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THE STRUCTURE OF EXCHANGE IN BARTER AND MONETARY ECONOMIES

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by

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"They that will be rich fall
into a temptation and a snare...
"For the love of money is the
root of all evil."

I Timothy 6:9, 10

"Money corrupts every human re-
lationship it touches."

Jerry Rubin, Do It!

In this essay I am trying to focus, with a general equilibrium view-
point, on two of the more rudimentary points in the monetary theory literature.
These are the "veil of money" thesis [3, 8] and the contention that monetary
exchange is more efficient than barter exchange because the latter required
a "double coincidence of wants [6, 12]." The first of these is not a very
interesting proposition for purposes of monetary theory inasmuch as it denies
significant effect to monetary variables. It's use, both in the literature
and here, is to simplify the analysis by eliminating one source of variation.

The principal substantial and interesting question on the "veil of
money" thesis is whether it is true. Clearly the hypothesis is not going
to be falsified in a model which assumes it. A test of the veil of money
theory would take place in a model in which relative prices and quantities
might or might not be affected by the presence of money. In such a model
one could develop necessary and sufficient conditions for money to leave

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prices and quantities unaffected. These would then be necessary and sufficient conditions for the validity of the "veil of money" approach.

The family of questions surrounding "double coincidence of wants" is rather more substantial.

The most precise statement of the problem can be found in Jevons:

"The earliest form of exchange must have consisted in giving what was not wanted directly for that which was wanted. This simple traffic we call barter..., and distinguish it from sale and purchase in which one of the articles exchanged is intended to be held only for a short time, until it is parted with in a second act of exchange. The object which thus temporarily intervenes in sale and purchase is money.

"The first difficulty of barter is to find two persons whose disposable possessions mutually suit each others wants. There may be many people wanting, and many possessed those things wanted; but to allow an act of barter, there must be a double coincidence which will rarely happen." ([6], p. 3)

Thus, for Jevons, barter is not merely the exchange of goods against goods, but rather the exchange of mutually desired goods. A barter transaction is one in which, for each trader, excess demand is not increased for any commodity and for some commodity is reduced and excess supply is not increased for any commodity and is reduced for some commodity.

The concept of double coincidence has two parts. The first is that all trade in a barter economy satisfies some ultimate want. When goats are traded for apples it is because the owner of the goats has an excess supply of goats and an excess demand for apples; the owner of the apples has an excess supply of apples and an excess demand for goats. The second part of the double coincidence condition is the idea that the only compensation a trader receives for supplying a second trader's wants is received from the second trader. One would suppose that this condition is an obvious

one except that it is somewhat at variance with the spirit and form of most general equilibrium models.

The double coincidence of wants requirement is a severe restriction on the trades which can take place. Indeed, it is very easy to generate examples of economies where trade is necessary to reach efficient allocations and yet in which no trade can take place because there is no trade satisfying this "double coincidence of wants" condition.

Some considerable prestidigitation is required to make difficulties arising from the absence of a double coincidence of wants a reason for the introduction of money. If we agree that to operate a system of exchange under such restrictive rules is awkward or ineffective, that hardly seems reason to further complicate the system by the introduction of another commodity. If exchange is made difficult and ineffective by a tiresome and restrictive rule then the reasonable response is to eliminate the rule. Inasmuch as no trader gets utility from money, as such, and hence money can satisfy no trader's ultimate wants, the introduction of such a good seems peculiarly pointless. The reason behind these contradictions is that when money is introduced to this family of models, it is defined to be the commodity to which the standard restrictions on desirability of commodities traded do not apply. Money is the only commodity which can be accepted in trade though the recipient has no excess demand for it; money is the only commodity which can be given in trade though the donor has no excess supply of it.

The obvious question at this point is "why bother?" Instead of introducing a single extra commodity for which the double coincidence condition

need not hold, why not simply eliminate the double coincidence condition? This would allow all commodities to change hands without necessarily satisfying ultimate wants. All goods would act as "money."

The answer is not clear. There is a definite feeling in the monetary literature that the number of media of exchange "money"s, should be small. In particular, not every commodity should be accepted in exchange, like money, only soon to be traded again. The reason behind this feeling is obscure.

Professor Tobin [11] posits a convincing analogy saying that money in a society is like language: it is very convenient to have one and only one. For both money and language this is because of costs of translation. But if money is just another commodity there seems little reason to translate from one money to another--no more reason than to translate from apple prices to orange prices. The reason is that Professor Tobin requires that a "money" act as both a unit of account and medium of exchange. In an economy with two units of account there is certainly reason to translate between them. Further, there may be substantial set-up costs to a system of accounts; an efficient solution may require that these costs be incurred only once.

An alternative rationale in the literature for keeping the number of media of exchange small seems to be an unspecified, but high, cost of search for trading partners. Here I am thinking of the family of arguments to the effect that monetary exchange is preferable to barter exchange because it saves traders' time and other resources looking for others with whom to trade. The argument apparently is that a real commodity used as a medium of exchange, should, unlike money, after a very few trades be acquired by its ultimate consumer. As I understand it, the feeling is, that if a commodity

is such that it is difficult to find an ultimate consumer, or he is unlikely to be found after only a few trades, then the uncertainty of finding him or the costs of searching him out are sufficiently high that the commodity will not be accepted as a medium of exchange. For very high search costs or very low probabilities of discovery this analysis gives double coincidence of wants as a condition of trade. Unfortunately this analysis explains too much. For if in order to assure acceptability of a commodity as a medium of exchange it is necessary to find, with small cost of search, an ultimate consumer for the commodity, then a fortiori if there is no ultimate consumer the good will not be accepted as a medium of exchange. A purely monetary commodity has no ultimate consumer, and hence by this argument the very commodity supposed to be universally acceptable is found to be universally unacceptable. Thus, the existence of a single, non-consumable monetary commodity cannot be explained by high costs of search for an ultimate consumer.

This is not really surprising. The trader who accepts a good in trade with the intention of trading it for something else is not interested in its ultimate consumption. He is interested only in his ability to pass it on to some other trader for a good which he desires or which will serve as a better medium of exchange. It is the cost of the search for this trader which is relevant, not the cost of search for an ultimate consumer. The upshot of this argument insofar as costs of search for a trading partner are relevant, is that any good which is found to be generally acceptable in exchange will be generally accepted as a medium of exchange.

It is very difficult in a general equilibrium model to discover why any commodity should be unacceptable in trade as a medium of exchange.

In a general equilibrium model all prices are known to all traders, thus eliminating price uncertainty as a rationale for unacceptability. We generally abstract from transactions costs, which if they differed among commodities might make one commodity preferred over another as media of exchange but only insofar as prices did not reflect this differential.

Yet one can generate some reasonably convincing models of barter in which search costs seem to be substantial and in which they can be reduced by the introduction of a common medium of exchange. Walras, for example, the prototypical general equilibrium theorist, suggests that an appropriate model of barter exchange is one in which there is a distinct market for each pair of goods to be traded against one another. (If this were barter in Jevons' sense, trade to an efficient allocation would require double coincidence of wants or the presence of arbitrageurs, the latter presumably being freed from the double coincidence condition. Walras requires arbitrageurs to avoid inequalities in the real prices of the same good in various markets.)

"We shall imagine that the place which serves as a market for the exchange of all the commodities (A), (B), (C), (D) ... for one another is divided into as many sectors as there are pairs of commodities exchanged. We should then have [for m commodities] $m(m-1)/2$ special markets, each identified by a signboard indicating the names of the two commodities exchanged there as well as their prices or rates of exchange." ([13], p. 158)

In this model the markets and prices are well defined. There would seem to be negligible transaction's costs. In equilibrium there will be no price differentials between markets for the same good. But with so many markets there seems to be a definite problem in bringing buyers and sellers together. If a trader has excess supplies of (A), (B) and excess demands

for (C), (D), (E), the markets the trader might go to are (A, C), (A, D), (A, E), (B, C), (B, D), (B, E). We are not assured of existence of equilibrium in this model, but even in equilibrium there seems to be a non-trivial problem of search for trading partners. The trader in question may use as few as three or as many as six of the pairwise markets to make satisfactory trades. If he goes to a market what assurance has he that there will be a purchaser there for what he wants to sell? Even if there is another trader with precisely complementary wants, this potential trading partner may have made a different choice of markets.

The interesting point here is not that the problem is, or can be made to look, complicated, but rather than it can be simplified by the introduction of money. Assume that all goods sold are sold for an $m+1^{\text{st}}$ good, all goods purchased are purchased with the $m+1^{\text{st}}$ good. Then there are m markets, one for each pair (A, $m+1$), (B, $m+1$), ..., instead of $m(m-1)/2$. Since there are many fewer markets to search, it seems reasonable to suppose that search costs incurred will be smaller. In equilibrium the difficulty of finding a trading partner should be considerably less. All traders with excess demands or supplies of a given commodity will meet at the same market, and in equilibrium, demand and supply on each market will be equal. An entirely distinct relevant argument is that if there diminishing marginal costs to the operation of markets as a function as a function of the volume of trade on the markets then it is considerably less costly to operate m markets at large volume than $m(m-1)/2$ markets each at comparatively low volume.

Clearly we cannot have diminishing marginal costs and still satisfy

convexity conditions. The presence or absence of trading partners is an indivisibility, again a violation of convexity. Inasmuch as general equilibrium analysis requires convexity in the material it studies, the fundamental aspects discussed here cannot be directly studied within a general equilibrium model. Consideration of these factors takes place, as above, outside the model, serving to inform the assumptions of the model, the aspects it emphasizes, and the questions posed.

I will consider a model of two closely related economies. In both, the focus is not on the existence and determination of equilibrium prices, the initial concern of most general equilibrium analysis, but rather on the nature of the transactions which take place once the prices have been determined and are taken as given. The two economies are nearly identical. One is a more or less traditional pure exchange barter economy. The second is an identical economy except that an additional commodity is introduced. This $N+1^{\text{st}}$ good is thought to behave like "money." The intention is, to compare the two economies and in some cases to see to what extent quantities determined in one economy can be adequately substituted into the other.

This substitution is designed as a use of the concept of the "veil of money." Working on the assumption that meaningful relative price determination is the result solely of real variables we can take a price vector p determined as an equilibrium for the barter economy and attach an arbitrary price of money p^m so that $\tilde{p} = (p, p^m)$ is an equilibrium for the corresponding monetary economy. Notations will be defined as needed. Generally, a notation of the form \hat{x} indicates that x is a monetary quantity and \tilde{x} is its barter counterpart. A notation of the form \tilde{x} indicates that x is a barter quantity and \hat{x} is a monetary counterpart of x . The process of converting a quantity

to its barter or monetary counterpart will usually consist simply in the deletion or insertion respectively of an $N+1^{\text{st}}$ co-ordinate.

Trades are described as a quantity of goods going from trader j to trader i , a_{ij} . In the barter economy a_{ij} will be an N dimensional vector; in the monetary economy a_{ij} will be an $N+1$ dimensional vector a_{ij}^n then denotes the amount of commodity n going from trader j to trader i . An array of a_{ij} for all possible pairs of traders i, j , then describes all trades taking place.

The Barter Economy

The economy consists of a finite set of traders T . A commodity bundle is an element of the nonnegative orthant of E^N . A transaction is an element of E^N ; a transaction is not generally nonnegative. A price system, usually denoted p , is an element of the nonnegative orthant of E^N . For each $t \in T$, there is an excess demand correspondence $d_t(p)$. Note that for any $x \in d_t(p)$, $p \cdot x = 0$.

A complex of transactions in the economy is represented as a rectangular array, a $|T|^2 \times N$ matrix. The notation $|T|$ denotes the number of elements in the set T . Each row of the matrix corresponds to a pair of traders. The N column entries of each row represent amounts of various goods being exchanged between the two traders. Each row of the matrix will be denoted by two indices. Each index indicates a trader. Thus we write that A is an exchange, $A = \|a_{ij}\|$, where $i, j \in T$ and a_{ij} is the transaction between i and j , an N dimensional vector.

Definition: An exchange, $A = \|a_{ij}^k\|$, is a $|T|^2 \times N$ matrix such that $a_{ij}^k = -a_{ji}^k$. a_{ij}^k is called transaction ij .

The restriction that $a_{ij} = -a_{ji}$ ensures that goods sent from i to j are received by j and understood to be from i . The sign convention indicates the direction in which the goods are going. A commodity whose component in a_{ij} is positive is going from j to i , commodities with negative entries in a_{ij} are going from i to j .

Definition: An exchange A is said to be price consistent at (price vector) p , if for each row of A , a_{ij} , $p \cdot a_{ij} = 0$.

Price consistency is a concept fundamental to the transactions analysis of a monetary economy. Because it applies to transactions and not directly to excess demands or consumptions this is a condition which does not appear in the general equilibrium literature (e.g. [1]). What price consistency requires is that all goods acquired must be paid for by sending goods of equal value from the trader acquiring the purchased goods to the trader supplying them. Price consistency is fulfilled whenever an exchange of goods involves a quid pro quo of equal value at the prices quoted. This is, of course, a considerably more stringent requirement than the usual condition on demand functions that the value of goods supplied to the market should equal the value of goods demanded from the market. Price consistency requires that the value of goods supplied to another trader equal the value of goods received from him. Absent some requirement of this sort there is no point in discussing media of exchange inasmuch as there is no need to pay the seller for goods purchased.

The price consistency condition is merely the abstraction of the obvious fact verified by casual empiricism that when one buys something, one pays the seller for it. Payment for goods purchased seems a concept

almost absent from general equilibrium theory. It is required there that the value of goods demanded equal the value of goods supplied but there is no requirement that the supplier of goods demanded be the recipient of goods supplied. One might conclude that in a general equilibrium model when a trader seeks to purchase goods from their owner he says to the owner "I wish to acquire from you k units of good n of which I understand you have an excess supply. I assure you that this acquisition will not cause a violation of my budget constraint at prevailing prices. You may of course consider that by supplying me with k units of n , your budget is enhanced by kp^n ." Since the world of general equilibrium theory is one of certainty, honest men making binding contracts in good faith with no possibility of default, the seller agrees to the above sale and delivery is made. The only payment for the goods consists in an addition to the seller's budget and a subtraction from the buyer's. These budgets seem to exist mainly in the memories or records of the agents in question. Needless to say, such a system will not long endure in a world of deceit, forgetfulness, and (honest) mistakes in arithmetic.

The following definition seeks to formulate part of the concept of double coincidence of wants in a market economy.

Definition: Let A be an exchange, and let p be a price vector. A is said to be monotonically excess demand diminishing at prices p if for each $i \in T$ there is $w_i \in d_i(p)$ so that

$$(i) \text{ sign } a_{ij}^k = \text{sign } w_i^k \text{ or } a_{ij}^k = 0, \text{ for all } j \in T, k = 1, \dots, N,$$

and

$$(ii) \left| \sum_{j \in T} a_{ij}^k \right| \leq |w_i^k|.$$

The sign restriction (i), says that each transaction of an exchange satisfying the definition reduces, or does not increase, the magnitude of excess demands and supplies of each commodity for both parties to the transaction. Condition (ii) ensures that in his zeal to fulfill his excess demands a trader does not overfulfill them, acquiring more than his demand for some good, delivering more than his excess supply.

One should note that monotone excess demand diminution is only half of Jevons' "double coincidence" of wants. Fulfillment of the former implies that goods are supplied by traders with excess supplies to traders with excess demands. It does not imply that these latter have excess supplies of goods for which the former have excess demands. If an exchange is price consistent and monotonically excess demand diminishing then I think it fulfills Jevons' concept of "double coincidence" of wants. In such a case each trader supplies others with goods of which he has an excess supply and receives from them each individually goods of an equal value of which they have an excess supply and for which he has an excess demand.

Definition: Let A be an exchange. A is said to be excess demand fulfilling at prices p if, for each $t \in T$, $(\sum_{i \in T} a_{ti}) \in d_t(p)$.

Definition: Let $p \in E^N$, $p \geq 0$. p is said to be an equilibrium price vector if for each $t \in T$ there is $x_t \in d_t(p)$ so that $\sum_{t \in T} x_t = 0$.

Theorem 1: Let p be an equilibrium price vector. There is an exchange A which is price consistent and excess demand fulfilling.

Proof of Theorem 1: Choose $x_t \in d_t(p)$ for each $t \in T$ so that

$\sum_{t \in T} x_t = 0$. Let $a_{ij} = -x_i$, $i \in T$, $i \neq 1$, $a_{ij} = 0$ for $i \neq 1 \neq j$.

Then we have $p \cdot a_{ij} = 0$ all i, j . $\sum_{j \in T} a_{ij} = x_i$ all $i \neq 1$.

$\sum_{j \in T} a_{1j} = \sum_{i \in T, i \neq 1} -x_i = \sum_{i \in T} -x_i + x_1 = 0 + x_1 = x_1$. Thus A is price

consistent and excess demand fulfilling.

QED

The proof of Theorem 1 is more interesting than the theorem itself.

The theorem makes the reasonably obvious statement that at equilibrium prices there is an exchange which fulfills all traders excess demands and relieves them of their excess supplies. Further, the exchange is price consistent; for every delivery of goods there is a quid pro quo of equal value. How is this achieved? In the proof, this is achieved by having all traders give their excess supplies to trader 1 and accept from trader 1 their excess demands. A single trader performs the function of a market clearinghouse familiar from general equilibrium theory. There is a clearinghouse function which will usually have to be performed. Clearly such an exchange will usually lack the monotone excess demand diminution property, there are large flows of goods through traders with neither excess demands nor supplies for them.

Theorem 2: Let p be an equilibrium price vector. There is an exchange A which is monotonically excess demand diminishing and excess demand fulfilling at p .

Proof of Theorem 2: For each $t \in T$ choose $x_t \in d_t(p)$ so that

$\sum_{t \in T} x_t = 0$. The proof proceeds by distributing excess supplies of a com-

modity among traders with excess demands for the commodity. Such an operation performed over all traders and all commodities yields an exchange satisfying

the two conditions. Without loss of generality let trader $x_1^i < 0$. That is, trader 1 has an excess supply of commodity i . Survey traders $2, 3, \dots, |T|$ in order; if $x_2^i > 0$ let $a_{21}^i = \min(|x_1^i|, |x_2^i|)$ if not, let $a_{21}^i = 0$. If $x_3^i > 0$, let $a_{31}^i = \min(|x_3^i|, |x_1^i - a_{21}^i|)$, if not let $a_{31}^i = 0$ and so on for all commodities i and all trading pairs $(1, t)$, $t \in T$. For all i so that $x_1^i < 0$, $a_{t1}^i = \min(|x_t^i|, |x_1^i - \sum_{r \in T, r < t} a_{r1}^i|)$ if $x_t^i > 0$ and $a_{t1}^i = 0$ if $x_t^i \leq 0$. Let $x_2^i < 0$ some i . Then if $x_1^i > 0$ set $a_{12}^i = \min(|x_2^i|, |x_1^i|)$, if not set $a_{12}^i = 0$. If $x_3^i > 0$ set $a_{32}^i = \min(|x_2^i - x_{12}^i|, |x_3^i - a_{31}^i|)$, $a_{32}^i = 0$ otherwise. Since $\sum_{t \in T} x_t^i = 0$ this distribution will exhaust all excess supplies and fill all excess demands.

QED

According to Theorem 2, for any equilibrium price vector, there is an exchange which satisfies all traders excess demands, involves them in no transaction which would increase the magnitude of any excess demand or supply, but which does not involve payment to the supplier by the recipient for goods received. I think it is just such exchanges which are at the back of one's mind in most general equilibrium analysis.

Note that the exchange all of whose entries are 0 is, for any p , price consistent and monotonically excess demand diminishing. Thus, using theorems 1 and 2, we have that if p is an equilibrium price then there is an exchange satisfying any two of the three conditions: price consistency, monotone excess demand diminution, excess demand fulfillment. They cannot generally all three be satisfied. A useful example of this is the case of three goods and three traders. Let prices be $(1, 1, 1)$ and suppose $d_1(p) = (1, 0, -1)$, $d_2(p) = (-1, 1, 0)$, $d_3(p) = (0, -1, 1)$. This is

typical of the cases where, though equilibrating trades are obvious, there is no transaction between any pair of traders which diminishes excess demands, increases no excess supplies and gives payment of equal value for all goods received.

The relation of the three concepts adduced to the double coincidence of wants now becomes clear. What does a double coincidence of wants condition look like in a general equilibrium model of a pure exchange economy? According to the arguments developed earlier, double coincidence holds at equilibrium prices p if there is an exchange A such that:

(i) Goods delivered to trader i from trader j are paid for with goods of equal value sent from i to j . That is, the exchange is price consistent.

(ii) Only goods for which trader i has an excess demand and of which trader j has an excess supply are sent from j to i . That is, the exchange is monotone excess demand diminishing.

(iii) Trade proceeds to equilibrium; all excess demands are satisfied. Thus, the exchange is excess demand fulfilling.

(i) is implicit in Jevons. If (i) were not required there would be no point to the insistence on a double coincidence, a single coincidence of demand and supply would be sufficient for trade to take place. (ii) is explicit. (iii) brings us into a meaningful general equilibrium framework.

The Money Economy

I am about to perform a bit of sleight of hand which has unfortunately fallen into disrepute of late, the trick of converting a barter economy to a monetary economy by the introduction of an $N+1^{\text{st}}$ good. As Shapley and

Shubik note,

"Economies using gold or paper or special credit relations bear about as much relation to an economy without some institutional form of money (even though it might have a price system) as do noneuclidean geometries to euclidean geometry. They contain different axioms; they have different rules for the game. The use of a monetary system by society amounts to the acceptance by the members of that society of extra conventions or 'rules of play' in their economic behavior." ([9], Ch. 5, p. 4.)

In the case below, the difference between the monetary and barter economies is the interpretation of excess demand diminution. In the monetary economy, the constraints of that definition are not applied to the $N+1^{\text{st}}$ good.

Definition: A monetary exchange is a $|T|^2 \times (N+1)$ matrix, $\|a_{ij}^k\|$, such that $a_{ij} = -a_{ji}$, a_{ij} is called transaction ij .

In keeping with the "veil of money" approach to monetary economies we arbitrarily set the price of money, $p^{N+1} \equiv 1$. Also, all traders' excess demands and supplies of the $N+1^{\text{st}}$ good are taken to be zero.

Definition: Let $A = \|a_{ij}^k\|$, $k = 1, \dots, N+1$, be a monetary exchange. The real counterpart of A , denoted \hat{A} , is $\|a_{ij}^k\|$, $k = 1, \dots, N$. That is, the real counterpart of a monetary exchange is the same exchange with all $N+1^{\text{st}}$ elements of the monetary exchange deleted.

Let p be a price vector for the barter economy. Then $\tilde{p} \equiv (p, 1)$, is a price vector for the monetary economy. Let $p = (p^1, p^2, \dots, p^{N-1}, p^N, 1)$ be a price vector for the monetary economy. Then $\hat{p} = (p^1, p^2, \dots, p^{N-1}, p^N)$ is a price vector for the barter economy.

The following definition embodies the special status of the $N+1^{\text{st}}$ good.

Definition: Let A be a monetary exchange, p be a monetary price

vector. A is said to be monotone excess demand diminishing at p if \hat{A} is monotone excess demand diminishing at \hat{p} .

The implication here is that, unlike most goods, money will be accepted in exchange whether it is desired or not.

Definition: Let A be a monetary exchange and p be a price system for the monetary economy. A is said to be excess demand fulfilling at p if \hat{A} is excess demand fulfilling at \hat{p} .

Definition: Let $A = \|\| a_{ij}^k \|\|$ be a monetary exchange. A is said to be money clear if for each $t \in T$ $\sum_{i \in T} a_{ti}^{N+1} = 0$.

Theorem 3: Let p be an equilibrium price vector for the monetary economy. Let A be a monetary exchange which is price consistent and excess demand fulfilling at p . Then A is money clear.

Proof of Theorem 3: By assumption we have

$$(1) \quad p \cdot a_{ij} = 0, \quad \text{all } i, j \in T$$

$$(2) \quad \left(\sum_{j \in T} \hat{a}_{ij} \right) \in d_i(\hat{p}) \quad \text{all } i \in T.$$

By (1) $\left(\sum_{j \in T} p \cdot a_{ij} \right) = 0$ all i , but

$$(3) \quad \sum_{j \in T} p \cdot a_{ij} = p \cdot \sum_{j \in T} a_{ij} = \hat{p} \cdot \sum_{j \in T} \hat{a}_{ij} + \sum_{j \in T} a_{ij}^{N+1}.$$

By (2) $\hat{p} \cdot \sum_{j \in T} \hat{a}_{ij} = 0$ so by (1) and (3) $0 = \sum_{j \in T} p \cdot a_{ij} = \hat{p} \cdot \sum_{j \in T} \hat{a}_{ij} + \sum_{j \in T} a_{ij}^{N+1} =$

$0 + \sum_{j \in T} a_{ij}^{N+1}$ so $\sum_{j \in T} a_{ij}^{N+1} = 0$ for all $i \in T$. QED

Theorem 3 makes the reasonably elementary point that in an economy where no trader has an excess supply or demand for money holdings, exchanges which fulfill excess demands and are consistent with prices will make no change in money holdings.

The following theorem, Theorem 4, constitutes the fundamental reason for the introduction of money in this family of models. Theorem 4 asserts the existence in the monetary economy of exchanges having characteristics discussed as desirable earlier in this essay. As shown earlier, such exchanges do not generally exist for the barter economy.

Theorem 4: Let p be an equilibrium price vector for the monetary economy. There is a monetary exchange A which, at p , is price consistent, monotonically excess demand diminishing, and excess demand fulfilling.

Corollary to Theorem 4: Let p be an equilibrium price vector for the monetary economy. There is a monetary exchange A which, at p , is price consistent, monotonically excess demand diminishing, excess demand fulfilling, and money clear.

Proof of Theorem 4 and Corollary: The corollary follows from the theorem directly by application of Theorem 3. To prove the theorem, choose $x_t \in d_t(p)$ for each $t \in T$ so that

(1) $\sum_{t \in T} x_t = 0$. For $k = 1, \dots, N$, choose a_{ij}^k so that $\text{sign } a_{ij}^k = \text{sign } x_i^k = -\text{sign } x_j^k$ or $a_{ij}^k = 0$ and so that

(2) $\sum_{j \in T} a_{ij}^k = x_i^k$ all $i \in T$, $k = 1, \dots, N$. (1) ensures the existence of such a_{ij}^k . Let

(3) $a_{ij}^{N+1} = -\sum_{k=1}^N p^k a_{ij}^k$. (3) gives price consistency. Sign restrictions on a_{ij}^k imply excess demand diminution. (2) implies excess demand fulfillment. QED

Theorem 4 reiterates the fundamental point discussed earlier. In

monetary economy all excess demands can be fulfilled by trades each of which satisfies excess demand of the trader accepting goods, alleviates an excess supply of the trader furnishing same, and includes payment in full to the supplier for goods received. This is not generally true of a barter economy when we include double coincidence of wants as a condition of barter exchange.

The following theorem notes that, at least in a "veil of money" world, money may facilitate, and certainly does not impede commerce. One accomplishes this by the ingenuous approach of describing a barter exchange and simply noting that a monetary exchange identical to the barter exchange except that there is an appropriate $N+1^{\text{st}}$ element in each row is a monetary exchange which has all the qualities (e.g. price consistency, excess demand fulfillment, ...) of the barter exchange from which it was derived. The implication of this is twofold. First, since for every acceptable barter exchange there is a corresponding acceptable monetary exchange and the converse is false, there are more acceptable monetary exchanges. This suggests that if one is seeking an extremum of some function over exchange--minimizing search or transactions costs for example--the extremum over the monetary exchanges will be, at least as good as that over barter exchanges.

Next, the correspondence between barter and monetary exchanges gives rise to a somewhat disingenuous "money doesn't matter" argument, since, serving no function in some cases, it can be dropped entirely. This provides some substance for an argument that the price (value in exchange) of money may be zero (Hahn in [4]). I certainly don't want to make anything of this point here.

Theorem 5: Let A be a barter exchange which is monotonically excess

demand diminishing and excess demand fulfilling at prices p . Then there is a monetary exchange B which is price consistent, monotonically excess demand diminishing, and excess demand fulfilling at \tilde{p} such that $\hat{B} = A$.

Proof of Theorem 5: Let $B = \{b_{ij}^k\}$. For $k = 1, 2, \dots, N$ let $a_{ij}^k \equiv b_{ij}^k$. Let $b_{ij}^{N+1} = -p \cdot a_{ij}$. Then $\tilde{p} \cdot b_{ij} = p \cdot \hat{b}_{ij} + b_{ij}^{N+1} = p \cdot a_{ij} - p \cdot a_{ij} = 0$. Thus, B is price consistent. Since A is excess demand fulfilling and monotonically excess demand diminishing, so is B . QED

Corollary 1 to Theorem 5: Let A be a barter exchange which is monotonically excess demand diminishing, excess demand fulfilling, and price consistent at prices p . Then there is a monetary exchange B with the same properties at \tilde{p} so that $b_{ij}^{N+1} = 0$ for all $i, j \in T$.

Proof of Corollary: Let B be as in the proof of the theorem. $b_{ij}^{N+1} = -p \cdot a_{ij}$. By price consistency of A , $b_{ij}^{N+1} = -p \cdot a_{ij} = 0$.

QED

Corollary 2 to Theorem 5: Let $M(p)$ be the class of all barter exchanges, A which are monotonically excess demand diminishing, excess demand fulfilling, and price consistent at prices p . Let $N(p)$ be the family of \hat{B} where B is a monetary exchange having those properties at \tilde{p} . Then $M(p) \subseteq N(p)$.

Proof of Corollary 2: Let $B = \tilde{A}$ be as constructed in the proof of Corollary 1, then $A \in M(p)$ implies $A \in N(p)$. QED

Corollary 3 to Theorem 5: Let g be a real valued function defined on barter exchanges. Then

$$\min_{A \in N(p)} g(A) \leq \min_{A \in M(p)} g(A)$$

$$\max_{A \in N(p)} g(A) \geq \max_{A \in M(p)} g(A) .$$

Proof of Corollary 3: Follows directly from corollary 2.

One might note that corollaries 2 and 3 are of somewhat limited interest inasmuch as $M(p)$ is nonempty only if the economy is excess demand complementary at p .

Returning now to a starting point of this essay, we can analyze part of the veil of money thesis. Speaking roughly, the thesis means that nothing real is changed by the presence of money in the system [3]. As an assumption the thesis has been built into the analysis mainly by assuming that excess demand for money is identically zero and that demand for goods depends only on the relative prices of goods.

What does the veil of money property mean in this family of models? Clearly it does not mean that transactions are unaffected by the introduction of money. The emphasis of this study is money's effect on transactions. I would suggest that a reasonable interpretation of the veil of money property in the present model is that the introduction of money does not affect the total net trade (i.e. final consumption) achieved by any trader. That is,

Definition: Let A be a monetary exchange. Let A be price consistent at prices p . A is said to have the veil of money property if

$$\sum_{j \in T} \hat{a}_{tj} \in d_t(\hat{p}) \text{ for all } t \in T .$$

This leads us to a rather straightforward theorem.

Theorem 6: Let A be a monetary exchange. Let A be price consistent at prices p . Let A have the veil of money property. Then A is money clear.

Proof: Price consistency implies $p \cdot a_{ij} = 0$ all $i, j \in T$. The veil of money property implies $\sum_{j \in T} \hat{a}_{ij} \in d(\hat{p})$. Hence $\hat{p} \cdot (\sum_{j \in T} \hat{a}_{ij}) = 0$.

$$p \cdot \sum_{j \in T} a_{ij} = p \cdot \sum_{j \in T} a_{ij} + \sum_{j \in T} a_{ij}^{N+1}, \text{ so } \sum_{j \in T} p \cdot a_{ij} = 0 + \sum_{j \in T} a_{ij}^{N+1}, \quad 0 = \sum_{j \in T} a_{ij}^{N+1}$$

for all $i \in T$.

QED

Theorem 6 suggests that money clarity is not an extremely strong condition in models of exchange. There may be some interest in models which assume money clarity where it is convenient to do so.

Coincidence of Wants in Large Economies

One way of conceiving of the problem of absence of coincidence of wants in a barter economy is to say that for some traders there are no other traders who are their mirror images, who have just the opposite excess demands. For example, in a three commodity economy, traders with excess

demands $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ would be, in this sense, "mirror images" of

one another. The non-coincidence problem would arise in the absence of such a pair of traders, for example if excess demands were $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. More generally, there will be coincidence if there are several

traders each of which form part of a mirror image. Thus if one trader's

excess demands are $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ then traders with demands $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ between them form the first trader's "mirror image."

From this point of view, coincidence depends on diversity among the traders. If for every trader his opposite or a group of partial opposites can be found then there is double coincidence of wants. When is such diversity likely to occur? One might reasonably guess that the larger the number of traders in the economy the more diverse a collection they are likely to be. If this were the case one would have a result something like the following:

Hypothetical Theorem: Let $f(T)$ be a measure of the proportion of excess demands not fulfilling double coincidence of wants. Then as $|T| \rightarrow \infty$, $P(f(T) = 0) \rightarrow 1$. I would certainly be very pleased to prove this result. Specifying the function f is a little tricky; actually characterizing values of f involves finding the equilibria--a task whose solution is at present only approximate [10]--and then solving the combinatorial problems of finding maximal sets of trading partners, a task whose complexity must vary roughly as $N^2|T|^2$. The proof must meaningfully take into account a probability measure on traders or sets of traders.

This last point would be probably mathematically the trickiest aspect of the proof. Very little is known concerning probabilistic statements on families of economies [5]. The emphasis on diversity of traders in this discussion points out why the hypothetical theorem is stated in probabilistic rather than deterministic form. This is because there are a good many cases where $|T| \rightarrow \infty$ and the extent of non-coincidence of wants grows in proportion.

Worse yet, these are just the cases where the behavior of the economy as $|T|$ becomes large are easy to analyze, the cases where T grows by replication (see, for example [2]). For in the case of proportional replication, if traders of a certain type cannot find their mirror image in the original economy, then there will be twice as many of them looking for twice as many non-existent mirror images when the economy is doubled in scale.

I would like now to take up one of the few reasonably tractable examples where the hypothetical theorem can be proved. Consider an economy with three traders and three goods. For convenience I will use excess demand functions rather than correspondences. Take $p = (1, 1, 1)$ to be an equilibrium price vector. The only loss of generality here is that we exclude the possibility of a commodity's price being zero. This example breaks down in such a case.

The first step in the analysis is to break down each excess demand vector into two parts, that which can be satisfied by price consistent monotonically excess demand diminishing trades, and the remainder. The first parts can be eliminated by simple barter transactions. It is the behavior of the remainders that interests us. We will not discuss excess demand vectors, but rather remainder vectors, those portions of excess demand which cannot be fulfilled by price consistent monotonically excess demand diminishing trades (except perhaps by eliminating some other such trade). Denote trader t 's remainder x_t . Thus the remainders for all three traders are represented by the array

$$\{x_1, x_2, x_3\} = \left\{ \begin{pmatrix} 1 \\ x_1^1 \\ x_1^2 \\ x_1^3 \end{pmatrix}, \begin{pmatrix} 1 \\ x_2^1 \\ x_2^2 \\ x_2^3 \end{pmatrix}, \begin{pmatrix} 1 \\ x_3^1 \\ x_3^2 \\ x_3^3 \end{pmatrix} \right\}.$$

$$(1) \quad p \cdot x_t = 0, \quad t = 1, 2, 3.$$

$$(2) \quad \sum_{t=1}^3 x_t = 0.$$

An important technical point, one that makes this example tractable, is that no trader can have non-zero remainder terms in all commodities. To see this, suppose, contrary to the assertion, for example that

$$(3) \quad x_1^1 > 0;$$

$$(4) \quad x_1^2, x_1^3 < 0.$$

(2) and (3) imply that

$$(5) \quad x_2^1 < 0$$

or

$$(6) \quad x_3^1 < 0.$$

Without loss of generality, take (5) to hold. Then by (1),

$$(7) \quad x_2^2 > 0$$

or

$$(8) \quad x_2^3 > 0.$$

In either case there is a price consistent monotonically excess demand diminishing trade to be made between traders 1 and 2: good 1 for good 2 if (7) holds or good 1 for good 3 if (8) holds. Thus there are at most two non-zero elements in each remainder vector. Further in order to avoid

unexploited trade opportunities, no two traders can have non-zero remainders in the same two goods.

Without loss of generality let trader 1 be the trader with non-zero elements in goods 1 and 2 (if he has non-zero remainders in any goods), let trader 2 have 2 and 3, and trader 3 have 1 and 3. (1) implies $x_1^1 = -x_1^2$, $x_2^2 = -x_2^3$, $x_3^1 = -x_3^3$. (2) implies $x_1^1 = -x_3^1$, $x_2^2 = -x_1^2$, $x_3^3 = -x_2^3$. Thus all non-zero x_t^i are of the same absolute value. This gives us that

$$\{x_1, x_2, x_3\} = \left\{ k \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, k \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, k \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\},$$

where k is real. The important point to note here is that the same k serves as scalar multiple for all traders. The remainder array for the economy can be completely characterized by specifying k . k is a characterization of the extent of absence of coincidence of wants in the economy.

Now consider a sequence of three man, three good economies T_i such that p is an equilibrium price vector for each of them. Let $J_i = \bigcup_{j=1}^i T_j$. What can we say about coincidence of wants or absence thereof in an economy consisting of J_i ? The thought experiment we have in mind here is to allow each T_i to trade separately at prices p . When all that is left of their excess demands is the remainder terms, the traders are brought together in J_i to see if there are any further price consistent monotonically excess demand diminishing trades to be made. The answer to this question depends completely on the k_j , the constants characterizing T_j 's remainder array.

For example, consider $J_2 = T_1 \cup T_2$. If $k_1 = -k_2$ then for each $t \in T_1$ there is $r \in T_2$ such that $x_t = -x_r$. r is t 's mirror image. All excess demands can be satisfied by price consistent monotonically excess demand diminishing trades. This is the situation for J_1 when $\sum_{j=1}^i k_j = 0$. In the opposite case $k_1 = k_2 \neq 0$, there is no increase in the volume of satisfactory trades. For each $t \in T_1$ there is $r \in T_2$ such that $x_r = x_t$. This is no help in finding price consistent monotonically excess demand diminishing trades. This is the case where $\text{sign } k_1 = \text{sign } k_2 = \dots = \text{sign } k_i$ and hence $\sum_{j=1}^i k_j = \text{sign } k_j \sum_{j=1}^i |k_j|$. This is roughly equivalent to replication. The important point here is that the extent of absence of coincidence of wants in J_1 is measured by the magnitude of $\sum_{j=1}^i k_j$. What is the behavior of this sum?

Consider the T_j to be drawn at random from some probability distribution which we will characterize by the distribution of k_j . Let k_j be distributed with mean μ and finite variance. Then as i becomes large, the law of large numbers [7] gives $\sum_{j=1}^i k_j / i \rightarrow \mu$. In particular, in the case where $\mu = 0$, $\sum_{j=1}^i k_j / i \rightarrow 0$. That is, if the mean of the distribution from which the k_j are drawn is zero the extent of absence of coincidence of wants is arbitrarily small in proportion to the size of the economy as the economy becomes large. $\mu \neq 0$ implies that the extent of absence of coincidence of wants tends to a non-zero constant in proportion to the size of the economy as the economy becomes large.

The case $\mu = 0$ is especially interesting because it suggests that in large economies there is almost complete coincidence of wants among traders.

As an assumption, $\mu = 0$ recommends itself primarily on some very tenuous grounds of neutrality. There seems very little a priori reason to say whether a trader with a positive excess demand remainder for commodity 1 is more likely to have an excess supply remainder for commodity 2 or for commodity 3. This involves some very complicated probabilistic statement not only about the trader in question, but also about those he has to trade with in his original three man market. Arguing along these lines, one might conclude that a quantity which there seemed no good reason to suppose more likely to be positive than negative or vice versa should have a mean of zero. Unfortunately this sounds like the principle of insufficient reason, an insufficient basis for any argument.

Whatever its merits as a realistic assumption, the case $\mu = 0$ poses an interesting family of questions as to the role of money in the economy. The argument suggests that small economies whose traders are drawn from such a distribution will require money to overcome the absence of double coincidence of wants. For a large economy, however, the absence of double coincidence should be negligible. Will the economy cease to require money? I suspect the answer is that the economy will still find money very useful. The reason for this suspicion is that in so large an economy, though problems of double coincidence may be small, the amount of search required to find trading partners may be large since there is so large a population in which to search. If search is costly, the cost of the search posited may be prohibitive. In that case, money can again serve a function as a medium of exchange not because of the absence of double coincidence of wants, but rather because even though they are there, it is costly to find pairs of

traders fulfilling the double coincidence condition.

Such an argument is far from the analysis of this essay. What it suggests is that there may be a very substantial theory of the role of money based on costs of search. Such a theory will depend essentially on the technology of search and communications mechanisms by which offers to buy and sell are made. Unfortunately this has a familiar ring. "Costs of information" is a term frequently heard when arguments get to the hand waving stage.

Conclusion

The intent in this essay has been to analyze the structure of transactions and the use of money in an economy with emphasis on coincidence of wants as a condition for barter exchange. Stating this family of questions in a form susceptible of a rigorous abstract analysis is itself a substantial innovation. Theorems 1 and 2 and the discussion surrounding them emphasize that three conditions on exchange are closely related to the desirability of money in the economy. Of the conditions on exchange--monotone excess demand diminution, price consistency, excess demand fulfillment--there is always a barter exchange satisfying any two but only if there is double coincidence of all wants (excess demand complementarity) will there be a barter exchange satisfying all three.

Theorem 4 makes the fundamental point that in a monetary economy all three conditions can be satisfied in general. Theorem 5 and its corollaries assert--roughly--that anything a barter economy can do a monetary economy can do better (or as well), at least in the case where the monetary system itself is costless. Theorem 6 points out that if money doesn't matter ex

ante, then in equilibrium money doesn't matter ex post. Finally, the analysis of the three man-three good economy suggests that if money doesn't matter at the mean of the distribution of the economy, money doesn't matter in the limit as the economy becomes large.

REFERENCES

- [1] Debreu, G., Theory of Value, New York: John Wiley and Sons, 1959.
- [2] _____ and H. Scarf, "A Limit Theorem on the Core of an Economy," International Economic Review, Vol. 4 (1963), pp. 235-246.
- [3] Greenwall, D., et al., The McGraw-Hill Dictionary of Modern Economics, New York: McGraw-Hill, 1965.
- [4] Hahn, F.H. and F.P.R. Brechling, The Theory of Interest Rates, London: Macmillan; New York: St. Martin's Press, 1965.
- [5] Hildenbrand, W. I'm told Professor Hildenbrand has an essay on probabilistic measures on spaces of economies though I haven't managed to get my hands on it yet.
- [6] Jevons, W.S., Money and the Mechanism of Exchange, London: D. Appleton and Co., 1875.
- [7] Parzen, E., Modern Probability Theory and its Applications, New York: John Wiley and Sons, 1960.
- [8] Pigou, A.C., The Veil of Money, London: Macmillan and Co., Ltd., 1949.
- [9] Shapley, L.S. and M. Shubik., Competition, Welfare and the Theory of Games, unpublished manuscript.
- [10] Scarf, H., "On the Computation of Equilibrium Prices," in Ten Economic Studies in the Tradition of Irving Fisher, John Wiley and Sons, Inc., New York, 1967.
- [11] Tobin, J., unpublished manuscript on money.
- [12] Wallace, N., "An Approach to the Study of Barter and Money Market Structure," unpublished.
- [13] Walras, L., Elements of Pure Economics, W. Jaffe, trans., Homewood, Illinois: Richard D. Irwin, 1954.