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EQUILIBRIUM AND DEMAND FOR MEDIA OF EXCHANGE IN A PURE EXCHANGE ECONOMY WITH TRANSACTIONS COSTS

Ross M. Starr

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Mathematical general equilibrium analysis has to date developed almost completely without reference to monetary phenomena or money as a commodity. Clearly this leaves us somewhat at sea in looking for a mathematical general equilibrium analysis of a monetary economy. This gap in the theory reflects actually a more fundamental lacuna, the absence of a theory of transactions.

Money's primary function--the one from which its other roles derive --is as a medium of exchange, a facilitator of transactions. But what do transactions look like in a general equilibrium model? The story that is usually told to describe the market of a general equilibrium model is something like this: All the traders get together in one place with an auctioneer. The latter eventually calls out equilibrium prices. All traders then put their excess supplies at those prices into a central pile of goods and withdraw from that pile their excess demands. Because the prices are equilibrium prices, supplies and demands balance and no trader is left with an unsatisfied demand.

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For a monetary analysis the main point to note here is that no one trades with anyone else. All trade is between individuals and "the market." In this story--indeed in all our abstract general equilibrium analysis--buyer and seller never meet. They confront not each other but the market.

A less sympathetic but more accurate story to go with this model is to say that the market is a black box with numbers, representing prices, posted on the outside of the box. Excess supplies are put into the black box, deliveries of excess demands are made from it. Our existence of equilibrium theorems merely tell us that there are numbers which could be posted on the outside of the box so that at those numbers the box will not be in net deficit of any commodity. Two points should be emphasized here. As noted before, there are no transactions between buyers and sellers. All exchange of goods takes place instantaneously in a single act. One does not first receive one good, then supply another, and then receive a third. Goods trade for other goods and they do so all at once. Clearly this eliminates an important function of media of exchange, bridging the gap between receipts and expenditures. For each trader the market constitutes itself as a trading partner with exactly complementary wants. In the terms of [14], for each trader the market sets itself up as that trader's mirror image. But as demonstrated there, in an economy where the mirror image of every trader is present, there is no need for media of exchange. Such a model of economic equilibrium gives us not so much no theory of money as a theory of no money.

The burden of this argument is of course that the reason it is difficult to get general equilibrium theory to give us a monetary theory is that it embodies no theory of transactions. Generating such a theory is an important part of the analysis below. Another part, one I suspect, with deeper implications, is a theory trading group size and formation. The use of a medium of exchange allows a sequence of pairs of traders to achieve a net trade through the exchange of goods and the medium of exchange (acting as quid pro quo) which could otherwise be achieved in pairwise trade only by foregoing quid pro quo or by trade of all the participants simultaneously rather than pairwise. An adequate theory along these lines would explain the two observations of casual empiricism that almost all transactions in our economy involve the transfer of money and most transactions involve no more than two parties. By contrast the traditional general equilibrium analysis assumes that all traders trade together at once and use no media of exchange.

An example of the point at issue is the three man-three good case of [14]. Let excess demands be $x_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. There are fundamentally two ways for trade to proceed. If the three traders wish to follow the procedure of the general equilibrium model they will constitute themselves as a market and trade individually with the market. Let $y_{i\mu}$ denote the flow of goods from the market, μ, to trader i. Positive elements indicate goods received by i, negative elements indicate goods delivered by i. Then if we let $y_{i\mu} = x_i$ for $i = 1, 2, 3$, excess demands will be satisfied and the market will clear.

Alternatively the three traders may engage in bilateral exchange. In this case let y_{ij} denote the flow of goods from j to i. Positive elements indicate goods received by i from j, negative elements indicate goods received by j from i. Dealing bilaterally then traders 1 and 2 agree on $y_{12} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ so that trader 1's excess demands are satisfied. In exchange for one unit of commodity 1 trader 2 receives one unit of commodity 2 for which he has no more use than he had for the commodity 1 he gave up. Commodity 2 here is used as a medium of exchange. It acts as money.

Next, traders 2 and 3 get together and agree on $y_{23} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Trader 2 uses the commodity 2 he received from trader 1 as money to purchase a unit of commodity 3 from trader 3. In bilateral exchange two pairs of traders using a good as a medium of exchange achieve an efficient allocation which would otherwise have required multilateral exchange, that is, the use of the three man market. Why two two-sided exchanges rather than one three-sided exchange? In general why does exchange not take place--as per the general equilibrium model--in one large market rather than in the many two-sided exchanges we observe? The answer seems to be that there are very significant costs associated with bringing large numbers of persons together. These costs are so high as to make it preferable to have large--and presumably also costly--flows of goods acting as media of exchange rather than incur the costs of bringing people together.

The deepest study along these lines is Professor Ostroy's [10]. The model there is an elaboration of the game theoretic model of a pure exchange economy. As noted above, one concept of the role of money is that money allows a sequence of exchanges among small groups of traders to achieve an end which would otherwise require a larger trading group. Ostroy notes that a game theoretic formulation of this phenomenon would be limitation on coalition (trading group) size in a market game. This has something of a sledge hammer air to it since it amounts to imposing a cost of 0 on the formation of small trading groups and an arbitrarily high cost on the formation of large groups. From this analysis we get an interesting and elaborate structure.

In the analysis below I set up several general equilibrium models of exchange. These point up explicitly: choice and cost of trading group size, choice and volume of flows of media of exchange, transactions costs,[1] effects of transactions costs on choice of medium of exchange. I fear this analysis is subject to the same criticism as Professor Ostroy's, being more elaborate in structure than in results. Nevertheless, it does give one a vocabulary and a model in which one can meaningfully and rigorously discuss transactions and media of exchange in a general equilibrium context.

Bilateral Exchange

The first step is to construct a model of a pure exchange economy in which all trade takes place in two sided exchanges. Let there be a finite set of traders T. For convenience, the elements of T will occasionally be numbered $1, 2, \ldots, |T|$, where $|T|$ denotes the number of elements in T.

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[1] Unfortunately taken to be linear to preserve convexity.

There are N commodities, denoted $1, 2, \ldots, N$. Commodity bundles are elements of the nonnegative orthant of E^N, E^{N+}. Each trader $t \in T$ has an endowment $\bar{x}_t$ and a utility function on his consumption bundles $u_t(y)$, $y \in E^{N+}$. u_t is real valued, continuous, semi-strictly quasi-concave, and increasing in its arguments. Semi-strict quasi-concavity means that u_t is quasi-concave and if $u_t(x_2) > u_t(x_1)$ then for $1 > \alpha > 0$ $u_t((1-\alpha)x_2 + \alpha x_1) > u_t(x_1)$. That is, indifference curves have no thickness.

In an economy where all trade takes place bilaterally, what do traders t's trading opportunities look like? In the Arrow-Debreu economy trading opportunities are subsets of N-dimensional space. This follows inasmuch as there are N commodities and only one party with whom they can be traded, the "market." In bilateral exchange there are $|T|-1$ parties with whom trader t may trade the N commodities, so the trading space is $(|T|-1)N$ dimensional. It turns out to be slightly more convenient to use a $|T|N$ dimensional space. Thus, we say t makes trades $y^t \in Y_t$. $Y_t \subseteq E^{|T|N}$. $y^t = (y^{t1}, y^{t2}, \ldots, y^{ti}, \ldots, y^{t|T|})$. $y^{ti} \in E^N$. The superscripts $1, 2, \ldots, i, \ldots, |T|$ denote the party with whom t is trading. The initial physical limitation on t's trades is his endowment. For any commodity he cannot deliver more in trade than he receives in trade plus his endowment. Thus,

$$Y_t = \{y^t | y^t \in E^{|T|N}, \sum_{i \in T} y^{ti} \geq -\bar{x}_t\} .$$

$\sum_{i \in T} y^{ti}$ is the sum of all trader t's deliveries and receipts from trade y^t.

In the absence of transactions costs the analysis here would be identical to that of [14]. Questions of choice of media of exchange, efficient coalition size are meaningful only in the presence of transactions costs.

There are convincing arguments to the effect that transactions should have diminishing marginal costs. To write a check for $1,000 requires less than 1,000 times the effort required to write a check for $1. Thus, transactions costs are likely to be strictly concave, giving rise to non-convexities in the opportunity sets. But in the absence of convexity it is not generally possible to prove the existence of equilibrium. This embarassment is partially eliminated by [12]. Unfortunately that analysis relies on the dimensionality of the choice space being held constant while the number of traders becomes large. Clearly this is going to be difficult to apply to in model where the dimension of the choice space varies linearly with the number of traders.

Another partial solution to problems of non-convexities in market economies is Shapley and Shubik's [11]. This approach depends much less on a given dimensionality of the commodity space. This may be a promising approach thought it is a little difficult to include transactions costs in a coalitional, non-market model. One approach would be to write individual utility functions as $u_t(\{x_1\}, \{x_2\}, \ldots, \{x_N\})$ where $\{x_i\} = \sum_{j \in T} [x_i^j]$, where x_i^j is the amount of commodity i received from trader j and the notation $[\ ]$ is defined as $[x] = \max(0, x - x^o)$, x^o being some predetermined level of the commodity in question. Such a utility function embodies the idea that in the presence of non-convex transactions costs it is better to receive a quantity of a good from one other trader than to receive the same quantity in small amounts from each of several traders.

Whatever the merits in versimilitude of strictly concave transactions costs, they are far outweighed by the disadvantages of intractability. I suspect that at this level one misses little of interest in the study of transactions by taking them to be divisible.

Transactions costs are resources consumed in the process of trade. They depend on the goods transacted, the transactors, and vary linearly in the quantity changing hands. Largely as a technical convenience I suppose that costs incurred by a trader are independent of whether that trader receives or delivers the goods. To do otherwise would require distinguishing between buying and selling transactions, and hence doubling the dimensionality of the space (see [13]). I think the loss of generality is more apparent than real inasmuch as for any pair of traders and any good traded, one will be the buyer, the other the seller. Only if they were both buyers and sellers of the same good to one another (apparently engaging in wash sales) does the possibility arise that different transactions costs of buying and selling might both be incurred in a single transaction of a given commodity. Some problems are being swept under the run but on the whole, I think the exclusion of wash sales above is the primary loss of generality.

Let $\ell_t^{in\nu}$ be the amount of trader t's commodity ν expended in making a unit trade of commodity n with trader i . Then the vector $\ell_t^{in} = (\ell^{tin1}, \ell^{tin2}, \ldots, \ell^{tinN})$ is a list of the goods used up by trader t in exchanging (delivering or receiving) one unit of commodity n with trader i . The list probably looks like: 3 minutes of t's time, 0.1 gallon gasoline, 0.05 unit of n lost in handling, and so forth. There is a similar list ℓ_i^{tn} of costs born by trader i in a unit exchange of commodity n with trader t. For each $t \in T$ there is an N dimensional transactions cost vector ℓ_t^{in} for each possible trading partner i and each commodity traded in n. Variations in these vectors across traders and commodities indicate that some traders are more convenient than others to trade with or that it is convenient to deal with some traders in some commodities and other traders in other commodities.

Trader t has $(|T|-1)N$ vectors ℓ_t^{in}. For bookkeeping convenience we assume that it is costless for t to trade with himself $\ell_t^{tn} = 0$ all n.

We place the transactions cost coefficients in a $|T|N \times |T|N$ rectangular array L_t. Let the typical element of this array be $\lambda_t^{inj\nu}$, $i, j \in T$, $n, \nu = 1, \ldots, N$. For $i = j$, $\lambda_t^{inj\nu} \equiv \lambda_t^{ini\nu} \equiv \ell_t^{in\nu}$. For $i \neq j$, $\lambda_t^{inj\nu} \equiv 0$. This gives us quite a few zero entries in the matrix, which certainly make it bulky and may cause confusion. They act as a conceptual and technical convenience below. Using the matrix L_t we will be able to subtract out from each trade t makes with another trader the transactions costs induced and attribute them to that trade. Use of a smaller matrix, an $N \times |T|N$ matrix for example, could subtract the same costs but would lack the attribution to the trades from which they arose. Let transactions $y_t \in Y_t$ be column vectors, having $|T|N$ row entries.

We want trader t to incur transactions costs both as buyer and seller. But if, for example, we characterize transactions costs incurred as $L_t y^t$ we would have t incurring costs for receiving goods but receiving transactions costs (a sort of commission in kind) for delivery goods. This is clearly not what we want. We require both sides of a transaction to incur transactions costs. Thus let A be an operator on vectors which replaces each co-ordinate of the vector with its absolute value. Then

$$Ay^t = \begin{pmatrix} |y^{t11}| \\ |y^{t12}| \\ \vdots \\ |y^{tin}| \\ \vdots \\ |y^{t|T|N}| \end{pmatrix}.$$

Transactions costs incurred by trader t in the transactions y^t will then be $L^t Ay^t$, a $|T|N$ dimensional column vector.

What do prices look like in the model of bilateral exchange? The Arrow-Debreu world with a single market and essentially a single--though many sided--exchange gives rise to a single price for each commodity, N prices in all. In the model of bilateral exchange, there are essentially as many "markets" as there are pairs of traders. Each such "market" has its own prices. Since transactions costs differ for every pair of traders, it seems reasonable that prices should do so as well. Further, each trader is already making choices in a $|T|N$ dimensional space, so there is no added complication introduced by letting him face $|T|$ sets of N prices.

Prices facing $t \in T$ are $p^t \in P_t$. $P_t = \{p^t | p^t \geq 0,\ p^t \in E^{|T|N},$ for each $i \in T,\ \sum_{n=1}^{N} p_n^{it} = 1\}$. That is, prices for a pair of traders are subject to the traditional restriction to the simplex.

Conceivable prices for the economy, all possible pairs of traders, are elements of

$$P = \{p | p \in \bigtimes_{t \in T} P_t , \quad p^{ij} = p^{ji} \quad \text{for all} \quad i, j \in T\} .$$

The restriction here is merely that traders i and j should agree on the prices at which they are trading.

From the utility function on consumption we can derive a well behaved utility function on transactions. Let $\sigma(\bar{x}_t, y^t) = \sum_{i \in T} y^{ti} + \bar{x}_t$. σ is merely a summation operator taking $|T|N$ dimensional trades, N dimensional endowments and summing them to an N-dimensional consumption choice bundle. Thus let $v_t(\bar{x}_t, y^t) = u_t(\sigma(\bar{x}_t, y^t))$. v_t is a synthetic utility function on transactions. To any transaction it imputes the utility t would derive from consuming the results of the transaction in addition to what is left of his endowment after the transaction is concluded. u_t is real-valued, continuous, quasi-concave, and increasing; v_t retains these properties.

Pairwise trades are subject to bilateral price constraints. Thus t's trading opportunities are a subset of the price constraint set

$$W_t(p) = \{y' | y' \in E^{|T|N}, \; p^{ti} \cdot y'^{i} = 0 , \quad \text{all} \quad i \in T\}$$

for $p \in P_t$. $W_t(p)$ is a choice of goods which t may market consistent with prices p . However, $W_t(p)$ includes neither restriction of feasibility nor subtraction of transactions costs. These are included in $\eta_t(p)$.

Let $\eta_t(p) = \{(y', y) | y' \in W_t(p), \; y = y' - L_t A y', \; y \in Y^t\}$ for $p \in P_t$.

The distinction here is between goods actually changing hands in trade, and what is left from those trades after transactions costs have been subtracted out. In $\eta_t(p)$, y' is a vector of goods exchanged in bilateral trade consistent with prices p . y is the corresponding vector of trades net of transactions costs. $\eta_t(p)$ is trader t's opportunity set. Given prices p he can choose any element of $\eta_t(p)$. The criterion he will use is the utility he will receive from available trades net of transactions costs.

Thus we derive t's choice correspondence $\gamma_t(p) = \{(y', y) | (y', y) \in \eta_t(p),\ y \text{ maximizes } v_t(\bar{x}_t, y) \text{ for all } (y', y) \in \eta_t(p)\}$. For any prices $p \in P_t$, $\gamma_t(p)$ tells us what transactions, y' , trader t will seek to make and what this transaction yields net of transactions costs, y . $\gamma_t(p)$ is the equivalent in a transactions theory of the traditional household excess demand functions of general equilibrium theory. In the standard general equilibrium theory global excess demand is derived from household excess demand by summation over households. In a transactions theory where every pair of traders constitutes a separate "market" a global configuration of excess demands is derived from the individual excess demands by Cartesian multiplication.

Let $\gamma'_t(p) = \{y' | (y', y) \in \gamma_t(p)\}$. γ' is merely the marketed part of γ ; these are goods actually demanded or offered on the pairwise markets without subtraction for transactions costs. Let $\Gamma(p) = \bigtimes_{i \in T} \gamma'_t(p^t)$ where $p \in P$, and p^t is the t^{th} part of p . An element of the correspendence $\Gamma(p)$ is a list at prices $p \in P$ of the trading plans of all traders at those prices. For each pair of traders it shows how much of each good each member of the pair expects to get from trading with the other, how much of each good each member of a pair expects to deliver to the other.

We are almost ready to discuss equilibrium in the bilateral exchange economy. We have to bound the variables under consideration by a technical constraint which is never binding in equilibrium. Let M be as in Lemma 6 below. Then let $\Psi = \bigtimes_{|T|N} [-2M, 2M]$, where the notation $\bigtimes_{|T|N}$ denotes $|T|N$-fold Cartesian product. Ψ is a cube in $E^{|T|N}$ with side $4M$. Ψ is a compact subset of $E^{|T|N}$ containing the achievable set as a proper subset. It will often be convenient to limit our attention to Ψ rather than consider all of $E^{|T|N}$. Let

$$\Omega = \bigtimes_{|T|} \Psi .$$

Let $\hat{\eta}_t(p) = \eta_t(p) \cap (\Psi \times \Psi)$ let $\hat{\gamma}_t(p) = \{(y', y) | (y', y) \in \hat{\eta}_t(p), y$ maximizes $v_t(\bar{x}_t, y)$ for all $(y', y) \in \hat{\eta}_t(p)\}$. The circumflex notation here indicates restriction to Ψ, a bounded subset of $E^{|T|N}$. Let $\hat{\gamma}_t'(p^t) = \{y' | (y', y) \in \hat{\gamma}_t(p^t)\}$ and let $\hat{\Gamma}(p) = \bigtimes_{t \in T} \hat{\gamma}_t'(p^t)$ where $p \in P$ and $p^t \in P_t$ is the t^{th} part of p.

There are substantial problems in this model in insuring that choice functions be continuous for prices in the neighborhood of zero. The well known difficulty here is that opportunity sets-- η and $\hat{\eta}$ in this model --generally have discontinuities when incomes $(p \cdot \bar{x}_t)$ are zero and some prices are zero. The way to avoid the difficulties this discontinuity poses is to keep incomes from being zero. There are essentially two ways to do this. The earliest and simplest approach (used for example in Debreu's [3]) is to make the unfortunately unrealistic assumption that all traders have positive endowments of all goods. Then no matter what p on the simplex is chosen as prices, the value of each trader's endowment is positive. The alternative, more aesthetic, and far more complex, approach is to make assumptions insuring that

(i) at least one trader has positive income, and

(ii) preferences are such that if one trader has positive income then all traders have positive income. (See McKenzie's [7], or Arrow-Hahn [1]) The latter approach, though it recommends itself by the weakness of its assumptions, requires a substantially enlarged analytic structure.

In this study positivity of income is rather less important than in the traditional analysis. It is also harder to achieve. The reduced importance comes inasmuch as at a zero value of endowment with some prices zero the trader may be unable to trade. He must have the commodity inputs to cover transaction costs. If the trader is not already endowed with these inputs he will have to acquire them in trade and may be unable to do so because the inputs may have positive price. This may eliminate what would otherwise be a large class of discontinuities. The increased complication comes from the multiplicity of small "markets," one for each pair of traders. Positive income on one need not assure positive income on any other. Such an inference would be justified only in the case of costless arbitrage--that is, no transactions costs--between the two "markets." Hence, stronger assumptions of coinciding wants would be required to assure the absence of discontinuities at zero prices for each pair of traders. The necessary assumption would probably be something like: For all $t \in T$ and for each $i \in T$ there is $\bar{x}^{tn_i} > 0$ for some n_i so that no matter what i's consumption y^i , y^i plus some positive amount of good n_i less transactions costs incurred on accepting the n_i is strictly preferred to y^i .

This is a strong assumption: weaker than strict positivity of endowment--though how much weaker depends on the structure of preferences--but much stronger than the indirect productive relatedness or irreducibility used by the standard analysis. If the above assumption really is necessary to insure equilibrium then there may be a moderately interesting family of disequilibria to be deduced from that necessity. Unemployment in such a model would be the result of failure of endowment to cover transactions costs.

Since the alternative seems to be the investigation of a rather substantial can of worms, I will assume strict positivity of endowment. Thus assume for all $t \in T$, $\bar{x}^t \gg 0$.

Lemma 1: $\eta_t(p)$, $\hat{\eta}_t(p)$ are continuous, convex, and nonnull for each $t \in T$, all $p \in P_t$.

Proof of Lemma 1: Convexity and nonemptiness are obvious. Continuity requires upper and lower semi-continuity. Continuity of the marketed goods (the first of the two $|T|N$ dimensional entries in η , $\hat{\eta}$) implies continuity of the trade net of transactions costs (the second entry) by continuity of the transformation $y = y' - L_t A y'$. Therefore it is sufficient to show continuity of y' , the first entry.

Upper semi-continuity follows directly from continuity of $p \cdot y'$. To show lower semi-continuity let $p^\nu \to p^o$, $y^o \in W_t(p^o) \cap Y_t$. We seek

$y^\nu \in W_t(p^\nu) \cap Y_t$ so that $y^\nu \to y^o$. Let $y' \gg 0$, $y'' \ll 0$, $y', y'' \in Y_t$. Note that the existence of such y', y'' depends on strict positivity of the trader's endowment. Let $\overline{y', y^o}$ denote the line segment between y' and y^o. Convexity of Y_t implies that $\overline{y', y^o} \subset Y_t$, $\overline{y'', y^o} \subset Y_t$. Note that $p^\nu \cdot y' > 0$ all ν, $p^\nu \cdot y'' < 0$ all ν. For ν such that $p^\nu \cdot y^o < 0$ let y^ν be the point of intersection of $\overline{y', y^o}$ and $W_t(p^\nu)$. Such a point always exists by the intermediate value theorem. For ν so that $p^\nu \cdot y^o > 0$ let y^ν be the point of intersection of $\overline{y'', y^o}$ and $W_t(p^\nu)$. If $p^\nu \cdot y^o = 0$ let $y^\nu = y^o$. Then y^ν exists and is unique for each ν. $y^\nu \in W_t(p^\nu) \cap Y_t$. $y^\nu = \alpha_1^\nu y' + \alpha_2^\nu y'' + \alpha_3^\nu y^o$; $\alpha_1^\nu, \alpha_2^\nu, \alpha_3^\nu \geq 0$, $\alpha_1^\nu \cdot \alpha_2^\nu = 0$. $p^\nu \to p^o$, $p^\nu \cdot y^\nu = 0$ implies $p^o \cdot y^\nu \to 0$.

$$p^o \cdot (\alpha_1^\nu y' + \alpha_2^\nu y'' + \alpha_3^\nu y^o) \to 0 ,$$

$$\alpha_1^\nu p^o \cdot y' + \alpha_2^\nu p^o \cdot y'' + \alpha_3^\nu p^o \cdot y^o \to 0 .$$

Hence $\alpha_1^\nu, \alpha_2^\nu \to 0$, $y^\nu \to y^o$.

QED.

Lemma 1 says that trading opportunities are a continuous function of prices. They are also convex and nonnull. When suitably bounded--by Ψ, as in $\hat{\eta}$ --they are compact. The next step is to show that in the face of these well behaved opportunities, choices are well behaved.

Lemma 2: $\hat{\gamma}_t(p)$ is nonnull, convex, and upper semi-continuous for each t, all $p \in P_t$.

Proof of Lemma 2: $\hat{\eta}_t(p)$ is compact so $v_t(\bar{x}_t, y)$ achieves a maximum on $\hat{\eta}_t(p)$. Hence $\hat{\gamma}_t(p)$ is nonnull. Convexity is trivial. Upper semi-continuity follows from [Berge [2], p. 116; or Debreu [3], 1.8.k(4)].

QED.

The reader who has mastered the intricacies of the notation here will note that Lemma 2 tells us very little about the behavior of the choice function γ_t . Rather, it suggests that if the potentially unruly (unbounded or perhaps empty) choice function γ can be held within bounds (Ψ) it will be quite well behaved. What will happen of course is that in equilibrium the restriction to Ψ will be redundant, and can be dropped. For convenience --but mainly to give a spurious impression of authenticity--I will refer to $\hat{\gamma}_t$ and γ_t indifferently as choice functions though the former is more correctly a truncated choice function.

The individual choice functions consist in a list, for given prices, of goods that each trader seeks to trade with each of the other traders, and a corresponding trade net of transactions costs which he anticipates receiving as a result. One of the two traders of a pair may plan on making very different trades with the other than the latter had anticipated. Trader i may seek to supply trader j with more of commodity n than trader j wants at the price prevailing for the pair. It seems reasonable to call this an excess supply of commodity n for pair ij . Similarly if trader j seeks to acquire more of commodity m from trader i then the latter had planned to deliver, there is said to be an excess demand for commodity m for the pair i, j . Note that a commodity may be in excess supply for one pair and in excess demand for another. If trader i seeks to make trade y^{ij} and trader j seeks to make y^{ji} then the excess demand for pair i, j is $z^{ij} = z^{ji} = y^{ij} + y^{ji}$. Thus if i seeks to supply j with

3 units of m, $y^{ijm} = -3$, and j demands 4 units, $y^{ijm} = 4$, then, $z^{ijm} = z^{jim} = y^{ijm} + y^{jim} = -3 + 4 = 1$. Thus, for the economy as a whole, excess demand is a list for each pair of traders of excess demand or supply of each good for that pair.

Let $\bar{\Omega} = \{x | \frac{1}{2} x \in \Omega\}$. That is $\bar{\Omega}$ is the set of points in $E^{|T|^2 N}$ no more than twice as large co-ordinate wise as elements of Ω. Then let $\zeta: \Omega \to \bar{\Omega}$.

<u>Definition</u>: Let $y \in \Omega$ and let $z = \zeta(y)$, then $z^{ijn} \equiv y^{ijn} + y^{jin}$.

ζ is a simple arithmetic operation. We will think of it as mapping anticipated trade vectors of all traders (elements of Ω) into the corresponding excess demand vectors for each pair of traders. Note that if $z = \zeta(y)$, $z^{ij} = z^{ji}$.

For proving the existence of equilibrium the function ρ mapping excess demands into prices is traditional and convenient $\rho: \bar{\Omega} \to P$, $\rho(z) = \{p^o | p^o$ maximizes $p \cdot z$ for all $p \in P\}$. Note without proof that ρ is an upper semi-continuous nonnull convex correspondence.

Our family of functions describing relations in the economy is now complete. Γ describes supplies and demands as a function of prices, ζ describes excess supplies and demand as a function of supplies and demands and ρ describes prices as a function of excess supplies and demands.

Consider $(p, w, \bar{w}) \in P \times \Omega \times \bar{\Omega}$.

Let
$$\hat{\Gamma}^*(p, w, \bar{w}) = \hat{\Gamma}(p)$$
$$\zeta^*(p, w, \bar{w}) = \zeta(w)$$
$$\rho^*(p, w, \bar{w}) = \rho(\bar{w}).$$

Then $\rho^* \times \hat{\Gamma}^* \times \zeta^*: P \times \Omega \times \bar{\Omega} \to P \times \Omega \times \bar{\Omega}$. $\rho^* \times \hat{\Gamma}^* \times \zeta^*$ is nonnull upper-semicontinuous, and convex.

Having set up rather an elaborate model, it is a bit late to inquire what is an equilibrium in this model. Better late than never. In traditional general equilibrium models equilibrium consists in prices and demands consistent with those prices such that excess demand for all goods is non-positive. I will adopt the same criterion here. The main difference between equilibria of this model and of the traditional model is that here excess demands are non-positive on each of the $|T|(|T|-1)/2$ pairwise markets. Thus

<u>Definition</u>: Let $(p^o, y^o, z^o) \in P \times E^{|T|N} \times E^{|T|N}$.

(p^o, y^o, z^o) is said to be an equilibrium for the economy if $w^t = y^{ot} - L_t A y^{ot}$ and w^t maximizes $v_t(\bar{x}_t, w^t)$ for all w that $(y, w) \in \eta_t(p^{ot})$ for all $t \in T$; if $z^{oij} = y^{oij} + y^{oji}$ for all $i, j \in T$, and if $z^o \leq 0$.

Note that the definition of equilibrium neither requires nor implies boundedness of demand or excess demand functions. This will appear as a conclusion in equilibrium.

<u>Lemma 3</u>: Let $(y', y) \in \hat{\gamma}_t(p)$, $p \in P_t$ and $(y', y) \in$ Interior $\Psi \times \Psi$. Then $(y', y) \in \gamma_t(p)$.

<u>Proof of Lemma 3</u>: Suppose not. Then there is $(w', w) \in \eta_t(p)$ so that $v_t(\bar{x}_t, w) > v_t(\bar{x}_t, y)$ but $(w', w) \notin \Psi \times \Psi$. Then for $\alpha > 0$ sufficiently small $(\alpha(w', w) + (1-\alpha)(y', y)) \in$ Interior $\Psi \times \Psi$ and by semi-strict quasi-concavity $v_t(\bar{x}_t, (\alpha w + (1-\alpha)y)) > v_t(\bar{x}_t, y)$. But $(\alpha(w', w) + (1-\alpha)(y', y))$

$\in \hat{\eta}_t(p)$ so contrary to hypothesis $(y', y) \notin \hat{\gamma}_t(p)$. The contradiction proves the lemma.

QED.

<u>Lemma 4</u>: Let (p^o, y^o, z^o) be a fixed point of $\rho^* \times \hat{\Gamma}^* \times \zeta^*$, and let $(y^{ot}, y^{ot} - L_t A y^{ot}) \in$ Interior $\Psi \times \Psi$ for all $t \in T$. Then (p^o, y^o, z^o) is an equilibrium for the economy.

<u>Proof of Lemma 4</u>: Definition of $\hat{\Gamma}^*$ gives that $(y^{ot}, y^{ot} - L_t A y^{ot}) \in \hat{\gamma}_t(p^{ot})$ for all $t \in T$. By Lemma 3, $(y^{ot}, y^{ot} - L_t A y^{ot}) \in$ Interior $\Psi \times \Psi$ implies that $(y^{ot}, y^{ot} - L_t A y^{ot}) \in \gamma_t(p^{ot})$. $z^{oij} = y^{oij} + y^{oji}$ all $i, j \in T$ by definition of ζ^* . It remains to show only that $z^o \leq 0$.

We have p^o maximizes $p^o \cdot z^o$ for all $p \in P$. But $p^{oij} \cdot z^{ij} = p^{oij} \cdot y^{oij} + p^{oij} \cdot y^{oji} = \rho + 0 = 0$. Hence $p^o \cdot z^o = 0$. But $p^o \geq 0$. Therefore $z^o \leq 0$.

QED.

<u>Lemma 5:</u> Let (p^o, y^o, z^o) be a fixed point of $\rho^* \times \hat{\Gamma}^* \times \zeta^*$. Then $\sum_{i\in T} y^{oti} \leq \sum_{\substack{i\in T\\ i\neq t}} \bar{x}^i$, $\sum_{i\in T} y^{oti} < \sum_{i\in T} \bar{x}^i$ for each $t \in T$.

<u>Proof of Lemma 5</u>: Just as in the proof of Lemma 4, we note that p^o maximizes $p^o \cdot z^o$ for all $p \in P$ implies that $z^o \leq 0$. Thus $y^{oij} + y^{oji} \leq 0$ for all $i, j \in T$. $y^{oi} \in Y_i$ all i. Hence $\sum_{j\in T} y^{oij} \geq -\bar{x}^i$.

Consider $t \in T$. $\sum_{\substack{i\in T\\ i\neq t}} \sum_{j\in T} y^{oij} \geq -\sum_{\substack{i\in T\\ i\neq t}} \bar{x}^i$, $\sum_{\substack{i\in T\\ i\neq t}} \sum_{\substack{j\in T\\ j\neq t}} y^{oij} = \sum_{\substack{i,j\in T\\ i,j\neq t}} y^{oij} + y^{oji}$ ≤ 0. But $\sum_{\substack{i\in T\\ i\neq t}} \sum_{j\in T} y^{oij} = \sum_{i\in T} y^{oit} + \sum_{\substack{i\in T\\ i\neq t}} \sum_{\substack{j\in T\\ j\neq t}} y^{oij} \leq \sum_{i\in T} y^{oit}$. $y^{oit} + y^{oti} \leq 0$ implies $\sum_{i\in T} y^{oit} \leq -\sum_{i\in T} y^{oti}$. Thus $-\sum_{i\in T} y^{oti} \geq -\sum_{\substack{i\in T\\ i\neq t}} \bar{x}^i$, $\sum_{i\in T} y^{oti} \leq \sum_{\substack{i\in T\\ i\neq t}} \bar{x}^i$, and $\sum_{i\in T} y^{oti} < \sum_{i\in T} \bar{x}^i$.

QED

Though the above lemmas imply the boundedness of final trades (the sum of individual trades) they tell us very little about the size of individual trades. Bounded sums could be the sum of arbitrarily large purchases and sales. With the aid of the following assumption we can use the boundedness of final trades to prove the boundedness of individual trades.

<u>Assumption</u>: For any n, $\ell_t^{in\nu} > 0$ for some ν, all $t, i \in T$. The assumption states that a flow of any good between any two traders imposes a transaction cost of a non-zero amount of some commodity. The following lemma notes that in the light of the above assumption an economy with finite

resources can sustain only a bounded volume of individual transactions.

Lemma 6: Let

$$M \equiv \sup_{t\in T} \sup_{n=1,\ldots,N} \{ \sum_{i\in T} y^{tin} | \sum_{i\in T} L_t^i A y^{tin} \leq \sum_{i\in T} \bar{x}^i,\ y^t \in Y_t,\ \sum_{i\in T} y^{ti} \leq \sum_{\substack{i\in T\\ i\neq t}} \bar{x}^i ,$$

$$y^t - L_t A y^t \in Y_t \} .$$ M exists and is finite.

$$M \geq \sup_{t\in T} \sup_{n=1,\ldots,N} \{ |y^{tin}| \Big| \sum_{i\in T} L_t^i A y^{tin} \leq \sum_{i\in T} \bar{x}^i,\ y^t \in Y_t,\ \sum_{i\in T} y^{ti} \leq \sum_{\substack{i\in T\\ i\neq t}} \bar{x}^i,$$

$$y^t - L_t A y^t \in Y_t \} .$$

Proof of Lemma 6:

$$-\bar{x}^t \leq \sum_{i\in T} y^{ti} \leq \sum_{\substack{i\in T\\ i\neq t}} \bar{x}^i$$

$$\sum_{i\in T} (y^{ti} - L_t^i A y^{ti}) \geq -\bar{x}^t$$

$$-\sum_{i\in T} L_t^i A y^{ti} \geq -\sum_{i\in T} y^{ti} - \bar{x}^t \geq -\sum_{i\in T} \bar{x}^i$$

$$0 \leq \sum_{i\in T} L_t^i A y^{ti} \leq \sum_{i\in T} \bar{x}^i$$

Let $\sup_{t,i\in T} \sup_{\substack{\nu=1,\ldots,N\\ n=1,\ldots,N}} \dfrac{\sum_{i\in T} \bar{x}^{i\nu}}{\ell_t^{in\nu}} = k$. Then $\sum_{i\in T} y^{tin} \leq k$ all n so the set in question is bounded and hence has a least upper bound M . The inequality follows by noting that $L_t A y_t = L_t A(-y_t)$ so that

$$M = \sup_{t \in T} \sup_{n=1,\ldots,N} \{ \sum_{i \in T} |y^{tin}| \, \Big| \, \sum_{i \in T} L_t^i A y^{tin}, \; y^t \in Y_t, \; \sum_{i \in T} y^{ti} \leq \sum_{\substack{i \in T \\ i \neq t}} \bar{x}^i, \; y^t - L_t A y^t \in Y_t \}$$

$$\geq \sup_{t \in T} \sup_{n=1,\ldots,N} \{ |y^{tin}| \, \Big| \, \sum_{i \in T} L_t^i A y^{tin} \leq \sum_{i \in T} \bar{x}^i, \; y_t \in Y_t, \; \sum_{i \in T} y^{ti} \leq \sum_{\substack{i \in T \\ i \neq t}} \bar{x}^i,$$

$$y^t - L_t A y^t \in Y_t \}$$

QED.

Lemma 7: Let (p^o, y^o, z^o) be a fixed point of $\rho^* \times \hat{\Gamma}^* \times \zeta^*$. Then for all $t \in T$, $\sup_{\substack{i \in T \\ n=1,\ldots,N}} |y^{oitn}| \leq M$, $(y^{ot}, y^{ot} - L_t A y^{ot}) \in$ Interior $\Psi \times \Psi$.

Proof of Lemma 7: From Lemma 5 we have $\sum_{i \in T} y^{oti} \leq \sum_{\substack{i \in T \\ i \neq t}} \bar{x}^i$. By hypothesis $y^{ot} - L_t A y^{ot} \in Y_t$, $y^{ot} \in Y_t$, so by Lemma 6 $\sum_{i \in T} y^{otin} \leq M$, $n = 1, \ldots, N$ and $|y^{otin}| \leq M$. The inequality implies inclusion in the interior.

QED.

Theorem 1: There is an equilibrium for the bilateral exchange economy.

Proof of Theorem 1: $\rho^* \times \hat{\Gamma}^* \times \zeta^* : P \times \Omega \times \bar{\Omega} \to P \times \Omega \times \bar{\Omega}$. $P \times \Omega \times \bar{\Omega}$ is compact, convex. $\rho^* \times \hat{\Gamma}^* \times \zeta^*$ is upper-semicontinuous and convex. Hence by the Kakutani fixed point theorem there is a fixed point (p^o, y^o, z^o). By Lemma 7, $(y^{ot}, y^{ot} - L_t A y^{ot}) \in$ Interior $\Psi \times \Psi$ for all $t \in T$. So by Lemma 4 (p^o, y^o, z^o) is an equilibrium.

Theorem 1 says that there are prices for every pair of traders (perhaps different for each pair) so that excess demands for all goods are non-positive for each pair of traders. This is an equilibrium.

This model also gives a well defined and explicit demand for and flow of media of exchange, "money"'s. To discover this, revert to the concept of "money" in [14]. Goods exchanged not to fulfill wants in themselves but in payment for other goods are media of exchange. How can we isolate them? One rather approximate approach--the one I will use--is to say that any flow of goods which does not result in a commodity being consumed by its recipient is a flow as medium of exchange.* In order to distinguish these flows, a modest increase in the mathematical structure is required:

Definition: Let $x \in E^m$, k be a positive integer. A k-fold non-redundant additive decomposition of x is a set $\{x_1, x_2, \ldots, x_i, \ldots, x_k\}$ so that $x_i \in E^m$, $\Sigma x_i = x$, and sign x_i^j = sign x^j for all $j = 1, 2, \ldots, m$ all $i = 1, \ldots, k$.

Let $D(x,k) = \{(x_1, x_2, \ldots, x_k) | \{x_1, \ldots, x_k\}$ is a k-fold non-redundant additive decomposition of $x\}$.

Consider a trade $y^t \in Y_t$. Those of y^t's flows fulfilling "ultimate wants" are merely those which are not balanced out by another flow of opposite sign. That is, "ultimate wants" are $\sum_{i \in T} y^{ti}$. Further, a $(|T|-1)$-fold non-redundant additive decomposition of $\sum_{i \in T} y^{ti}$ is a list of minimal flows of goods to and from t so that his net trade would be $\sum_{i \in T} y^{ti}$. A list summing to the same quantity but not a non-redundant additive decomposition would contain some flows which could be deleted or reduced in magnitude leaving the sum unchanged.

We can decompose any element $y^t \in Y_t$ into two parts: an element of $D(\sum_{i \in T} y^{ti}, |T|-1)$ and a remainder. The former represents minimal flows

*This approach purposefully ignores some flows whose purpose is arbitrage, not mediation of exchange. So far as I can tell, at this level of abstraction the distinction is without substance.

to achieve the net trade, the remainder is taken to be flows in mediation of exchange. Thus we define a correspondence φ which for any trade identifies the flows in mediation of exchange which are part of the trade. Generally there will be some ambiguity as to whether a specific flow is for mediation or satisfaction of wants. If a trader receives two apples, keeps one and re-trades the other, which of the two apples received is a medium of exchange? The ambiguity prompts us to let φ be a correspondence. Thus $\varphi: \Psi \to \Psi$,

$$\varphi(y) = \{x | x = y - y^*, \; y^* \in D(\sum_{i \in T} y^i, \; |T|) \text{ and } y^* = w \text{ minimizes } |y - w| \text{ for all } w \in D(\sum_{i \in T} y^i, \; |T|)\} .$$

$\varphi(y)$ represents the volume of use of media of exchange involved in trade y.

A trader's demand for media of exchange is a function of prices. Denote this demand $\theta(p)$. Let $\theta_t(p) = \{x | x \in \varphi(y), \; y \in \gamma_t'(p)\}$, for $p \in P_t$. In order to get the social demand for media of exchange we merely take the Cartesian product of θ_t across traders. Thus, the social demand for media of exchange is $\Theta(p) = \bigtimes_{t \in T} \theta_t(p^t)$, $p \in P$. $\theta: P \to \Omega$.

<u>Lemma 8</u>: $D(x,m)$ is a continuous convex correspondence.

<u>Proof of Lemma 8</u>: $w, z \in D(x,m)$ implies $\sum w^i = \sum z^i = x$. Thus $\sum(1-\alpha)w^i + \sum \alpha z^i = (1-\alpha)\sum w^i + \alpha \sum z^i = x$. This shows convexity.

Upper and lower semicontinuity follow directly from the continuity of the addition operation.

QED.

Lemma 9: $\varphi(y)$ is continuous and convex.

Proof of Lemma 9: Convexity follows from convexity of $D(\sum_{i \in T} y^i, |T|)$ and minimization of length. Continuity follows from continuity of D and minimization of length.

QED.

Theorem 2: For each $t \in T$, all $p \in P_t$, $\theta_t(p)$ is upper semi-continuous and convex. For all $p \in P$, $\Theta(p)$ is upper semi-continuous and convex.

Proof of Theorem 2: $\theta_t(P) = \{x | x \in \varphi(y), y \in \gamma_t^0(P)\}$. Upper-semicontinuity follows from continuity of φ, upper semi-continuity of γ_t^0. Let x', x'' $\in \theta_t(p)$ then there are y', y'' so that $y', y'' \in \gamma_t^0(p)$, $x' \in \varphi(y')$, $x'' \in \varphi(y'')$. Convexity of γ_t^0 implies for $0 \leq \alpha \leq 1$, $\alpha y' + (1-\alpha)y''$ $\in \gamma_t(p)$. But then $\alpha x' + (1-\alpha)x'' \in \varphi(\alpha y' + (1-\alpha)y'')$ so $\theta_t(p)$ is convex.

QED.

Theorem 2 notes that demand for media of exchange is an upper semi-continuous convex function of prices. Once transactions are completed there is no final demand, in this model for media of exchange: $\sum_{i \in T} x^i = 0$ for $x \in \theta_t(p)$.

Multilateral Exchange

The analysis above is a study of the equilibrium of an economy where all trade takes place between pairs of traders. This is distinct from the standard general equilibrium model where all trade takes place between individual traders and "the market." In the standard model there is no role for media of exchange in the fashion of the bilateral model. In the latter, media of exchange allow traders to purchase their wants from some traders and render their excess supplies to others, but in the standard model there is only one other "trader" with whom a trader can trade--the market. The use of media of exchange in bilateral trade permits the achievement of net trades which in the absence of media of exchange could be achieved only by the use of the single market of the standard model. Clearly the use of media of exchange and bilateral exchange is not costless. There are large costs incurred in exchanging the media of exchange which would not be incurred in a single market economy with no medium of exchange. Nevertheless, most transactions in modern economies take place bilaterally and use media of exchange. One would hope that this indicates some technical superiority of multiple over unitary markets. The most direct way to investigate this point is in a model containing both bilateral exchange and the standard model as special cases.

Clearly the cases of trade between pairs of traders and trade among all traders at once are polar. We may consider as well trade among triples of traders, quadruples, and so forth. Thus, let the set of possible trading groups be $\mathcal{S} = \{S | S \subseteq T, |S| \geq 2\}$. That is, let the list of groups which can form to trade be the power set of T excluding the empty set and singletons. Just as above we generated a theory of trading in pairs, we can here find a theory of trading in elements of $\mathcal{S}$. Such a theory includes as special cases both the theory of bilateral exchange developed above, and the standard theory of a unitary market. This follows since $T \in \mathcal{S}$ and $\{i, j\} \in \mathcal{S}$ for any $i, j \in T$, $i \neq j$.

The analysis now proceeds much as before. I will use the tilde notation, $\tilde{x}$, to denote an element of the multilateral exchange argument corresponding to the element without tilde, x, in the bilateral exchange analysis.

Thus for each $t \in T$, t chooses trades from $\tilde{Y}_t = \{y^t | y^t \in E^{N|\mathcal{A}|}$, $y^t = (y^{tS_1}, y^{tS_2}, \ldots, y^{tS}, \ldots, y^{tT})$, $S_1, S_2, \ldots, T \in \mathcal{A}$, $y^{tS} \equiv 0$ for all $S \in \mathcal{A}$ such that $t \notin S$; $\sum_{S \in M} y^{tS} \geq -\bar{x}^t\}$. $\tilde{Y}_t$ is merely Y_t with elements of $\mathcal{A}$ replacing the pairs of traders considered in Y_t. For each trader the transactions cost matrix is $\tilde{L}_t$. Whereas L_t is $|T|N \times |T|N$, $\tilde{L}_t$ is $|\mathcal{A}|N \times |\mathcal{A}|N$. Prices are elements of $\tilde{P} = \{p | p \in E^{|\mathcal{A}|N}$, $p \geq 0$, $\sum_{n=1}^{N} p_n^S = 1$, for all $S \in \mathcal{A}\}$. $\tilde{\Psi} = \bigtimes_{|\mathcal{A}|N} [-2\tilde{M}, 2\tilde{M}]$, $\tilde{\Omega} = \bigtimes_{|\mathcal{A}|} \tilde{\Psi}$. $\tilde{\eta}_t$ and $\tilde{\gamma}_t$ can be readily defined and the analysis goes through as before. Thus

Theorem 3: There is an equilibrium for the multilateral exchange economy.

Theorem 4: For each $t \in T$, all $p \in \tilde{P}_t$, $\tilde{\theta}_t(p)$ is upper semicontinuous and convex. For all $p \in \tilde{P}$, $\tilde{\theta}(p)$ is upper semicontinuous and convex.

It is unfortunately very much easier to say where the theory should go from here than to get it to go there. In fact we have a structure for a theory of markets and a theory of media of exchange. But substantial qualitative results seem to require heroic assumptions. One result we would like to get out of the bilateral exchange model is that most transactions use the same commodity--"money"--as medium of exchange. The straightforward way to achieve this in the model of bilateral exchange is to take the transactions cost associated with a given commodity to be zero (contrary to previous assumption) or very small compared to costs associated with other goods (which may be taken to be quite large). There are several objections to this approach. The use of sledge hammers rather than subtlety in the formulation of assumptions is to be frowned upon; further, this approach obscures fundamental questions. For what we seek is not a restatement but an explanation of our casual observation. It is tautological to say that money is a good medium for exchange because it has low transactions costs. Rather, we seek a consistent rationale for those low transactions costs.

From the model of multilateral exchange we might well seek an explanation of why most transactions take place bilaterally. The explanation ought to be somewhat more illuminating than the assertion that transactions costs are lower that way.

The models above provide a meaningful general equilibrium framework in which media of exchange play an important role. One may be skeptical as to the extent to which they illuminate this role.

In summary, the models above give a theory of general equilibrium in a pure exchange economy with transactions costs, a theory of transactions and bilateral exchange, a theory of media of exchange and choice among them.

The growing literature on exchange with transactions costs includes essays by Foley [4], Hahn [5], Hirshleifer [6], Niehans [8], and Ostroy [9, 11]. The first two are quite similar and represent the deepest general equilibrium analysis of this group. Their focus is on behavior of the economy over time rather than at a point in time, the emphasis of this essay and most general equilibrium analysis. Ostroy's analysis focuses on costs and benefits of varying trading group size, a fundamental point in the analysis of the structure of exchange.

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