

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

**Box 2125, Yale Station
New Haven, Connecticut**

COWLES FOUNDATION DISCUSSION PAPER NO. 293

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than mere acknowledgment by a writer that he has access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

**ON THE EXISTENCE OF A COOPERATIVE SOLUTION
FOR A GENERAL CLASS OF N-PERSON GAMES**

Herbert Scarf

April 16, 1970

ON THE EXISTENCE OF A COOPERATIVE SOLUTION
FOR A GENERAL CLASS OF N-PERSON GAMES

by

Herbert Scarf*

I. Introduction

It is remarkable that the two game theoretic concepts of greatest applicability to economics, antedate the publication of The Theory of Games and Economic Behavior by at least fifty years. The concept of a Nash equilibrium for a general n person game is a direct descendant of the procedure used by Cournot in 1838 to analyze the effects of oligopolistic behavior. The other major technique of solution, the core, was introduced by Edgeworth under the name of the contract curve as early as 1886.

In the modern terminology of game theory, the core is a cooperative solution involving, as it does, the behavior of coalitions of players; the Nash equilibrium is regarded as a non-cooperative solution in which the joint interest of groups of players is not explicitly considered. The two concepts are quite distinct, and have traditionally been applied to games whose basic characterizations are very different.

*The research described in this paper was carried out under grants from the National Science Foundation and from the Ford Foundation.

The Nash equilibrium requires the game to be given in normal form; the independent strategies available to each player must be described completely, along with a specification of each player's utility for joint strategic choices. A selection of strategies, one for each individual, is then in equilibrium if no player can increase his utility by making an alternative choice, under the assumption that the remaining players make no change in their strategies.

In order for a cooperative concept, such as the core, to be applicable, the game must be described by specifying the set of utility vectors which can be achieved by each coalition. A utility vector is then in the core if, first of all, it is achievable by all of the players acting collectively, and secondly, no coalition can achieve a higher utility for each of its members.

The successful application of cooperative game theory in economics has been largely to those models of exchange, and possibly production, in which the collection of achievable utility vectors can readily be defined for each coalition. In a model in which each consumer begins the trading period with a stock of commodities, and has a utility function for final consumption, the utility vectors achievable by a coalition are most naturally taken to be those arising from an arbitrary redistribution of that coalition's assets.

This simplicity disappears, however, if the model of exchange is generalized even slightly. If, for example, some of the goods are undesirable and require the use of real resources for their disposal, the players not

included in a given coalition may, by their actions, modify the distribution of utilities within the coalition. This complicates substantially the determination of what a coalition can achieve by itself. Similar difficulties arise if external effects in consumption are introduced into the model of exchange, and in many other variations of the neo-classical model as well.

These examples illustrate the general proposition that the possibilities open to a coalition may best be viewed as derived from a prior specification of the game in its normal form; that is, in terms of the strategic choices open to the individual players, and their evaluations of the outcomes. von Neumann and Morgenstern attempted to do precisely this, but based their analysis upon the arbitrary assumption of transferable utility. For them, the question of whether a utility vector $\{u_i\}$, for $i \in S$, is achievable for the coalition S depends only on the sum $u_S = \sum_{i \in S} u_i$, rather than the individual utilities themselves. Moreover--and here I am simplifying somewhat--the coalition can achieve this vector if and only if its members have a fixed strategy which produces a vector of utilities summing to at least u_S , independently of what the members of the complementary coalition choose to do.

Aumann's suggestion [1] for passing from the normal form of a game to a cooperative description is completely ordinal and avoids the addition of utilities which modern welfare economics finds so distasteful. His definition is essentially identical with that given above, but based on the utilities themselves rather than their sums. For Aumann the utility vec-

tor $\{u_1\}$ is achievable by the coalition S , if and only if its members have a fixed strategy which guarantees this vector independently of the actions of the complementary coalition.

The present paper, using Aumann's approach, will describe a simple and general set of conditions in terms of the normal form of an n person game, which are sufficient to guarantee the existence of a nonempty core.

Let us then consider an n person game in which the first player has a set of possible strategies X^1 , the second player X^2 , etc. A typical strategy for player i will be denoted by X^i . We shall make the following assumption about these strategy sets:

- I. Each X^i is assumed to be a closed bounded convex set in a finite dimensional Euclidean space.

The significant aspect of this assumption is the requirement that each strategy space be convex; the remaining qualifications are technical and in particular the assumption that each strategy space be contained in a finite dimensional Euclidean space can be relaxed.

The convexity assumption is a familiar one in game theory, and is usually obtained by permitting the players to randomize independently over their strategy choices. There are, however, many n person games in which the convexity requirement is satisfied without the introduction of mixed strategies. For example, a model of exchange can be formulated as a game by permitting each player to allocate arbitrarily his initial stock of commodities among all of the players. Production may be incorporated into

the game by having several of the players represent firms, who select production and distribution plans consistent with their technology and with the resources furnished by the remaining players. In each of these examples the convexity requirement is either immediate or follows as a simple consequence of conventional economic assumptions.

To complete the description of the game, the i^{th} player will be assumed to have a utility function $u_i(x^1, \dots, x^n)$ defined on the product space $X = X^1 \times X^2 \times \dots \times X^n$, which describes his preferences for joint strategic choices. Some specific assumptions about the preferences will be made, in order for the solution discussed in the next section to be applicable.

II. The Solution to the Game

The particular type of solution to be discussed shares with other proposed solutions for n-person games the property of being based on a concept of equilibrium. Each of the players is assumed to select a strategy; the state which arises is then examined to see whether there are compelling arguments leading to alternative strategies. The characteristics of the solution are therefore defined by the selection of admissible arguments for departing from a given joint-strategy choice.

For example, a Nash equilibrium is a set of strategies x^1, \dots, x^n , with the property that no player can improve his utility by the selection of an alternative strategy, assuming that the strategies of the remaining players are not modified. In this solution concept the arguments for a

departure from a potential equilibrium are based in the isolated action of individual players, and assume that no subsequent response will be made by the $n-1$ remaining players.

Our proposed solution will differ in two major respects from the Nash equilibrium. First, an arbitrary coalition of players will be permitted to modify their strategies collectively, with the hope of improving the utility levels of every member of the coalition. And secondly, the complementary coalition will be permitted a subsequent modification of its strategic choices so as to deter the coalition initiating the change.

A joint strategic choice x^1, \dots, x^n , which provides the i^{th} player with utility u_i , will therefore be in equilibrium if for every selection of strategies by the complementary coalition which prevents some player in S from achieving a utility larger than u_i . In other words, a state will be in equilibrium if no coalition can insist on higher utility levels for all of its members independently of the actions of the complementary coalition.

The excessively conservative treatment of threats implied in this solution is, of course, troublesome. A coalition S , in attempting to obtain an improved position for all of its members must confront the entire range of strategic possibilities open to the players not in S , including those which lead to disastrous consequences for the complementary coalition and would in all probability not be undertaken. This inability to discriminate among counter-responses reduces the opportunity for a coalition to object to the status quo, and results in the inclusion of more outcomes

in the solution than might seem reasonable. In order to overcome this problem, the responses of the complementary coalition would have to be restricted by considerations of plausibility, and there seems to be no clear way of doing this within the general framework discussed here.

The major result of the present paper is that assumption I about the convexity of each player's strategy space, and the following assumption on the utility functions are sufficient to guarantee the existence of at least one set of strategies which are in equilibrium in the sense described above.

II. Each $u_i(x)$ is a continuous quasi-concave function. In other words, if $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$ are two joint strategy selections, and $0 \leq \alpha \leq 1$, then

$$u_i[\alpha x + (1-\alpha)y] \geq \min[u_i(x), u_i(y)] .$$

The major theorem of the paper may now be stated.

Theorem: In an n person game satisfying I and II there is at least one joint strategy choice which is in equilibrium in the sense that no coalition has an alternative strategy which guarantees higher utility levels for all of its members, independently of the actions of the complementary coalition.

The quasi-concavity assumption required in this theorem is quite strong in comparison with assumptions typically made in n person game theory. For example in order to demonstrate the existence of Nash equilibrium strategies it can be replaced by the far weaker condition that each player's

utility be a quasi-concave function of his strategy alone. The severity of the assumption is illustrated by the fact that it is generally not satisfied in the case in which each player has a finite number of pure strategies, and is permitted to randomize over these strategies, independently of the choice of the remaining players.

On the other hand, n person games derived from economic models do lead to preferences with this property if the conventional assumptions of convex indifference curves and production sets are made. Consider for example a typical model of exchange, in which the i^{th} consumer has a stock of commodities w^i prior to trade, and a utility function h_i for consumption. We model this as a game in normal form by permitting player i to allocate his stock of commodities

$$w^i = x^{i1} + x^{i2} + \dots + x^{in}$$

in an arbitrary fashion, possibly retaining some commodities for himself. The result of a joint selection of strategies will be that reallocation of the stocks initially owned, in which player j receives the commodity bundle

$$\sum_{i=1}^n x^{ij} .$$

If we take his utility for this joint selection of strategies to be his utility

$$h_j \left(\sum_{i=1}^n x^{ij} \right)$$

of consumption, then the assumption of quasi-concavity will be satisfied if the conventional indifference curves are convex from above.

The theorem may therefore be applied to the classical case of a market with the customary convexity assumptions about preferences. The conclusion states the existence of an allocation of society's initial stocks, which no coalition can improve upon by an alternative reallocation of its own assets, i.e., an allocation in the core of this market.

This example has at least one interesting generalization which seems not to have appeared before in the economic literature. Consider the case in which there are external effects in consumption, so that each individual's utility depends not only on his own vector of consumption, but is a function of the bundles consumed by some or all of the remaining consumers. If this generalized utility function has indifference surfaces which are convex from above when viewed in the product of each consumer's space of consumption bundles, then the theorem of this paper is applicable.¹ We conclude that there is an allocation of society's initial holdings, which cannot be improved upon by a specific redistribution of the assets of any coalition, if the complementary coalition is subsequently permitted to redistribute its own initial assets in an arbitrary fashion.

If external effects in consumption are not present, and there are

¹ Jacques Dreze has suggested to me that convexity of this sort may be related to a preference for equity.

no costs of disposal, this subsequent redistribution of the complementary coalition's assets can have no influence on the utility levels of the coalition attempting to block. But the presence of external effects permits the members of the complementary coalition, either by refusing to consume, or by a maldistribution of their own resources, to influence, in a possibly substantial fashion, the utilities in the blocking coalition. It is this aspect of external effects on consumption which requires the theorem of the present paper rather than a more conventional approach.

III. A Proof of the Main Theorem

The reader not versed in the intricacies of game theory may prefer to avoid this section in which a technical proof is given for the main theorem of the paper. The proof depends on the notion of a "balanced" n person game, discussed by Bondareva [2] and Shapley [4] for the case of transferable utility, and by the author [3] in the general case.

Consider an n -person game in which it is possible to define precisely for each coalition $S \subseteq N$, the set V_S of those utility vectors (u_1, \dots, u_n) which can be achieved by S , with the understanding that the components of u with subscripts referring to players not in S are arbitrary. We make the following assumptions on these sets:

1. For each S , V_S is closed and non empty.
2. If $u \in V_S$ and $u' \leq u$, then $u' \in V_S$.
3. V_N is bounded from above.

A vector u is in the core of this game if $u \in V_N$ and is not interior to V_S for any coalition S . The notion of a balanced collection of coalitions is introduced in order to obtain sufficient conditions for the existence of a vector in the core.

Definition. A collection of coalitions $T = \{S\}$ is balanced if there are non negative weights δ_S , equal to zero for these coalitions not in T , such that the equations

$$\sum_{S \ni i} \delta_S = 1$$

hold for $i = 1 \dots n$.

Definition. An n -person game is defined to be balanced if

$$\bigcap_{S \in T} V_S \subseteq V_N$$

for all balanced collections T .

The major result demonstrated in [3], is that a balanced n -person game always has a non empty core. In order to use this theorem in the present context we must pass from the definition of the game in terms of strategies and utility functions, to the sets V_S in such a way that the core based upon these sets corresponds to the solution proposed in the previous section. This will be true if we follow Aumann and define the sets V_S as follows:

Definition. A vector $u = (u_1, \dots, u_n)$ is in V_S , if there is a joint strategy of the members of S , which provides player i (for $i \in S$) with a utility of at least u_i for all strategy choices of the players not in S .

As we see, the components of a vector $u \in V_S$, corresponding to players not in S , are arbitrary under this definition.

The non-trivial part of the proof of our theorem is to demonstrate that the assumption of quasi-concavity of the utility functions implies that the game is balanced. Let $T = \{S\}$ be a balanced collection of coalitions with weights δ_S and let $u \in \bigcap_{S \in T} V_S$. We must therefore demonstrate that $u \in V_N$.

For each $S \in T$, the fact that $u \in V_S$ means that there is a joint strategy of the players in S , yielding player i (for $i \in S$) a utility of at least u_i for all strategies of the complementary coalition. Let this joint strategy be denoted by $x^i(S)$ for all $i \in S$. The proof that $u \in V_N$ proceeds by showing that the strategies

$$x^1 = \sum_{S \supset \{1\}} \delta_S x^1(S)$$

$$x^2 = \sum_{S \supset \{2\}} \delta_S x^2(S)$$

$$\vdots$$

$$x^n = \sum_{S \supset \{n\}} \delta_S x^n(S)$$

are each feasible for the respective players, and if used collectively guarantee the i^{th} player a utility at least u_i . But for each i , $x^i \in X^i$ since it is a convex combination of strategies in X^i , and this disposes of the first question.

In order to show that the state (x^1, \dots, x^n) yields the i^{th} player a utility at least u_i it is sufficient to restrict our attention to the special case $i = 1$. We shall express

$$(x^1, \dots, x^n) = \sum_{\substack{S \in T \\ S \ni 1}} \delta_S (y^1(S), \dots, y^n(S))$$

with $u_1(y^1(S), \dots, y^n(S)) \geq u_1$ for all coalitions $S \in T$ which contain player one. The assumption of quasi-concavity will then be sufficient to show that $u_1(x^1, x^2, \dots, x^n) \geq u_1$.

We now define the vectors $(y^1(S), \dots, y^n(S))$ for each $S \in T$ containing player one. Let us fix our attention on a particular S . We consider two cases for $y^i(S)$. If $i \in S$, set

$$y^i(S) = x^i(S).$$

If i is not in S , then we define

$$y^i(S) = \frac{\sum_E \delta_E x^i(E)}{\sum_E \delta_E},$$

where in both the numerator and the denominator the summation is taken over all $E \in T$ which contain player i but not player 1. (It should be noticed that $y^i(S)$ is the same for all coalitions S which do not contain player i .)

Still fixing our attention on the coalition S , we see that we have defined a joint strategy choice in which the players in S use the strategy $x^i(S)$, and the players not in S use a specific strategy $y^i(S)$. But since $\{x^i(S)\}$ guarantees player 1 a utility of at least u_1 , regardless of the strategy choices of the players not in S , we see that $u_1(y^i(S), y^2(S), \dots, y^n(S)) \geq u_1$ and this is true for all S in T which contain player 1.

In order to finish the proof it is necessary to demonstrate that the representation of (x^1, \dots, x^n) in terms of the δ 's and y 's is correct. We must therefore show that

$$\begin{aligned} x^i &= \sum_{\substack{S \in T \\ S \supset \{1\}}} \delta_S y^i(S) \\ &= \sum_{\substack{\delta_S > 0 \\ S \supset \{1, i\}}} \delta_S x^i(S) \\ &\quad + \sum_{\substack{\delta_S > 0 \\ S \text{ contains} \\ 1 \text{ but not } i}} \delta_S \left(\frac{\sum_E \delta_E x^i(E)}{\sum_E \delta_E} \right). \end{aligned}$$

with the range of summation of the index E taken over all E containing player i but not player 1 .

This is equivalent to showing that

$$x^i = \sum_{\substack{\delta_S > 0 \\ S \ni \{1, i\}}} \delta_S x^i(S) + \sum_{\substack{\delta_E > 0 \\ E \text{ contains } i \text{ but not } 1}} \delta_E x^i(S) \cdot c,$$

with the constant c given by

$$c = \frac{\sum_{\delta_S > 0} \delta_S}{\sum_{\delta_E > 0} \delta_E} \cdot \frac{S \text{ contains } 1 \text{ but not } i}{E \text{ contains } i \text{ but not } 1}$$

The representation is therefore correct, and the proof complete if

$c = 1$ or

$$\frac{\sum_{\delta_S > 0} \delta_S}{S \text{ contains } 1 \text{ but not } i} = \frac{\sum_{\delta_S > 0} \delta_S}{S \text{ contains } i \text{ but not } 1}$$

But this result may be obtained quite simply from the equality

$$\sum_{\substack{\delta_S > 0 \\ S \ni \{1\}}} \delta_S = \sum_{\substack{\delta_S > 0 \\ S \ni \{i\}}} \delta_S = 1$$

by subtracting

$$\sum_{S \supset \{1, i\}} \delta_S > 0$$

from both sides. This completes the proof of the main theorem.

IV. The β -Core

The particular type of solution described in the present paper is what Armann has termed the α -core, in distinction to an alternative concept designated as the β -core. For a utility vector to be in the α -core two conditions are necessary--first, the utility vector must arise from some joint selection of strategies by all of the players, and secondly no coalition can do better for all of its members by selecting an alternative set of strategies, independently of what the complementary coalition chooses to do.

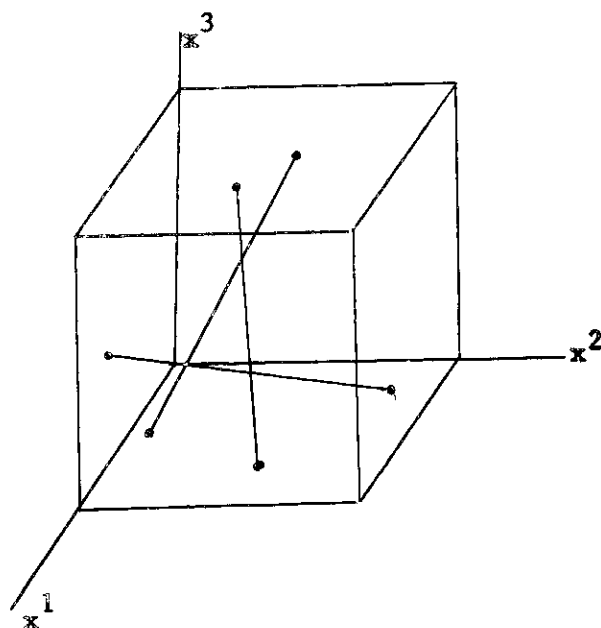
The second of these conditions is modified in the definition of the β -core. A blocking coalition is no longer required to select a specific strategy independently of the remaining players, but rather is permitted to vary its blocking strategy as a function of the complementary coalition's choice. It is as if a blocking coalition announces its intention to block, forces the complementary coalition to move first, and then responds, rather than the reverse order of moves. Clearly the blocking possibilities are larger, for any coalition, under the second of these conditions, and the β -core correspondingly smaller than the α -core. In this sense the β -core

is a sharper concept than the α -core, and it is of interest to see whether assumptions I and II are also sufficient for the β -core to be non-empty.

The conjecture is, however, incorrect. We shall construct a relatively simple three person game in normal form satisfying both I and II, whose β -core is empty.

Let player i ($i = 1, 2, 3$) have the strategy space $0 \leq x^i \leq 1$. We shall begin our counter-example by constructing utility functions $u_i(x^1, x^2, x^3)$, continuous and quasi-concave on the unit cube, with the following specific property. Every two player coalition can achieve a utility pair $(1, 1)$ by selecting a pair of strategies which are functionally dependent on the third player's choice; whereas the utility triple $(1, 1, 1)$ cannot be achieved by the three players acting collectively. While this first example does have a non-empty β -core, it can be modified slightly so as to produce another example in which the β -core is empty.

Consider the unit cube



each of whose points represents a specific joint strategy choice. For $i = 1, 2, 3$ let L_i be an arbitrary straight line from the face $x^i = 0$ to the face $x^i = 1$. Let C_i be the tetrahedron which is the convex hull of L_2 and L_3 ; C_2 the convex hull of L_1 and L_3 , and C_3 the convex hull of L_1 and L_2 . Finally let $u_i(x^1, x^2, x^3)$ be a piecewise linear concave function which is strictly less than unity outside of the tetrahedron C_i and greater than or equal to unity inside.

It should be clear that for any strategy choice of player 3, players 1 and 2 have strategies which yield each of them a utility of at least unity --they need only select a pair (x^1, x^2) such that $(x^1, x^2, x^3) \in C_1 \cap C_2$, and that point on the line L_3 with the third coordinate equal to x^3 will do. A similar remark holds for the other two player coalitions.

On the other hand a strategy triple in which all three players obtain at least unity must be in $C_1 \cap C_2 \cap C_3$. It is quite easy however to construct the three lines such that this intersection is empty, and we assume this to have been done. By a continuity argument the utility vector $(1-\epsilon, 1-\epsilon, 1-\epsilon)$ also cannot be obtained, for ϵ sufficiently small.

Let us also consider a second game in which player i has a strategy space

$$Y^i = \{(y^{i1}, y^{i2}, y^{i3}) \geq 0 \mid \sum_{j=1}^3 y^{ij} = 1\} \text{ and}$$

the utility function

$$v_i(y^1, y^2, y^3) = (1-\epsilon)(y^{1i} + y^{2i} + y^{3i})^{1/a},$$

with ϵ a small positive number and $a > 1$. This example is actually that of a market involving a single commodity and 3 players, each of whom owns a single unit of this commodity prior to trade. It is a simple matter to verify that all three players can collectively achieve any utility triple with

$$u_1^a + u_2^a + u_3^a \leq 3(1-\epsilon)^a,$$

and that any two player coalition $S = (i, j)$ can achieve the utility pairs (u_i, u_j) with

$$u_i^a + u_j^a \leq 2(1-\epsilon)^a.$$

It may be shown that the unique vector in the core of this second game is given by $(u_1, u_2, u_3) = (1-\epsilon, 1-\epsilon, 1-\epsilon)$.

To obtain the counter-example, consider the game in which player i selects (x^i, y^i) and has a quasi-concave utility function given by

$$\min[u_i(x^1, x^2, x^3), v_i(y^1, y^2, y^3)].$$

In this composite game, a utility vector can be achieved by a given coalition if and only if the same vector can be achieved by the coalition in each of the two earlier games.

In the composite game therefore the utility triple $(1-\epsilon, 1-\epsilon, 1-\epsilon)$ cannot be achieved by all three players acting collectively. But if the parameter a is sufficiently large, any member of the set $\{(u_i, u_j) \mid u_i^a + u_j^a \leq 2(1-\epsilon)^a\}$ must have u_i and $u_j \leq 1$. This implies that in the composite game the two player coalition (i, j) can achieve any utility pair (u_i, u_j) with $u_i^a + u_j^a \leq 2(1-\epsilon)^a$; therefore if (u_1, u_2, u_3) is to be in the β -core we must have

$$u_i^a + u_j^a \geq 2(1-\epsilon)^a \quad \text{for all}$$

pairs (i, j) .

Adding these three inequalities together we obtain

$$u_1^a + u_2^a + u_3^a \geq 3(1-\epsilon)^a.$$

But to be feasible for all three players in the composite game we must have equality here, from which we deduce that

$$(u_1, u_2, u_3) = (1-\epsilon, 1-\epsilon, 1-\epsilon)$$

a contradiction which demonstrates that the β -core is empty.

REFERENCES

- [1] Aumann, Robert J., "The Core of a Cooperative Game without Side Payments," Trans. Amer. Math. Soc. 98 (1961), pp. 539-552.
- [2] Bondareva, O., "The Core of an n-Person Game," Vestnik Leningrad Univ. 17 (1962), pp. 141-142.
- [3] Scarf, Herbert, "The Core of an N-Person Game," Econometrica 35 (1967), pp.
- [4] Shapley, Lloyd, "On Balanced Sets and Cores," RM-4601-PR, June, 1965, RAND Corporation, Santa Monica, California.