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COWLES FOUNDATION DISCUSSION PAPER NO. 299

DIFFERENTIAL TAXATION, PUBLIC GOODS, AND ECONOMIC EFFICIENCY

J.E. Stiglitz and P. Dasgupta

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by

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p. 6, **line 16** change $g_1 \ldots g_s$ to $\underline{g}^1 \ldots \underline{g}^m$.

p. 7, line 2
$$\sum_k \frac{\partial V^k}{\partial q_i} = \sum_{\ell=0}^{n} \rho_\ell \frac{\partial C_\ell}{\partial q_i}, \quad i = 1, \ldots, n \qquad (2.1.10)$$

line 3
$$\sum_k \frac{\partial V^k}{\partial g_u} = \gamma_u + \sum_{\ell=0}^{n} \rho_\ell \frac{\partial C_\ell}{\partial g_u}, \quad u = 1, \ldots, s \qquad (2.1.11)$$

p. 12, line 5 $(\eta^d)^{-1}\eta_i^s E^{-1} \cdot \nu$

p. 15, line 4
$$\sum_k \frac{\partial V^k/\partial g_u}{\epsilon} = \frac{[\gamma_u + \sum_{\ell=0}^{n} \rho_\ell \frac{\partial C_\ell}{\partial g_u}]}{\rho_0[\sum_{\ell=1}^{n} \frac{t_\ell}{C_\ell} \frac{\partial C_\ell}{\partial q_i} + 1]} \qquad (2.4.1)$$

line 7
$$\sum_k \frac{\partial V^k/\partial g_u}{\epsilon} = \frac{\left[\frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_0^j} - \sum_{\ell=0}^{n} t_\ell \frac{\partial C_\ell}{\partial g_u}\right]}{\sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i} + C_i} \qquad (2.4.2)$$

line 11
$$\sum_{\ell=0}^{n} p_\ell \frac{\partial C_\ell}{\partial g_u} = -\sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial g_u} \qquad (2.4.3)$$

p. 16, line 1 $$\sum_k \frac{\partial V^k/\partial g_u}{\xi} = \frac{\left[\frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_0^j} - \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial g_u}\right] C_i}{\sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i} + C_i} \qquad (2.4.4)$$

p. 17, line 10 $$\frac{dg_u}{dt_i} = \frac{dR}{dt_i} \frac{\partial F^j/\partial y_0^j}{\partial F^j/\partial g_u^j} = \left[C_i + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i} + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial g_u} \frac{dg_u}{dt_i} \right] \frac{\partial F^j/\partial y_0^j}{\partial F^j/\partial g_u^j}$$

line 12 $$\frac{dg_u}{dt_i} = \frac{C_i + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i}}{\left[\frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_0^j} - \Sigma t_\ell \frac{\partial C_\ell}{\partial g_u}\right]} = \frac{dR}{dt_i} \frac{\partial F^j/\partial y_0^j}{\partial F^j/\partial g_u^j} \qquad (2.4.6)$$

line 14 $$\sum_k \frac{\partial V^k/\partial g_u}{\xi} = \frac{C_i}{dR/dt_i} \frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_0^j} = \frac{s_i}{d \ln R/d \ln t_i} \frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_0^j} \qquad (2.4.7)$$

p. 27, line 13 $$\theta = -\frac{\xi}{\lambda_1} + 1 - \Sigma \frac{\partial C_i}{\partial I} \tilde{t}_i - (\Gamma^* - 1) w_2 \frac{\partial L_2}{\partial I} \qquad (4.2.20)$$

line 19 change $\partial C_i/\partial q_j = 0$ to $(\partial C_i/\partial q_j)_{\bar{u}} = 0$

p. 29, line 3 $$\frac{\Gamma_{2j} - \Gamma^*}{\Gamma_{2j}} = \frac{\theta}{\alpha_j \sigma_j + \hat{\eta}_{jj}^d (1 - \alpha_j)}$$

line 8 $$\frac{\tilde{t}_j}{q_j} = \frac{\theta(1 - \alpha_j)}{\alpha_j \sigma_j + \hat{\eta}_{jj}^d (1 - \alpha_j)} < \frac{\theta}{\hat{\eta}_{jj}^d} \quad \text{if} \quad \sigma_j > 0, \quad j = b+1, \ldots, d \qquad (4.2.22a)$$

line 17 $$t_j = \tilde{t}_j + (\Gamma^* - 1)(1 - \alpha_j)(q_j - t_j) \quad j = b+1, \ldots, d$$

p. 33, line 7 should read "to scale private sectors; the extent to which it does depends on four"

p. 40, line 15

$$\frac{t_i}{p_i} = \frac{\theta''\left(\frac{1}{\hat{\eta}^d_{ii}} + \frac{1}{\eta^s_i} + \frac{\tau}{\eta^d_{ii}C_i}\frac{\partial C_i}{\partial I}\sum_j \frac{p_j C_j}{\eta^s_j}\right)}{\left(1 - \frac{\theta''}{\eta^d_{ii}}\left(1 + \frac{\tau}{C_i}\frac{\partial C_i}{\partial I}\sum\frac{p_j C_j}{\eta^s_j}\right)\right)} \qquad (4.4.23)$$

line 16 should read "Comparison of (4.4.22) and (4.4.23) yields Rule 13d.

p. 44, line 20 (footnote)

$$\Xi - \lambda_1 + \lambda_1 \sum_1^{m_1} \tilde{t}_j \frac{\partial C_j}{\partial I} + (\lambda_1 + B)\sum_{m_1}^{m} \tilde{t}_j \frac{\partial C_j}{\partial I}, \quad i = 1, \ldots, m_1$$

p. 45, line 5 should read "tively some argue that the marginal rate of transformation gives the correct"

p. 46, line 17 should read "factor taxation when account is taken of (a) realistic restrictions"

p. 48, line 16 should read "[9] J. Mirrlees, 'A Note on Producer Taxation,' (mimeo, 1970).

DIFFERENTIAL TAXATION, PUBLIC GOODS, AND ECONOMIC EFFICIENCY*

by

J.E. Stiglitz and P. Dasgupta

1. Introduction

Most economies make extensive use of differential (distortionary) taxes. Three classes of taxes, corresponding to the three necessary conditions for Pareto Optimality, may be distinguished:

(1) Taxing different commodities and factors at different rates introduces a distortion between the marginal rate of substitution and the marginal rate of transformation.

(2) Differential factor taxes[1] make the marginal rates of substitution of different factors in different industries different, and hence interfere with productive efficiency. Examples include the corporate income tax, which differentiates between capital used in the corporate and non-corporate sectors; the selective employment tax; and the differential treatment of gasoline used in road transportation and in agriculture.

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[1]They are also sometimes called "discriminatory," "selective" or "partial" factor taxes. The four terms are often used synonymously to denote non-uniform taxes, although they have slightly different connotations. We prefer to use simply the term "differential taxes."

(3) Differential tax treatment of different individuals makes the marginal rate of substitution of different commodities differ among individuals and hence results in "exchange inefficiency." Two examples of such taxes are the progressive income tax and subsidies to housing and food of the poor.

The purpose of this paper is to answer the following three questions:

(1) What kinds of distortions are "best to have," if as appears to be the case, you must have them? We shall show that if no restrictions are imposed on the set of commodity and factor taxes (in particular, if 100% profits tax can be levied) then no distortionary factor taxes should be imposed, but as soon as any of the innumerable restrictions which in fact obtain are taken into account, this is no longer true. More precisely, we shall analyze the optimal structure of factor and commodity taxation under a number of alternative sets of restrictions on the set of feasible taxes.

(2) What is the optimal supply of public goods, when revenue for these goods must be raised by distortionary taxation? We not only shall show that the conventional rule, that the sum of the marginal rates of substitution (over all individuals) should equal the marginal rate of transformation, no longer obtains, but also shall ascertain under what conditions the conventional rule represents an oversupply and when it represents an undersupply.

(3) What is the correct relationship between shadow prices in the public sector, and marginal rates of transformation and substitution in the private sector, when the sources of the divergences are (optimally

chosen) distortionary taxes. We shall show that none of the prevalently held current views on this subject is completely correct.[1]

In this paper, we are primarily concerned with the efficiency (distortionary) aspects of taxation, rather than with their redistributive implications, although the two cannot really be separated out. We therefore focus on the simplified case of identical individuals, and hence on the first two types of differential taxation discussed above.

2. The Socialist Economy

We begin our analysis by considering a government which, as in many socialist economies, controls production directly but purchases factors from individuals and sells to them the commodities that it produces for private consumption. The problem faced by the government will be to choose a set of consumer prices and a production plan, so that all markets clear and so that social welfare is maximized. In the next section we shall show that this socialist economy is essentially equivalent to a mixed economy with no restrictions on the set of feasible taxes.[2]

2.1. The Model: Let there be $(n+1)$ privately consumable commodities (including factors supplied) (labelled $i = 0, \ldots, n$) and s public goods (labelled $u = 1, \ldots, s$). We shall treat factors supplied by consumers

[1]The second and third questions may be put slightly differently: What is the optimal level of production of public goods and what is the optimal choice of technique in the public sector when revenue for the public sector is raised by distortionary taxes?

[2]Except, of course, that it cannot impose lump sum taxes.

as negative demands and factors demanded by production units as negative supplies. Let there be r identical individuals (labelled $k = 1, \ldots, r$) and m production functions available to the government (labelled $j = 1, \ldots, m$). We write

$\{C_i^k\} = \underline{C}^k = (C_0^k, C_1^k, \ldots, C_n^k)$ as the net consumption vector of private goods by the k^{th} (representative) consumer.

$\{g_u\} = \underline{g} = (g_1, \ldots g_u, \ldots g_s)$ as the consumption vector of public goods.

$\{q_i\} = \underline{q} = (q_0 \ldots q_i \ldots q_n)$ as the price vector faced by the consumer.

$\{y_i^j\} = \underline{y}^j = (y_0^j, \ldots y_i^j, \ldots y_n^j)$ as the net output vector of private goods by the j^{th} production unit.

and $\{g_u^j\} = \underline{g}^j = (g_1^j, \ldots, g_u^j, \ldots, g_s^j)$ as the output vector of public goods by the j^{th} production unit.

We suppose that the k^{th} individual's utility function can be written as

$$U^k(C_0^k, \ldots, C_n^k, g_1 \ldots g_s) = U^k(\underline{C}^k, \underline{g})$$

The individual maximizes U^k with respect to $\underline{C}^k$, subject to his budget constraint

$$\underline{q}\cdot\underline{C}^k = I^k, \tag{2.1.1}$$

where I^k is his income from fixed sources (i.e. those that do not depend directly on his supply of factors, e.g. in a capitalist economy, his share of profits). If there were to exist lump sum transfers, taxes and subsidies, they would be included in I^k. In our socialist economy, in which no such lump sum transfer is allowed, we assume $I^k = 0$. That is

$$\underline{q}\cdot\underline{C}^k = 0 \tag{2.1.2}$$

Now, corresponding to his utility function is his indirect utility function, $V^k(\underline{q}, \underline{g}, I^k)$, giving the maximum level of utility attainable by the individual faced with the consumer price vector $\underline{q}$, when the government supplies public goods in the amount $\underline{g}$. That is to say,

$$V^k(\underline{q}, \underline{g}, I^k) = \max_{\{\underline{C}^k\}} U^k(\underline{C}^k, \underline{g})$$

$$\text{subject to} \quad \underline{q}\cdot\underline{C}^k = I^k \tag{2.1.3}$$

Clearly V^k is homogeneous of degree zero in consumer prices $(\underline{q})$ and income (I^k). In our socialist economy, with $I^k = 0$, it is, therefore, homogeneous of degree zero in $\underline{q}$ alone.

If we assume that the social welfare function W is individualistic, i.e.

$$W = W(U^1, \ldots, U^r) = W(\underline{U}) \tag{2.1.4}$$

then it is clear that social welfare is homogeneous of degree zero in $\underline{q}$. Hence, without loss of generality, we set

$$q_0 = 1$$ [1]

[1] Provided, of course, that the 0^{th} commodity (factor) does not have a zero price in the optimal solution. The analysis does not depend on the normalization; rather, it is made simply for expositional convenience.

As all consumers are assumed identical, we shall let W take on the special form

$$W = \sum_k U^k = \sum_k V^k(\underline{q}, \underline{g}, I^k) \tag{2.1.5}$$

We assume the government has available to it m production processes, described by

$$F^j(y_0^j \ldots y_n^j, g_1^j \ldots g_s^j) = F^j(\underline{y}^j, \underline{g}^j) = 0 , \quad j = 1, \ldots, m . \tag{2.1.6}$$

In equilibrium all markets must clear. That is, we have

$$\sum_{j=1}^{m} y_i^j = \sum_{k=1}^{r} C_i^k \equiv C_i , \quad i = 0, \ldots, n \tag{2.1.7}$$

and

$$\sum_{j=1}^{m} g_u^j = g_u , \quad u = 1, \ldots, s \tag{2.1.8}$$

where C_i is the total demand for the i^{th} private good.

The planning problem, therefore, is to maximize social welfare (2.1.5) subject to the production functions (2.1.6) and the market clearing equations (2.1.7) and (2.1.8); and we have as our controls the outputs of private goods (inputs of private factors), $\underline{y}^1, \ldots, \underline{y}^m$; the output of public goods $g_1 \ldots g_s$; and the consumer prices of the private goods and factors, $q_1 \ldots q_n$. We write the Lagrangian of the problem as

$$\mathcal{L} = V^k(\underline{q}, \underline{g}, 0) + \sum_{i=0}^{n} \rho_i \Big(\sum_{j=1}^{m} y_i^j - C_i \Big) + \sum_{u=1}^{s} \gamma_u \Big(\sum_{j=1}^{m} g_u^j - g_u \Big) + \sum_{j=1}^{m} \mu_j F^j \tag{2.1.9}$$

and obtain the following first order conditions:

$$\sum_k \frac{\partial V^k}{\partial q_i} = \sum_{i=0}^{n} \rho_i \frac{\partial C_i}{\partial q_i}, \quad i = 1, \ldots, n \tag{2.1.10}$$

$$\sum_k \frac{\partial V^k}{\partial g_u} = \gamma_u + \sum_{i=0}^{n} \rho_i \frac{\partial C_i}{\partial g_u}, \quad u = 1, \ldots, s \tag{2.1.11}$$

$$\rho_i = -\mu_j \, \partial F^j/\partial y_i^j, \quad i = 1, \ldots, n \tag{2.1.12}$$

$$\gamma_u = -\mu_j \, \partial F^j/\partial g_u, \quad u = 1, \ldots, s \tag{2.1.13}$$

Equations (2.1.10) yield the optimal structure of consumer prices (supply and demands for private commodities); (2.1.11) give the optimal supply of public goods; while equations (2.1.12) and (2.1.13) give the optimal production conditions. The remaining three subsections of this section are concerned with interpreting these three sets of equations.

2.2. Efficiency in Production: We turn first to the first order conditions (2.1.12) and (2.1.13). Together they imply overall production efficiency at the optimum:

Rule 1: The marginal rate of transformation of commodity i into commodity j must be the same for all production processes.

The production decisions at the optimum may be decentralized by the government instructing the managers of the different production processes to maximize their profits on the basis of the shadow producer prices, ρ_i/ρ_0 (for the private goods) and γ_u/ρ_0 (for the public goods).[1]

2.3. <u>Optimal Pricing Policy</u>: (2.1.10) has a very natural interpretation:

<u>Rule 2a</u>: <u>The social cost of raising the price of the i^{th} commodity by a unit</u> (the LHS of (2.1.10)) <u>must equal the net social evaluation of the resources freed by raising the price by a unit for all commodities</u> (the RHS of (2.1.10)).

The right hand side of (2.1.10) may be rewritten as follows: $p_i = \rho_i/\rho_0$ are the normalized producer prices (again letting the 0^{th} commodity by the numeraire). Define the commodity tax vector t,

$$\underline{t} = \underline{q} - \underline{p} \qquad (2.3.1)$$

as the difference between consumer and producer prices.[2] Substituting (2.3.1)

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[1] Actually, (2.1.12) and (2.1.13) only assure us that, given the set of operating processes, the economy is efficient; we must also ensure that the economy is efficient with respect to the choice of plants (production processes) which it operates. But this follows immediately if we take proper account of the appropriate inequalities: for a factor

$$\rho_i + \mu_j \frac{\partial F^j}{\partial y_i^j} \geq 0$$

while for a commodity

$$y_i + \mu_j \frac{\partial F^j}{\partial y_i^j} \leq 0$$

with inequalities holding only when the given factor (commodity) is not used (produced) by the given production process. Then a process is not used only if the marginal rate of transformation of every factor into any commodity is less than the ratio of the shadow prices, at a zero level of production. A more complete analysis of these efficiency considerations is contained in [5]. The argument that even with divergences between marginal rates of substitution and marginal rates of transformation production should be efficient is originally due to Boiteux. It should be clear that the argument for efficiency does not depend on the strong assumptions of differentiability which we have employed, as Diamond and Mirrlees have established [5].

[2] It is obvious that $t_0 = 0$. All of this is simply a normalization procedure.

into (2.1.10) and using the fact (by differentiating the budget constraint, (2.1.2)) that

$$\sum_{\ell=0}^{n} q_\ell \frac{\partial C_\ell}{\partial q_i} - C_i = 0 \qquad (2.3.2)$$

we obtain

$$\sum_{\ell=0}^{n} \rho_\ell \frac{\partial C_\ell}{\partial q_i} = \rho_0 \sum_{\ell=0}^{n} p_\ell \frac{\partial C_\ell}{\partial q_i} = \rho_0 \sum_{\ell=1}^{n} (q_\ell - t_\ell) \frac{\partial C_\ell}{\partial q_i} = -\rho_0 \left(\sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i} - C_i \right) \qquad (2.3.3)$$

Thus the optimal tax structure requires

<u>Rule 2b</u>: <u>The change in the "tax revenue" from the increase in the "tax" on the i^{th} commodity be proportional to the loss in utility from raising the price of the i^{th} commodity.</u>

(2.3.3) is still not the optimal tax rule originally derived by Ramsey and Boiteux. To obtain this, we substitute Roy's formula[1] into the left hand side of (2.1.10)[2] and make use of the Slutsky equation and the symmetry of the Slutsky terms in (2.3.2)[3] to obtain

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[1] Roy's formula states that the amount by which an individual needs to be compensated for an increase in the price of the i^{th} commodity is equal to his consumption of that commodity:

$$\frac{\partial V^k}{\partial q_i} \Big/ \frac{\partial V^k}{\partial I^k} = -\left(\frac{\partial I^k}{\partial q_i} \right)_{\bar{V}} = -C_i^k \qquad (2.3.4)$$

[2] Thus the left hand side of (2.1.10) becomes

$$\Sigma - \frac{\partial V^k}{\partial I^k} C_i^k = -C_i \frac{\partial V_i^k}{\partial I} \qquad (2.3.5)$$

[3] Substituting the Slutsky equation

$$\frac{\partial C_i}{\partial q_\ell} = \left(\frac{\partial C_i}{\partial q_\ell} \right)_{\bar{V}} - C_\ell \frac{\partial C_i}{\partial I}$$

$$\frac{1}{C_i} \sum_{\ell=1}^{n} t_\ell \left(\frac{\partial C_i}{\partial q_\ell}\right)_{\bar{V}} = \frac{\xi}{\rho_0} - 1 + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial I} = -\theta \text{ all } i \qquad (2.3.7)$$

where[1, 2, 3]

$$\xi = \xi^k \equiv \frac{\partial V^k}{\partial I^k}$$

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into (2.3.3), we obtain

$$-\rho_0 \sum_{\ell=1}^{n} t_\ell \left[\left(\frac{\partial C_i}{\partial q_\ell}\right)_{\bar{V}} - C_\ell \frac{\partial C_i}{\partial I}\right] - \rho_0 C_i$$

Because of the symmetry of the Slutsky terms

$$\left(\frac{\partial C_i}{\partial q_\ell}\right)_{\bar{V}} = \left(\frac{\partial C_\ell}{\partial q_i}\right)_{\bar{V}}$$

this may be rewritten as

$$-\rho_0 \sum_{\ell=1}^{n} t_\ell \left[\left(\frac{\partial C_\ell}{\partial q_i}\right)_{\bar{V}} - C_\ell \frac{\partial C_i}{\partial I}\right] - \rho_0 C_i \qquad (2.3.6)$$

Setting (2.3.4) equal to (2.3.6) and dividing by C_i we obtain the desired result.

[1]Actually, the argument so far has only established this for $i = 1, \ldots, n$, but the argument for $i = 0$ follows from this, making use of the budget constraints.

[2]One case to which special attention needs to be drawn is that when one of the factors is supplied inelastically. A tax on that factor is like a lump sum tax in that it results in no distortions. If the value of these inelastically supplied factors, at producer prices, is greater than the deficit of the government sector, then only these are taxed; if it is less, then these are taxed at 100% and distortionary taxes are imposed in addition. To see this we must take explicitly into account the constraint that q_n (letting the n^{th} commodity be an inelastically supplied factor) be non-negative. Letting v be the shadow price on this constraint, we obtain, instead of (2.1.10),

$$\Sigma \frac{\partial V^k}{\partial q_n} = \Sigma_\ell \rho_\ell \frac{\partial C_\ell}{\partial q_n} + v \qquad (2.1.10')$$

where

$$v q_n = 0$$

(2.3.7) asserts:

Rule 2c: The percentage reduction, along the compensated demand curve, of the consumption of all commodities be the same, relative to what they would have been had the consumer prices been equal to the producer prices.

But except in the most unusual of circumstances, the producer prices in the optimal tax structure are not the pre-tax producer prices, even when the government sector is very small.

It is of some interest to know how the after tax levels of consumption, and producer and consumer prices compare with those before tax. If there is a single factor (the o^{th} commodity), and no joint production, the supply of each commodity in competitive equilibrium will be a function only of its price. Letting

$$\eta_i^s = \text{elasticity of the supply curve}$$

$$\eta_{ij}^d = \frac{\partial \ln C_i}{\partial \ln q_j} = \text{elasticity of the demand schedules}$$

Then, if the government revenue is small and labor alone is used to produce public goods,

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Using (2.3.7), we have

$$\frac{1}{C_n} \Sigma\, t_\ell \left(\frac{\partial C_n}{\partial q_\ell}\right)_{\bar{V}} = \frac{v}{C_n} - \theta \qquad (2.3.7')$$

If $v = 0$, $q_n > 0$ and $t_\ell = 0$, $\ell = 1, .. n-1$ satisfies (2.3.7')

$$\frac{1}{C_i} \Sigma\, t_\ell \left(\frac{\partial C_i}{\partial q_\ell}\right)_{\bar{V}} = 0, \quad i = 1, \ldots, n$$

If $v > 0$, $q_n = 0$, and if $t_n = p_n$, $v = C_n\theta$, then trivially (2.3.7') holds and, for $i = 1, n-1$, (2.3.7) holds as before.

[3]This formula, as well as several other of the formulae of this section, are, of course, not new. See, e.g. [3, 11].

the percentage change in the i^{th} producer price is proportional to[1]

$$E^{-1}\cdot\nu$$

where ν is the unit vector, the percentage change in the i^{th} consumer price is proportional to

$$\eta_i^s(\eta^d)^{-1}\eta_i^s E^{-1}\cdot\nu$$

and the percentage change in consumption is proportional to

$$\eta_i^s E^{-1}\cdot\nu$$

where

η^s is the vector of supply elasticities

η^d is the matrix of demand elasticities

$E = \{e_{ij}\} = \eta^s I + \eta^d$ is a matrix the elements of which consist of sums of demand and supply elasticities where I is the identity matrix).

[1]Equilibrium after taxes requires

$$C^d(\underline{p} + \underline{t}) = C^s(\underline{p})$$

while before taxes

$$C^d({}^o\underline{p}) = C^s({}^o p)$$

Taking a Taylor series expansion around ${}^o\underline{p}$, we obtain

$$\left.\frac{\partial C_i^s}{\partial p_i}\right|_{\underline{p}={}^o\underline{p}} \Delta p_i = \sum_{\ell} \left.\frac{\partial C_i^d}{\partial q_\ell}\right|_{\underline{p}={}^o\underline{p}} (\Delta p_\ell + t_\ell)$$

or (using 2.3.3 and 2.1.10)

$$[\eta^s I + \eta^d]\left[\frac{\Delta \underline{p}}{{}^o p}\right] = \left(1 + \frac{\underline{t}}{p_0}\right)\nu$$

Note that the expressions derived above depend on uncompensated elasticities.

A special case of some interest is that where demands are independent; the reduction in consumption is simply proportional to $1/1+(\eta^d_{ii}/\eta^s_i)$ a function of the ratio of the demand elasticity to the supply elasticity. Thus, in the very special case of constant results to scale, where $\eta^s_i = \infty$, the percentage reduction in output is the same in all industries; more generally,

Rule 2d: Of two industries with the same demand elasticity, the one with the greater supply elasticity will have the greater reduction in output;

this is as it should be, for the more inelastic supply the greater the loss to the government in tax revenues from the "producer surplus" from a given percentage reduction in output.

These results differ markedly from those, for instance, of Ramsey, who argued that the consumption of all commodities ought to be reduced by the same percentage. The reason for this is that he implicitly assumed that profits were distributed to individuals rather than being taxed.[1, 2]

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[1] See below, section 4.

[2] Actually, for Ramsey's result to hold, in addition to the assumption that all industries have the same supply elasticities, one must also assume that there is a single factor. Since different factors are taxed at different rates, even with constant returns to scale, the producer prices of different commodities will be changed by different amounts. For instance, if there are two factors no joint production, constant returns to scale, and demands are independent, then if the price of the second factor has increased, the larger the share in cost of the second factor the greater the reduction in output. This has the following immediate implication: if government is neither a net supplier nor a net purchaser of the second factor, the producer price of the second factor must be decreased; thus the percentage reduction in the consumption of every commodity is less than the percentage increase in the supply of the second factor. For assume that were not the case. If the producer price is unchanged or (increased), the demand for every commodity is reduced, and the demand for the second factor per unit of output is unchanged (reduced) so that the total demand for the factor has been reduced; but its supply has been increased and hence markets could not have cleared.

Thus, the percentage reduction in consumption will depend both on supply elasticities on and demand elasticities; on the other hand, conventional wisdom, which argues that the greater the supply elasticity, the smaller the tax rate, has been shown to be incorrect--relative tax rates depend simply on properties of the demand functions.

2.4. Public Goods: Pigou long ago recognized that the optimal supply of public goods depended on how the revenue of those goods was to be raised. He argued that optimality required the marginal benefit from an increase in a unit of the public good be equal to its marginal social cost, including any deadweight loss from the extra taxes required to finance the increment. This is similar to what (2.1.11) says. The left hand side is the social benefit. γ_u is the direct social cost of production, while the remaining terms on the right hand side represent the social value of the private goods released (absorbed) as a result of the change in the supply of the public good.

The conventional rule (which assumes lump sum taxes) must accordingly be modified to read:

Rule 3a: The sum of the marginal rate of substitution must be equal to the marginal rate of economic transformation

recognizing the fact that to transform a unit of private goods into a unit of public goods may require distortionary taxation.[1]

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[1] In our socialist economy, we use the term "distortionary taxation" as a shorthand for the fact that consumer prices differ from producer prices.

To see that this is implied by our first order conditions we substitute Roy's formula[1] and (2.3.3) into (2.1.10), the condition for optimal taxation, and divide (2.1.11) by the result to obtain

$$\sum_k \frac{\partial V^k/\partial g_u}{\xi} = \frac{[\gamma_u + \sum_{i=1}^{n} \rho_\ell \frac{\partial C_\ell}{\partial g_u}]}{\rho_0[\sum_{\ell=1}^{n} \frac{t_\ell}{C_i} \frac{\partial C_\ell}{\partial q_i} + 1]} \tag{2.4.1}$$

We then substitute (2.1.12) and (2.1.13), the conditions for optimal production, into the result, to obtain

$$\sum_k \frac{\partial V^k/\partial g_u}{\xi} = \frac{\left[\frac{\partial F^j/\partial g_i^j}{\partial F^j/\partial y_i^j} - \sum_{i=0}^{n} t_i \frac{\partial C_i}{\partial g_u}\right]}{\sum_{i=i}^{n} t_\ell \frac{\partial C_i}{\partial q_i} + C_i} \tag{2.4.2}$$

The overall consumer's budget constraint reads as:

$$(\underline{p} + \underline{t})\cdot\underline{C} = 0 ,$$

which, on differentiation with respect to g_u yields

$$\sum_{i=1}^{n} p_i \frac{\partial C_i}{\partial g_u} = -\sum_{i=1}^{n} t_i \frac{\partial C_i}{\partial g_u} \tag{2.4.3}$$

Substituting into (2.4.2) we obtain

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[1] See above, footnote 1, p. 9.

$$\sum_k \frac{\partial V^k/\partial g_i}{\xi} = \frac{\left[\frac{\partial F^j/\partial g_i^j}{\partial F^j/\partial y_i^j} - \sum_{i=0}^{n} t_i \frac{\partial C_i}{\partial g_u}\right] C_i}{\sum_{i=1}^{n} t_i \frac{\partial C_i}{\partial q_i} + C_i} \tag{2.4.4}$$

The LHS is simply the sum of the marginal rate of substitution. The first term in the numerator of the RHS is the marginal physical rate of transformation; the second term in the numerator gives the change in the tax revenue resulting from the change in the supply of the public goods. Thus the numerator gives the total change in government revenue to be raised from new taxes required to finance the given increment in public expenditure. The value of the change in consumption from the required additional tax is

$$\frac{\Sigma q_\ell \frac{\partial C_\ell}{\partial q_i}}{\partial R / \partial t_i} = \frac{-C_i}{C_i + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i}} \tag{2.4.5}$$

which is just the denominator of (2.4.7).

Pigou thought that when the revenue had to be raised by distortionary taxation, the optimal supply of public goods would be smaller than if lump sum taxes were used, since each extra unit of a public good not only displaces directly the production of private goods but also causes an additional distortion due to the additional taxes required to raise the requisite additional revenue.

Common sense is, as usual, somewhat ambiguous; one might also have argued as follows: The optimal supply of public goods may be described as if a 'tax' on the production of public goods were imposed which reduced its supply from what it would have been if there were lump sum taxation by the same percentage that the private goods consumption is reduced. Whether the conventional rule implies an under-supply or over-supply of public goods would then depend on the relative size of the 'public goods tax' and the taxes on the private goods. To see whether in fact the conventional rule represents an under or over supply of public goods (or, equivalently, whether the marginal economic rate of transformation is smaller or larger than the marginal physical rate of transformation) we must calculate the total change in g_u due to a change in t_i:[1]

$$\frac{dg_u}{dt_i} = \frac{dR}{dt_i} \frac{\partial F^j/\partial y_i^j}{\partial F^j/\partial g_u^j} = \left[C_i + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i} + \sum_{i=1}^{n} t_i \frac{\partial C_\ell}{\partial g_u} \frac{dg_u}{dt_i} \right] \frac{\partial F^j/\partial y_i^j}{\partial F^j/\partial g_u^j}$$

or

$$\frac{dg_u}{dt_i} = \frac{C_i + \sum_{i=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i}}{\left[\frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_i^j} - \Sigma t_\ell \frac{\partial C_\ell}{\partial g_u} \right]} = \frac{dR}{dt_i} \frac{\partial F^j/\partial y_i^j}{\partial F^j/\partial g_u^j} \tag{2.4.6}$$

Using (2.4.9) we obtain

$$\sum_k \frac{\partial V^k/\partial g_u}{\xi} = \frac{C_i}{dR/dt_i} \frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_i^j} = \frac{s_i}{d \ln R/d \ln t_i} \frac{\partial F^j/\partial g_u^j}{\partial F^j/\partial y_i^j} \tag{2.4.7}$$

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[1] These are total changes, taking into account the effect of the taxes as well as the change in g_u. Thus

$$\frac{dR}{dt_i} = C_i + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial q_i} + \sum_{\ell=1}^{n} t_\ell \frac{\partial C_\ell}{\partial g_u} \frac{dg_u}{dt_i}$$

where s_i is the share of total tax revenue raised by tax on i^{th} commodity. Thus, the conventional rule represents an under or over supply of public goods as the share of tax revenue from the i^{th} commodity is less than or greater than the elasticity of tax revenue from an increase in the tax rate on the i^{th} commodity;[1] or equivalently, as the marginal revenue from raising the tax on the i^{th} commodity by a unit is greater or less than C_i.

If the government expenditure does not affect consumption patterns, whether the marginal economic rate of transformation is greater or less than the marginal physical rate of transformation depends on whether as a result of the changed tax, there is a change in consumption patterns to, on average, higher or lower taxed commodities so that tax revenue from already existing taxes increases or decreases. Thus, if there is only one consumption good and one factor, labor, whether the conventional rule represents an under or over supply depends simply on whether the supply curve of labor is backward bending or upward sloping.[2]

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[1] Because these taxes are chosen optimally, it makes no difference which commodity's tax is raised (at the margin). Similarly, since the supplies of different public goods are chosen optimally, these results are valid for all g_u.

[2] It should be emphasized that these formulae for the optimal supply of public goods (as well as the earlier formulas for optimal taxes) depend critically on the assumption of identical individuals. Not surprisingly, no simple formulas seem to emerge when the possibly quite complex distributional effects of distortionary taxation are taken into account. See [7], [5], [15] for a discussion of the structure of taxes with heterogeneous individuals and for a discussion of the supply of public goods. The result that, provided there is at least one commodity which is demanded or supplied by all consumers, it is still desirable to remain productively efficient does not depend on identical individuals, as Diamond and Mirrlees have demonstrated.

3. The Mixed Economy

In many countries the government controls directly the production of only a few industries; the rest are affected by government action primarily through tax policy. One then wants to know the extent to which such indirect controls are a substitute for the direct control of production.

We have already pointed out that the Lagrange multipliers in our Socialist Planning Problem could be used for decentralized production. Thus, if the private sector had constant returns to scale,[1] the socialist equilibrium could be sustained by a competitive market in which commodity and factor taxes, equal to the divergences between consumer and producer prices observed in the previous section for the socialist economy, were imposed. The two economies are essentially identical.

If, on the other hand, the private sector has non-constant returns to scale, there are now "real profits" or "deficits" (instead of just accounting profits or deficits) which must be dealt with. If in the socialist equilibrium, the levels of consumption are less than they would have been had producer prices been charged (that is, there is a deficit to be financed by taxation) then for the mixed economy to be equivalent to the socialist economy, 100% profits taxes, or lump sum taxes (subsidies) on corporations (franchise taxes) equal to the accounting profit (losses) in the socialist equilibrium, must be imposed.[2]

An immediate implication of the analysis of the previous section to this case (as well as to the case with constant returns to scale in the private sector) is that there should be no differential taxation of factors by use and that shadow prices in the government sector should be equal to producer prices in the private sector. On the other hand, when the socialist economy

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[1] This is the case discussed in [5].

[2] With 100% profits taxes, although firms would not mind producing the amount required to obtain the socialist equilibrium, they have no incentive to do so.

runs an over-all surplus and the private sector has decreasing returns to scale, the profits provide a method by which the "surplus of the economy" may be distributed in a non-distortionary manner. Hence, in that case the level of welfare attained in the mixed economy will be higher than in the socialist economy which is not allowed to make lump sum transfer payments.[1,2]

Two further anomolies are associated with this situation: it may prove optimal for the government to levy taxes to limit the production of a constant returns to scale producer in favor of a less efficient decreasing returns to scale producer; and there may in fact exist no optimal tax policy.[3]

4. Restricted Taxation

4.1. Introduction: In the previous section, we argued that, in order for the mixed economy to imitate the centrally controlled economy, we needed 100% profits taxes (franchise taxes) for all industries and commodity (factor) taxes on all commodities (factors). Moreover, these taxes would differ from commodity to commodity. The purpose of this section is to show that some of

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[1] One might well argue that, even if the restriction that the government not levy lump sum taxes is a natural one, the seemingly symmetric assumption that it cannot make lump sum transfer payments is not. Indeed, many governments do seem to provide payments which are very close to lump sum transfers, e.g. child allowances. We shall not pursue this question further here because the situation where it arises is probably not of great relevance in countries, like most western economies, where a large proportion of national income is spent on public goods, including defense.

[2] We note one further case: it seems possible that even when the socialist equilibrium has a deficit, the optimal policy in a mixed economy--because of the possibilities of, in effect, making lump sum transfers--entails the private sector having a surplus greater than the deficit of the public sector.

[3] These questions are discussed in greater length in [4] and [9].

the basic results of the previous sections, in particular, the desirability of productive efficiency, no longer obtain as soon as these restrictions are taken into account; nonetheless, at least in the simple model presented below, simple formulae giving the optimal structure of taxes in the presence of these restrictions may be derived.

4.2. Limited Commodity and Factor Taxes: Some commodities and factors cannot (as least without great expense) be taxed directly. The services of privately owned automobiles is one important example of this.[1] Other commodities are those the value of which the national income accountant must impute, e.g. owner occupied housing, consumption of domestic services, consumption of non-marketed agricultural produce.

In this section we shall establish the following tax and production rules under these circumstances:

Rule 4: For those industries in which factor and commodity taxes may be imposed, consumption should be reduced by the same percentage (along the compensated demand curve) from what it would have been had producer prices been charged,[2] if the demand for these commodities does not depend on prices of the untaxed or partially taxed commodities.

[1] Although purchases of automobiles may be taxed and purchases of gasoline used in automobiles is taxed, neither are a correct measure of the output.

[2] In other words, exactly the same tax formulae discussed above in Section 2 apply in the fully taxed (including the government) sectors. Hence, Rule 4 can take any one of the alternative forms presented in Section 2 for optimal taxation. This rule is a correction of the rule originally due to Ramsey.

<u>Rule 5</u>: <u>In those industries in which a commodity tax cannot be imposed, differential taxes on the factors used in those industries should be imposed--productive efficiency should be abandoned. The magnitude of the tax (the extent to which the factor tax may serve as a partial substitute for the commodity tax) depends on the elasticity of substitution and the factor shares.</u>

<u>Rule 6</u>: <u>The public sector should use as its shadow prices producer prices in the fully taxed sector, and the public sector should be efficient.</u>

<u>Rule 7</u>: <u>In those industries in which a factor tax cannot be imposed on, say, the second factor, the direct tax on the commodity is increased from what it would have been otherwise; the amount by which it is increased is proportional to the share of the second factor in the cost of production.</u>

To see these results, we consider in this section the special case where there is no joint production. For simplicity, we assume there are only two factors of production, L_1 and L_2, but a large number of commodities.

We divide the industries in the private sector into four categories:

(1) Those industries that can be fully taxed (i.e. in which both commodity and factor taxes may be imposed), we label as $1, \ldots, a$.

(2) Those industries in which no taxes can be imposed at all; we label as $a+1, \ldots, b$.

(3) Those industries in which factor taxes can be imposed but not commodity taxes, we label as $b+1, \ldots, d$.

(4) Those industries in which commodity taxes may be imposed but not factor taxes we label as $d+1, \ldots, e$.

In order to avoid the special problems raised by restricted taxation with non-constant returns to scale (to which we turn in Section 4.3) we let C_j, the output of the j^{th} industry be a linear homogeneous function of L_{1j} and L_{2j}, the amount of the two factors used in the j^{th} industry:

$$C_j = \mathcal{F}_j(L_{1j}, L_{2j}) \equiv L_{1j} f_j\left(\frac{L_{2j}}{L_{1j}}\right) \equiv L_{1j} f_j(\ell_j) \tag{4.2.1}$$

$$j = 1, \ldots, a, a+1, \ldots, b, b+1, \ldots, d, d+1, \ldots, e .$$

where $\ell_j \equiv L_{2j}/L_{1j}$ and where we assume as usual that $f_j' > 0$, $f_j'' < 0$.

The industries being competitive, the price of any commodity will be equal to the minimum cost at the given factor prices (including any tax on the factors). Let w_1 and w_2 be the before tax prices of the two factors. Then cost minimization requires the marginal rate of substitution to equal the after tax ratio of factor prices:

$$\frac{f_j'}{f_j - \ell_j f_j'} = \frac{w_2 \Gamma_{2j}}{w_1 \Gamma_{1j}} \tag{4.2.2}$$

where $(\Gamma_{ij} - 1)$ is the ad-valorem tax rate on the i^{th} factor $(i = 1, 2)$ in the j^{th} industry.

(4.2.2) may easily be solved for ℓ_j as a function of $w_2\Gamma_{2j}/w_1\Gamma_{1j}$ (since $f_j - \ell_j f_j'/f_j'$ is a monotone function of ℓ_j):

$$\ell_j = \ell_j\left(\frac{w_2 \Gamma_{2j}}{w_1 \Gamma_{1j}}\right), \quad \ell_j' < 0 . \tag{4.2.3}$$

We note for later reference that the absolute value of the elasticity of ℓ_j with respect to the factor price ratio is the elasticity of substitution:

$$-\frac{d \ln \ell_j}{d \ln \frac{w_2\Gamma_{2j}}{w_1\Gamma_{1j}}} \equiv \sigma_j = -\frac{f_j'(f_j - \ell_j f_j')}{f_j f_j'' \ell_j} . \tag{4.2.4}$$

Since the amount of the two factors used in the j^{th} industry are given by

$$L_{1j} = \frac{C_j}{f_j}, \quad L_{2j} = \frac{C_j \ell_j}{f_j} \tag{4.2.5}$$

respectively, the minimum cost of producing a unit of C_j is

$$\frac{w_1\Gamma_{1j}}{f_j} + \frac{w_2\Gamma_{2j}\ell_j}{f_j} = \phi^j(w_1\Gamma_{1j}, w_2\Gamma_{2j}) . \tag{4.2.6}$$

ϕ^j is clearly homogeneous of degree one, so

$$\phi^j = \psi^j\left(\frac{w_2\Gamma_{2j}}{w_1\Gamma_{1j}}\right) w_1\Gamma_{1j} . \tag{4.2.7}$$

Now if $(\hat{t}_j - 1)$ is the tax (as a percentage of cost) on commodity j, then the competitive consumer price q_j is

$$q_j = \psi^j\left(\frac{w_2\Gamma_{2j}}{w_1\Gamma_{1j}}\right) w_1\Gamma_{1j}\hat{t}_j , \qquad j = 1, \ldots, e \tag{4.2.8}$$

Since profits are zero, individual's utility depends only on relative consumer prices; producers' outputs (inputs) depend only on relative producer prices, so without loss of generality, we can set

$$w_1 = \Gamma_{1j} = 1 \; .$$

(4.2.8) then becomes

$$q_j = \psi^j (w_2 \Gamma_{2j}) \hat{t}_j \; . \qquad (4.2.9)$$

Assume further that the government produces public goods g_u by means of the production processes

$$g_u = g_u(L_{1g_u}, L_{2g_u}) \, , \quad u = 1, \ldots, s \qquad (4.2.10)$$

where L_{ig_u} is the amount of the i^{th} factor used in the production of the u^{th} public good.[1]

When prices of private goods are given by (4.2.9), the supply schedules of all private commodities are perfectly horizontal. Thus for all markets to clear we require that

$$L_1 = \sum_{j=1}^{e} L_{1j} + \sum_{u=1}^{s} L_{1g_u} = \sum_{j=1}^{e} \frac{C_j}{f_j} + \sum_{u=1}^{s} L_{1g_u}$$

$$L_2 = \sum_{j=1}^{e} L_{2j} + \sum_{u=1}^{s} L_{2g_u} = \sum_{j=1}^{e} \frac{C_j \ell_j}{f_j} + \sum_{u=1}^{s} L_{2g_u} \qquad (4.2.11)$$

where L_{1j} and L_{2j} are given by (4.2.5), and where C_j is the quantity of the commodity demand.

The government wishes to maximize social welfare, which, as before, can simply be written as a function of consumer prices and the supplies of public goods:

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[1]The assumption that all private goods are produced by the private sector and public goods by the public sector is a simplifying assumption which, at the cost of some increase in notational complexity, may easily be removed.

$$\max \sum_k V^k(\underline{q}, \underline{g}) \tag{4.2.12}$$

subject to the market clearing equations (4.2.11). Thus, forming the Lagrangian

$$\mathcal{L} = \sum_k V^k(\psi^j(w_2\Gamma_{2j})\hat{t}_j, w_2, \underline{g}(L_{1g_u}, L_{2g_u})) + \lambda_1(L_1 - \sum_j \frac{C_j}{f_j} - \sum_u L_{1g_u})$$
$$+ \lambda_2(L_2 - \sum_j \frac{C_j \ell_j}{f_j} - \sum_u L_{2g_u}) \,. \tag{4.2.13}$$

we can write down the first order conditions

$$\frac{\partial \mathcal{L}}{\partial \hat{t}_j} = \psi^j \left\{ \sum_k \frac{\partial V^k}{\partial q_j} + \lambda_1 \left(\frac{\partial L_1}{\partial q_j} - \sum_{i=1}^{e} \frac{\partial C_i}{\partial q_j} \frac{1}{f_j} \right) + \lambda_2 \left(\frac{\partial L_2}{\partial q_j} - \sum_{i=1}^{e} \frac{\partial C_i}{\partial q_j} \frac{\ell_i}{f_i} \right) \right\} = 0$$

$$j = 1, \ldots, a, d+1, \ldots, e \,. \tag{4.2.14}$$

$$\frac{\partial \mathcal{L}}{\partial \Gamma_{2j}} = \frac{\partial \mathcal{L}}{\partial \hat{t}_j} \left(\frac{\psi^{j'} w_2 \hat{t}_j}{\psi^j} \right) + \frac{C_j}{f_j^2} (\lambda_1 f_j' - \lambda_2(f_j - \ell_j f_j')) \frac{d\ell_j}{d\Gamma_{2j}} = 0$$

$$j = 1, \ldots, a, b+1, \ldots, d \,. \tag{4.2.15}$$

$$\frac{\partial \mathcal{L}}{\partial w_2} = \sum_{j=1}^{e} \frac{\partial \mathcal{L}}{\partial \Gamma_{2j}} \frac{\Gamma_{2j}}{w_2} + \sum_k \frac{\partial V^k}{\partial w_2} + \lambda_1 \left(\frac{\partial L_1}{\partial w_2} - \sum_{i=1}^{e} \frac{\partial C_i}{\partial w_2} \frac{1}{f_i} \right)$$
$$+ \lambda_2 \left(\frac{\partial L_2}{\partial w_2} - \sum_{i=1}^{e} \frac{\partial C_i}{\partial w_2} \frac{\ell_i}{f_i} \right) = 0 \tag{4.2.16}$$

$$\frac{\partial \mathcal{L}}{\partial L_{1g_u}} = \frac{\partial \mathcal{L}}{\partial g_u} \frac{\partial g_u}{\partial L_{1g_u}} - \lambda_1 = 0 \,, \qquad u = 1, \ldots, s \,. \tag{4.2.17a}$$

$$\frac{\partial \mathcal{L}}{\partial L_{2g_u}} = \frac{\partial \mathcal{L}}{\partial g_u} \frac{\partial g_u}{\partial L_{2g_u}} - \lambda_2 = 0 \,, \qquad u = 1, \ldots, s \,. \tag{4.2.17b}$$

where $$\frac{\partial \mathcal{L}}{\partial g_u} = \sum_k \frac{\partial V^k}{\partial g_u} - \lambda_1\left(\frac{\partial L_1}{\partial g_u} - \sum_{j=1}^{e} \frac{\partial C_j}{\partial g_u}\frac{1}{f_j}\right)$$

$$- \lambda_2\left(\frac{\partial L_2}{\partial g_u} - \sum_{j=1}^{e} \frac{\partial C_j}{\partial g_u}\frac{\ell_j}{f_j}\right)$$

From (4.2.14)-(4.2.17) we derive our Rules 4-7 as follows:

(a) Rule 4. We define

$$\tilde{t}_j = q_j - \left(\frac{1}{f_j} + \frac{\ell_j}{f_j}\frac{\lambda_2}{\lambda_1}\right) \tag{4.2.18}$$

as the difference between the consumer price and the cost of production using shadow prices, and $\Gamma^* - 1$ as the ad valorem tax on factor 2 in the fully taxed sectors $(\Gamma^* w_2 = \lambda_2/\lambda_1)$. We use the Slutsky equation, Roy's formula, the symmetry of the Slutsky terms, and the budget constraint to rewrite (4.2.14) as

$$\frac{1}{C_i}\left(\sum_\ell \tilde{t}_\ell \left(\frac{\partial C_i}{\partial q_\ell}\right)_{\bar{u}} + (\Gamma^* - 1)w_2\left(\frac{\partial C_i}{\partial w_2}\right)_{\bar{u}}\right) = -\theta \tag{4.2.19}$$

where

$$\theta = -\frac{\xi}{\lambda_2} + 1 - \sum \frac{\partial C_i}{\partial I}\tilde{t}_i - (\Gamma^* - 1)w_2\frac{\partial L_2}{\partial I} \tag{4.2.20}$$

From (4.2.15), for those commodities for which commodity taxes may be levied, $\frac{\partial \mathcal{L}}{\partial t_i} = 0$, so

$$\frac{f_j'}{f_j \ell_j f_j'} = \frac{\lambda_2}{\lambda_1} \tag{4.2.21}$$

which implies, using (4.2.2), that producer prices in those industries which may be fully taxed are equal to the shadow prices. Hence $t_i = \tilde{t}_i$. Thus if $\partial C_i/\partial q_j = 0$, $i = 1, \ldots, a$, $j > a$ we immediately obtain Rule 4. More generally, we have

Rule 4': For all industries in which commodity taxes are imposed, consumption should be reduced by the same percentage (along the compensated demand curve) from what it would have been had the consumers been charged the cost of production using shadow prices.

(b) Rule 6 follows immediately from (4.2.20) and by dividing (4.2.17b) by (4.2.17a).

(c) Rule 5 follows from rewriting (4.2.15)

$$\frac{\lambda_2}{\lambda_1} - \frac{f_j'}{f_j - \ell_j f_j'} = \frac{\partial \mathcal{L}}{\partial \hat{t}_j}\left(\frac{\psi^{j'} w_2 \hat{t}_j}{\psi^j}\right)\frac{f_j^2}{C_j} \Big/ \frac{d\ell_j}{d\Gamma_{2j}} \lambda_1 (f_j - \ell_j f_j') \qquad (4.2.22)$$

Using (4.2.2), (4.2.4) and the fact that (from (4.2.6))

$$\psi^{j'} = \ell_j / f_j$$

we obtain

$$\frac{\Gamma_{2j} - \Gamma^*}{\Gamma_{2j}} = \frac{\theta + \left[\sum_i \tilde{t}_i \left(\frac{\partial \ln C_j}{\partial q_i}\right)_{\bar{u}} + (\Gamma^* - 1) w_2 \left(\frac{\partial \ln C_j}{\partial w_2}\right)_{\bar{u}}\right]}{\alpha_j \sigma_j}$$

$$j = b+1, \ldots, d ,$$

where

$$\alpha_j = \frac{f_j - \ell_j f_j'}{f_j} , \qquad (4.2.21a)$$

the share of the first factor in the cost of production. Thus, of two industries which are identical on the demand side (i.e. $(\partial \ln C_i / \partial q_k)_{\bar{u}} = (\partial \ln C_j / \partial q_k)_{\bar{u}}$ all k) we impose a higher differential tax on the one with the smaller elasticity of substitution and on the one in which the share of factor two is smaller.[1] To see this another way, assume the com-

[1] Assuming, as will normally be the case, that $\frac{\partial \mathcal{L}}{\partial \hat{t}_i} > 0$, $\frac{\partial \mathcal{L}}{\partial \hat{t}_j} > 0$.

pensated demand curves were independent, i.e. $(\partial \ln C_i/\partial q_k)_{\bar{u}} = 0$, $i \neq k$, then

$$\frac{\Gamma_{2j} - \Gamma^*}{\Gamma_{2j}} = \frac{\theta}{\sigma_j + \hat{\eta}^d_{jj}\alpha_j} \qquad (4.2.20)$$

where

$$\hat{\eta}^d_{jj} = (d \ln C_j/d \ln q_j)_{\bar{u}} \ .$$

The difference between the cost of production at shadow prices and the market price is

$$\frac{\tilde{t}_j}{q_j} = \frac{\theta(1-\alpha_j)}{\alpha_j\sigma_j + \hat{\eta}^d_{jj}(1-\alpha_j)} < \frac{\theta}{\hat{\eta}^d_{jj}} \quad \text{if} \quad \sigma_j > 0 \ , \quad j = b+1, \ldots, d \qquad (4.2.22a)$$

while for the fully taxed commodities

$$\frac{\tilde{t}_j}{q_j} = \frac{t_j}{q_j} = \frac{\theta}{\hat{\eta}^d_{jj}} \qquad j = 1, \ldots, a \qquad (4.2.22b)$$

so that only for a zero elasticity of substitution is the factor tax a complete substitute for the commodity tax.

(d) Rule 7 follows from (4.2.19). The only difference between the industries in which factor two can be taxed and those in which it cannot is in the interpretation of $\tilde{t}_j$. For the former, we have noted that $\tilde{t}_j = t_j$, while for the latter

$$t_j = \tilde{t}_j + (\Gamma^* - 1)(1 - \alpha_j) \qquad j = b+1, \ldots, d \qquad (4.2.24b)$$

Thus of two industries identical in all respects except that in one no factor tax can be imposed, the <u>total</u> tax per unit of production is identical if the elasticity of substitution is unity (since then α_j will be the same), but if $\sigma > 1$, the fully taxed sector pays a higher total tax, and if $\sigma < 1$, the converse is true.

4.3. <u>Taxing Different Commodities at Identical Rates</u>. One of the most important restrictions arises when different commodities must be taxed at the same rate.

There are two major reasons for this restriction: (a) It is administratively difficult to have separate tax rates for every commodity; the fact that from a production point of view two commodities are similar does not mean that they have identical demand functions, and the latter is what is required (in general) if they are to be taxed at the same rate in the unrestricted optimal tax structure. Thus, almost all tax systems group commodities into fairly wide classes, 'children's clothing,' 'food,' 'dairy products,' etc. (b) It is often impossible for tax authorities--or national income accountants--to differentiate between different kinds of income. In unincorporated enterprises, it is impossible to differentiate between returns to the labor of the owner, returns to his capital, and pure profits. Hence, they are all taxed at the same rate, even though the optimal tax structure almost certainly would instruct us to tax them differentially. In incorporated enterprises, it is impossible to differentiate that part of the return to equity which is a return to capital, and that part which is pure profits.[1]

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[1]It would be desirable to introduce explicitly the administrative costs of alternative tax structures in deciding on the optimal one, but we have not done this here.

In this case, we obtain the following rules:

Rule 8: The percentage reduction of the composite good, consisting of the commodities which must be taxed at the same rate, weighted by the consumer prices (in equilibrium) must be the same as for those commodities for which no restrictions are imposed.[1]

Rule 9: If it is feasible to impose differential taxation on factors in those industries in which uniform commodity taxes must be imposed then it is desirable to do so. (The optimal factor tax is given by (4.2.15) and described by Rule 5.) The weighted average marginal rate of substitution between factors is equal to the marginal rate of substitution between factors in the government sector (or sectors with unrestricted taxation) where the weights are

$$\frac{\alpha_j \sigma_j L_{2j}}{1 - \alpha_j} \Big/ \Sigma \frac{\alpha_j \sigma_j L_{2j}}{1 - \alpha_j}$$

Rule 10: If uniform taxes must be imposed both on the factors and the commodities in a group of industries, the producer price of factors in the private sector may be greater or less than in the public sector. The economy is efficient if factor shares are identical in all the industries which must be taxed at the same rate.

The derivation of these rules follows along the line of subsection 4.2 and the calculations are omitted.

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[1] As in the previous subsection, the tax formulae involve differences between consumer prices and cost of production using shadow prices.

4.4. Limited Profits Taxation. No government has imposed on a regular basis[1] 100% taxes on profits and the income of fixed factors, in spite of the long standing advice of economists (e.g. Henry George) of the desirability of such non-distortionary taxes. When there is a limitation on the maximum rate at which profits may be taxed,[2] it is again no longer true that production efficiency is desirable. Taxes on factors reduce the profits of the firm; unlike the previous cases, where either there were no profits or they were taxed at 100%, this change in profits affects individual welfare directly. This might suggest that the greater the share of profits in the given industry, the smaller the factor tax should be. On the other hand, the tax on the factor may serve as a partial substitute for the tax on profits (just as in the previous section, the tax on the factor was a substitute for a tax on the commodity); and this would suggest that the greater the share of profits, the greater the factor tax should be.

It turns out that the latter effect always dominates the former. More precisely, we examine below a modification of the model presented in section 4.2. We assume no joint production, two factors, and a homothetic

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[1] Although at war time a few governments have imposed 100% surtax rates, and a few governments have employed on occassion capital levies at rates equivalent to more than a 100% profits tax.

[2] Two possible explanations for this limitation suggest themselves. (1) It is difficult if not impossible (particularly in the presence of uncertainty) to separate out pure profits from, say, income to capital, and few if any governments--or national income accountants--have even attempted the task. (2) In at least some western economies, where the rights of private property are considered to be very important, a 100% profits tax would be considered equivalent to nationalization of the fixed factors. (The imposition of the tax would, moreover, involve great inequities; for before the tax, in the absence of risk, individuals would be indifferent between holding fixed and quasi fixed factors; those who happened to hold their wealth in the former would lose everything, while those who held their wealth in the latter would not.)

If there are heterogeneous individuals, it is possible to show that it may not be desirable to impose 100% profits tax. See [4].

production function with decreasing returns to scale.[0] We let the first factor be our (untaxed) numeraire.[1] We then obtain the following

<u>Rule 11</u>: <u>Differential taxes on the second factor should be imposed in all industries with decreasing returns to scale (or incompletely taxed fixed factors)[2] so that the producer price of the second factor is greater than the shadow price in the public sector or in constant returns to scale private sectors; the extent to which it depends on four factors: (i) the maximum rate at which profits may be taxed; at a tax rate of 100 per cent, the private and public sector use the same factor prices in production; the smaller the tax rate, the greater the difference between the public and private sectors; (ii) the share of profits; if the share of profits were zero (constant returns to scale), again the producer prices in the public and private sectors would be identical; the greater the share of profits, the greater the difference between the public and private sector prices; (iii) the share of the second factor in the cost of production; the greater the share, the greater the extent to which the factor tax can serve as a substitute for a profits tax and hence the higher will be the optimal factor tax in that industry (the greater will be the difference between public and</u>

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[1] As in the preceding sections, this is just a normalization rule. See above, pp. 24-25.

[2] Here as throughout this section, we assume that the revenue from the taxes on fixed factors and profits are not sufficiently large to cover the deficit of the government sector, so that distortionary taxes must be used.

[0] Equivalently, we could assume there exists a fixed factor which cannot be taxed at 100%.

private producer prices); (iv) the elasticity of substitution; the smaller the elasticity of substitution, the greater the divergence between the marginal rate of substitution in the private sector and in the public sector.

As a special case of Rule 11, we obtain

Rule 12: The constant returns to scale sectors and public sectors use the same producer prices.

The structure of commodity taxes as well as the structure of factor taxes is changed when there are limitations on profits taxes. There are two reasons for this. (1) Now there is a divergence between costs of production at shadow prices and at market prices, (2) Changes in consumer prices change demands, and thus profits, some of which now go directly to consumers. We thus obtain

Rule 13a: The greater the surtax on the factor use in the industry the lower is the commodity tax.

If there is only one factor, we obtain a generalization of Rule 2d.

Rule 13b: If there are no taxes on profits, there is an equiproportionate reduction in the output of all commodities (relative to the pretax situation, for small taxes). (This is the Ramsey Rule.)

Finally, if profits are taxed, and if in addition, demands are independent, we obtain the following two rules:

Rule 13c: The percentage tax rate is proportional to the sum of the inverse of the elasticity of demand and $(1 - \tau)$ (where τ is the tax rate on profits) times the inverse of elasticity of supply. (In the special case of $\tau = 0$, this is simply Ramsey's result.)

Rule 13d: Output in inelastically supplied industries is reduced less than in elastically supplied industries. The extent of the difference is an increasing function of the tax rate.

To see these results, we modify the model presented in the previous section to take account of our homothetic decreasing returns to scale production function. We can write the minimum average cost of producing C_j as

$$\left[\frac{1}{f_j(\ell_j)} + \frac{\Gamma_{2j} w_2 \ell_j}{f_j(\ell_j)}\right] \frac{H_j(C_j)}{C_j} \tag{4.4.1}$$

where $H_j'' > 0$, $H_j(1) = 1$ and $L_{1j} f(\ell_j) = 1$.

Cost minimization implies, just as before, that

$$\frac{f_j'}{f_j - \ell_j f_j'} = w_2 \Gamma_{2j} \tag{4.4.2}$$

Marginal cost pricing implies that

$$q_j = \hat{t}_j H_j' \left[\frac{1}{f_j(\ell_j)} + \frac{\Gamma_{2j} w_2 \ell_j}{f_j(\ell_j)}\right] \equiv \hat{t}_j p_j \tag{4.4.3}$$

Profits are given by

$$\begin{aligned} \Pi_j(p_j, w_2\Gamma_{2j}) &= \frac{q_j}{\hat{t}_j} C_j - L_{1j} - w_2 \Gamma_{2j} L_{2j} \\ &= (C_j H_j' - H_j)\left(\frac{1}{f_j(\ell_j)} + \frac{\Gamma_{2j} w_2 \ell_j}{f_j(\ell_j)}\right) \end{aligned} \tag{4.4.4}$$

i.e. the difference between the marginal and average costs, times the output.

Since all individuals are assumed to be identical, the profits received by each of the r individuals in the absence of taxation is just $\sum_j \Pi_j / r$.

It is easy to show that

$$\frac{\partial \Pi_j}{\partial p_j} = C_j \tag{4.4.5a}$$

$$\frac{\partial \Pi_j}{\partial w_2} = -L_{2j} = -\frac{\ell_j}{f_j} H_j(C_j) \tag{4.4.5b}$$

In the discussion below, we write C_j^s when we are treating C_j as a function of (p_j, w_2), i.e. the supply curve, and C_j^d when C_j is viewed as a function of consumer demand prices and income (i.e. the demand curve).

Assume the government can impose a tax rate on profits at τ. Since profits taxes are non-distortionary, it will clearly want to impose the maximum possible rate. Now,

$$\begin{aligned}\mathcal{L} = \sum V^k[\underline{q}, w_2, \underline{g}, \sum_j \frac{\Pi_j}{r}(p_j, w_2\Gamma_{2j})(1-\tau)] \\ + \lambda_1 \left[L_1 - \sum_j \frac{H_j(C_j^d)}{f_j[\ell_j(w_2\Gamma_{2j})]} - \sum_u L_{1g_u} \right] \\ + \lambda_2 \left[L_2 - \sum_j \frac{H_j(C_j^d)\ell_j}{f_j[\ell_j(w_2\Gamma_{2j})]} - \sum_u L_{2g_u} \right] + \sum_i \rho_i (C_i^s - C_i^d)\end{aligned} \tag{4.4.6}$$

We choose as our controls $(\underline{q}, \underline{p}, w_2, \Gamma_{2j}, \underline{L}_{1g_u}, \underline{L}_{2g_u})$. Then

$$\frac{\partial \mathcal{L}}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial I} \frac{\partial \Pi_i}{\partial p_i} \frac{(1-\tau)}{r} + \rho_i \frac{\partial C_i^s}{\partial p_i} = 0 \tag{4.4.7}$$

$$\frac{\partial\mathcal{L}}{\partial q_i} = \sum_k \frac{\partial V^k}{\partial q_i} + \lambda_1\left[\frac{\partial L_1}{\partial q_i} + \frac{\lambda_2}{\lambda_1}\frac{\partial L_2}{\partial q_i} - \sum_j H_j' \frac{\partial C_j^d}{\partial q_i}\left(\frac{1}{f_j} + \frac{\lambda_2}{\lambda_1}\frac{\ell_j}{f_j}\right)\right] - \sum_j \rho_j \frac{\partial C_j^d}{\partial q_i} = 0 \tag{4.4.8}$$

$$\frac{\partial\mathcal{L}}{\partial \Gamma_{2i}} = \frac{\partial\mathcal{L}}{\partial I}\frac{\partial \Pi_i}{\partial \Gamma_{2i}}\frac{(1-\tau)}{r} + H_i \frac{\lambda_1}{f_i^2}\left[f_i' - \frac{\lambda_2}{\lambda_1}(f_i - \ell_i f_i')\right]\frac{d\ell_i}{d\Gamma_{2i}} + \rho_i \frac{\partial C_i^s}{\partial \Gamma_{2i}} = 0 \tag{4.4.9}$$

$$\frac{\partial\mathcal{L}}{\partial w_2} = \sum_k \frac{\partial V^k}{\partial w_2} + \lambda_1\left[\frac{\partial L_1}{\partial w_2} + \frac{\lambda_2}{\lambda_1}\frac{\partial L_2}{\partial w_2} - \sum_j H_j' \frac{\partial C_j^d}{\partial w_2}\left(\frac{1}{f_j} + \frac{\lambda_2}{\lambda_1}\frac{\ell_j}{f_j}\right)\right]$$

$$- \sum_j \rho_j \frac{\partial C_j^d}{\partial w_2} + \sum_i \frac{\partial\mathcal{L}}{\partial \Gamma_{2i}}\frac{\Gamma_{2i}}{w_2} = 0 \tag{4.4.10}$$

$$\frac{\partial\mathcal{L}}{\partial L_{1g_u}} = \frac{\partial\mathcal{L}}{\partial g_u}\frac{\partial g_u}{\partial L_{1g_u}} - \lambda_1 = 0 \tag{4.4.11a}$$

$$\frac{\partial\mathcal{L}}{\partial L_{2g_u}} = \frac{\partial\mathcal{L}}{\partial g_u}\frac{\partial g_u}{\partial L_{2g_u}} - \lambda_2 = 0 \tag{4.4.11b}$$

where

$$\frac{\partial\mathcal{L}}{\partial I} = \sum_k \frac{\partial V^k}{\partial I} + \lambda_1\left[\frac{\partial L_1}{\partial I} - \Sigma H_j' \frac{\partial C_j^d}{\partial I}\left(\frac{1}{f_j} + \frac{\lambda_2}{\lambda_1}\frac{\ell_j}{f_j}\right) + \frac{\partial L_2}{\partial I}\frac{\lambda_2}{\lambda_1}\right] - \sum_i \rho_i \frac{\partial C_i^d}{\partial I} \tag{4.4.12}$$

$$\frac{\partial\mathcal{L}}{\partial g_u} = \sum_k \frac{\partial V^k}{\partial g_u} + \lambda_1\left[\frac{\partial L_1}{\partial g_u} - \Sigma H_j' \frac{\partial C_j^d}{\partial g_u}\left(\frac{1}{f_j} + \frac{\lambda_2}{\lambda_1}\frac{\ell_j}{f_j}\right) + \frac{\partial L_2}{\partial g_u}\frac{\lambda_2}{\lambda_1}\right] - \sum_j \rho_j \frac{\partial C_j^d}{\partial g_u} \tag{4.4.13}$$

Note that (4.3.8)-(4.4.12) are very similar to the corresponding equations in subsection 4.2, with the modifications necessary to take account of the changes in profits resulting from changes in prices.

The derivation of the Rules is now fairly straightforward. From (4.4.7) and (4.4.9) we obtain Rule 11:

$$\frac{(\Gamma_{2i} - \Gamma^*)w_2}{\Gamma_{2i}} = \frac{w_2}{\lambda_1} \frac{\partial \mathcal{L}}{\partial I} \frac{\gamma_i(1-\tau)}{r\alpha_i\sigma_i} \qquad (4.4.14)$$

where

$\Gamma^* w_2 = \lambda_2/\lambda_1$, the shadow price of factor two (in terms of the numeraire factor 1).

α_i = share of the first factor in the cost of production: $\alpha_i = f_i - \ell_i f_i'/f_i$.

σ_i = the elasticity of substitution.

and

γ_i = ratio of profits to costs = $H_i' C_i - H_i/H_i$

= elasticity of supply if $H_i = C_i^{\gamma+1}$ (constant elasticity).

In the perhaps more normal case, where $\Gamma^* > 1$ (i.e. the second factor is 'taxed' in public production) it should be noted that

$$\Gamma_{2i} w_2 > \Gamma^* w_2 > w_2 ,$$

the shadow price in the government sector lies between the marginal rate of transformation and the marginal rate of substitution in the private sector. On the other hand, it is possible that $\Gamma^* < 1$, in which case the shadow price in the public sector need not be between the marginal rate of transformation and substitution in the public sector.[1]

[1]For an earlier example illustrating the same point in the context of the question of the choice of the discount rate in public investment see Stiglitz [14].

Dividing (4.4.9b) by (4.4.9a) we obtain

$$\frac{\partial g_u/\partial L_{1g_u}}{\partial g_u/\partial L_{2g_u}} = \frac{\lambda_2}{\lambda_1} \tag{4.4.15}$$

which, together with (4.4.14) with $\gamma_i = 0$ implies Rule 12.

To obtain Rules 13a-d, we substitute (4.4.5) into (4.4.7) and the result into (4.4.8), and use the properties of the demand functions employed earlier to derive Rule 2 (see above pp. 9- 10) to obtain

$$\frac{\frac{1}{C_i}\left(\Sigma\tilde{t}_i\left(\frac{\partial C_i}{\partial q_j}\right)_{\bar{u}} + w_2(\Gamma^* - 1)\left(\frac{\partial C_i}{\partial w_2}\right)_{\bar{u}}\right)}{1 - (1-\tau)\sum_j p_j \frac{(\partial \ln C_i^d/\partial q_j)_{\bar{u}}}{\eta_j^s}} = -\theta' \tag{4.4.16}$$

where

$$\theta' = \frac{-\frac{\xi}{\lambda_1} + 1 - \left(\sum_j \tilde{t}_j\left(\frac{\partial C_j}{\partial I}\right) + w_2(\Gamma^* - 1)\ \frac{\partial L_2}{\partial I}\right)}{1 - (1-\tau)\sum_i \frac{\partial C_i^d}{\partial I}\ \frac{p_i}{\eta_i^s}} \tag{4.4.17}$$

Rule 13a follows immediately from (4.4.16) upon observing that

$$\frac{\tilde{t}_i}{p_i} = \frac{t_i}{q_i} - \frac{\Gamma^* - \Gamma_{2i}}{\Gamma_{2i}}\ (1 - \alpha_i) \tag{4.4.18}$$

Rule 13b follows upon observing that, with a single factor, profits in the i^{th} sector are a function only of the producer price in that sector, p_i . Thus(assuming that the government only purchases the single factor), the market clearing conditions may be written

$$C_i = C_i^d(\underline{p}(C_i) + \underline{t};\ (1-\tau)\Sigma\Pi_i\ (p_i(C_i)) \qquad (4.4.19)$$

As above (pp. 12), we take a Taylor Series expansion around $t = 0$ and divide by C_i to obtain

$$\frac{\Delta C_i}{C_i} \approx \sum_j \frac{\partial C_i^d}{\partial q_j}\left\{\left(\frac{1}{\partial C_j/\partial p_j}\right)\frac{\Delta C_j}{C_i} + \frac{t_j}{C_i}\right\} + \frac{(1-\tau)}{C_i}\frac{\partial C_i}{\partial I}\sum_j \frac{C_j}{(\partial C_j/\partial p_j)}\Delta C_j$$

$$= \Sigma\left(\frac{\partial \ln C_i}{\partial \ln q_i}\right)_{\bar{u}}\left(\frac{p_j}{q_j}\frac{\Delta C_j}{C_j\eta_j^s} + \frac{t_j}{q_j}\right) - \frac{\tau\partial C_i}{C_i\partial I}\Sigma\frac{p_jC_j}{\eta_j^s}\frac{\Delta C_j}{C_j} \qquad (4.4.20)$$

If $\frac{\Delta C_i}{C_i} = \frac{\Delta C_j}{C_j} = -\theta''$, we obtain

$$\frac{\sum_j\left(\frac{\partial \ln C_i^d}{\partial \ln q_j}\right)_{\bar{u}}\frac{t_j}{q_j}}{1 - \Sigma\left(\frac{\partial \ln C_i^d}{\partial \ln q_j}\right)_{\bar{u}}\frac{q_j}{p_j}\frac{1}{\eta_j^s} + \frac{\tau}{C_i}\frac{\partial C_i}{\partial I}\sum_j\frac{p_jC_j}{\eta_j^s}} = -\theta'' \qquad (4.4.21)$$

(4.4.21) is identical in form to (4.4.16) when $\tau = 0$.

Rule 13c is obtained directly from (4.4.16) which, after some manipulation, yields

$$\frac{t_i}{p_i} = \theta'\left(\frac{1}{\hat{\eta}_{ii}^d} + \frac{(1-\tau)}{\eta_i^s}\right)\Big/ 1 - \theta'/\hat{\eta}_{ii}^d \qquad (4.4.22)$$

By contrast if we wanted to reduce outputs by the same percentage, we would have imposed taxes at the rates

$$\frac{t_i}{p_i} = \frac{\theta''\left(\frac{1}{\hat{\eta}_{ii}^d} + \frac{1}{\eta_i^s} + \frac{\tau}{C_i}\frac{\partial C_i}{\partial I}\sum_j\frac{p_jC_j}{\eta_j^s}\right)}{1 - \theta''\left(\frac{1}{\hat{\eta}_{ii}^d} + \frac{\tau}{C_i}\frac{\partial C_i}{\partial I}\Sigma\frac{p_jC_j}{\eta_j^s}\right)} \qquad (4.4.23)$$

Comparison of (4.4.22) and (4.4.23) yields Rule 12d.

A slight modification of the model presented above can be used to provide the answers to these questions.[1]

The structure of factor taxes is identical in form to that given by Rule 11 (eq(4.4.14)), but now, γ_i the ratio of profits to costs, is given by

$$\gamma_i = \frac{p_i C_i - H_i}{H_i} = \frac{H_i' C_i}{(1 - 1/\eta_{ii}^d)} - 1$$

instead of simply $H_i' C_i / H_i - 1$.[2]

If the government could raise revenue by lump sum taxes, a subsidy at the rate $1/\eta_{ii}^d$ would result in price equalling marginal cost. When distortionary factor taxes are imposed, the subsidy is changed to take account of the deviation between private and social marginal cost.[3]

If 100% profits taxes were imposed the subsidy would be reduced to make the producer price greater than the social marginal cost by an amount proportional to the inverse of the elasticity of demand. This results, as before, in the same percentage reduction in consumption from the price equal social marginal cost situation (not from the original pre-tax situation).

The subsidy in the case of $\tau = 0$ may either be smaller or larger than when $\tau = 1$, depending on whether

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[1] We assume in the discussion below that demands are independent. In the discussion of the structure of commodity subsidies, we also assume constant marginal utility; this assumption may, however, easily be dropped.

[2] The formula for optimal factor taxes is

$$\frac{\Gamma_{2i} - \Gamma^*}{\Gamma_{2i}} = \frac{\theta \gamma_i (1-\tau)}{\alpha_i \sigma_i}$$

[3] The formula for optimal commodity taxes is

$$\frac{t_i}{p_i} = \frac{\theta\left(\frac{2-\tau}{\eta_{ii}^d} + (1-\tau)\frac{C_i H_i''}{H_i'}\right) + \left(\frac{\Gamma^* - \Gamma_{2i}}{\Gamma_{2i}}\right)(1 - \alpha_i) - \frac{1}{\eta_{ii}^d}}{1 - \theta\left(\frac{2-\tau}{\eta_{ii}^d} + \frac{C_i H_i''}{H_i'}\right)}$$

4.5. Monopolies and Increasing Returns to Scale Industries. Increasing returns to scale, and the monopolies and divergence between price and marginal cost which results, has been one of the primary arguments (of economists) for nationalization of these industries. In that case, the pricing and production structure of the public sector as described above applies, and no special consideration needs to be given to these industries (as opposed to constant or decreasing returns to scale industries in the public sector, except as already noted in section 2.[1])

In the United States, these industries are usually (although not always) regulated; whether, however, the regulations significantly change the behavior of the firm from what it would have been otherwise is a moot question. Consider first the case of an unregulated monopolist; the question is, what is the optimal structure of commodity and factor taxes (or subsidies)? If the firm is making a profit which cannot be wholly taxed away, should one impose differential factor taxes, as in the previous section, or will this simply reduce output still further below its already sub-optimal level? One can induce the firm to produce the optimal level of output by giving commodity subsidies, but if to raise revenue for these subsidies, one must impose distortionary factor taxes, is it desirable to equate marginal cost to price? If not, how large of a subsidy should be given?

[1]In fact, however, even after nationalization, governments seem to be reluctant to provide lump sum subsidies to meet their deficits, and the nationalized industries sometimes seem to act little differently from regulated private industries.

This constraint was originally studied by Boiteux. He ignored, however, the taxes imposed on commodities in the private sector used to provide the revenue for the deficit of the public sector, and accordingly, the solution he proposed must be modified.

In order to focus on the peculiar problems raised by the budget constraint, we shall for simplicity assume constant returns to scale in the private sector. The budget constraint of the government may now be written

$$\sum_u L_{1g_u} + w_2 \sum_u L_{2g_u} + \sum_{m_1+1}^{m} (L_{1j} + w_2 L_{2j} - q_j C_j) \leq b \tag{4.6.1}$$

where, as before $m_1+1 \ldots m$ represent the set of industries in the government sector.

To see what this implies for the structure of taxes, we form the Lagrangian

$$\begin{aligned}\mathcal{L} \equiv \Sigma V^k &+ \lambda_1 (L_1 - \sum_{i=1}^{m_1} \frac{C_i}{f_i} - \sum_{m_1+1}^{m} L_{1i} - \sum_u L_{1g_u}) \\ &+ \lambda_2 (L_2 - \sum_{i=1}^{m} \frac{C_i}{f_i} \ell_i - \sum_{m_1+1}^{m} L_{2i} - \sum_u L_{2g_u}) \\ &+ \sum_{m_1+1}^{m} \rho_j (f_j(L_{1j}, L_{2j}) - C_j) \\ &+ B(b + \sum_{m_1}^{m} q_i C_i - \sum_u (L_{1u} + w_2 L_{2u}) - \sum_{m_1+1}^{m} (L_{1i} + w_2 L_{2i}))\end{aligned} \tag{4.6.2}$$

From the first order conditions, we can derive the following Rules.

Rule 14: The governmental sector is efficient, i.e.

$$\frac{\partial f_i}{\partial L_{1j}} \Big/ \frac{\partial f_j}{\partial L_{2j}} = \frac{\lambda_1 + B}{\lambda_2 + w_2 B} = \frac{\partial g_u / \partial L_{1g_u}}{\partial g_u / \partial L_{2g_u}} \qquad u = 1, \ldots, s \quad j = m_1+1, \ldots, m \tag{4.6.3}$$

$$\frac{1}{\eta_{ii}^{d}} + \frac{C_i H_i^{\prime\prime}}{H_i^{\prime}} \gtrless 0$$

i.e. whether the demand elasticity is greater than the supply elasticity of a subsidized firm which set price equal to marginal cost. (Unless there are very strong increasing returns to scale, the above expression will be positive, i.e. the subsidy will be reduced.) This means that normally (assuming a single factor, so we can ignore the deviation between private and social costs of production) the percentage reduction in consumption from the price equal marginal cost situation is greater for increasing returns industries than for decreasing returns industries.

In the case of the regulated monopoly, where prices are set to allow the firm just to break even, factor taxes should be reduced (or eliminated). In any case, the rate of commodity taxation should be lower than for a commodity of the same demand elasticity in the public sector.

4.6. Budget Constraint. The restrictions introduced thus far concerned primarily the possibility of introducing taxes on particular commodities or factors.

Another class of restrictions pertains to groups of commodities or factors. One such restriction is that discussed in 4.2: the inability to differentiate rates within a group of commodities. This section is concerned with another "group constraint": a limitation on the size of the over-all budget deficit which public enterprises may incur. We examine this constraint in some detail both because of its apparent importance in practice and because the results are representative of those of similar constraints, such as that, in a particular period, there is a limit on the size of the trade deficit, which also are of some importance.

Rule 15: The private sector is efficient; no differential factor taxes (across industries) should be imposed:

$$\frac{f_i'}{f_i - \ell_i f_i'} = \frac{f_j'}{f_j - \ell_j f_j'} = \frac{\lambda_2}{\lambda_1} \equiv w_2 \Gamma^* \qquad (4.6.4)$$

Comparing (4.6.3) and (4.6.4), we observe that the economy overall is not efficient; the marginal rate of substitution of one factor for another will not be the same in the private and public sectors. Indeed, we have

Rule 16: The shadow prices in the government sector are between the marginal rate of transformation and the marginal rate of substitution in the public sector.

$$w_2 \gtrless \frac{\lambda_2 + Bw_2}{\lambda_1 + B} \gtrless \frac{\lambda_2}{\lambda_1} \quad \text{as} \quad w_2 \gtrless \frac{\lambda_2}{\lambda_1} \qquad (4.6.5)$$

Not surprisingly, in the structure of commodity taxes, we must take account of the difference between shadow and market prices in the private sector. We can obtain

Rule 17: The size of the reduction in consumption (along the compensated demand schedule, from what it would have been had consumers been charged marginal social cost) depends on the relative sensitivity of the demand for the commodity on prices of privately produced versus publicly produced commodities, as well as on whether the good is produced in the private or public sector. Consumption of publicly produced commodities whose demand depends only on the price of publicly produced commodities is reduced more than that of privately produced commodities whose demand depends only on prices of privately produced commodities.[1]

It should be observed that this model reduces to that of section 2 if either the constraint (4.6.1) is not binding or if the entire economy is embraced in the public sector. In this interpretation, the results of economic efficiency and optimal taxation derived there, as well as those derived in, e.g. [5] and [9] may be viewed as special cases of the above analysis.

5. Conclusions:

5.1. Shadow Prices for Cost-Benefit Analysis in Second-Best Economies:

There has been much controversy of late over the relationship between the shadow prices to be used in the public sector (e.g. in cost benefit analysis) and certain market prices. If the marginal rates of substitution between different commodities (factors) were equal to the marginal rates of transformation, there seems to be a fair amount of agreement that it is these marginal rates which should be used as the relative shadow prices. The problem arises, it is alleged, when the two sets of rates differ. The reasons why the two may differ have been extensively discussed. For our purposes, we note only that one of the primary sources of divergence is the tax system.

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[1] The optimal tax formula may be written

$$\frac{1}{C_i}\left\{\lambda_1 \sum_1^{m_1}\left[\tilde{t}_i\left(\frac{\partial C_i}{\partial q_j}\right)_{\bar{u}} + (\Gamma^* - 1)w_2\left(\frac{\partial C_i}{\partial w_2}\right)_{\bar{u}}\right] + (\lambda_1 + B)\sum_{m_1}^{m}\tilde{t}_j\left(\frac{\partial C_i}{\partial q_j}\right)_{\bar{u}}\right\}$$

$$= \begin{cases} \xi - \lambda_1 - \sum_1^{m_1}\tilde{t}_j\frac{\partial C_j}{\partial I} + (\lambda_1 + B)\sum_{m_1}^{m}\tilde{t}_j\frac{\partial C_j}{\partial I}, & i = 1, \ldots, m_1 \\ \xi - (\lambda_1 + B) + \lambda_1\sum_1^{m}\tilde{t}_j\frac{\partial C_j}{\partial I} + (\lambda_1 + B)\sum_{m_1}^{m}\tilde{t}_j\frac{\partial C_j}{\partial I}, & i = m_1, \ldots, m \end{cases}$$

where shadow producer prices in the public sector (relative to the shadow price of the first factor, in the public sector) are just $\rho_i/\lambda_i + B \equiv P_i \equiv q_i - t_i$.

When the marginal rates of transformation differ from the marginal rates of substitution, some argue that since the preferences of individuals are paramount, the marginal rate at which two commodities (factors) are substituted for one another (consumer prices) should be used. Alternatively some argue that the marginal rate transformation gives the correct opportunity cost of using the given factor (commodity) in the public sector as opposed to the private sector. Thus the marginal rates of transformation ought to be used as shadow prices. Finally, there are the 'compromisers,' who argue that the appropriate shadow price is a number between the marginal rate of transformation and the marginal rate of substitution.

The analysis of the preceding sections shows that there is no reason that any of these three arguments need be correct; the correct answer depends on the exact specification of the constraints imposed on the government. We have shown that

(i) If the only constraint on the government is the imposition of lump sum taxes, then the shadow prices to be used in the public sector are equal to the marginal rate of transformation in the private sector. (Section 2.2.)

(ii) If a constraint is imposed on the taxation of some factors (commodities) in the private sector, the shadow prices to be used in the public sector are equal to the marginal rate of transformation in the fully taxed sectors (i.e. the sectors in which both factor and commodity taxes may be imposed). (Section 4.2.)

(iii) If, say, a tax cannot be imposed on L_1 and L_2 in any sector of the economy, then the marginal rate of substitution in the private sector will equal the marginal rate of transformation; nonetheless, the government in the public sector should <u>not</u> use this marginal rate as its shadow price. (Section 4.2.)

(iv) If there is an overall budget constraint on the government, but no other constraint on the fiscal powers of the government then the shadow price in the government sector is a weighted average of the marginal rate of substitution and the marginal rate of transformation. (Section 4.6.)

(v) When other constraints are imposed on the taxation powers of the government, e.g. the government cannot impose 100 per cent profits tax, the shadow price in the government sector need not lie between the marginal rate of substitution and the marginal rate of transformation in the private sector. (Section 4.4.)

5.2. Concluding Comments: This paper has attempted to extend and develop the framework for analyzing problems of taxation and public production in a general equilibrium context originally developed by Ramsey and Boiteux. In doing so, we have seen how the conclusions of previous studies must be modified significantly both with respect to the formulae for the structure of the optimal commodity taxes, with respect to the interpretation of these formulae, and with respect to the utilization and structure of differential factor taxation when account is taken of (a) the reasonable restrictions which are imposed on the tax structure and (b) non-constant returns to scale industries in the private sector.

We end with a note of caution: when Pigou discussed differential taxation, he raised three kinds of considerations: (a) the 'efficiency' considerations with which this paper has been almost exclusively concerned; (b) distributive considerations and (c) administrative problems.[1] The equity and

[1] In contrast say, to the value added tax, which has recently received extensive support for its simplification of the tax structure, the kind of analysis given above would seem to entail a significant increase in the complexity of the tax structure.

distributional considerations are likely to run directly counter to one another, and although tax formulae may be derived taking these into account, the tax structure seems to be dependent critically on the social weights assigned to the various groups.[1] Included in the administrative problems of instituting this system are (1) difficulties in obtaining reliable estimates of the relevant parameters and (2) the likelihood of attempts by pressure groups to obtain differential tax advantages for themselves that are not necessarily in accord with the prescriptions of our analysis (and which they would not be able to obtain if the government adopted a more simplified tax structure). What is required is a more detailed study of the welfare gains from differentiation to ascertain whether they are sufficiently large to offset these other considerations.

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[2] See, e.g. [1], [5], [15].

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