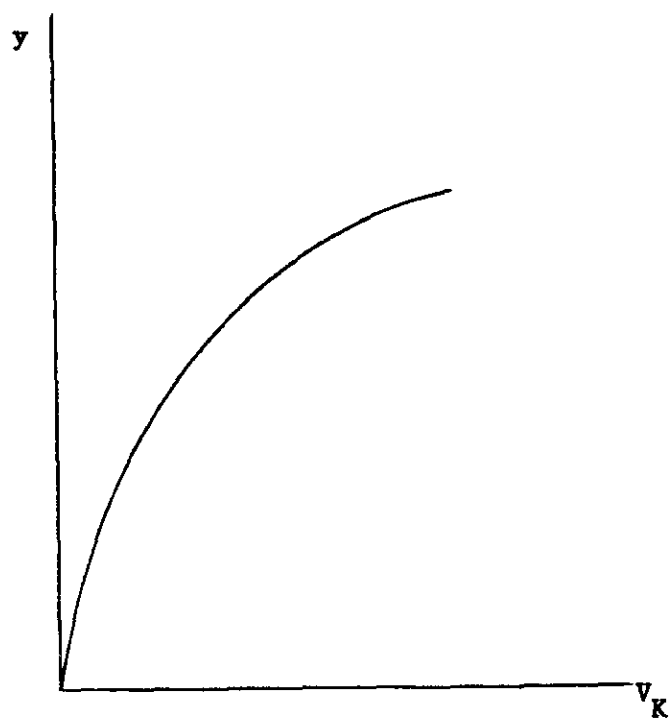FIGURE 7.3

The Pseudo-Production Function

is maximized, i.e.

(7.2) $b'(\ell) = w$

The competitive interest rate, i.e. that interest rate which would discount quasi-rents back to costs, is given by

(7.3) $r + \mu = \max_{\ell} \dfrac{b(\ell) - w}{w} = \dfrac{b(\ell) - \ell b'(\ell)}{b'(\ell)}$

which is just the marginal product of capital $b(\ell) - \ell b'(\ell)$ measured in labor numeraire. (7.1) and (7.3) define parametrically the factor price frontier. It is clear that it is "well-behaved," i.e. downward sloping. Unlike, however, the Samuelson [1962] example, where the capital intensities in the two sectors are identical, the slope of the factor price frontier,

(7.4) $-\dfrac{dw}{dr} = -\dfrac{dw/d\ell}{dr/d\ell} = \dfrac{w^2}{b} = \dfrac{w}{r+\ell}$

is not equal to the value of the capital stock per capita,[44]

(7.5) $V_K = \dfrac{w}{(n+\mu) + \ell}$

where n is the rate of growth of population, except if $n+\mu = r$, i.e. at the Golden Rule.[45]

On the other hand, the factor price frontiers corresponding to any two techniques can only intersect once. For a given technique, the factor price frontier is defined by

(7.6) $r + \mu = \dfrac{b(\ell)}{w} - \ell$

r is a linear function of $1/w$. See Figure 7.2.

The pseudo-production function, giving the value of net output corresponding to different values of capital (per capita) in steady state is given parametrically by (7.5) and

$$(7.7) \qquad\qquad y(\ell) = w(\ell) + rV_K(\ell)$$

It is easy to confirm that Y is a monotonically increasing, concave function of V_K, i.e. the pseudo production function is well behaved. (Figure 7.3)

7.3. Optimal Growth with a Linear Objective Function

The optimal trajectory of a socialist economy with the technology discribed in the previous subsections wishing to maximize the discounted value of consumption has been studied by Srinivasan [1962], Bruno [1967], and Stiglitz [1968]. A striking feature of the trajectory is that only one type of machine is ever constructed; machines of other capital intensities would be used if they happened to be around--if output per man on these machines were sufficiently high--but they would not be constructed on the optimal trajectory. Although the number of workers working in the consumption goods sector increases monotonically (capital "widening" occurs in a smooth way) output of consumption goods need not be monotonically increasing. These results have been shown to hold in the slightly more general Samuelson two sector canonical model, where even though these is reswitching along the frontier, only one type of machine is ever constructed (Bruno [1967]).

By comparison, in the analogous malleable capital good economy consumption is always monotonic. On the other hand, just as in the ex post

fixed coefficients economy, the wage rate and price of capital goods in terms of consumption goods and the capital labor ratio in the consumption goods sector are constant.[46]

7.4. These results depend on two crucial assumptions: (1) the linearity of the utility function and (2) the assumption that each process uses only one kind of capital good. When either of these are removed, the Wicksellian story begins to look even less plausible. Sub-section 7.5 considers the effects of imposing a minimum consumption constraint on the optimal growth trajectory, and section 8 presents a simple example of reswitching with a linear utility function.

7.5. Minimum Consumption Constraint

The easiest non-linearity to introduce into the utility function is to assume that there is a minimum consumption constraint. (This also removes the objection raised by Professor Robinson in her note [69].)

$$(7.8) \qquad \text{Maximize} \quad \int_0^\infty C e^{-\delta t} dt$$

subject to the constraint

$$(7.8a) \qquad C \geq \underline{C}$$

where $\underline{C}$ is the minimum level of consumption

and δ is the pure rate of time discount

The solution to this constrained maximization problem will involve

two stages, one in which the constraint is binding and the other in which it is not.[47] The latter is identical to that described above, and so we will limit ourselves to a detailed analysis of the former stage.

The value of national income (the Hamiltonian) will be given by

$$(7.9) \quad V = C + \lambda(C - \underline{C}) + \Sigma p(\ell, t)(N_I(\ell) - \mu K(\ell) + w(1 - \Sigma N_C(\ell) - \Sigma N_I(\ell))$$

where $p(\ell, t)$ is the (shadow) price of a machine of type ℓ at time t.

w is the shadow price of labor.

λ is the shadow price associated with the minimum wage constraint.

$K(\ell)$ is the number of machines of type ℓ.

$N_I(\ell)$ is the number of workers in the investment goods sector constructing machines of type ℓ.

$N_C(\ell)$ is the number of workers in the consumption goods sector working on machines of type ℓ.

N_I is the total number of workers in the investment goods sector.

N_C is the total number of workers in the consumption goods sector.

And as before,

μ is the exponential rate of depreciation.

For simplicity, we assume there is no growth in the labor force and the size of the labor force is normalized at unity.

The solution to the optimal growth problem can, as usual, be divided into two parts: the maximization of V given the shadow prices and the determination of the correct shadow prices.

Maximization of V requires that only the machine(s) with the highest value of $p(\ell, t)$ be constructed. We denote this type of machine with a caret:

$$(7.10) \quad p(\hat{\ell}(t), t) \geq p(\ell, t) \quad \text{all} \quad \ell$$

(7.11) $N_I(\ell) = 0$ $\ell \neq \hat{\ell}$

Machines with an output per man greater than $w/(1+\lambda)$ will be used to pro-
duce consumption goods. Since output per man is a monotonic decreasing func-
tion of ℓ, only those machines with the lowest ℓ are used. Denoting
by $\bar{\ell}$ the marginal machine used and by Z the proportion of those machines
used, we have

(7.12) $C = \underset{\ell < \bar{\ell}}{\Sigma} b(\ell)K(\ell) + Zb(\bar{\ell})K(\bar{\ell})$

(7.13) $N_C = \underset{\ell < \bar{\ell}}{\Sigma} \ell K(\ell) + Z\bar{\ell}K(\bar{\ell})$

(7.14) $w = (1+\lambda)b(\bar{\ell})/\bar{\ell} = p(\hat{\ell}(t), t)$

(7.15) $N_C + N_I = 1$

(7.16) $\lambda(C - \underline{C}) = 0$

(7.14) simply says that the wage rate is equal to the value of the output
per man on the marginal machine in use; (7.15) is the full employment con-
straint, and (7.16) says that either the constraint $C \geq \underline{C}$ is binding (so
$\lambda > 0$, and $C = \underline{C}$) or it is not (so $\lambda = 0$ and $C > \underline{C}$). Equations
(7.10)-(7.15) completely define the instantaneous allocation for the economy.[48]

The determination of the correct shadow prices is considerably more
difficult. At this point, it might be convenient to recall the values of
the various shadow prices in long run equilibrium (cf. Stiglitz [1968]).
In long run equilibrium, we have shown we choose the technique for which labor
costs per unit of output are minimized (taking account of depreciation and

time preference). The labor per unit output required directly in production on machines of type ℓ is $\ell/b(\ell)$ while it takes $\mu/b(\ell)$ workers to replace the depreciating machines. The total labor costs, taking account of the fact that the construction of the machines occurs before the output, is just

$$\frac{\ell}{b(\ell)} + \frac{\mu+\delta}{b(\ell)}$$

This is minimized where

(7.17)
$$\mu + \delta = \frac{b(\hat{\ell}^*) - b'(\hat{\ell}^*)\hat{\ell}^*}{b'(\hat{\ell}^*)}$$

where $\hat{\ell}^*$ is the machine constructed in long run equilibrium. Its price must equal the present discounted value of its quasi-rents

(7.18)
$$p^*(\ell^*) = \frac{b(\hat{\ell}^*) - \hat{\ell}^* w^*}{\mu + \delta}$$

as well as its cost of construction

(7.19)
$$p^*(\ell^*) = w^*$$

from which, using (7.17), we immediately derive

$$w^* = b'(\hat{\ell}^*) = b(\ell^*)/\ell^*$$

where ℓ^* is the marginal machine used in the long run equilibrium. (See Figure 7.4.)

Similarly, <u>at each moment of time</u>, we can calculate $p(\ell, t)$ as the present discounted value of the quasi-rents from the given type of machine:

$$(7.20) \quad p(\ell, t) \equiv \int_{A(\ell, t)} \{(1+\lambda)b(\ell) - \ell w\} e^{-(\mu+\delta)\tau} d\tau = \int_{A(\ell, t)} (1+\lambda)(b(\ell)$$

$$- \ell \left(\frac{b(\ell)}{\ell}\right) e^{-(\mu+\delta)\tau} d\tau \ .$$

where $A(\ell, t)$ is the set of dates at which the machine is used, i.e.

$$(7.21) \qquad A(\ell, t) \equiv \{\tau \mid \tau \geq t, \ b(\ell)/\ell > b(\ell(\tau))/\ell(\tau)\}$$

(at other times, the wage is sufficiently high that the machine is not used, i.e. it is, at least temporarily technologically obsolete.) Because $b(\ell)/\ell$ is a monotonically decreasing function of ℓ ,

$$A(\ell, t) \supseteq A(\ell + \Delta\ell, \ t) \ , \qquad \Delta\ell \geq 0$$

The machine which is constructed has the maximum p ; if $b(\ell)$ is twice differentiable

$$\frac{\partial p(\ell, t)}{\partial \ell} = \int_{A(\ell, t)} \{b'(\ell) - (b(\ell)/\ell)\}(1+\lambda)e^{-(\mu+\delta)\tau} d\tau = 0$$

or

$$(7.22) \qquad b'(\ell) = \frac{\int \frac{w}{1+\lambda} (1+\lambda)e^{-(\mu+\delta)\tau} d\tau}{\int (1+\lambda)e^{-(\mu+\delta)\tau} d\tau}$$

The marginal product of labor along the ex ante production function must equal the weighted average real wage over the life time of the machine.

Unfortunately, $p(\ell, t)$, viewed as a function of ℓ , may have several extremals. (See Figure 7.5.) Even for this well behaved technology, because of economic obsolescence, the present discounted value per dollar invested in a machine of type ℓ is not a "well-behaved" function of ℓ . (For
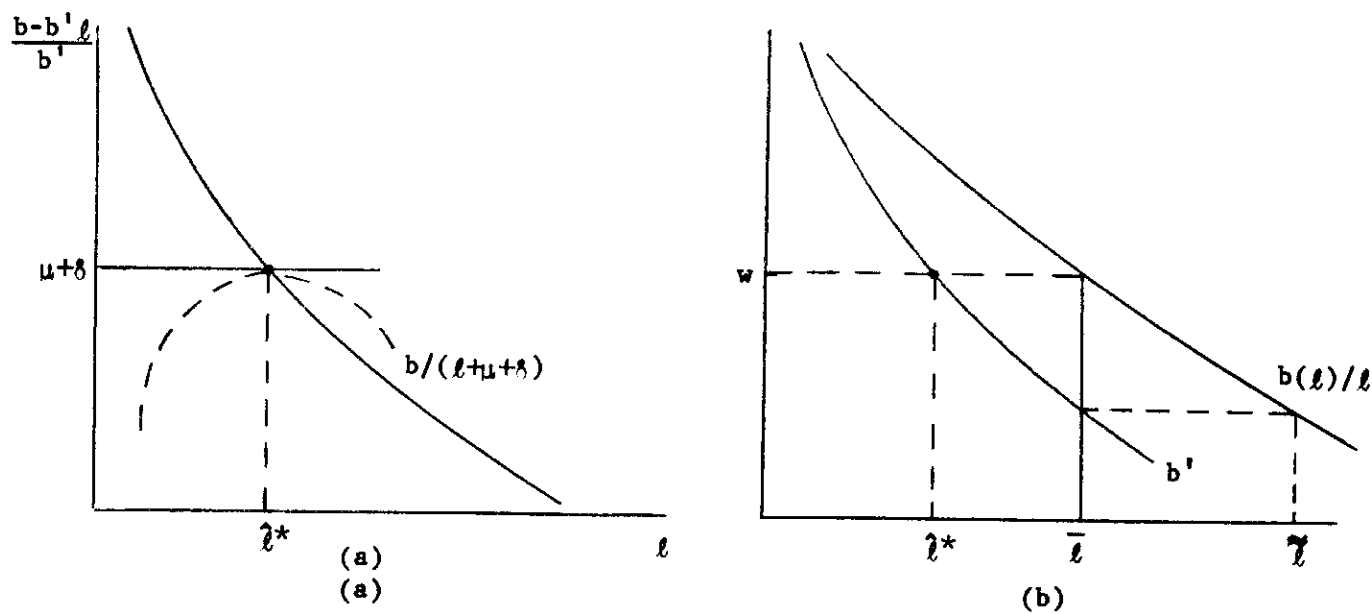
FIGURE 7.4

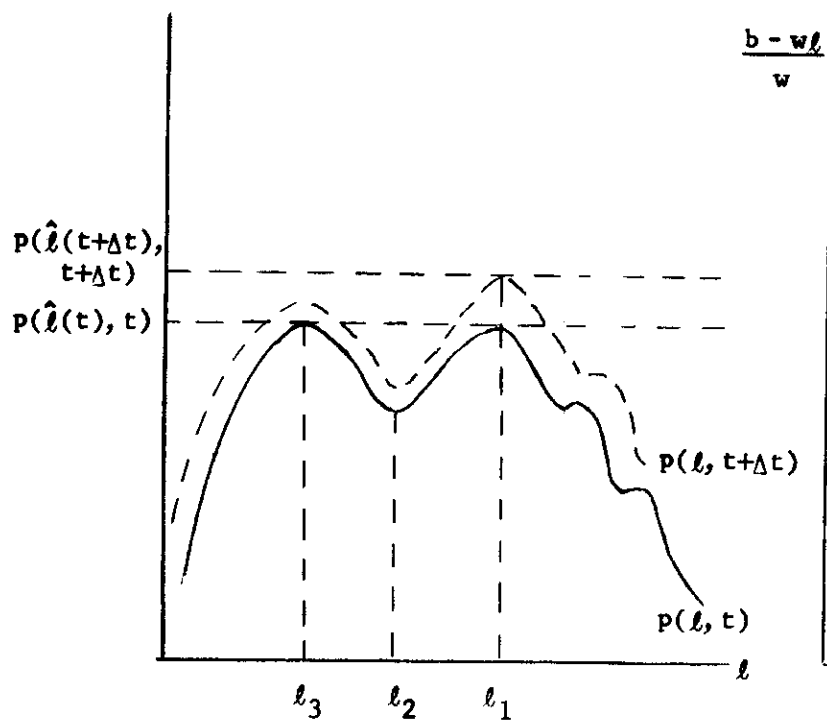Determination of Choice of Technique in Long Run Equilibrium

FIGURE 7.5

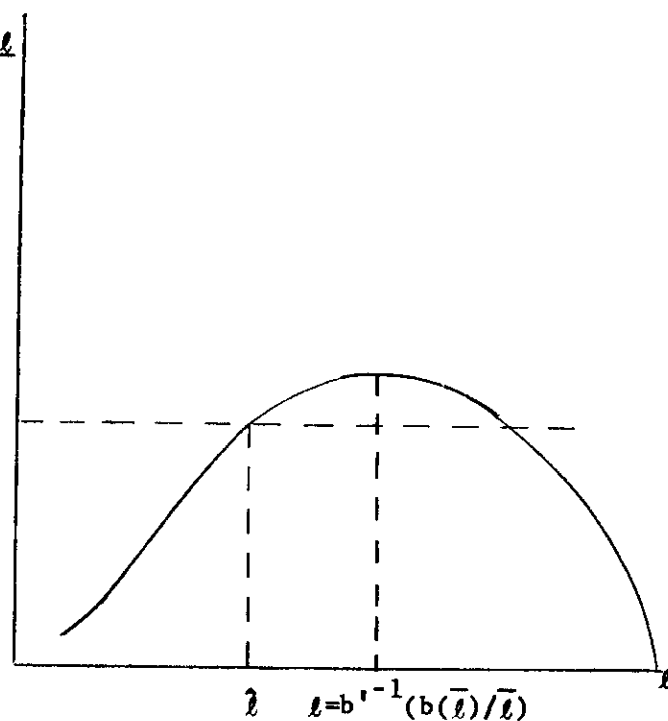Present Discounted Value of Quasi-Rents

FIGURE 7.6

Quasi-Rents on Newly Constructed Machines
(per dollar invested)

similar results, see Bliss [1968], Cass-Stiglitz [1969].) Discounted quasi-rents for $\ell + \Delta \ell$ are smaller than those for ℓ over the period which they are both used, but machines of type ℓ will be used longer.[49] By the same argument, since it can be shown that $\overline{\ell}$ will never fall below $\overline{\ell}^*$ (i.e. the wage will never rise above w^*)[50] $p(\ell, t)$ has at most one local maximum in the region $\ell < \overline{\ell}^*$. This information about $p(\ell, t)$ as a function of ℓ implies that

(i) more than one type of machine may be constructed at any moment of time;

(ii) there may be discontinuities in the choice of technique; that is, if the $p(\ell, t)$ function looks as depicted in Figure 7.5 at two nearby moments of time, machines of type ℓ_1 are constructed at one time, of type ℓ_3 at another, but an intervening type $\ell_2, \ell_3 < \ell_2 < \ell_1$ is not constructed at any intervening moment of time;[51]

(iii) only one type of machine with $\ell < \overline{\ell}^*$ is constructed at any moment of time and in the region $\ell < \overline{\ell}^*$ the choice of technique is continuous over time in the sense that if $\hat{\ell}(t) < \overline{\ell}^*$ and $\hat{\ell}(t + \Delta t) < \overline{\ell}^*$, for Δt sufficiently small, $|\hat{\ell}(t + \Delta t) - \hat{\ell}(t)|$ must be arbitrarily small.

To ascertain further properties of the choice of technique, we must differentiate the pricing equations (7.20) to obtain

$$(7.23) \qquad \partial p(\ell, t) / \partial t = (\mu + \delta) p(\ell, t) - w \left(\frac{b(\ell) \overline{\ell}}{b(\overline{\ell})} - \ell \right)$$

((7.23) are just the usual differential equations for shadow prices.) If t^* denotes the time at which the economy switches into the second stage (when the minimum consumption constraint is not binding), then the differential

equations must satisfy the boundary conditions

$$(7.24) \qquad p(\ell, t^*) \equiv p^*(\ell) = \begin{cases} (b(\ell) - \ell w^*)/(\mu + \delta) & \ell \leqq \overline{\ell}^* \\ \\ 0 & \text{if } \ell > \overline{\ell}^* \end{cases}$$

If we compare the rate of change of the price of the machine being constructed at time t and that of some other machine,

$$(7.25) \quad \frac{\partial(\ln p(\ell, t))}{\partial t} - \frac{\partial(\ln p(\hat{\ell}, t))}{\partial t} = -\frac{\overline{\ell}}{b(\ell)} \left[\left(b(\ell) - \ell \frac{b(\overline{\ell})}{\ell} \right) \frac{p(\hat{\ell}, t)}{p(\ell, t)} \right.$$

$$\left. - b(\hat{\ell}) - \hat{\ell} \frac{b(\overline{\ell})}{\ell} \right]$$

it is clear that the only types of machines which can possibly be constructed at the subsequent moment are those with lower quasi-rents, i.e.

$$\frac{b(\hat{\ell}(t + \Delta t))\overline{\ell}}{b(\ell)} - \hat{\ell}(t + \Delta t) < \frac{b(\hat{\ell}(t))\overline{\ell}}{b(\overline{\ell})} - \hat{\ell}(t)$$

Diagramatically, this is depicted in Figure 7.6. The maximum value of $b(\ell) - \ell \frac{b(\overline{\ell})}{\overline{\ell}}$ is at the point where $b'(\ell) = b(\overline{\ell})/\overline{\ell}$. This means that if in any interval of time there are no discontinuities in the choice of technique,[52] at most two kinds of capital goods can be constructed in that interval. It also means that if the wage is expected to rise monotonically, in fact the only possible types of machines which can be constructed are those for which

$$b'(\ell) > b(\overline{\ell})/\overline{\ell}$$

which in turn implies that the capital intensity (output per man) of the
newly constructed machines must rise monotonically if the wage is monotonic.
In this case, clearly there can be no recurrences of techniques.

Unfortunately, nothing assures us that the wage will rise monotonically
along the optimal trajectory. Indeed, the following example illustrates
an economy in which along the optimal path wages are not monotonic and re-
switching (recurrence) does occur. The technology and the initial endow-
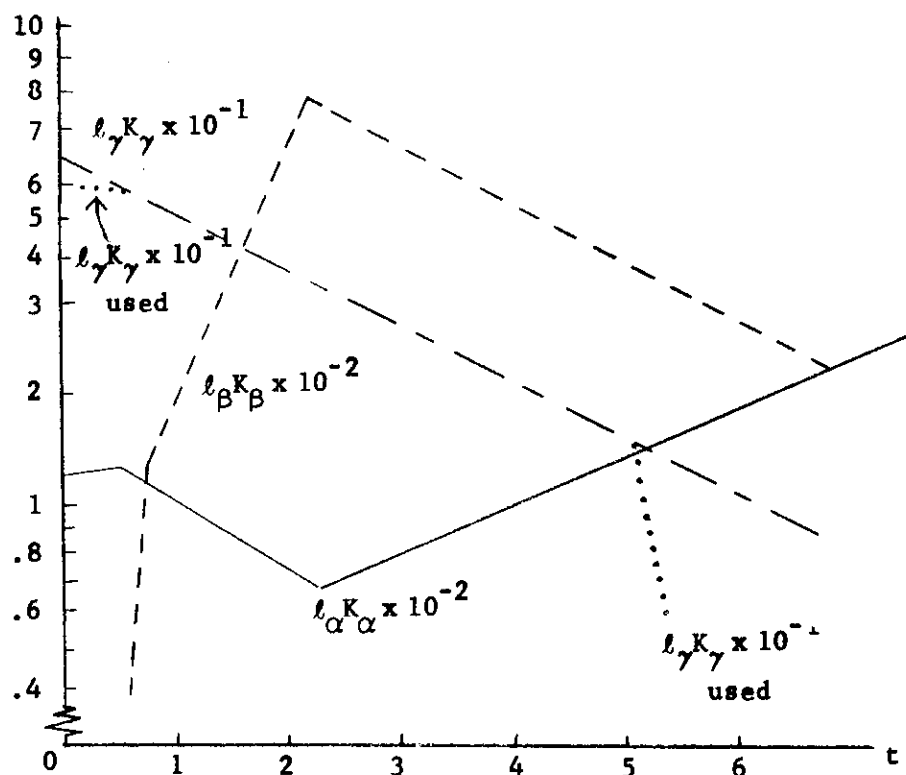ments are set forth in Table 1. [$\underline{C} = 1.5$, $\mu = .2$, $\delta = .102$]

TABLE 1

Type of Machine	Output per Machine $b(\ell)$	Output per Man $b(\ell)/\ell$	Initial Endowment of Machines
α	.50	50	1.22
β	.625	5	0
γ	.75	1.5	1.28
ϵ	.77	1.0	Not binding constraint

Phases of Development

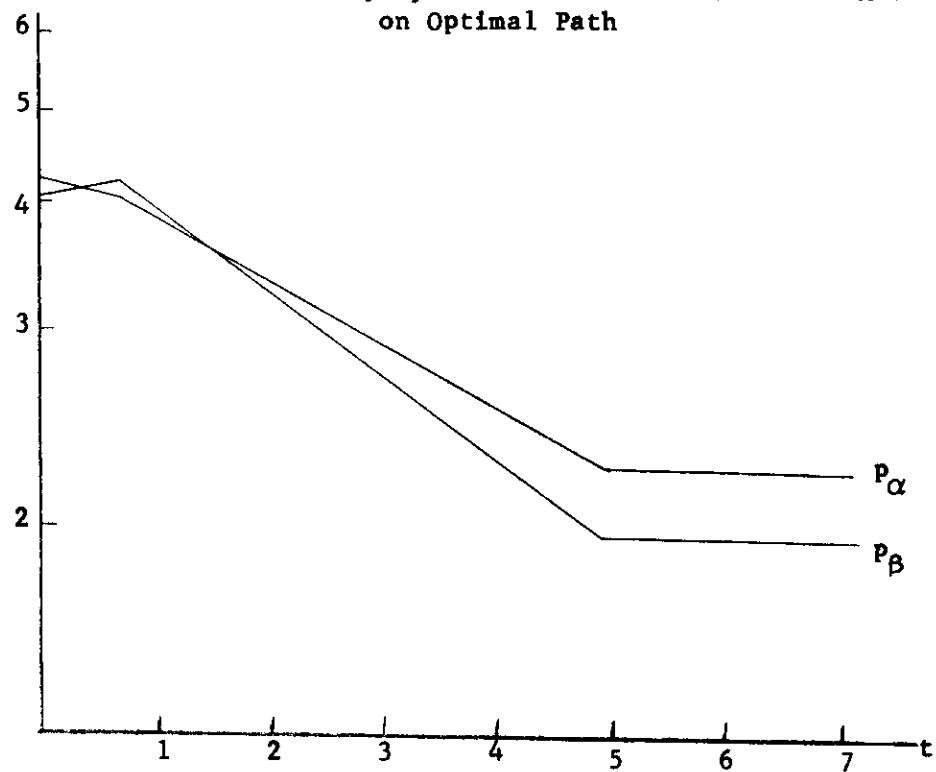| Phase	Approximate Duration	Type of Machine Constructed $(\hat{\ell})$	$\bar{\ell}$
1	.5	α	γ
2	.28	β	γ
3	1.35	β	ϵ
4	3.55	α	ϵ
5	1.1	α	γ
6	∞	α	β

(a)

FIGURE 7.7

Potential and Actual Employment on Different Kinds of Machines on Optimal Path

(b)

FIGURE 7.7

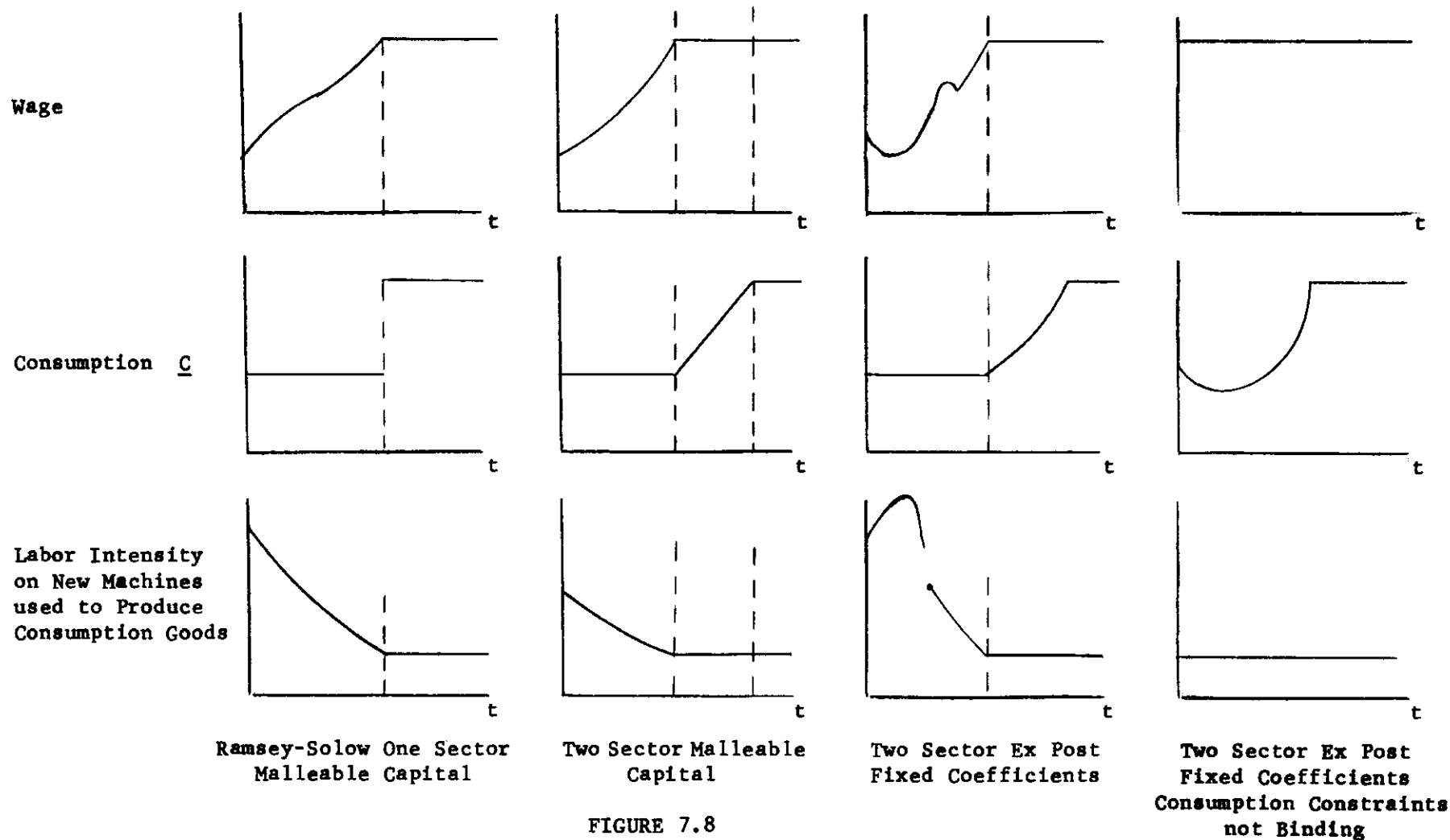Price of α and β Machines along Optimal Path

FIGURE 7.8

Comparative History of Some Economies Maximizing $\int Ce^{-\delta t}dt$ Subject to Minimum Consumption Constraint

Figure 7.7 shows diagrammatically the time paths of prices and capital stocks.
In the initial stage, type α machine is constructed, while in the next
stage type β is constructed, but eventually α is again constructed.

It should also be noted that, in this example, the rate of interest
(the own rate of return of consumption, or equivalently, the quasi-rent on
newly constructed capital, measured in consumption goods numeraire) falls
(when $\bar{\ell}$ falls), rises, and then falls again.

How can we explain these apparent anamolies? Under what circumstances
can they occur? As we noted above, we choose our technique so that the
marginal productivity of labor along the ex ante production function equals
the average wage over the life of the machine. It is possible that the
initial endowment of machines is such that the marginal machine used is
sufficiently capital intensive in relationship to the amount of labor free
to work in the capital goods sector so that, were machines of type
$\ell < b'^{-1}(b(\bar{\ell})/\bar{\ell})$ constructed, the output on those new machines would be
insufficient to replace the output lost from depreciating machines. Hence
the wage must fall, and as the wage falls, the capital intensity of the
newly constructed machines is reduced; but at the lower capital intensities,
the output on the new machines is greater than the output lost from depre-
ciating machines, and the wage rises.[53]

In Appendix A to this section, the conditions under which recurrences
can occur are set forth somewhat more rigorously. Two further properties
of the growth path of the economy are also established: So long as the
minimum consumption constraint is binding, the proportion of the labor force
working in the investment goods sector rises monotonically. There is a value

of N_I above which, even though the minimum consumption constraint is binding, the wage rate is monotonically rising so there can be no recurrences. Thus, recurrences are limited to the initial stages of development[54] (although in practice these may be the very stages in which it may be most important and most difficult to make the correct choices of technique).

8. Further Remarks on the Occurrence of Recurrences on Optimal Paths

The example of the previous section showed that recurrences could occur in (a) the type of new machine constructed and (b) some of the machines (processes) used in the consumption goods industry, i.e. a machine is used over one interval of time, becomes temporarily technologically obsolescent, and then is brought back into use. On the other hand, some of the processes (machines) continue to be used (if they are available) throughout the de-velopment program (i.e. those with very high output per man). Indeed, whenever capital is not malleable, it seems likely that some processes may be used all along the optimal trajectory. When, however, capital is malleable, it is possible that there be recurrences in the entire input-output matrix of the economy, as the following example illustrates.

There are two sectors in our economy. The first produces capital good 1 by means of labor and capital good 2; there are a number of different processes available. For simplicity, we shall assume that there are a continuum of techniques, described by the production function

(8.1) $\dot{K}_1 + \mu K_1 = F(K_2, L) = Lf(k_2)$ where $k_2 = K_2/L$

and μ is the exponential rate of depreciation of capital. F has constant

returns to scale. For simplicity, we shall assume the population is constant, and normalized at unity.

The second sector produces capital good two and the consumption good and uses only capital good one:

$$(8.2) \qquad \dot{K}_2 + \mu K_2 + C = \alpha K_1$$

Thus, the set of input-output matrices available to the economy is summarized by the following table:

Requirement per Unit of Output of	Sector 1 (produces K_1)	Sector 2 (produces C and K_2)
Labor	$\dfrac{L}{F(K_2, L)} = \dfrac{1}{f(k_2)}$	0
K_1	0	$1/\alpha$
K_2	$\dfrac{K_2}{F(K_2, L)} = \dfrac{k_2}{f(k_2)}$	0

Thus specifying k_2 completely specifies the technology used by the economy.

It is easy to show that for this economy there can be no reswitching of techniques along the factor price frontier (i.e. in steady states). Nonetheless, there may be recurrences in techniques even with the linear utility function. The government wishes to maximize

$$\int_0^\infty C e^{-\delta t} dt$$

where as before δ is the pure rate of time preference. It can be shown that during the initial stages of development $C = 0$. Hence, during this

phase, the dynamics of the economy are completely described by

$$\dot{K}_1 = f(K_2, 1) - \mu K_1$$

$$\dot{K}_2 = \alpha K_1 - \mu K_2$$

and are depicted in Figure 8.1. It is clear that if the initial endowments of K_1 and K_2 are both small and $K_1(0) < \mu K_2(0)/\alpha$, then it is possible that initially K_2 falls and then rises; accordingly we initially use successively less and less capital intensive techniques in the first sector ($k_2 = K_2/L$ decreases) and then more and more capital intensive techniques. It should be clear that similar results hold if the utility function is non-linear (e.g. there is a minimum consumption constraint).

The point of this and the preceding examples is that the reswitching phenomenon has nothing to say about the true dynamic behavior of the economy out of steady state. Indeed, the fact that the cost of using different kinds of capital goods includes capital gains (losses) as well as rentals makes the occurrence of recurrences all the more likely in a many capital good model; even were the real wage in consumption numeraire to increase monotonically, there clearly can be recurrences.[55]

9. Recurrences in Descriptive Models

9.1. Introduction

In this section, we continue our investigation of the dynamic behavior of the Wicksellian model of capital accumulation. We show that if we replace the sophisticated savings behavior of Section 7 (where the savings rate is chosen to maximize intertemporal utility) by the crude savings functions
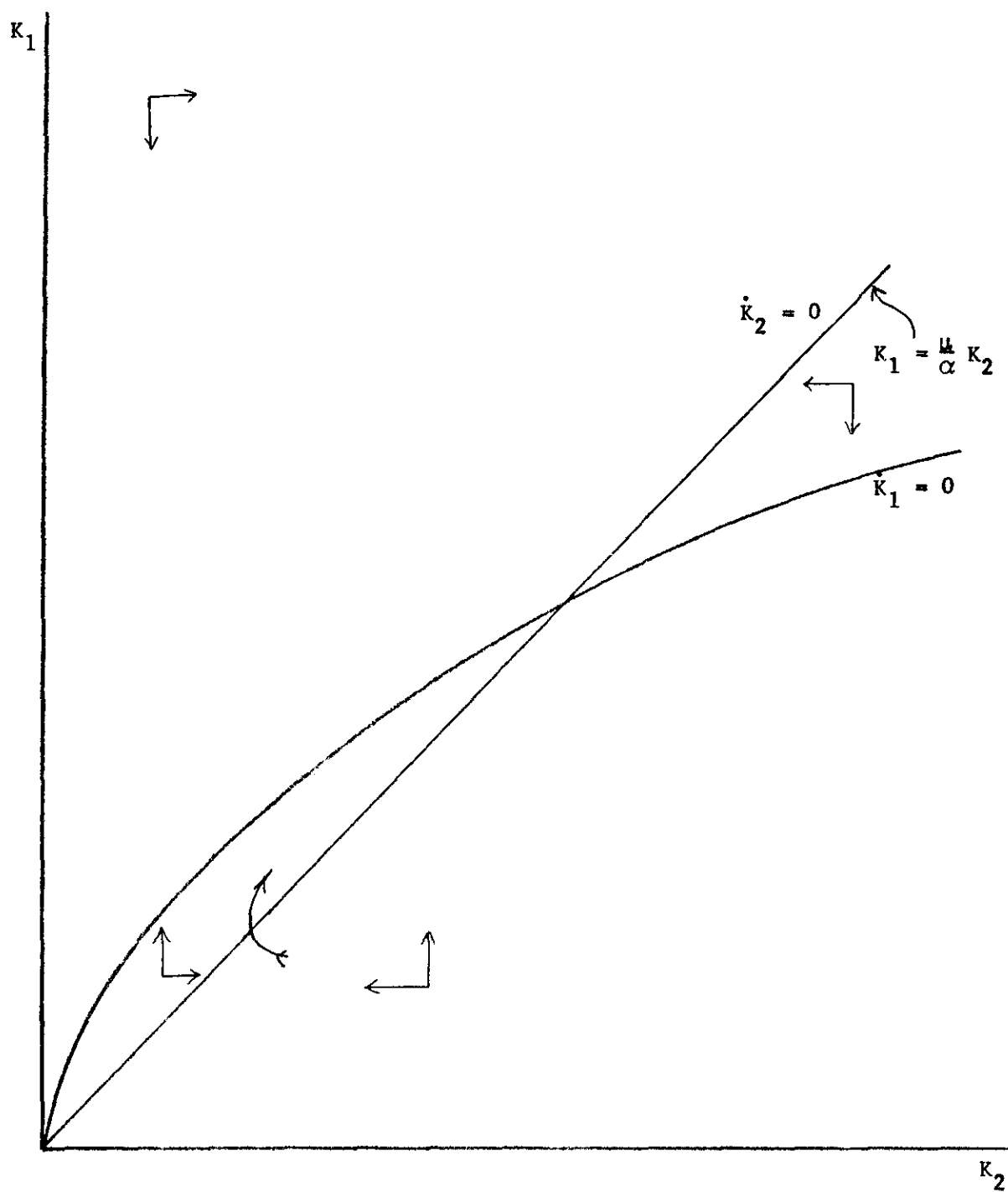
FIGURE 8.1

Recurrence in Input-Output Matrix along Optimal Path

common in descriptive models of economic growth, and if we replace the assumption of perfect foresight of future prices with alternative, perhaps more reasonable, expectations hypotheses, the economy is still likely to be badly behaved.

In Section 9.2, we assume that there are static expectations and that all of profits are saved but none of wages; although there are likely to be oscillations (in the wage rate, the allocation of labor between the two sectors, the choice of technique, output per man, etc.) all the oscillations are damped.

Sections 9.3 and 9.4 show that the stability observed in the previous subsection depended on the assumption that capital depreciates exponentially. We consider the alternative polar case of capital which lives only one period. Section 9.3 considers two simple savings rules: the Marxian savings assumption and the assumption, common in recent growth theory, that gross savings are a constant fraction of gross output. Section 9.4 considers the life cycle savings model with overlapping generations.

Although the assumption of one period capital goods is clearly unsatisfactory, Cass and I have been able to show that the qualitative results set forth here hold in more complex models with finitely lived capital goods. Indeed, it can be shown in such models that there also exist efficient oscillations with unemployment of, say labor, in alternate periods.

9.2. The Wicksell-Solow Model with Marxian Savings

We first recall the technology and notation described in Section 7. There are two sectors; machines are produced by labor alone; machines and

labor produce consumption goods. One man in one year can construct one machine. A machine of type ℓ produces an output (per year) of $b(\ell)$ and requires ℓ workers to man it. We shall assume that b is twice differentiable:

$$b'(\ell) > 0 , \quad b''(\ell) < 0$$

All machines depreciate at the constant exponential rate μ. For simplicity, we shall assume that there is no labor growth and no embodied or disembodied technological change. (Alternatively, we can interpret μ as the sum of pure labor augmenting disembodied technological change, the rate of depreciation, and the rate of growth of the labor force.) The labor force is normalized at unity.

$\hat{\ell}$ is the type of machine constructed at time t

$\overline{\ell}$ is the marginal machine used at time t

$\kappa(\hat{\ell}, t)$ is the number of machines of type $\ell \leq \tilde{\ell}$ at time t

$N_C(t)$ is the number of workers in the consumption goods sectors

$N_I(t)$ is the number of workers in the investment goods sector

$w(t)$ is the wage rate at time t

$\pi(t)$ are profits at time t

It is clear that in competitive equilibrium, the wage in consumption numeraire will equal the average product of a worker on the marginal machine in use:

$$(9.1) \qquad \frac{b(\overline{\ell})}{\ell} = w$$

Only the best machines (those with the highest output per man, i.e. lowest ℓ) will be used in the consumption goods sector. The output of consumption

goods, C , is thus given by

$$(9.2) \qquad C = \int_0^{\bar{\ell}} b(\ell) d\kappa(\ell, t)$$

Similarly, employment in the consumption goods sector is given by

$$(9.3) \qquad N_C = \int_0^{\bar{\ell}} \ell\kappa(\ell, t) = 1 - N_I$$

Profits (quasi-rents) are simply the difference between total output and wage payments in the consumption goods sector:

$$(9.4) \qquad \pi(t) = \int_0^{\bar{\ell}} (b(\ell) - w\ell) d\kappa(\ell, t)$$

To close the model we need two further behavioral assumptions (which distinguish it from the optimal growth models presented earlier). First, instead of assuming, as we did earlier, that there is perfect foresight (i.e. the planner knows prices at all times in the future), we shall assume that individuals have static expectations: the wage expected to prevail over the future is today's wage. Thus a profit maximizing firm maximizes quasi-rents in each period:

$$(9.5) \qquad \max \; b(\ell) - w\ell$$

i.e.

$$(9.6) \qquad b'(\hat{\ell}) = w$$

or

$$\hat{\ell} = b'^{-1}(w)$$

Second, instead of assuming that the savings rate is chosen to maximize intertemporal utility, we now assume that all of profits are saved and none of wages. Thus

$$(9.7) \qquad wN_I = \pi = \int_0^{\bar{\ell}} (b(\ell) - w\ell)d\kappa(\ell, t)$$

$$(9.8) \qquad w = C = \int_0^{\bar{\ell}} b(\ell)d\kappa(\ell, t)$$

The integral equations (9.7) and (9.8) with the associated equations defining w (9.1), and the choice of technique, $\hat{\ell}$, (9.6) completely describe the behavior of the economy. The analysis of the dynamics is, however, somewhat simpler if we use the difference-differential form of (9.7) and (9.8). To obtain these, we first observe that

$$(9.9) \qquad \frac{\partial \kappa(\ell, t)}{\partial t} = \begin{cases} -\mu\kappa(\ell, t) & \ell < \hat{\ell}(t) \\ -\mu\kappa(\ell, t) + N_I & \ell \geq \hat{\ell}(t) \end{cases}$$

The number of machines of any given type decreases exponentially except for the new machines. N_I machines of type $\hat{\ell}$ are constructed at time t . Differentiating (9.7) and (9.8), using (9.9) we obtain[56]

$$(9.10) \qquad \frac{d\pi}{dt} = \frac{dwN_I}{dt} = -\mu wN_I + (b(\hat{\ell}) - w\hat{\ell})N_I$$

$$(9.11) \qquad \frac{dC}{dt} = \frac{dw}{dt} = -\mu w + b(\hat{\ell})N_I + \dot{\ell}b(\bar{\ell})d\kappa(\bar{\ell}, t)$$

$$= \frac{-\mu w + b(\hat{\ell})N_I}{1 - (d\bar{\ell}/dw)b(\bar{\ell})d\kappa(\bar{\ell}, t)} \sim -\mu w + b(\hat{\ell})N_I$$

The interpretation of these equations is clear. (9.10) says that the change in profits (investment) is equal to the decrease in profits from depreciating machines plus the increase in profits from the new machines. (9.11) says that if the wage rate is to be unchanged, the decrease in output of consumption goods from depreciation $-\mu C$, must be exactly offset by the increased output from new machines, $b(\hat{\ell})N_I$.[57]

The value of investment, wN_I , is constant along the curve

$$-\mu w + b(\hat{\ell}) - w\hat{\ell} = -\mu b'(\hat{\ell}) + b(\hat{\ell}) - b'(\hat{\ell})\hat{\ell} = 0$$

In (wN_I, w) phase space, this is a vertical straight line. Increasing w decreases $\hat{\ell}$, and hence to the right of the $\dot{\pi} = 0$ curve, the value of investment is decreasing, to the left increasing. w is constant along the curve

$$wN_I = \frac{\mu w^2}{b(\hat{\ell})} = \frac{\mu b'(\hat{\ell})^2}{b(\hat{\ell})}$$

which is upward sloping. Since increasing N_I increases $\dot{w}$, above the curve $\dot{w}$ is positive, below it negative.

It is clear from Figure 9.1 that oscillations are possible if not likely. To see that these oscillations must be damped, consider the economy at some time t' with wage w' and profit π' , where $\dot{w}(t') = 0$, $\dot{w}'(t') < 0$. If the oscillation were periodic or undamped, we must sometime later, say at t'' , return to the wage w' with $\pi'' = \pi(t'') \leq \pi'$. We shall now show that this is impossible.

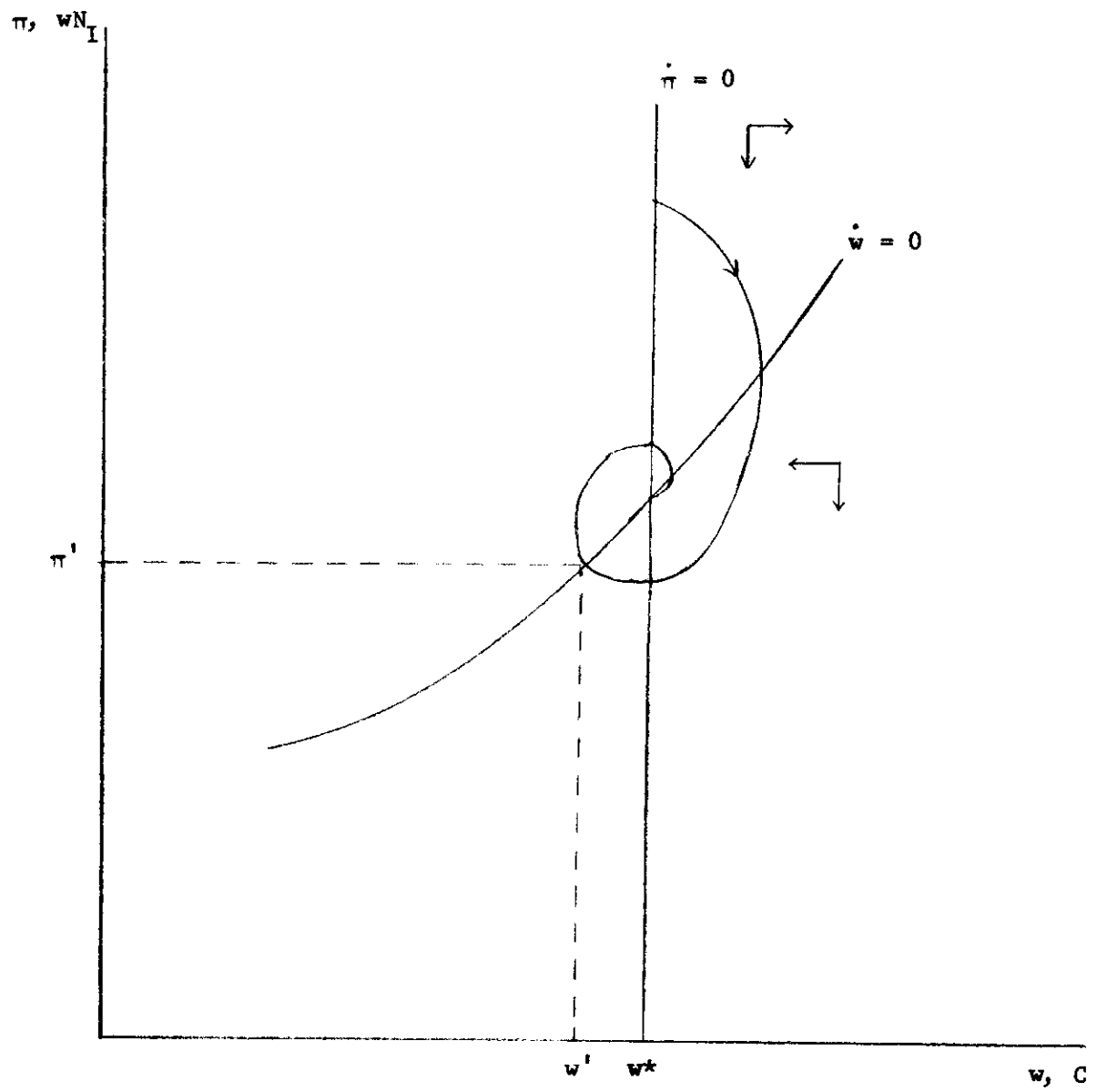When $\dot{w} = 0$, $\dot{\pi} > 0$, which implies that

FIGURE 9.1

Marxian Savings and Static Expectations

$$b - b'\hat{\ell} > N_I b = (1 - N_C)b$$

or

$$(9.12) \qquad N_C > b'(\hat{\ell}(t'))\hat{\ell}(t')/b(\hat{\ell}(t'))$$

Now

$$N_C(t'') = \int_0^{\bar{\ell}} \ell d\kappa(\ell, t'') \equiv \int_0^{\bar{\ell}} \frac{b(\ell)\ell}{b(\ell)} [d\kappa(\ell, t'') - d\kappa(\ell, t')e^{-\mu(t''-t')}]$$

$$+ \int_0^{\bar{\ell}} \ell d\kappa(\ell, t')e^{-\mu(t''-t')}$$

$$\leq \frac{\hat{\ell}(t')}{b(\hat{\ell}(t'))} \int_0^{\bar{\ell}} b[d\kappa(\ell, t'') - d\kappa(\ell, t')e^{-\mu(t''-t')}] + N_C(t')e^{-\mu(t''-t')}$$

$$= \frac{\hat{\ell}(t')}{b(\hat{\ell}(t'))} w(1 - e^{-\mu(t''-t')}) + N_C(t')e^{-\mu(t''-t')}$$

$$\leq N_C(t')$$

The first inequality follows from the fact that in the interval (t', t'') only machines of type $\ell < \hat{\ell}(t')$ were constructed. Hence

$$d\kappa(\ell, t'') = d\kappa(\ell, t')e^{-\mu(t'', t')} \qquad \ell > \hat{\ell}(t')$$

For $\ell \leq \hat{\ell}(t')$, $\ell/b(\ell) < \hat{\ell}(t')/b(\hat{\ell}(t'))$. The last inequality follows from (9.12) (recalling (9.6)). Finally, we note that the fact that $N_C(t'') \leq N_C(t')$ implies that $\pi(t'') \geq \pi(t')$. Hence oscillations must be damped.

9.3. A One Period Capital Model

It turns out, however, that the stability of the ex post fixed coefficients model described in the previous section depends crucially on the assumption that capital is infinitely lived (or decays only exponentially). To demonstrate this, we consider the other polar case, that where capital lives only one period. As in the preceding sections, we assume there is ex ante substitutability but ex post fixed coefficients and that capital is produced by labor alone. We shall assume, moreover, that population grows at the constant rate of n per period. In the ensuing discussion, all variables are expressed in per capita terms, as of time t .

Thus, if machines are to be fully employed, which they will be along equilibrium paths, the output of consumption goods is

$$(9.13) \qquad C(t) = b(\ell(t))K(t)$$

where $\ell(t)$ is the type of machine used at time t (constructed at time $t-1$) , and $K(t)$ is the number of such machines. Employment[58] in the consumption goods industry is

$$(9.14) \qquad N_C(t) = \ell(t)K(t)$$

Full employment of labor implies that

$$(9.15) \qquad N_I(t) = 1 - N_C(t)$$

so that the value of investment in consumption numeraire is

$$(9.16) \qquad w(t)N_I(t)$$

since investment goods require only labor to be produced. The number of machines per capita at time t is just equal to the number constructed per capita at t-1 divided by 1 plus the rate of reproduction.

$$(9.17) \qquad K(t) = \frac{N_I(t-1)}{1+n} = \frac{1 - \ell(t-1)K(t-1)}{1+n}$$

The type of machine constructed at t-1 is that which maximizes expected quasi-rents, i.e.

$$(9.18) \qquad \max \; b(\ell) - w^e(t)\ell$$

where $w^e(t)$ is wage expected to prevail at time t .

$$(9.19) \qquad b'(\ell(t)) = w^e(t)$$

Under the assumption of perfect foresight, we obtain

$$(9.20) \qquad b'(\ell(t)) = w(t)$$

Finally, we assume, as in the preceding subsection, that (gross) investment is equal to (gross) profits; i.e. wages are equal to consumption:

$$(9.21) \qquad w = C$$

Defining the variable

$$\alpha(\ell) = b'\ell/b(\ell)$$

(the share of labor along the ex ante production function) and substituting (9.13)-(9.20) into (9.21) we obtain the first order difference equation

describing the growth of the economy:

$$(9.22) \qquad \alpha\left(\frac{N_C(t)(1+n)}{1 - N_C(t-1)}\right) = N_C(t)$$

It is easy to show that there is a unique balanced growth path, that momentary equilibrium is uniquely determined, but nonetheless, the balanced growth path may be unstable. Local stability requires

$$(9.23) \qquad -1 < \left.\frac{dN_C(t)}{dN_C(t-1)}\right|_{N_C(t) = N_C(t-1) = N_C^*} = \frac{\dfrac{\alpha'\ell}{1 - \alpha}}{1 - \alpha'\ell/\alpha} < 1$$

Letting σ be the elasticity of substitution along the ex ante production function,

$$\sigma = \frac{d \ln \ell}{d \ln (b'/b - b'\ell)}$$

we obtain the result that a necessary and sufficient condition for local stability is that

$$(9.24) \qquad \sigma > 1 - \frac{1}{2\alpha}$$

Thus, if the share of labor is sufficiently large relative to the elasticity of substitution, the balanced growth path is unstable. Note that if $\sigma < 1$,

$$\lim_{N_C(t-1) \to 1} N_C(t) = 0$$

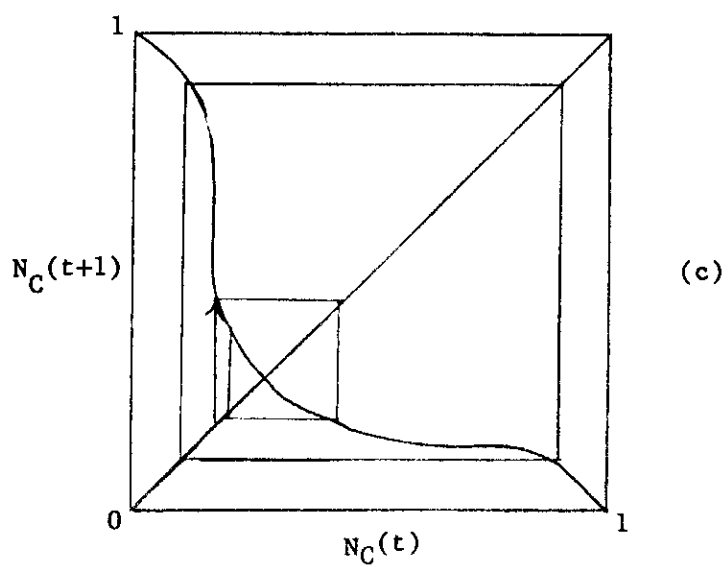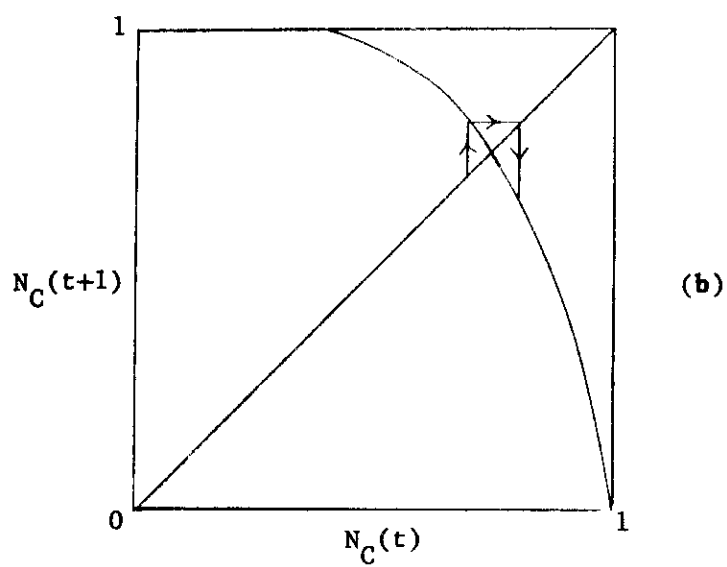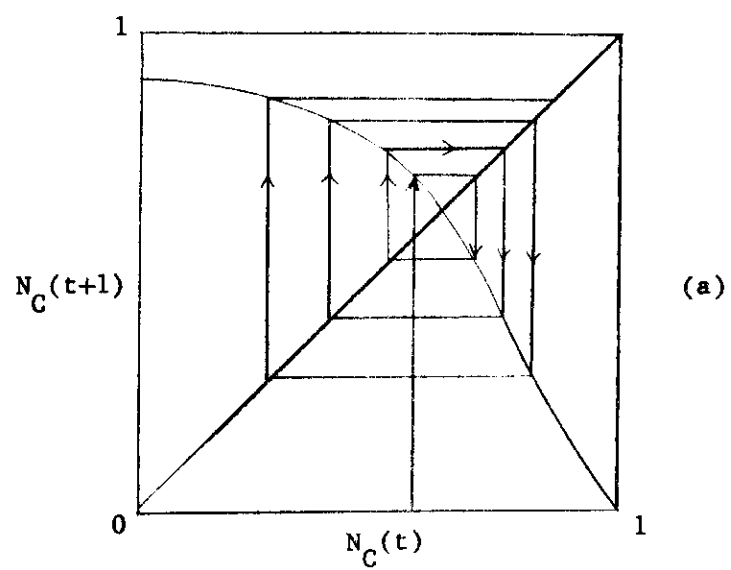If we define x^* as the solution to

$$\alpha(x^*(1+n)) = x^*$$

FIGURE 9.2

Dynamics of One Period Capital Model: Marxian Savings

then, if $x^* < 1$,

$$\lim_{N_C(t-1) \to 0} N_C(t) < 1$$

(See Figure 9.2a), while if $x^* > 1$, for sufficiently small $N_C(t-1)$, $N_C(t) = 1$, i.e. all of labor is allocated to the consumption good sector. In either case, the economy settles down to a limit cycle (see Figure 9.2b).

Even when the balanced growth path is stable, of course, it is possible that there be limit cycles, as illustrated in Figure 9.2c.

The explanation of this instability is the following: Consider a situation where initially there are more workers in the consumption goods sector and less in the capital goods sector than in long run equilibrium. This will have two consequences: Fewer machines will be constructed and they will be more labor intensive. If the elasticity of substitution is very small, a large change in the wage rate induces only a very small change in the labor intensity of the machines constructed; hence the number of workers in the consumption goods sector next period will actually be below the long run equilibrium value. The economy has "overshot" equilibrium. If the elasticity of substitution is sufficiently small, these successive oscillations may be (near the balanced growth path) undamped.

It should be clear that these cycles are not replacement cycles of the usual variety, since every period all capital is replaced. Rather, they appear to be much more akin to the capital intensity cycles extensively discussed in the 1930's.

In order to make sure that these cycles did not depend on the particular savings assumption employed, and in particular, that they were not "distribution"

cycles, we investigated the case where a constant fraction of gross output was saved.[59] This implies that

$$(9.21') \qquad w(t)N_I(t) = s(w(t)N_I(t) + C)$$

$$= \frac{sC}{1-s}$$

The first order difference equation describing the growth of the economy is

$$(9.22') \qquad \left(\frac{1 - N_C(t)}{N_C(t)}\right)\left(\frac{1-s}{s}\right)\alpha\left(\frac{N_C(t)(1+n)}{1 - N_C(t-1)}\right) - 1 = \Psi(N_C(t), N_C(t-1))$$

Again, balanced growth as well as momentary equilibrium are uniquely determined, but local stability requires

$$(9.23') \qquad -1 < \frac{dN_C(t)}{dN_C(t-1)}\bigg|_{N_C(t) = N_C(t-1) = N_C^*} < 1$$

a necessary and sufficient condition for which is that

$$(9.24') \qquad \sigma > \frac{(1-\alpha)(2N_C^* - 1)}{1 + (1-\alpha)(2N_C^* - 1)}$$

Again, low elasticities of substitution lead to instability in the growth path.

In order to make sure that these results did not depend on the particular expectations assumption (perfect foresight) we have employed thus far, we considered the alternative polar case of static expectations. Again, it turns out that the balanced growth path may be unstable. Since the calculations are straightforward but tedious, they are omitted.

9.4. Life Cycle Savings

Expectations are crucial for determining not only the pattern of investment, but also the level of savings (and hence of investment). Except under very special circumstances, savings will depend on expectations of future rates of return and future wage incomes. These in turn are likely to depend on wages and interest rates prevailing today. On the other hand, the distribution of income--wages and interest rates--today will depend on savings today, except in the special case of a one sector growth model. As a result of these strong interactions between the present and the future, even when expectations are static, momentary equilibrium need not be unique, and the balanced growth path will not in general be stable. To illustrate these points, we consider the life cycle model in which individuals live only two periods, working in the first, and living off the proceeds of their savings in the second. For simplicity, we shall assume that the in- difference map between consumption in the two periods is homothetic, so that the savings rate, s , may be written simply as a function of the expected rate of return on savings between this period and the next, $r^e(t)$

$$(9.25) \qquad\qquad s(t) = s(r^e(t))$$

Thus if savings are to equal investment at full employment,[60]

$$s(r^e(t))w(t) = N_I(t)w(t) = (1 - N_C(t))w(t)$$

i.e.

$$(9.26) \qquad\qquad s(r^e(t)) = 1 - N_C(t)$$

Note that full employment cannot be attained through flexibility of the current wage rate, except in so far as changes in current wage rates affect expectations of future rates of return.

What determines the expected rate of return on capital? Clearly, the rate of return on capital depends on expectations of wage rates:

$$(9.27) \qquad 1 + r^e(t) = \max \frac{b(\ell) - w^e(t+1)\ell}{w(t)}$$

where $w^e(t)$ is the wage expected to prevail at time t. Thus the type of machine constructed for use at time t is $\ell(t)$:

$$(9.28) \qquad \ell(t) = b'^{-1}(w^e)$$

so

$$(9.29) \qquad 1 + r^e(t) = \frac{b(\ell(t)) - b'(\ell(t))\ell(t)}{w(t)}$$

Consider first the case of static expectations, i.e., where

$$(9.30) \qquad w^e(t) = w(t-1)$$

Then the growth of the economy is described by the first order difference equation

$$(9.31) \qquad s\left(\frac{b - b'(\ell(t))\ell(t)}{b'(\ell(t))}\right) = 1 - N_C(t-1)$$

where, along a full employment path

$$(9.32) \qquad \ell(t) = \frac{N_C(t)(1+n)}{1 - N_C(t-1)}$$

In this model, it is quite possible that there be multiple balanced growth paths. A necessary and sufficient condition for uniqueness is that

$$(9.33) \qquad (1+n) \frac{s'(1-\alpha)}{\sigma\alpha} + (1 - N_C)^2$$

be one signed, where, as before, α is the share of labor along the ex ante production function and σ the elasticity of substitution. Hence, if the savings rate increases as the rate of return increases, there will be at most one balanced growth path. As in the previous model even when the balanced growth path is unique, it may not be stable, as illustrated in Figure 9.4a. Thus static expectations do not ensure the stability of the economy

Somewhat more interesting, from our present point of view, is the possibility that momentary equilibrium not be uniquely determined. Indeed, it is easy to see that a necessary and sufficient condition for uniqueness of momentary equilibrium is that the savings rate be a monotonic function of the rate of return on capital. Although that will be the case if the utility function is additive and of constant elasticity, this in general will not be true. One might well expect that at low rates of return, the substitution effect dominates the income effect, so $s' > 0$ while at higher levels of r (and hence higher levels of utility) the reverse holds, and $s' < 0$. See Figure 9.3.

Similar results hold if, instead of assuming static expectations, we had assumed perfect foresight. The growth path of the economy would then be described by the second order difference equation:

$$(9.32) \qquad s\left(\frac{b(\ell(t)) - b'(\ell(t))\ell(t)}{b'(\ell(t-1))}\right) = 1 - N_C(t-1)$$

FIGURE 9.3

Savings Rate as a Function of Expected Rate of Return

(a) (b)
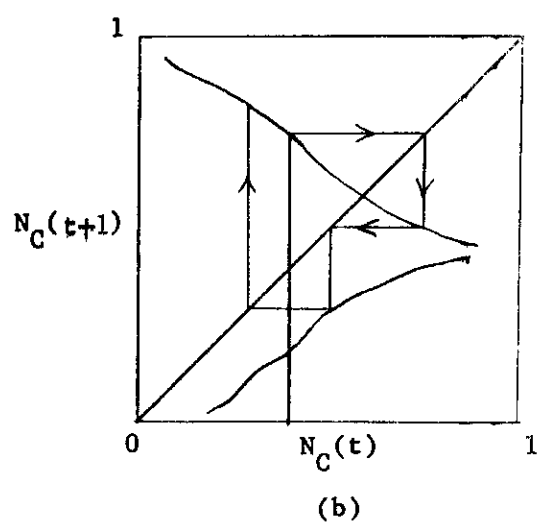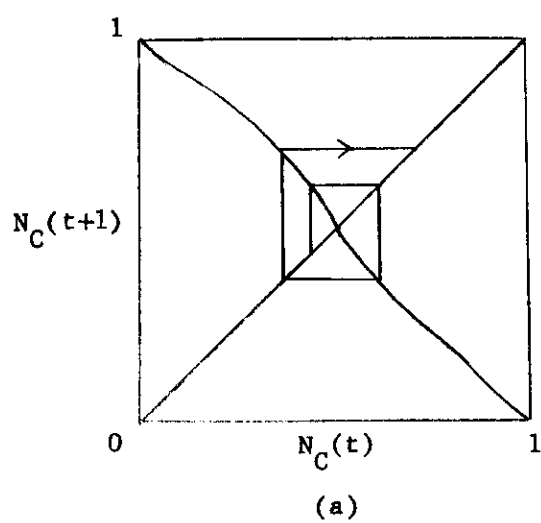
FIGURE 9.4

Life Cycle Dynamics

$$N_C(t+1) = \frac{(1 - N_C(t))}{1+n} w(\ell_2)s^{-1}(1 - N_C(t))$$

$$N_C(t+1) = \ell_1(1 - N_C(t))$$

$$N_C(t+1) = \frac{1 - N_C(t)}{1+n} w(\ell_1)s^{-1}N_C(t+1)$$
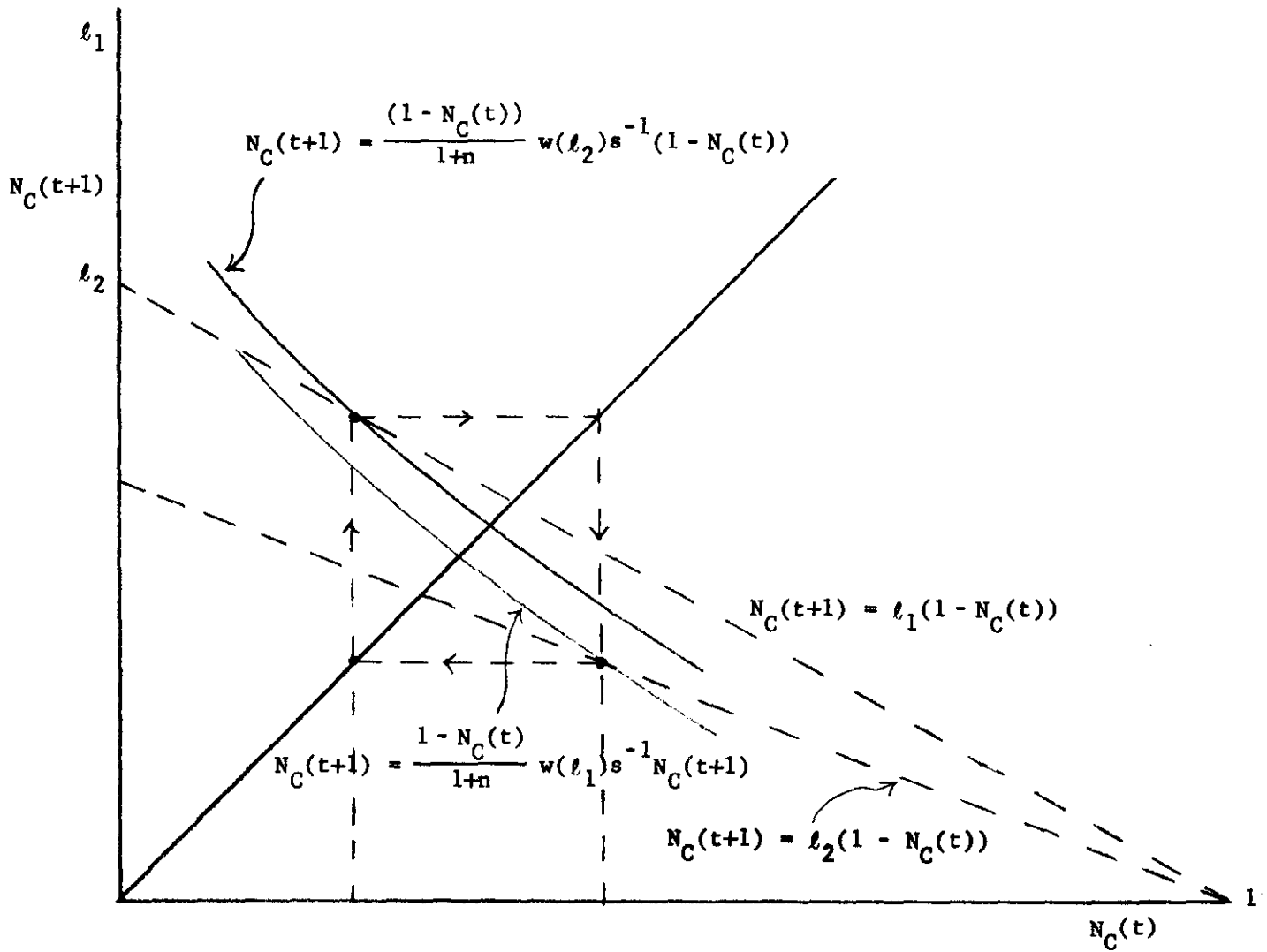
$$N_C(t+1) = \ell_2(1 - N_C(t))$$

FIGURE 9.5

Limit Cycle in Life Cycle Model
with Perfect Foresight $(s' > 0)$

where now

$$(9.33) \qquad \mathcal{Z}(t) = b^{0-1}(w(t)) = \frac{N_C(t)(1+n)}{1 - N_C(t-1)}$$

As before, even when the balanced growth path is uniquely determined, it may not be stable; rather, it is possible that the economy converges to a limit cycle, thus again illustrating the fact that perfect foresight paths need not converge to balanced growth.[61] And again, momentary equilibrium is uniquely determined if and only if the savings rate is a monotonic function of the rate of return.

Whenever momentary equilibrium is not uniquely determined, the economy may "wobble"; it may neither converge to balanced growth or to a limit cycle, simple going from one short run equilibrium to another. It may well be argued that this model of the "wobbling" economy is far more descriptive of the behavior of at least some capitalist economies than the conventional neo-classical models, in which the economy approaches smoothly and steadily the balanced growth path.

This indeterminacy in the growth path is very different from that which arises out of the conventional two sector growth models. It has nothing to do with the relationship between the distribution of income today, the output of capital goods today, and savings today. Indeed, the output of capital goods and consumption goods today is the same in all (full employment) equilibria. Rather, it has to do with the relationship between the wage rate today, ex-pectations of wages tomorrow, and the type of machine constructed today for use tomorrow. In the case of static expectations, there is one equilibrium where the wage is low today and hence is expected to be low tomorrow (the

rate of return is expected to be high), and another equilibrium where the wage is high today. In the case of perfect foresight, there is one equilibrium where we expect the wage to be low tomorrow, and in fact it will turn out to be low tomorrow, and another in which we expect it to be high and it in fact will be high.

It should be noted that expectations today of the distribution of income tomorrow affect, in general, both the distribution of income today and the distribution of income which will actually prevail tomorrow.

Both static expectations and perfect foresight are polar cases. No matter what expectations formation process is assumed, to ensure full employ- ment

$$(9.26) \qquad\qquad s(r^e(t)) = 1 - N_C(t)$$

If for some reason there were a spontaneous increase in the wage rate expected to prevail next period, in order for the rate of return to be such as to ensure full employment (i.e. to satisfy (9.26)) the wage rate today would have to rise. It is in this sense that we can say that the distribution of income today is determined by expectations of the distribution of income tomorrow. Without a theory of expectations i.e. the determination of the wage expected to prevail next period, there is no theory of the determination of the distribution of income today. This, I take it, is one of the major criticisms of the "Cambridge" economists of neoclassical theory: the simpler neoclassical models are formulated in such a way that expectations play no role, whereas in the "real world" they clearly do.

On the other hand, this model is consistent with marginal productivity

theory (correctly interpreted): each period, the technique (for the next period) which is chosen , maximizes expected quasi-rents, i.e. the marginal productivity of labor (along the ex ante production function) is equal to the expected wage. If the expected wage were equal to the wage that turns out to prevail next period (as it reasonably would if the economy converged to balanced growth) then the wage today would correctly reflect the marginal productivity of labor. On the other hand, in the "wobbling economy" even with static expectations, although the economy is consistently wrong in its expectations, it does not necessarily consistently underestimate or overestimate the expected wage. In that case the marginal productivity of labor (along the ex ante production function) and the actual wage (as opposed to the expected wage) may have no systematic relationship with one another.

It is by affecting the choice of technique for the capital that will be used tomorrow that expectations today of the wage tomorrow affect the wage tomorrow.

So far, we have assumed that all individuals have the same expectations. In fact, there will undoubtedly be a diversity of expectations of future wages. Since the wage today depends on expectations of what the wage will be tomorrow, estimating what the wage tomorrow will be is equivalent to guessing what individuals tomorrow will expect the wage will be the day after. Hence, individuals who guess better than average what other individuals are guessing the wage rate to be in the future will make better than average returns on their capital: pure profits are simply a return to guessing well.

If there are rigidities in (a) the adjustment of wages,[62] (b) the
adjustment of expectations to changes in wages or (c) the adjustment of
savings to changes in expectations of interest rates, it is easy to see
how the model we have depicted may lead to cyclical unemployment.

9.5. Concluding Remarks

Solow began his classic 1956 paper with the remarks:

"All theory depends on assumptions which are not quite true.
That is what makes it theory. The art of successful theorizing
is to make the inevitable simplifying assumptions in such a
way that the final results are not very sensitive. A "crucial"
assumption is one on which the conclusions do depend sensi-
tively, and it is important that crucial assumptions be rea-
sonably realistic. When the results of a theory seem to flow
specifically from a special crucial assumption is dubious,
the results are suspect."

It now appears--in the perspective of some fifteen years of subsequent re-
search--that the theory developed by Solow, the picture of an economy smoothly
converging to balanced growth in an economy in which expectations play no
explicit part, is as suspect in this respect as the earlier theory of Harrod.[63]

Solow's 1956 growth model had three important assumptions which allowed
him to ignore completely the role of expectations in the growth process:
(a) a single, malleable (shiftable) capital good, (b) constant savings rates,
(c) instantaneous adjustment of all markets to equilibrium. If any of these
assumptions are dropped, the characteristics of the dynamic path of the economy
are significantly altered. Moreover, Solow's assumption that all sectors
have identical production functions is necessary for the result that the
distribution of income depends only on factor supplies. We have set forth
a model in which expectations of future wages and interest rates are crucial
for the determination of the distribution of income today.

Thus the difficulties with the conventional neoclassical models of economic growth do not lie so much in the capital theoretic issues of re-switching as in the questions arising from the heterogeneity of capital goods, from the dependence of savings on the distribution of income and the expectations of future wage and interest rates, and, perhaps most important, from the crucial role of expectations formation in the development of economies without perfect futures markets.

REFERENCES

[1] Akerlof, G.A. and J.E. Stiglitz. "Capital, Wages and Structural Un-employment," Economic Journal, 72 (June, 1969), pp. 269-281.

[2] Bhaduri, A. "The Concept of the Marginal Productivity of Capital and the Wicksell Effect," Oxford Economic Papers, 18 (1966), pp. 284-288.

[3] Bliss, C.J. "On Putty-Clay," Review of Economic Studies, 35 (April, 1968), pp. 105-122.

[4] _____. "Rates of Return in a Linear Model," Cambridge, 1968, mimeo.

[5] Bruno, M. "Optimal Accumulation in Discrete Capital Models," in K. Shell, ed., Essays on the Theory of Optimal Economic Growth (Cambridge: M.I.T. Press, 1967).

[6] _____. "Fundamental Duality Relations in the Pure Theory of Capital and Growth," Review of Economic Studies, 36 (1969), pp. 39-53.

[7] _____; E. Burmeister and E. Sheshinski. "The Nature and Implications of the Reswitching of Techniques," Quarterly Journal of Economics, 80 (November, 1966), pp. 526-553.

[8] Burmeister, E. and K. Kuga. "The Factor Price Frontier, Duality, and Joint Production," Review of Economic Studies, 37 (1970), pp. 11-20.

[9] Cass, D. and J.E. Stiglitz. "The Implications of Alternative Saving and Expectations Hypotheses for Choices of Technique and Patterns of Growth," Journal of Political Economy, 77 (July-August, 1969).

[10] Champernowne, D.G. "The Production Function and the Theory of Capital: Comment," Review of Economic Studies, 31 (1953-54), pp. 112-35.

[11] _____. "Luigi Pasinetti's Note on 'Changes in the Rate of Profit'" Cambridge, February, 1966 (mimeographed).

[12] Das Gupta, P.S. "On the Optimum Rate of Accumulation in a Labor-Surplus Economy," Carnegie-Mellon, October, 1968 (mimeographed).

[13] Diamond, P.A. "National Debt in a Neoclassical Growth Model," American Economic Review, 55 (December, 1965), pp. 1126-50.

[14] Garegnani, L. "Switching of Techniques," Quarterly Journal of Economics, 80 (November, 1966), pp. 554-567.

[15] Goodwin, R.M. "A Growth Cycle," in C.H. Feinstein, ed., <u>Socialism,</u> <u>Capitalism, and Economic Growth,</u> (Cambridge: Cambridge University Press, 1967).

[16] Hahn, F.H. "Equilibrium Dyanmics with Heterogeneous Capital Goods," <u>Quarterly Journal of Economics,</u> 80 (November, 1966).

[17] _____. "On Warranted Growth Paths," <u>Review of Economic Studies,</u> 35 (April, 1968), pp. 175-184.

[18] _____. "On Some Adjustment Problems," Presidential Address, Econometric Society, December, 1968.

[19] _____, and R.C.O. Matthews. "The Theory of Economic Growth: A Survey," in <u>Surveys of Economic Theory,</u> Vol. II (New York: St. Martin's Press, 1966).

[20] Harcourt, G.C. "Some Cambridge Controversies in the Theory of Capital," <u>Journal of Economic Literature,</u> 7 (June, 1969), pp. 369-405.

[21] Hicks, J.R. <u>Capital and Growth</u> (Oxford, 1965).

[22] Kurz, M. "The General Instability of a Class of Competitive Growth Processes," <u>Review of Economic Studies,</u> 35 (April, 1968), pp. 155-174.

[23] Levhari, "A Nonsubstitution Theorem and Switching of Techniques," <u>Quarterly Journal of Economics,</u> 79 (February, 1965), pp. 98-105.

[24] Malinvaud, E. "Capital Accumulation and Efficient Allocation of Resources," <u>Econometrica,</u> April, 1953.

[25] McManus, M. "Process Switching in the Theory of Capital," <u>Economica,</u> 30 (May, 1963).

[26] Meade, J. <u>A Neo-Classical Theory of Economic Growth</u> (London: Allen and Unwin, 1961).

[27] Mirrlees, J. "The Dynamic Nonsubstitution Theorem," <u>Review of Economic Studies,</u> 36 (January, 1969).

[28] Morishima, M. <u>Equilibrium, Stability and Growth</u> (Oxford, 1964).

[29] _____. "Refutation of the Nonswitching Theorem," <u>Quarterly Journal of Economics</u> (November, 1966), 80, pp. 520-525.

[30] Pasinetti, L.L. "Changes in the Rate of Profit and Switches of Techniques," <u>Quarterly Journal of Economics,</u> 80 (November, 1966), pp. 503-517.

[31] _____. "Switches of Techniques and the 'Rate of Return' in Capital Theory," <u>Economic Journal,</u> 79 (September, 1969).

[32] Ramsey, F.P. "A Mathematical Theory of Saving," _Economic Journal_ (September, 1928), pp. 543-559.

[33] Robinson, J. "The Production Function and the Theory of Capital," _Review of Economic Studies_, 21 (1953-54), pp. 81-106.

[34] _____, _The Accumulation of Capital_ (London: Macmillan, 1956).

[35] _____, _Exercises in Economic Analysis_, 1960.

[36] _____, "A Model for Accumulation Proposed by J.E. Stiglitz," _Economic Journal_, 79 (June, 1969), pp. 412-413.

[37] Samuelson, P.A. "Parable and Realism in Capital Theory: The Surrogate Production Function," _Review of Economic Studies_, 30 (June, 1962), pp. 193-206.

[38] _____, "A Summing Up," _Quarterly Journal of Economics_, 80 (November, 1966), pp. 568-583.

[39] _____, _et al._ (1967a) "Symposium on Switching of Techniques," _Quarterly Journal of Economics_, 81 (November, 1967).(See [7, 14, 30, 38])

[40] _____, (1967b) "Indeterminacy of Development in a Heterogeneous Capital Model with Constant Saving Propensity," in K. Shell, ed., _Essays on the Theory of Optimal Growth_ (Cambridge: M.I.T. Press, 1967).

[41] Shell, K. and J.E. Stiglitz. "The Allocation of Investment in a Dynamic Economy," _Quarterly Journal of Economics_, 81 (November, 1967), pp. 592-609.

[42] Shell, K.; M. Sidrauski; and J.E. Stiglitz. "Capital Gains, Income, and Saving," _Review of Economic Studies_, 36 (January, 1969), pp. 15-26.

[43] Solow, R.M. "A Contribution to the Theory of Economic Growth," _Quarterly Journal of Economics_, 70 (February, 1966), pp. 65-94.

[44] _____, "Substitution and Fixed Proportions in the Theory of Capital," _Review of Economic Studies_ (June, 1962), pp. 207-218.

[45] _____, "The Interest Rate and Transition between Techniques," in C.H. Feinstein, ed., _Socialism, Capitalism, and Economic Growth_, (Cambridge: Cambridge University Press, 1967).

[46] _____; J. Tobin; C.C. von Weizsacker; and M. Yaari. "Neoclassical Growth with Fixed Proportions," _Review of Economic Studies_ (April, 1966), pp. 79-115.

[47] Sraffa, P. _Production of Commodities by Means of Commodities_ (Cambridge: Cambridge University Press, 1960).

[48] Srinivasan, T.N. "Investment Criteria and Choice of Technique of Production," _Yale Economic Essays_, 2 (1962), pp. 59-115.

[49] Starrett, D.A. "Switching and Reswitching in a General Production Model," _Quarterly Journal of Economics_, 79 (November, 1969), pp. 673-87.

[50] Stiglitz, J.E. "Accumulation in a Leontief-Sraffa Economy and the Reswitching of Techniques," Cambridge, 1966 (mimeographed).

[51] _____. "A Note on Technical Choice under Full Employment in a Socialist Economy," _Economic Journal_, 78 (September, 1968), pp. 603-609.

[52] _____. (1970a) "Factor Price Equalization in a Dynamic Economy," _Journal of Political Economy_ (March-April, 1970).

[53] _____. (1970b) "On Transitions between Steady States," _Economic Journal_ (forthcoming).

[54] _____. (1970c) "Non Substitution Theorems with Durable Capital Goods," _Review of Economic Studies_ (1970).

[55] Swan, T. "Economic Growth and Capital Accumulation," _Economic Record_, 32 (November, 1956).

[56] Uzawa, H. "On a Two-Sector Model of Economic Growth," _Review of Economic Studies_, 29 (October, 1961).

[57] _____. "Optimal Growth in a Two-Sector Model of Capital Accumulation," _Review of Economic Studies_, 31 (January, 1964), pp. 1-24.

[58] Weitzman, M., mimeographed note on reswitching of techniques, M.I.T., 1966.

[59] Wicksell, K. _Lectures on Political Economy_ and Appendix 2, "Real Capital and Interest," tr. by E. Classen (London: George Routledge and Sons, Ltd., 1934).

FOOTNOTES

[1]This as well as several other of the results of Sections 2 and 7 below have been pointed out elsewhere in the literature. They are presented here in order to complete the discussion; the exact references to the earlier literature will be presented in the text.

[2]In fact, in the Shell-Stiglitz [1967] model, there was, for any initial endowment of capital goods, a unique set of initial prices which allowed the economy to converge to balanced growth (i.e. in price-capital space, the equilibrium was shown to be a saddle point).

[3]The literature on reswitching is extensive and has been growing probably faster than exponentially. Moreover, much of the literature remains not only unpublished, but circulated only at the institution at which it was written, e.g. Champernowne's comments of 1966. As a consequence, no attempt at a complete survey of the literature is attempted here. Champernowne [1953] was the first to notice not only that reswitching could occur, but also that consumption per man could move directly (rather than inversely) with the rate of profit. Subsequently these paradoxes were discussed by Robinson [1958] and Sraffa [1960], and still later by Morishima [1964] and McManus [1963]. (Cambridge tradition has it that Sraffa had observed the possibility of reswitching considerably earlier; although Sraffa and Robinson only talked about the possibility of reswitching for a single industry, it is clear that if it is possible for an industry, it is possible for the economy; we need only assume that the given industry is the only industry for which there is a choice of technique.) Interest in the matter initially was minimal. Champernowne described these situations as "anomolies" and Robinson wrote, "The following paragraphs are concerned with a somewhat intricate piece of analysis which is not of great importance" (Robinson [1956]). Interest seems to have become widespread only when it was denied that reswitching could occur if the economy were indecomposable (Levhari [1965]) Passinetti [1966] not only detected some of the errors in the Levhari proof, but also provided an elaborate counterexample (in spite of the fact that a much simpler counterexample had already been provided in Champernowne's original discussion; subsequently, Champernowne provided an even simpler example [1966]). Stiglitz [1966] and the papers in the Quarterly Journal of Economics symposium [1966] presented further counter-examples and pointed out further errors in the Levhari proof. Bruno, Burmeister, and Sheshinski [1966], Hicks [1965], Solow [1967], Starrett [1969], Stiglitz [1966] and Weitzman [1966] provide conditions under which reswitching cannot occur. Pasinetti [1966, 1969], Garegnani [1966], and Harcourt [1969] have argued that reswitching has fundamental implications for conventional neoclassical analysis, while Bruno, Burmeister, and Sheshinski [1966], Samuelson [1966], Solow [1966], and Stiglitz [1966, 1970b] have discussed the implications of reswitching from a more neoclassical point of view. For a survey of the subject from a considerably different view point than that taken here, see Harcourt [1969].

[4]The dynamic non-substitution theorem asserts that, for economies in steady state, for any rate of interest there is a unique set of relative prices (and an associated set of techniques) which will support all competitive equilibria (independent of preferences).

The notion of reswitching can be extended to economies in which the dynamic non-substitution theorem does not apply in the following manner. Corresponding to any interest rate, there will be a <u>set</u> of technologies which are efficient. Then reswitching is defined as the situation where a given technology is in the set of efficient technologies for two interest rates and not for some intervening interest rate.

[5]In this example we have assumed that wages get paid at the end of the period of production. The results of (2.3) and (2.4) do not depend on this.

[6]To ensure that the values of r at which the two techniques are equally profitable are both positive, we require

$$\frac{1 - a_{11}}{a_{01}} < \frac{1 - b_{11}}{b_{01} + b_{21}b_{02}}$$

[7]This economy is not indecomposable. But it is obvious that this is not a crucial assumption: if we place small $\epsilon > 0$ in our input output matrix where before we had zeros, the equations describing the cost of production are altered only infinitesimally.

[8]This theorem was independently proved by Bruno-Burmeister-Sheshinski [1966] and Stiglitz [1966].

[9]Reswitching requires

$$a_{0j}^i - a_{0j}^k + (1+r)(p_1 + p_2)\left\{\frac{p_1}{p_1 + p_2}\left(a_{1j}^i - a_{1j}^k\right) + \frac{p_2}{p_1 + p_2}\left(a_{2j}^i - a_{2j}^k\right)\right\}$$

to change sign twice. Without loss of generality, we let

$$a_{0j}^i > a_{0j}^k , \qquad a_{1j}^i > a_{1j}^k , \qquad a_{2j}^i < a_{2j}^k .$$

If p_1/p_2 is a monotonically increasing function of r, the above expression is a monotonically increasing function of r. If p_1/p_2 is a monotonically decreasing function of r, the derivative of the above expression is negative when the above expression is zero.

[10] Proof: Our pricing equations may be written

$$P_1 = w + (1+r)(a_{11}P_1 + a_{21}P_2)$$

$$P_2 = w + (1+r)(a_{12}P_1 + a_{22}P_2)$$

Dividing both equations by P_1, and letting $p = P_2/P_1$, we obtain

$$\begin{bmatrix} 1 - (1+r)a_{11} \\ - (1+r)a_{12} \end{bmatrix} = \begin{bmatrix} 1 & (1+r)a_{21} \\ 1 & (1+r)a_{22} - 1 \end{bmatrix} \begin{bmatrix} w/P_1 \\ p \end{bmatrix}$$

so

$$p = \frac{-1 + (1+r)(a_{11} - a_{12})}{-1 + (1+r)(a_{22} - a_{21})}$$

Thus

$$\frac{dp}{dr} \sim (a_{11} + a_{21}) - (a_{12} + a_{22})$$

[11] The requirement that we be able to order the goods by their (unweighted) capital requirements, independently of the interest rate, is a sufficient condition for this to obtain. A special case where relative prices are independent of the rate of interest is that in which the unweighted capital requirements are identical for all industries (where units are chosen as above). This is known as the case of "equal organic composition."

[12] In a continuous time model, we require, if there is no depreciation,

$$\varphi(C, X, nX, L) = 0$$

while if capital goods depreciate exponentially at the rate μ, we require

$$\varphi(C, X, (n+\mu)X, L) = 0$$

[13] Unless the two are identical for every value of $P_i(1+r)$, which means that whenever A is used, it is equally profitable to use B. It is easy to show that in fact A can only be used at isolated values of r except in the special case of "equal organic compositions of capital" referred to above (fn.11), since as r changes, relative prices will change except in that special case. Indeed, it appears that in general A will be used only at a single value of r.

Somewhat more formally, recall that when $r = n = 0$, the non-substitution

theorem implies that, if A is a technology which is competitive at $r = 0$, and B is not, the net production possibilities set corresponding to A is a hyperplane which lies everywhere above that corresponding to B, i.e. there exist vectors $\underset{\sim}{X}_A$, $\underset{\sim}{X}_B$, $\underset{\sim}{C}_A$, $\underset{\sim}{C}_B$ such that

$$A\underset{\sim}{X}_A + \underset{\sim}{C}_A = \underset{\sim}{X}_A, \quad a_0\underset{\sim}{X}_A = 1$$

$$B\underset{\sim}{X}_B + \underset{\sim}{C}_B = \underset{\sim}{X}_B, \quad b_0\underset{\sim}{X}_B = 1$$

and $\underset{\sim}{X}_A - A\underset{\sim}{X}_A > \underset{\sim}{X}_B - B\underset{\sim}{X}_B$

By exactly the same argument we can show that when $n, r > 0$, if A is a technology competitive at r and B is not, then there exist vectors $\underset{\sim}{X}_A$, $\underset{\sim}{X}_B$, $\underset{\sim}{C}_A$, and $\underset{\sim}{C}_B$ satisfying the first two equations above and such that $\underset{\sim}{X}_A - (1+r)A\underset{\sim}{X}_A > \underset{\sim}{X}_B - (1+r)B\underset{\sim}{X}_B$. (Indeed, we simply redefine our technology matrix appropriately.) Thus, if A is competitive at r_1 and B is not, but B is competitive at r_2 and A is not, and if $p(r_1)$ is the steady state price vector at r_1, then

$$p(r_1)(\underset{\sim}{X}_B - B\underset{\sim}{X}_B(1 + r_2)) > p(r_1)(\underset{\sim}{X}_A - A\underset{\sim}{X}_A(1 + r_2))$$

while, since $p(r_1) = a_0 + (1+r_1)p(r_1)A$ and $p(r_1) \leq b_0 + (1 + r_1)p(r_1)B$,

$$p(r_1)(\underset{\sim}{X}_A - A\underset{\sim}{X}_A(1 + r_1)) = a_0\underset{\sim}{X}_A = 1 = b_0\underset{\sim}{X}_B \geq p(r_1)(\underset{\sim}{X}_B - (1 + r_1)B\underset{\sim}{X}_B)$$

Subtracting, we obtain

$$(r_1 - r_2)p(r_1)(A\underset{\sim}{X}_A - B\underset{\sim}{X}_B) < 0$$

i.e.

$$p(r_1)(A\underset{\sim}{X}_A - B\underset{\sim}{X}_B) > 0$$

Similarly, if A is competitive at $r_3 > r_2$ but B is not,

$$p(r_3)(A\underset{\sim}{X}_A - B\underset{\sim}{X}_B) < 0$$

But if, as required here, $p(r_3)$ is proportional to $p(r_1)$, this is impossible. Starrett [1969] has presented a generalization of this proof for the case where joint production is allowed.

[14] Since all inputs and outputs are non-negative, when $X_i = 0$ only the right hand derivative of the expression in (2.7) or (2.10) are defined. But to ensure that the equalities hold in (2.8) or (2.11), we need both left and right hand derivatives to be defined (and equal to each other). Hence, in the first theorem, we require $(X_{i_1} \cdots\cdots X_{i_n}, X_{i_1+m}) > 0$.

Similarly for the second theorem.

[15] It is easy to establish that an interest rate is a switch point between two technologies when wages are paid at the end of the period of production if and only if it is a switch point when wages are paid at the beginning of the period.

[16] All three theorems of this section carry over in a straightforward manner to technologies involving joint production, where we redefine reswitching as in footnote 2.

[17] In the Uzawa-Meade two sector model [26, 56] there is an unambiguous meaning to the stock of "real" capital.

[18] Akerman's problem, it will be recalled, was the description of the competitive equilibrium for an economy in which the durability of capital was variable; by increasing the labor used to construct a machine, it may be made more durable.

[19] It should be emphasized, however, that these valuation perversities may occur even when reswitching does not occur, as Champernowne pointed out in his classic paper [1953]. The valuation perversities are sometimes referred to as the Ruth Cohen curiosum.

[20] This assumption is only made to simplify the calculations. It plays no crucial role in the analysis.

[21] $(1 - x - e^{-x}) < 0$ for $x > 0$ since at $x = 0$, it equals zero, and its derivative with respect to x , $-1 + e^{-x} < 0$ for $x > 0$.

[22] Every year $\frac{C}{T}$ machines wear out. Hence employment in the capital goods industry is NC/T . Since one machine requires one man to operate it, employment in the consumption goods industry. Under our normalization, total employment is $1 = C + NC/T$.

[23] $xe^{-x} - 1 + e^{-x} < 0$ for $x > 0$ since at $x = 0$ it equals zero, and its derivative with respect to x , $-xe^{-x} < 0$.

[33]Pasinetti's [1969] criticism of Solow's [1967] analysis of the relationship between the rate of return and the rate of interest (he argues that they will not in general be equal) is based on two elementary confusions: (a) As we noted above, he used the change in the value of capital to measure the "cost" of going from one steady state to another; what we should be concerned with, however, is the change in consumption along the transition path. (b) If one steady state requires less of some capital good than another, in going from the second steady state to the first, he assumes capital must become redundant. If fact, it can be shown, under the weak assumption that capital goods are finitely lived, that it is always possible to make the transition in a finite number of periods keeping full employment of all capital goods and labor along the transition path. (See Stiglitz, [1970b].) Solow establishes the existence of full employment transition paths under the assumption that all capital goods can also be used for consumption; instead of simply attacking the unrealism of that assumption, Pasinetti would have been much better advised to use his time (and that of his readers) to establish whether that assumption was or was not essential.

[34]Under suitable restrictions (in particular, assuming that the present discounted value of the capital stock on the given program goes to zero, cf. Malinvaud [1953]) (a) corresponding to every efficient path there is (at least) one set of prices such that present discounted value of consumption is greater than that on any other feasible path, (b) If P_{it} is the (shadow or competitive) price of commodity i at time t, C_{it} the output of consumption good i at time t, $\partial C_{i0}^+/\partial C_{it}$ the left handed derivative of C_{i0} as C_{it} increases, when all other C_{jt} are not decreased, and similarly if $\partial C_{i0}/\partial C_{it}^-$ is the corresponding right handed derivative (the existence of the right and left handed derivatives is easy to establish), then if $r_{it,t'}$ is the rate of interest in terms of the ith good numeraire between t and t', $t' > t$,

$$\frac{\partial C_{i0}^+}{\partial C_{it}} \leq \frac{P_{it}}{P_{i0}} = (1 + r_{i0,t}) = \prod_{v=0}^{v=t-1} (1 + r_{i,v,1+v}) \leq \frac{\partial C_{i0}^-}{\partial C_{it}}$$

For an excellent discussion of these issues, see Bliss [1968].

[35]For an economy not in steady state with a single consumption good, there is an easily proven theorem which has some semblance to the dynamic non-substitution theorems for economies in steady state; the set of technologies which maximizes the real wage depends only on the real rates of return on the n capital goods. But since the essence of the non-substitution theorem is that the choice of technique be independent of preferences, assuming that there is only one consumer good assumes away the essence of the non-substitution theorem. (Cf. Bruno [1969], Burmeister-Kuga [1970].)

[36] Readers may wonder at the juxtaposition of "Robinson" and "neoclassical;" but it is clear that when she addresses herself to the question of accumulation (as opposed to comparing islands) she tells an essentially Wicksellian story of the deepening of techniques. See, e.g., Robinson [1956, 1960].

[37] There are several other senses in which we might interpret reswitching in a dynamic context. (a) If there were a single consumption good, we could consider the set of efficient technologies at any real wage w ; reswitching is said to occur if some technology is in the set for $\tilde{w}$ and $\tilde{\tilde{w}}$ but not $\tilde{\tilde{\tilde{w}}}$, where $\tilde{w} < \tilde{\tilde{w}} < \tilde{\tilde{\tilde{w}}}$. (b) If there were a single consumption good, we could consider the set of efficient technologies at any set of real rates of return on capital, $\rho_1, \ldots, \rho_i, \ldots, \rho_n$. Then reswitching may be said to occur if some technology is in the set for $(\rho_1, \ldots, \tilde{\rho}_i, \ldots, \rho_n)$ and $(\rho_1, \ldots, \tilde{\tilde{\rho}}_i, \ldots, \rho_n)$ but not $(\rho_1, \ldots, \tilde{\tilde{\tilde{\rho}}}_i, \ldots, \rho_n)$, where $\tilde{\rho}_i < \tilde{\tilde{\rho}}_i < \tilde{\tilde{\tilde{\rho}}}_i$. (c) We could consider two economies with the same initial conditions but differing in their pure time rate of discount; we maximize

$$\sum_0^\infty C(t)(\frac{1}{1-\delta})^t \ ;$$

reswitching is said to occur if, at any (or every) t , the technology used if $\delta = \tilde{\delta}$ is the same as that used if $\delta = \tilde{\tilde{\delta}}$, but a different technology is used at $\tilde{\tilde{\tilde{\delta}}}$, $\tilde{\delta} < \tilde{\tilde{\delta}} < \tilde{\tilde{\tilde{\delta}}}$. (This interpretation was suggested to me by P. Diamond.)

All three interpretations have the same unattractive character that the reswitching problem has in its original context of steady states; in particular, answering these questions gives us little insight into the behavior of an economy along any path it might actually follow. For the third interpretation, it is easy to show that reswitching can occur whenever there is reswitching in the conventional steady-state sense; for consider the initial endowment corresponding to the steady state of a technology A which has two switch points with technology B , at δ_1 and δ_2 ; A is used if $r < \delta_1$ or $r > \delta_2$ (where r is the rate of interest in steady state). Let $U(C) = C$. Then if $\delta < \delta_1$ or $\delta > \delta_2$, we remain using A , while if $\delta_1 < \delta < \delta_2$ we eventually go to B . (Indeed, similar results hold asymptotically for any initial conditions.) Similarly, it is easy to show that reswitching can occur in the second interpretation. The real wage, w , can be written e.g. for the case of two capital goods as (for technology A)

$$w = 1 - \rho_1 a_{11} - \rho_2 a_{22} + \rho_1\rho_2\hat{A}_{11}/\rho_1\rho_2|A| - \rho_1\hat{A}_{32} - \rho_2\hat{A}_{22} - \rho_1\rho_2\hat{A}_{11} + a_{0c}$$

where $\hat{A}_{ij}$ is the cofactor of the ith element of the input-output matrix

$$A \equiv \left\{ \begin{array}{ccc} a_{0c} & a_{01} & a_{02} \\ a_{1c} & a_{11} & a_{12} \\ a_{2c} & a_{21} & a_{22} \end{array} \right\} ,$$

where a_{ij} is the input requirement of factor i (0 stands for labor) required to produce a unit of commodity j (c is the consumption good). Similarly for technology B. The set of values of ρ_1 for given ρ_2 for which the real wage corresponding to technology B is equal to that for A is given by the solution to a quadratic equation; it is clear then that "reswitching" can occur. On the other hand, reswitching could never occur in the first interpretation, since at every real wage, we allow ρ_1/ρ_2 to vary from zero to infinity.

[38] The analysis of this section is simply a generalization of the analysis presented in the appendix to [52].

[39] From a formal point of view, it is unnecessary to distinguish (in this case) between intermediate capital goods and durable capital goods.

[40] In the simple Wicksell model, there is no ambiguity in defining which is the more capital intensive sector, since one of the two sectors requires no capital at all.

[41] In the Solow-Swan one sector model, there is no room for capital widening as defined here; more generally, whether the process of accumulation is accompanied by capital widening will depend, at least in part, on the relative capital intensities of the capital goods and consumption goods sectors. If capital goods were more capital intensive, then in the early stages of development, there is likely to be capital widening, subsequently there may be capital "narrowing" (cf. Uzawa [1964]).

[42] This form of the Wicksellian-model is due to Solow [1962].

[43] Throughout this section, we assume b' exists. The modification for the other case is straightforward.

[44] If there are M machines per capita, $(\mu+n)M$ laborers must be employed in building new machines for the growing population and replacing depreciating machines. Since each machine employs ℓ laborers, we have

$$\ell M + (\mu+n)M = 1$$

or

$$M = 1/(\mu + n + \ell)$$

The price of a machine is equal to its cost of production, w. Equation (7.5) follows immediately.

[45] This in fact is a general proposition in an economy with a single consumption good. Assume a discrete technology, with net output C, wage w, value of capital V_K (in consumption good numeraire). Then

$$C = w + rV_K - nV_K$$

Since C is fixed

$$\frac{dw}{dr} = -V_K + (r-n)\frac{dV_K}{dr}$$

$$= -V_K \quad \text{only if} \quad \frac{dV_K}{dr} \text{ is zero (no Wicksell effects)}$$

$$\text{or} \quad r = n \quad \text{and} \quad \frac{dV_K}{dr} \text{ is finite}$$

(cf. Bhaduri [1966]).

[46] For reference, we note the corresponding properties of the one sector growth model. If capital were malleable, the optimal trajectory would exhibit the "bang-bang" characteristic: below the long run equilibrium capital labor ratio, consumption remains at zero; all of output is invested. The demand price of capital is (for a capital poor economy) monotonically decreasing. If there is ex post fixed coefficients, near equilibrium, only one type of machine is constructed; the wage is constant; investment is constant and output increases monotonically. (Away from equilibrium, none of these properties need hold.) See Cass-Stiglitz [1969].

[47] Throughout the remainder of the discussion we assume there exists an optimal trajectory, and that $\underline{C} < C^*$, the level of consumption in long run equilibrium. For certain initial endowments of capital, the constraint may not be binding initially (in the sense that, if ℓ^* is the marginal machine used in the long run equilibrium, $\int^{\ell^*} b(\ell)K(\ell) > \underline{C}$) but if $\ell < \hat{\ell}^*$ were constructed, (where $\hat{\ell}$ is the type of machine constructed in long run equilibrium), it would become binding eventually. Along the optimal path, the constraint will eventually become binding; once the constraint is binding, it will remain binding until the economy switches into the second stage. In the following analysis we shall assume that the constraint is binding initially. The extension to the other case is straightforward. It can also be shown that if it is feasible to produce $\underline{C}$ initially, with $N_C < 1$, then there exists paths for which it is always possible to produce $\underline{C}$.

[48] If there were a continuum of techniques, (7.9) may be written

$$V = C + \lambda(C-\underline{C}) + \int p(\ell, t)d\tilde{N}_I(\ell) - \mu d\kappa(\ell) + w(1 - \int d\tilde{N}_I(\ell) - N_C)$$

where $\widetilde{N}_I(\widetilde{\ell})$ is the number of workers employed in constructing machines of type $\ell \le \widetilde{\ell}$,

$\kappa(\widetilde{\ell})$ is the number of machines of type $\ell \le \widetilde{\ell}$,

$$N_C(\ell) = \int_0^\ell \ell \, d\kappa(\ell)$$

and where all integrals are understood to be Stieljes integrals; see Cass-Stiglitz [1969].

[49] When (7.22) is satisfied

$$\frac{\partial P(\ell + \Delta\ell, t)}{\partial \ell} - \frac{\partial P(\ell, t)}{\partial \ell} = \int_{A(\ell+\Delta\ell, t)} \{b'(\ell + \Delta\ell) - b'(\ell)\}(1+\lambda)e^{-(\mu+\delta)\tau}d\tau$$

$$- \int_{A(\ell, t)-A(\ell+\Delta\ell, t)} (b'(\ell) - b(\ell)/\ell)(1+\lambda)e^{-(\mu+\delta)\tau}d\tau$$

If $\Delta\ell > 0$, both terms are negative.

[50] Under the assumption set forth in footnote 47.

[51] For a more complete discussion of this point as well as those of the following paragraphs, see Cass-Stiglitz [1969].

[52] For a more precise statement of this, the reader is again referred to Cass-Stiglitz [1969].

[53] It has been suggested (perhaps somewhat facetiously) that this may be the true economic explanation of China's policy of backyard furnaces. (In contrast, the policy of the preceding sub-section, of constructing from the beginning of the plan the type of machine which the economy will eventually use exclusively (the very capital intensive techniques) has some semblance to the policies pursued in the Soviet Union; this policy may be referred to as the "Stalin Plan" while the policy with recurrences as the "Mao Plan."

[54] In the more general case of a non-linear utility function, there is no reason to expect that recurrences be limited to the initial stages of development.

[55] Das Gupta [1968] has provided another detailed example of recurrences in an economy in which capital is non shiftable and in which the investment goods sector has two processes, one of which requires labor alone. Under the normal capital intensity hypothesis, that the capital goods sector is more labor intensive than the consumption goods sector, it is shown that,

if the objective function is linear, for certain initial endowments, the optimal trajectory requires the use of the primitive process initially, then a switch to using only the advanced process, a return to the primitive process, and finally a return to the advance process only. If the objective function is non-linear, clearly more complicated patterns are possible.

[56] The qualitative properties of the solution do not in fact require differentiability; if however, $d\kappa(\ell, t)/d\ell = 0$ over an interval, there is an essential indeterminacy in $w(t)$. Cf. Cass-Stiglitz [1969].

[57] Cf., eq. (A.1) and (A.2) and the discussion following in Appendix A.

[58] Because all variables are expressed in per capita terms, $N_C(t)$ is really the fraction of the population working in the consumption goods sector at time t.

[59] This is probably not a very reasonable assumption in spite of its prevalence in the recent growth literature. Almost the only exception to this rule is Solow's [1956] model. The two sector literature, as well as Solow's own later work with Tobin, et al [1966], use the assumption employed here.

[60] Throughout this discussion, we assume that the savings rate is sufficiently flexible so that there always exists a value of $r^e(t)$ satisfying (9.26).

[61] Figure 9.4 illustrates a limit cycle for this economy. For different values of $\ell(t-1)$, the solid lines plot $N_C(t)$ as a function of $N_C(t-1)$. The dotted lines plot for different values of $\ell(t) = (1+n)N_C(t)/1 - N_C(t-1)$, $N_C(t)$ as a function of $N_C(t-1)$.

[62] For a detailed discussion of the role of rigidities in wage adjustments in generating oscillatory behavior in the path of the economy (including recurrences in the choice of technique), see Akerlof and Stiglitz [1969] and Goodwin [1967].

[63] The basic point of Solow's article, that the "fundamental opposition of warranted and natural rates turns out in the end to flow from the crucial assumption that production takes place under conditions of fixed proportions," remains valid. This is a question of balanced growth, rather than of the nature of the dynamic path which an economy might actually follow.

APPENDIX A TO SECTION 7

In this appendix, we describe the phase diagram corresponding to the optimal growth problem of Section 7.5. This will enable us to determine conditions under which recurrences cannot occur.

By differentiating equation (7.12) and (7.13) it is possible to show that[1]

(A.1)
$$\dot{\ell} \sim \mu\underline{C} - N_I b(\hat{\ell})$$

(A.2)
$$\dot{N}_C = -\mu N_C + \hat{\ell} N_I + \dot{\bar{\ell}} d\kappa(\bar{\ell}(t),\ t)\bar{\ell}(t)$$

$$= -\mu N_C + \hat{\ell} N_I - (N_I b(\hat{\ell}) - \mu\underline{C})\,\frac{\bar{\ell}}{b(\bar{\ell})} = -\dot{N}_I$$

where $\sim$ denotes "is of the same sign as," and where $\kappa(\tilde{\ell}, t)$ is the number of machines at time t with $\ell \leq \tilde{\ell}$.

The interpretation of these equations is straightforward. If in this period we were to use for the production of consumption goods only the machine used last period plus the new machines, the change in output of consumption goods is $-\mu\underline{C}$ (depreciation on the old machines) $+N_I$ (the number of new machines) times $b(\ell)$ the output per machine on new machines. If this is positive, we are exceeding our minimum consumption constraint, so we can retire the poorest machines (lower $\bar{\ell}$) and conversely if it is negative. Similarly, (A.2) has the interpretation that the change in workers in the consumption goods industry is equal to the reduction of workers from depre-

---

[1] We assume throughout this section that there are a continuum of techniques available. For the derivation of the analogous equations for the Johansen one sector ex post fixed coefficients model, see D. Cass and J. Stiglitz [1969]. These results hold even if $\kappa(\ell, t)$ is not a differentiable function of ℓ . If there are flat regions in the $\kappa(\ell, t)$ function $\hat{\ell}$ may not exist. The LHS of (A.1) is to be read then as $\ell(t+\Delta t)-\ell(t)$ for Δt sufficiently small and positive. If there are discontinuities in $\kappa(\ell, t)$ (mass points), then ℓ may equal zero even though the right hand side of (A.1) is positive (or negative), but in this case, machines of type ℓ are being taken out of production (being put into production), i.e. Z is decreasing or increasing. If more than one type of machine is constructed at a time, with say $\eta_{\hat{\ell}}$ of N_I devoted to constructing type $\hat{\ell}$ then we replace $\hat{\ell}$ by $\Sigma\eta_{\hat{\ell}}\hat{\ell}$ and $b(\hat{\ell})$ by $\Sigma\eta_{\hat{\ell}}b(\hat{\ell})$.

ciation of machines upon which they have been working $-\mu N_C$ + the increase of workers to man the new machines $N_I \hat{\ell}$ plus an adjustment from taking out or putting into service old machines, $-\bar{\ell}(N_I b(\hat{\ell}) - \mu\underline{C})/b(\bar{\ell})$.

In the second stage of development, when all prices are at their long run values, $\dot{\bar{\ell}}$ is of course zero, but the differential equation for N_I is simply[2]

$$\dot{N}_I = \mu - N_I(\mu + \hat{\ell})$$

We can depict the motion of the economy most simply in terms of $(N_I, \hat{\ell})$ phase space. From (A.2), we have

$$(A.3) \qquad N_{I(\dot{N}_I=0)} = \frac{\mu(1 - \underline{C}\bar{\ell}/b(\bar{\ell}))}{\mu - \left(\dfrac{b(\hat{\ell})\bar{\ell}}{b(\bar{\ell})} - \hat{\ell}\right)}$$

In the relevant region,[3] this is a U-shaped curve which attains its minimum at the point where quasi-rents are maximized, i.e. $b'(\hat{\ell}) = b(\bar{\ell})/\bar{\ell}$. Above the curve, $\dot{N}_I > 0$, and conversely below it. From (A.1)

$$(A.4) \qquad N_{I(\dot{\bar{\ell}}=0)} = \mu\underline{C}/b(\hat{\ell})$$

which is a monotonically decreasing curve. Above it $\bar{\ell}$ is decreasing, below it, increasing. As we lower $\bar{\ell}$, the $\dot{N}_I = 0$ curve shifts down or up depending on whether $-\mu\underline{C} + N_I b(\hat{\ell}) < 0$ i.e. whether $\dot{\bar{\ell}} \gtrless 0$. This means that all the N_I curves (for different values of $\bar{\ell}$) intersect the $\dot{\bar{\ell}} = 0$ curve at the same point $(\tilde{N}_I, \tilde{\ell})$ where $(\tilde{N}_I, \tilde{\ell})$ are the simultaneous solutions to the equations[4]

$$(A.5) \qquad \mu\underline{C} = N_I b(\tilde{\ell})$$

$$N_I = \mu/(\tilde{\ell} + \mu)$$

---

[2] See Stiglitz, op.cit. This is identical to (A.2) except that no adjustment for taking machines out of (or putting into) production need be made.

[3] Assuming $-\mu + (b(\bar{\ell})\bar{\ell}/b(\bar{\ell}) - \hat{\ell})$ is positive, i.e. net quasi-rents on the given machine are positive. Since only machines with positive quasi-rents
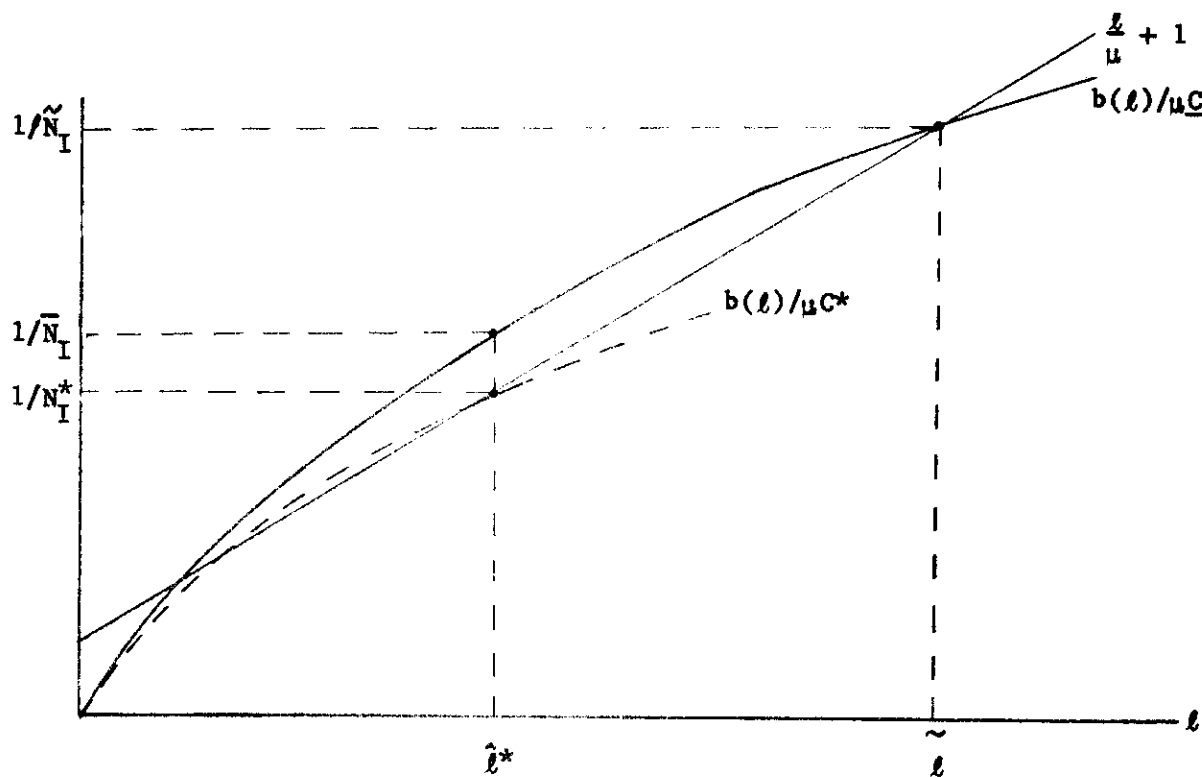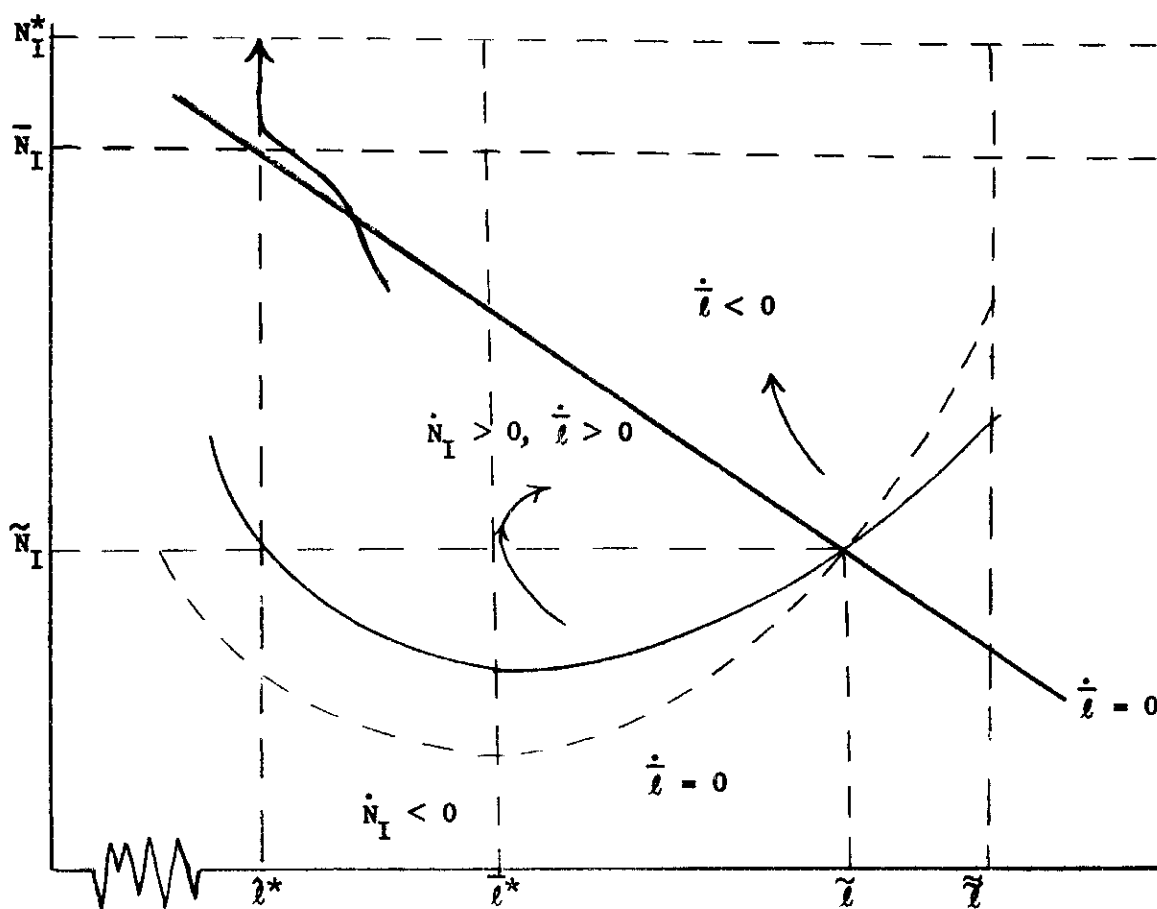
FIGURE A.1

FIGURE A.2

Phase Diagram of Economy with Minimum Consumption Constraint

(See Figure A.1.)

In Figure (A.2), the phase diagram of the economy, we have drawn two $\dot{N}_I = 0$ curves, one for $\bar{\ell} = \bar{\ell}^*$ [5] and the other for $\bar{\ell} > \bar{\ell}^*$. We have noted, in addition, the following values of N_I and ℓ :

(i) $\bar{N}_I$ is the value of N_I at which $\dot{\ell} = 0$ when $\hat{\ell} = \hat{\ell}^*$:

(A.6)
$$\mu \underline{C} = \bar{N}_I b(\hat{\ell}^*)$$

Since, the long run values of the economy must satisfy

(A.7)
$$\mu C^* = N_I b(\hat{\ell}^*)$$

$$N_I^* = \mu/(\hat{\ell}^* + \mu)$$

where

$$C^* > \underline{C}$$

it is clear that

$$N_I^* > \bar{N}_I > \tilde{N}_I$$

and

$$\tilde{\ell} > \hat{\ell}^*$$

(ii) $\tilde{\ell}$ is the value of $\bar{\ell}$ such that if, after some date, $\bar{\ell} < \tilde{\ell}$, the newly constructed machines never become even temporarily obsolescent (see Figure 7.4) :

$$\frac{b(\tilde{\ell})}{\tilde{\ell}} = b'(\bar{\ell}^*)$$

Two further properties of the behavior of the system may now be noted: First, at the time of the switch from the first stage, when the constraint $C \geq \underline{C}$ is binding, to the second stage, in which all prices are constant

---

at the time they are constructed will ever be constructed, we can safely ignore the other case.

[4] It is apparent that there will in general be two solutions to (A.5) but one of these occurs at a value of $\ell < \hat{\ell}^*$ and so will be ignored.

[5] Recall that astericks denote the long run equilibrium value of the variable.

and $C > \underline{C}$, the right hand side of (A.1) must be negative. (Otherwise, the minimum consumption constraint becomes binding.)

Second, if in the first stage, N_I is ever negative, it is always negative. To see this, observe that the derivative of N_I with respect to $\bar{\ell}$ has the same sign as $-\overset{\circ}{\ell}$, and the derivative of N_I with respect to the quasi-rents on the machine constructed is positive; if a different type of machine is constructed at time $t + \Delta t$ than at t, at fixed $\bar{\ell}$ quasi-rents of the newly constructed machines must be less than those of the machines constructed at t. This, in conjunction with the previous observations, implies that the optimal path must always lie above the $\overset{\circ}{N}_I = 0$ curve corresponding to $\bar{\ell}(t)$.

We can now distinguish two phases of the first stage (where $C = \underline{C}$). If $N_I < \bar{N}_I$ there may be recurrences in the techniques constructed as well as discontinuities. In the second phase, with $N_I > \bar{N}_I$, there can be no recurrences, since the wage must be rising monotonically. Moreover, once $\bar{\ell} < \tilde{\ell}$ and $N_I > \bar{N}_I$ there can be no discontinuities in the choice of technique. The approach to equilibrium is then monotonic and continuous.

APPENDIX B TO SECTION 7

The following example illustrates the interactions between the choice

of technique and intertemporal preferences in our simple technology. Assume

that there are only three types of machines, α, β and γ, with $\ell_\gamma >$

$\ell_\beta > \ell_\alpha$, and initially only γ is available. Assume also that the economy

wishes eventually to use type α. We wish to know, under what conditions

will β ever be produced.

If β is ever produced, there must occur a time when $p_\alpha = p_\beta$ and

$\dot{p}_\alpha > \dot{p}_\beta$. Thus, if

$$b_\alpha - \ell_\alpha \frac{b_\gamma}{\ell_\gamma} > b_\beta - \ell_\beta \frac{b_\gamma}{\ell_\gamma}$$

the economy will immediately produce α; if the reverse inequality holds,

the economy __may__ produce β.[1] Whether it will in fact produce β may easily

be determined.

Letting $v = p_\beta/p_\alpha$, and $m_{ij} = \dfrac{b_i \ell_j - \ell_i b_j}{b_j}$ we observe that when

γ is the marginal machine,

$$\dot{v} = -m_{\beta\gamma} + m_{\alpha\gamma} v$$

or

$$v(t) = \left(v_0 - \frac{m_{\beta\gamma}}{m_{\alpha\gamma}} \right) e^{m_{\alpha\gamma} t} + \frac{m_{\beta\gamma}}{m_{\alpha\gamma}}$$

At $t*$, $v(t) = p_\beta^*/p_\alpha^* = v*$, i.e. the ratio of their long run equilibrium

---

[1] For if β were ever produced, p_α could never subsequently exceed p_β
so long as γ were the marginal machine; and when β becomes the marginal
machine, $\dot{p}_\beta/p_\beta = \mu + \delta > \dot{p}_\alpha/p_\alpha$.

values. Solving backwards, this means that at

$$t^* - t_1 = t^* - \ln\left(\frac{m_{\beta\gamma}/m_{\alpha\gamma} - v^*}{m_{\beta\gamma}/m_{\alpha\gamma} - 1}\right) m_{\alpha\gamma}$$

$p_\alpha = p_\beta$, and for $t < t^* - t_1$, type β machine will be produced. But how long does it take to construct enough machines of type α to produce $\underline{C}$? The differential equation for K_α (if no β is produced) is simply

$$\dot{K}_\alpha = \left(1 - \frac{\underline{C}\ell_\gamma}{b_\gamma}\right) + (m_{\alpha\gamma} - \mu)K_\alpha$$

which can be immediately solved for $K_\alpha(t)$:

$$K_\alpha(t) = \frac{(1 - \underline{C}\ell_\gamma/b_\gamma)}{(m_{\alpha\gamma} - \mu)} (e^{(m_{\alpha\gamma}-\mu)t} - 1).$$

At t^*, $K_\alpha = \underline{C}/b_\alpha$, so the time required to build the requisite machines is

(7.23) $$t_2 = \frac{1}{m_{\alpha\gamma} - \mu} \ln\left(\frac{1 - \underline{C}/C^*}{m_{\alpha\gamma} - \mu}\right)$$

where C^* is steady state consumption.[2] β machines are built if and only if $t_2 > t_1$. Note that this does depend on _preferences_, for v^* depends on the pure rate of time preference δ : the greater δ, the smaller t_1, and hence the more likely β is to be constructed, as one might expect.

Thus the attempt by Robinson (e.g. [1960]) to derive a theory of the choice of technique based solely on technological considerations, without recourse to preferences, must inevitably be doomed to failure.

---

[2] Feasibility requires $C^* > \underline{C}$.