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AGGREGATION AND DISAGGREGATION IN THE PURE THEORY OF CAPITAL AND GROWTH: A NEW PARABLE

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AGGREGATION AND DISAGGREGATION IN THE PURE THEORY OF CAPITAL AND GROWTH: A NEW PARABLE*

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I. Introduction

One of the most important things to know about any economy is the available trade-off between present and future social welfare. The social rate of return is just a convenient way of representing the terms on which welfare alternatives are being offered to society over time. The relation between the social rate of return and certain other important economic concepts constitutes a classical problem of capital theory. As conventional wisdom has it, economic societies with lower rates of return ought to possess in some sense higher values of consumption, income, and capital.

To analyze this problem theoretically, the dynamic behavior of an economy has to be studied as a function of the rate of return, treated as an exogenously imposed parameter. However, the set of all possible dynamic economies with a given rate of return is far too rich to permit meaningful comparisons of what happens as the rate of return is varied. The traditional way of regularizing comparisons among economies with different rates of return is to consider only steady state regimes. Unfortunately the basic

*I owe a special debt to David Cass for starting me on this paper by clearly suggesting why the usual steady state approach to parable making was inadequate. The research was carried out under grants from the National Science Foundation and from the Ford Foundation.

problem ends up being miscast when it is forced into a steady state mold. As we will argue, it is more appropriate to normalize comparisons by limiting attention to dynamic economies with a given rate of return starting from the same historically given initial conditions.

Following this kind of a strategy will avoid some paradoxes of capital theory which plague the steady state approach. It will also yield a set of consistent national income relations interpretable as elements of a new parable of economic growth. In the course of working out such relations, several propositions concerning national income accounting under optimal growth will be developed. Such results may be interesting or useful in their own right.

II. Posing the Problem

The viewpoint taken is that of a central planner trying at time zero to map out future economic possibilities in a hypothetical socialist economy. This planned economy is presumed to contain a total of n economic commodities. Used in a broad sense, the term "economic commodity" is intended to cover consumption goods, investment goods, and factors of production.

Fixed factors of production are traditionally divided into reproducible and non-reproducible components, i.e., capital and natural resources (including labor). Here an alternative dichotomy is emphasized. Consistent with the viewpoint of a central planner at the present moment, all reproducible and non-reproducible factors inherited from the past at time zero are lumped together and treated as primary factors. From a planner's perspective at a time when long range plans are being contemplated, there is no inherent distinction between the existing inventory of man made capital and of natural resources. Both are at present non-controllable gifts of the past whose historical production costs are irrelevant to the purpose of planning future economic development.

The column n-vector of all factors inherited from the past at time zero is denoted H (for heritage). Those capital goods created after time zero but before time t make up the column n-vector of accumulated factors at time t , denoted A(t) . In contrast with the immobility of inherited factors, future capital accumulation can be controlled. Its regulation is the basic instrument of long range central planning.

To capture sharply the notion of a <u>fixed</u> factor, all factors of production are assumed to be permanent. This postulate will help to clearly differentiate the roles played by inherited and accumulated factors in the process of economic development. Adopting it amounts to assuming that capital is infinitely durable and that all natural resources are fixed. As a result, H is considered constant and A(t) is treated as non-decreasing for all times $t \geqq 0$.

At time t the economy produces non-negative column n-vectors of consumption, denoted C(t) , and of investment, denoted $\dot{A}(t)$.[1] Laws of production are embodied in a time independent n-dimensional production

[1]A dot over a variable denotes the derivative with respect to time--thus $\dot{A}(t) \equiv \frac{dA}{dt}$. As an alternative to assuming that capital lasts forever, we could allow capital to deteriorate exponentially and view the condition $\dot{A}(t) \geq 0$ as an artifically imposed allocation constraint which hopefully doesn't change the nature of the unconstrained problem very much. Note that at any time there are probably many zeros among the entries of C , $\dot{A}$, H or A . For example, non-reproducible natural resources would have a positive entry only as a component of H , whereas a consumption good which is worthless for investment purposes might have a positive entry only as a component of C .

set Ω . At any time $t \geqq 0$, Ω depends solely on total factor endowments $H + A(t)$. The pair $(C, \dot{A})$ is producible if and only if $C + \dot{A} \in \Omega(H+A)$ and $C, \dot{A} \geqq 0$. It is assumed that Ω is convex in $C + \dot{A}$ and displays constant returns to scale in $H + A$, with free disposal allowed.[2] The consumption stream $\{C(t)\}$ is <u>producible</u> for a given H if it is non-negative and if there exists a non-negative investment path $\{\dot{A}(t)\}$ satisfying $C(t) + \dot{A}(t) \in \Omega(H + A(t))$ with $A(0) = 0$.

The instantaneous social worth or value of consumption $C \geqq 0$ is measured by a time invariant utility index $U(C)$. It is assumed that $U(C)$ is a concave function known to the central planners.

A schedule of consumption changes $\{\Delta C(t)\}$ is <u>feasible</u> for a given producible $\{C(t)\}$ if the consumption stream $\{C(t)+\Delta A(t)\}$ is producible. At time t consumption change $\Delta C(t)$ yields utility change

$$\Delta U(t) \equiv U(C(t)+\Delta C(t)) - U(C(t))$$

The consumption stream $\{C(t)\}$ is said to have a social rate of return ρ if it is producible and if for any feasible change $\{\Delta C(t)\}$

$$\int_0^\infty \Delta U(t) e^{-\rho t} dt \leqq 0 .$$

A consumption stream with rate of return ρ is denoted $\{C(\rho, t)\}$. The

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[2] Convexity means that if $C_1 + \dot{A}_1 \in \Omega(H+A)$ and $C_2 + \dot{A}_2 \in \Omega(H+A)$, then $\lambda(C_1 + \dot{A}_1) + (1-\lambda)(C_2 + \dot{A}_2) \in \Omega(H+A)$ for all λ satisfying $0 \leqq \lambda \leqq 1$. Constant returns to scale means that if $C + \dot{A} \in \Omega(H+A)$, then $\mu(C+\dot{A}) \in \Omega(\mu(H+A))$ for all $\mu \geqq 0$. Free disposal means that if $C+\dot{A} \in \Omega(H+A)$, and if $C' + \dot{A}' \leqq C + \dot{A}$, then $C' + \dot{A}' \in \Omega(H+A)$.

standard of living index $U(C)$ serves as the appropriate numeraire base for calculating the social rate of return in real terms.[3]

An alternative way of looking at $\{C(\rho, t)\}$ is as a solution to the optimal growth problem

$$\text{maximize} \quad \int_0^\infty U(C(t))e^{-\rho t}dt \tag{1}$$

$$\text{subject to} \quad C(t)+\mathring{A}(t) \in \Omega(H+A(t)) \tag{2}$$

$$C(t),\ \mathring{A}(t) \geqq 0 \tag{3}$$

$$A(0) = 0 \tag{4}$$

In the optimal growth formulation ρ represents the social rate of time preference. This interpretation is actually equivalent to the social rate of return idea. Each concept emphasizes a different side of the same coin. The rate of return parameterizes transformation margins on the production side whereas the rate of time preference is a parameterization of substitution margins on the side of tastes. In both cases social utility is the natural unit of account for expressing real rates.

Solution n-vectors to the problem (1)-(4) are denoted $C(\rho, t)$, $\mathring{A}(\rho, t)$, and $A(\rho, t)$. $U(\rho, t)$ is defined as $U(C(\rho, t))$. The primary object of this study is to examine properties of solutions swept out as ρ is parametrically varied.

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[3] The problem of picking a numeraire base for calculating ρ is evaded when, as is often postulated, there exists but a single consumption good. What is exhibited as the social rate of return in such cases is in our terminology obtained by postulating a linear utility function defined over the single consumption good. In general there is no way of meaningfully parameterizing behavior of an entire economic system as a function of a single real rate of return without introducing some kind of a standard of living or utility index.

III. A Critique of the Steady State Approach

A steady or stationary state can be defined as an optimal solution of (1)-(4) having constant values of $C(t) = C \geqq 0$ and of $A(t) = \mathring{A}(t) = 0$. Such a solution is sometimes called a modified golden rule. It is the balanced path which would be voluntarily maintained forever as optimal. In general stationary solutions, if they exist at all, will exist only for certain initial heritage vectors whose values will typically depend on ρ. Such values of H need not coincide with the historically given heritage vector. In this sense the steady state is ahistorical. Worse than that, a stationary solution may not exist for <u>any</u> H in some well defined optimal growth problems. This will generally be the case, for example, in an economy without binding natural resource constraints.

As is well known, the standard one sector neoclassical model with output a constant returns to scale function of capital and labor alone yields "well behaved" comparative steady state results. For a given amount of labor, lower rates of return imply stationary states with higher consumption (equals output) and more capital. These conclusions do not generalize to multi-sector models where non-monotone behavior has been observed.[4]

A steady state, when it exists, is typically the ahistorical <u>limit</u> of an optimal trajectory which is attained independently of initial capital stocks.[5] Viewed in these terms, the steady state approach at best takes

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[4] There is by now a large literature on this subject. For a summary of the issues, see Samuelson [1966].

[5] Provided the optimal path for (1)-(4) goes to a limit--if not all the more reason why it is irrelevant for a planner to concentrate on steady states since the original problem may still be perfectly well defined.

account of only the very last part of an optimal growth path. To see clearly the nature of this drawback, consider the following example.

Assuming it can be done, set H to yield a stationary solution for a given value of ρ, say ρ'. Suppose now that ρ is increased from ρ' to ρ'' where $\rho'' > \rho'$ but that H remains at its previous $\rho = \rho'$ steady state level. Suppose also that the new optimal path for $\rho = \rho''$ (now not necessarily a stationary path) goes to some limit. In general there is nothing to prevent the limiting optimal utility along the non-stationary $\rho = \rho''$ path from becoming higher than the stationary $\rho = \rho'$ utility level. If this were to happen, however, utility would be gained in the limit only at the expense of lower utility in earlier periods. It is questionable whether it is not premature to label such a phenomenon as perverse solely on the basis of limiting steady state behavior.[6] Taking the trajectory as a whole, the utility of consumption may well be lower in some average sense along the second path.

For the planning purposes of analyzing the optimum possibilities open to an economy as a function of the rate of return, a relevant approach should take into account the whole path of what is to be achieved from the given initial conditions and not just what is arrived at (if at all) only eventually and usually independently of initial capital stocks.

In this paper we work with entire path averages of economic variables, called stationary equivalents. Such averages will be shown to exhibit a coherent pattern of dependence on ρ which can be rationalized in terms of a new parable.

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[6] Solow [1967] has a message which is somewhat similar.

IV. Stationary Equivalence

In order to be able to talk about what happens to utility as the rate of return changes, the time path $\{U(\rho, t)\}$ must be summarized as a function of ρ. How can the necessarily complicated behavior of $\{U(\rho, t)\}$ be described in simple terms? In the approach taken here, $\{U(\rho, t)\}$ is aggregated into a single number representing the entire path. The index summarizing $\{U(\rho, t)\}$ is chosen to be that hypothetical constant utility level which would yield exactly the same social welfare as the path $\{U(\rho, t)\}$. This stationary equivalent of $\{U(\rho, t)\}$ is denoted by $\bar{U}(\rho)$. It is defined as the solution to the equation

$$\int_0^\infty \bar{U}(\rho)e^{-\rho t}dt = \int_0^\infty U(\rho, t)e^{-\rho t}dt ,$$

which can be written as

$$\bar{U}(\rho) \equiv \frac{\int_0^\infty U(\rho, t)e^{-\rho t}dt}{\int_0^\infty e^{-\rho t}dt} = \rho \int_0^\infty U(\rho, t)e^{-\rho t}dt . \qquad (5)$$

Although it is intuitively appealing, there is no compelling *a priori* reason for working with a stationary norm as opposed to some other kind. The primary motive for employing an average value of $\{U(\rho, t)\}$ based on the weighting function $\{e^{-\rho t}\}$ is that it gives rise to nice results which are easy to interpret.

On a somewhat more abstract level, let X stand for some particular national income type of aggregate economic concept. Along an optimal program having rate of return ρ , let X be represented at time t by the variable $X(\rho, t)$. The basic principle underlying the exact definition of $X(\rho, t)$ is that it be measured in terms of utility at time t as numeraire. The basic principle underlying the creation of an index to represent the path $\{X(\rho, t)\}$ is the use of the weighting function $\{e^{-\rho t}\}$ based on the prevailing social rate of return to form an average value of X . Consistent with (11), the <u>stationary equivalent</u> of $\{X(\rho, t)\}$ is defined to be the hypothetical constant value of X , denoted $\bar{X}(\rho)$, which would yield the same present discounted value as $\{X(\rho, t)\}$ at rate of return ρ . In symbols,

$$\bar{X}(\rho) \equiv \frac{\int_0^\infty X(\rho, t)e^{-\rho t}dt}{\int_0^\infty e^{-\rho t}dt} = \rho \int_0^\infty X(\rho, t)e^{-\rho t}dt \ .$$

V. National Income Accounting under Optimal Growth

A multi-sector economy undergoing optimal growth as described by (1)-(4) yields a surprisingly rich collection of aggregate national income relations. The concept of stationary equivalence developed in the previous section will provide a unified way of looking at some of these relations.

The distinguishing feature of national income accounting under optimal growth is the availability of an explicit utility index which can be used for evaluating the social worth of consumption at any time. In practise the national income statistician works with his own implicit standard of living measure when he attempts to evaluate real consumption by a linear index. His standard can be considered a special case of the more general utility index $U(C)$.

Along an optimal growth path with rate of return ρ there will exist non-negative dual price variables associated with C, $\dot{A}$ and the pair (H, A) which satisfy certain well known necessary conditions. Let $P(\rho, t)$ be a row n-vector of imputed prices at time t for $\dot{A}(\rho, t)$ and let $q(\rho, t)$ be a row n-vector of imputed prices at time t for $C(\rho, t)$. Let $R(\rho, t)$ be a row n-vector of imputed rentals at time t for H and $A(\rho, t)$. The specific necessary conditions which $P(\rho, t)$, $q(\rho, t)$ and $R(\rho, t)$ must satisfy will be noted and discussed piecemeal in those parts of the present paper where the particular results are used. For the present it is noted only that the existence of dual variables satisfying the as yet not enumerated necessary conditions is guaranteed by Pontryagin's principle.[7]

The rest of this section is divided into three subsections. These deal with the valuation, under optimal growth, of accumulated factors, national product, and national income.

(a) <u>The Social Value of Accumulated Factors</u>

Let $\hat{A} \geqq 0$ be a vector of accumulated factors at time t. How much is $\hat{A}$ worth to society? Since the economy by definition starts off with no accumulated factors it is natural to choose $A(t) = 0$ as the benchmark of zero value. The value of $\hat{A}$ at time t is defined to be the maximum social welfare attainable from time t on starting with $A(t) = \hat{A}$ over and above

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[7] See Pontryagin and associates [1962]. At times we will implicitly assume strong regularity conditions for some functions--e.g. smooth differentiability and uniform convergence. These implicit regularity conditions could undoubtedly be considerably weakened without altering the main results of the paper. In keeping with our usual convention, all dual prices are expressed in terms of utility as numeraire.

the maximum social welfare obtainable from time t on starting with $A(t) = 0$. Social welfare is measured by the integral of discounted utility using the prevailing social rate of return as the discount rate. Consistent with the principle that $U(C(t))$ be the unit of account for evaluating the worth of $A(t) = \hat{A}$, all discounting is done back to time t.

More formally, let $\psi(\rho,\hat{A})$ be defined as follows:

$$\psi(\rho,\hat{A}) \equiv \max. \int_t^\infty U(C(s))e^{-\rho(s-t)}ds$$

$$\text{s.t.} \quad C(s)+\dot{A}(s) \in \Omega(H+A(s))$$

$$C(s),\ \dot{A}(s) \geqq 0$$

$$A(t) = \hat{A}$$

Note that ψ is independent of t due to the t-invariance of the problem defining it. The value (at <u>any</u> time) of accumulated capital $\hat{A}$ with rate of return ρ is defined to be

$$V(\rho,\hat{A}) \equiv \psi(\rho,\hat{A}) - \psi(\rho,0) .$$

From the principle of optimality and the factoring out properties of $e^{-\rho t}$,

$$\psi(\rho,A(\rho,t)) = \int_t^\infty U(\rho,s)e^{-\rho(s-t)}ds \qquad (6)$$

which implies

$$V(\rho,A(\rho,t)) = \int_t^\infty U(\rho,s)e^{-\rho(s-t)}ds - \int_0^\infty U(\rho,s)e^{-\rho s}ds \qquad (7)$$

For some purposes, a more useful expression for $V(\rho, A(\rho,t)) = \psi(\rho, A(\rho,t)) - \psi(\rho, 0)$ is

$$\int_0^t \left.\frac{\partial\psi}{\partial\hat{A}}\right|^+_{\rho, A(\rho,s)} \mathring{A}(\rho,s)ds \ .$$

Since[8]

$$P(\rho,s) = \left.\frac{\partial\psi}{\partial\hat{A}}\right|^+_{\rho, A(\rho,s)} , \tag{8}$$

it follows that

$$V(\rho, A(\rho,t)) = \int_0^t P(\rho,s)\mathring{A}(\rho,s)ds \ . \tag{9}$$

The right hand side of (9) is the historical cost book value at time t of capital stock produced since time zero. It is interesting to note that the old fashioned idea of evaluating capital at its original production cost is exactly what is appropriate in the present context as a measure of true social value in terms of utility. By evaluating capital created at time s, $\mathring{A}(\rho,s)$, with the prices of that time $P(\rho,s)$, the national wealth statistician is capturing the consumers' and producers' surplus created by adding the increment $\mathring{A}(\rho,s)$ to the existing stock $A(\rho,s)$. If $A(\rho,t)$ is evaluated entirely by its strictly marginal current reproduction cost

[8] Because ψ is concave in $\hat{A}$, it must possess right and left hand partial derivatives everywhere, and ordinary partial derivatives a.e.. The existence of investment prices equal to the right hand partial derivatives of ψ is part of the set of necessary conditions for an optimum.

$P(\rho, t)$, the measure of surplus value is lost. In Adam Smith's terminology, "value in use" (which is the appropriate concept for our purposes) is not necessarily the same thing as "value in exchange."

With (9) giving the value of the stock of accumulated factors $A(\rho, t)$ at time t , we define the stationary equivalent of the value of accumulated factors along an optimal path to be

$$\bar{V}(\rho) \equiv \rho \int_0^\infty V(\rho, A(\rho, t))e^{-\rho t}dt = \rho \int_0^\infty e^{-\rho t}dt \int_0^t P(\rho, s)\mathring{A}(\rho, s)ds$$

Reversing the order of integration,

$$\bar{V}(\rho) = \int_0^\infty P(\rho, t)\mathring{A}(\rho, t)e^{-\rho t}dt \tag{10}$$

(b) Social National Product

We seek a measure of national product at time t , denoted $Y(\rho, t)$, which is consistent with the spirit of defining a concept in terms of $U(C(t))$ as numeraire. A standard definition of national product is the <u>value of current consumption plus the accretion to wealth</u>. In terms of our model, this formulation yields

$$Y(\rho, t) \equiv U(C(\rho, t)) + \dot{V}(\rho, A(\rho, t)) \tag{11}$$

The notion of investment implicit in the above definition is "accretion to wealth." Defining national investment at time t as $I(\rho, t)$, (9) yields

$$I(\rho, t) \equiv \dot{V}(\rho, A(\rho, t)) = P(\rho, t)\dot{A}(\rho, t) \ . \tag{12}$$

Using (12), definition (11) can be written

$$Y(\rho, t) = U(\rho, t) + I(\rho, t) = U(\rho, t) + P(\rho, t)\dot{A}(\rho, t) \ . \tag{13}$$

One of the important Pontryagin necessary conditions is

$$\left.\begin{aligned} U(\rho, t) + P(\rho, t)\dot{A}(\rho, t) = \max. \quad & U(C) + P(\rho, t)\dot{A} \\ \text{s.t.} \quad & C+\dot{A} \in \Omega(H+A(\rho, t)) \\ & C, \ \dot{A} \geqq 0 \end{aligned}\right\} \tag{14}$$

An interpretation of (14) is that at any time along an optimal path national product is maximized given current factor endowments and the socially optimal prices of investment goods.

Expression (13) is conceptually analogous to the national income accountant's book-keeping practice of adding in investment goods to the value of consumption by weighting them with prices measuring their marginal rates of transformation with the numeraire. That $P(\rho, t)$ is indeed the appropriate list of marginal transformation rates with $U(C(t))$ is established by (14).

A slightly different viewpoint sees national product at any time as a proxy for the economy's power to consume at a constant rate from then on. Operating in the spirit of this idea suggests the following approach. At time t the optimal growth economy possesses accumulated factors $A(\rho, t)$. From (6) the value of an optimal trajectory starting at time t in terms of $U(C(t))$ as numeraire is $\psi(\rho, A(\rho,t)) = \int_t^\infty U(\rho,s)e^{-\rho(s-t)}ds$. The constant hypothetical utility level which yields from time t on the identical social welfare as the optimal path $\{U(\rho,s)\}$ is

$$\overline{U}_t(\rho) \equiv \rho \int_t^\infty U(\rho,s)e^{-\rho(s-t)}ds \tag{15}$$

Equating (7) with (9),

$$\int_0^t P(\rho,s)\dot{A}(\rho,s)ds = \int_t^\infty U(\rho,s)e^{-\rho(s-t)}ds - \int_0^\infty U(\rho,s)e^{-\rho s}ds$$

Differentiating both sides of the above equation with respect to t and substituting into (15) yields

$$\overline{U}_t(\rho) = U(\rho,t) + P(\rho,t)\dot{A}(\rho,t) \tag{16}$$

Equation (16) is an interesting result. A naive application of the largest permanently maintainable value of consumption notion might immediately identify national product at time t with $U(\rho,t) + I(\rho,t)$ for the following incorrect reason. If all investment goods were convertable at time t into utility at the transformation rate $P(\rho,t)$, the maximum attainable level of utility which could be maintained forever without running down capital stocks would appear to be $U(\rho,t) + I(\rho,t)$. Such reasoning is fallacious because marginal transformation rates cannot in general be used to change non-marginal amounts of investment into consumption. For this reason the utility level $U(\rho,t) + I(\rho,t)$ is undoubtedly not attainable at time t .

Nevertheless, equation (16) with definition (15) says that the maximum welfare in fact attainable from time t on is exactly the same as what would be obtained from the hypothetical constant utility level $U(\rho, t) + I(\rho, t)$. In this sense the naive interpretation of the current power to consume at a constant rate idea implies the right answer, although for the wrong reason.

The stationary equivalent of investment is

$$\bar{I}(\rho) \equiv \rho \int_0^\infty I(\rho, t) e^{-\rho t} dt$$

$$= \rho \int_0^\infty P(\rho, t) \dot{A}(\rho, t) e^{-\rho t} dt . \tag{17}$$

Comparing (10) with (17), we observe that

$$\bar{I}(\rho) = \rho \bar{V}(\rho) . \tag{18}$$

The stationary equivalent of national product is

$$\bar{Y}(\rho) \equiv \rho \int_0^\infty Y(\rho, t) e^{-\rho t} dt \tag{19}$$

$$= \bar{U}(\rho) + \bar{I}(\rho) . \tag{20}$$

(c) Social National Income

Define the function ϕ as follows:

$$\begin{aligned} \phi(\rho, t, M) \equiv \max. \quad & U(C) + P(\rho, t)\dot{A} \\ \text{s.t.} \quad & C + \dot{A} \in \Omega(M) \\ & C, \dot{A} \geqq 0 \end{aligned}$$

Given $P(\rho, t)$, $\phi(\rho, t, M)$ records the maximum attainable national product at time t as a function of total factors M . Due to concavity of U and convexity of Ω , ϕ will be concave in M . This implies the existence for all $M \geqq 0$ of right and left hand derivatives with respect to M . Using results from non-linear programming, there will exist at time t imputed rentals $R(\rho, t)$ on the fixed factors H and $A(\rho, t)$ satisfying for all $M \geqq 0$

$$\phi(\rho, t, M) \leqq \phi(\rho, t, H+A(\rho, t)) + R(\rho, t)[M-(H+A(\rho, t))] ,$$

and

$$\frac{\partial \phi}{\partial M}\bigg|^{+}_{H+A(\rho, t)} \leqq R(\rho, t) \leqq \frac{\partial \phi}{\partial M}\bigg|^{-}_{H+A(\rho, t)} .$$

The imputed rentals $R(\rho, t)$ are thus interpretable as the (identical) marginal national products of both H and A in an optimal program at time t .

Because $q(\rho, t)$ is defined to be the shadow price of consumption at time t in the national product maximizing problem (14), it must satisfy[9]

$$\frac{\partial U}{\partial C}\bigg|^{+}_{C(\rho, t)} \leqq q(\rho, t) \leqq \frac{\partial U}{\partial C}\bigg|^{-}_{C(\rho, t)} .$$

From programming theory, first degree homogeneity of Ω in the factors H and A implies

$$q(\rho, t)C(\rho, t) + P(\rho, t)\dot{A}(\rho, t) = R(\rho, t)[H + A(\rho, t)] . \qquad (21)$$

Equation (21) is interpretable as an exhaustion of the product condition along

[9] Note that if commodity i is simultaneously consumed and invested at time t , then $P_i(\rho, t) = q_i(\rho, t)$. If commodity i is valuable only as a consumption good, then $P_i(\rho, t) = 0$. If, on the other hand, it has no intrinsic value as consumption but is useful solely as investment, then $q_i(\rho, t) = 0$.

an optimal path at time t .

Convexity of Ω and concavity of U in condition (14) imply a separating hyperplane result of the form

$$U(\rho,t) + P(\rho,t)\dot{A}(\rho,t) = \max.\quad U(C) + P(\rho,t)\dot{A} \tag{22}$$

$$\text{s.t.}\quad q(\rho,t)C + P(\rho,t)\dot{A} = q(\rho,t)C(\rho,t) + P(\rho,t)\dot{A}(\rho,t)$$

$$C,\ \dot{A} \geqq 0$$

Having settled on an expression for national product at time t , it is natural to ask for a definition of national income. The money income competitively imputed to the factors H and A at time t is

$$R(\rho,t)[H + A(\rho,t)] \ .$$

The real value of this money income is the maximum amount (with $U(C)$ as numeraire) of real product $U(C)+P(\rho,t)\dot{A}$ which can be purchased with it. Denoting this real national income at time t by $N(\rho,t)$,

$$N(\rho,t) \equiv \max.\quad U(C) + P(\rho,t)\dot{A}$$

$$\text{s.t.}\quad q(\rho,t)C + P(\rho,t)\dot{A} = R(\rho,t)[H+A(\rho,t)]$$

$$C,\ \dot{A} \geqq 0$$

From (22), (21), and (13),

$$N(\rho,t) = Y(\rho,t) \ ,$$

i.e., at all times real national income equals real national product. It follows from (19) that $\bar{N}(\rho) \equiv \int_0^{\infty} N(\rho,t)e^{-\rho t}dt = \bar{Y}(\rho)$.

The imputed competitive returns to the accumulated factors at time t is

$$\pi(\rho, t) \equiv R(\rho, t)A(\rho, t) \ .$$

The stationary equivalent of total returns to A is

$$\bar{\pi}(\rho) \equiv \rho \int_0^\infty \pi(\rho, t)e^{-\rho t}dt \tag{23}$$

An important necessary condition is

$$R(\rho, t) + \dot{P}(\rho, t) = \rho P(\rho, t) \tag{24}$$

Equation (24) can be interpreted as saying that along an efficient path there will exist competitive imputations which everywhere equate interest payments to gross rentals plus capital gains.

Using (24) and integrating by parts,

$$\int_0^\infty [R(\rho, t)e^{-\rho t}]A(\rho, t)dt = \int_0^\infty - \frac{d[P(\rho, t)e^{-\rho t}]}{dt} A(\rho, t)dt$$

$$= -P(\rho, t)e^{-\rho t}A(\rho, t)\Big|_0^\infty + \int_0^\infty P(\rho, t)\dot{A}(\rho, t)e^{-\rho t}dt$$

Since $A(\rho, 0) = 0$ and it is assumed that the transversality condition

$$\lim_{t\to\infty} P(\rho, t)e^{-\rho t}A(\rho, t) = 0$$

is satisfied, we have

$$\int_0^\infty R(\rho, t)A(\rho, t)e^{-\rho t}dt = \int_0^\infty P(\rho, t)\dot{A}(\rho, t)e^{-\rho t}dt \ .$$

Employing definitions (17) and (23), we have shown that

$$\overline{I}(\rho) = \overline{\pi}(\rho) \tag{25}$$

The significance of (25) is that while it need not be true at any specific time that $I(\rho, t) = \pi(\rho, t)$, on the average investment equals the returns to accumulated factors in the sense that their stationary equivalents are identical. Conceptually, the financing of investment can be thought of as coming out of the returns to accumulation because the present discounted values of both are the same.

In completely analogous fashion it becomes natural to ask if it would be possible to think of financing social consumption out of the imputed returns to the inherited factors. From (21) and (25),

$$\int_0^\infty R(\rho, t)He^{-\rho t}dt = \int_0^\infty q(\rho, t)C(\rho, t)e^{-\rho t}dt \tag{26}$$

so that the answer to this question is yes. Relatedly, what is the real value of the stream of total returns to inherited factors $\{R(\rho, t)H\}$? If we are thinking of $\{R(\rho, t)H\}$ as financing consumption, that value ought to be measured by the maximum social welfare which can be "purchased" at prices $\{q(\rho, t)\}$ and interest rate ρ with the "consumption fund" $\int_0^\infty R(\rho, t)He^{-\rho t}dt$. Let $\overline{w}(\rho)$ be the stationary equivalent of the real purchasing power of the social income stream $\{R(\rho, t)H\}$. Then

$$\left.\begin{array}{l} {}_0\int^{\infty} \bar{w}(\rho)e^{-\rho t}dt \equiv \max. \quad {}_0\int^{\infty} U(C(t))e^{-\rho t}dt \\ \\ \qquad\qquad \text{s.t.} \quad {}_0\int^{\infty} q(\rho,t)C(t)e^{-\rho t}dt = {}_0\int^{\infty} R(\rho,t)He^{-\rho t}dt \\ \\ \qquad\qquad\qquad C(t) \geqq 0 \end{array}\right\} \quad (27)$$

Condition (26), concavity of $U(C)$, and the notion of $\{q(\rho,t)e^{-\rho t}\}$ as efficiency prices for $\{C(\rho,t)\}$ imply the separating hyperplane result that the right hand side maximand of (27) equals ${}_0\int^{\infty} U(\rho,t)e^{-\rho t}dt$. Thus,

$$\bar{w}(\rho) = \bar{U}(\rho). \quad (28)$$

VI. A Graphic Representation

In Figure 1 a graph of the points $\{\bar{V}(\rho), \bar{Y}(\rho) | 0 < \rho < \infty\}$ is depicted. Presuming for the moment that the implied correspondence between $\bar{Y}$ and $\bar{V}$ can be expressed as a functional relationship, we write

$$\bar{Y} = F(\bar{V}). \quad (29)$$

The "opportunities function" $F(\bar{V})$ depends not only on the composition of the production set $\Omega(H+A)$, but also on the shape of $U(C)$ and on the magnitude of the initial heritage H.

The relationships (18), (20), (25), (28) between the stationary equivalents $\bar{Y}$, $\bar{V}$, $\bar{U}$, $\bar{I}$, $\bar{w}$, $\bar{\pi}$ are depicted in Figure 1. We have yet to demonstrate the following two propositions:

$$\text{(i)} \quad F'(\bar{V})\Big|_{\bar{V}(\rho)} = \rho \quad (30)$$

(ii) $F''(\bar{V}) \leqq 0$ (concavity). (31)

When we have done so our justification for the way Figure 1 is drawn will be complete. The rest of this section is devoted to proving (i) and (ii).

From (13), (15), (16), (19),

$$\begin{aligned}\bar{Y}(\rho) &= \rho^2 \int_0^\infty e^{-\rho t} dt \int_t^\infty U(\rho, s) e^{-\rho(s-t)} ds \\ &= \rho^2 \int_0^\infty dt \int_t^\infty U(\rho, s) e^{-\rho s} ds \\ &= \rho^2 \int_0^\infty U(\rho, s) e^{-\rho s} ds \int_0^s dt \\ &= \rho^2 \int_0^\infty U(\rho, t) t e^{-\rho t} dt\end{aligned}$$

Using the above expression for $\bar{Y}(\rho)$ and (18), (20), (5),

$$\bar{V}(\rho) = \frac{\bar{Y}(\rho) - \bar{U}(\rho)}{\rho} = \int_0^\infty (\rho t - 1) U(\rho, t) e^{-\rho t} dt .$$

Differentiating[10] $\bar{Y}(\rho)$ and $\bar{V}(\rho)$ with respect to ρ,

$$\frac{d\bar{Y}}{d\rho} = \int_0^\infty (2\rho t - \rho^2 t^2) U(\rho, t) e^{-\rho t} dt + \rho^2 \left[\frac{\partial}{\partial \theta} \left(\int_0^\infty t U(\theta, t) e^{-\rho t} dt \right) \right]_{\theta = \rho} \tag{32}$$

$$\frac{d\bar{V}}{d\rho} = \int_0^\infty (2\rho t - \rho^2 t^2) U(\rho, t) e^{-\rho t} dt + \left[\frac{\partial}{\partial \theta} \left(\int_0^\infty (\rho t - 1) U(\theta, t) e^{-\rho t} dt \right) \right]_{\theta = \rho} \tag{33}$$

Define

$$h(\theta, \rho) \equiv \int_0^\infty U(\theta, t) e^{-\rho t} dt$$

[10] At this point we start implicitly assuming differentiability and/or uniform convergence for some functions.

Treating ρ as fixed, the function $h(\theta, \rho)$ attains a maximum at $\theta = \rho$. It follows that

$$\left.\frac{\partial h}{\partial \theta}\right|_{\theta=\rho} = 0 \tag{34}$$

$$\left.\frac{\partial^2 h}{\partial \theta^2}\right|_{\theta=\rho} \leq 0. \tag{35}$$

Using (34) in (33), we simplify $\frac{d\bar{V}}{d\rho}$ to

$$\frac{d\bar{V}}{d\rho} = \int_0^\infty (2t-\rho t^2)U(\rho, t)e^{-\rho t}dt + \rho\left[\frac{\partial}{\partial \theta}\left(\int_0^\infty tU(\theta, t)e^{-\rho t}dt\right)\right]_{\theta=\rho} \tag{36}$$

Comparing (32) and (36) we note that

$$\frac{d\bar{Y}}{d\rho} = \rho\,\frac{d\bar{V}}{d\rho},$$

proving (30).

To prove (31), we start by differentiating $\frac{d\bar{Y}}{d\bar{V}} = \rho$ with respect to $\bar{V}$:

$$F''(\bar{V}) = \frac{d^2\bar{Y}}{d\bar{V}^2} = \frac{\frac{d}{d\rho}\left(\frac{d\bar{Y}}{d\bar{V}}\right)}{\frac{d\bar{V}}{d\rho}} = \frac{1}{\frac{d\bar{V}}{d\rho}} \tag{37}$$

We note first that since (34) holds for all ρ, $\frac{\partial}{\partial \rho}\left(\left.\frac{\partial h}{\partial \theta}\right|_{\theta=\rho}\right) = 0$. This implies

$$\left.\frac{\partial^2 h}{\partial \rho \partial \theta}\right|_{\theta=\rho} + \left.\frac{\partial^2 h}{\partial \theta^2}\right|_{\theta=\rho} = 0.$$

Comparing to the above expression with (35),

$$\left.\frac{\partial^2 h}{\partial\rho\partial\theta}\right|_{\theta=\rho} \geqq 0 \ . \tag{38}$$

Changing the order of differentiation of $\left.\frac{\partial^2 h}{\partial\rho\partial\theta}\right|_{\theta=\rho}$,

$$\left.\frac{\partial^2 h}{\partial\rho\partial\theta}\right|_{\theta=\rho} = \left.\frac{\partial^2 h}{\partial\theta\partial\rho}\right|_{\theta=\rho} = \left[\frac{\partial}{\partial\theta}\left(\int_0^\infty -t\ U(\theta,t)e^{-\rho t}dt\right)\right]_{\theta=\rho}$$

Since (38) holds, substitution of the above expression into (36) yields

$$\frac{d\bar{V}}{d\rho} \leqq \int_0^\infty (2t-\rho t^2)U(\rho,t)e^{-\rho t}dt$$

$$= U(\rho,t)t^2e^{-\rho t}\Big|_0^\infty - \int_0^\infty \dot{U}(\rho,t)t^2e^{-\rho t}dt$$

(integrating by parts). Assuming a reasonable boundedness condition like $U(\rho,t) \leqq Be^{gt}$ for some $B > 0$, $g < \rho$ causes $U(\rho,t)t^2e^{-\rho t}\Big|_0^\infty$ to vanish, implying

$$\frac{d\bar{V}}{d\rho} \leqq -\int_0^\infty \dot{U}(\rho,t)t^2e^{-\rho t}dt \tag{39}$$

To evaluate the right hand side integral of (39) we need to take a slight detour through two lemmas.

<u>Lemma 1</u>: $\int_0^\infty U(\rho,g(t))e^{-\rho t}dt \leqq \int_0^\infty U(\rho,t)e^{-\rho t}dt$ for any continuously differentiable function $g(t)$ satisfying $g(0) = 0$, $0 \leqq g'(t) \leqq 1$.

<u>Proof</u>: Consider the program

$$\tilde{C}(\rho, t) \equiv C(\rho, g(t))$$

$$\tilde{A}(\rho, t) \equiv A(\rho, g(t))$$

$$\dot{\tilde{A}}(\rho, t) \equiv g'(t)\dot{A}(\rho, g(t))$$

We first show that the $\sim$ program is attainable. For any time $t \geqq 0$, $\tilde{C}(\rho, t)$ and $\dot{\tilde{A}}(\rho, t)$ are both non-negative and

$$\tilde{C}(\rho, t) + \dot{\tilde{A}}(\rho, t) \in \Omega(H + A(\rho, t))$$

(assuming free disposal of investment goods) because

$$\tilde{C}(\rho, t) + \dot{\tilde{A}}(\rho, t) \leqq C(\rho, g(t)) + \dot{A}(\rho, g(t)) \in \Omega(H + A(\rho, g(t)) = \Omega(H + \tilde{A}(\rho, t)) .$$

Finally, note the required consistency in our definitions of $\tilde{A}$ and $\dot{\tilde{A}}$:

$$\tilde{A}(\rho, t) = \int_0^t \dot{\tilde{A}}(\rho, s)ds = \int_0^t g'(s)\dot{A}(s, g(s))ds = \int_0^{g(t)} \dot{A}(\rho, r)dr = A(\rho, g(t)) .$$

Since $\{\tilde{C}(\rho, t)\}$ is attainable, it cannot yield higher welfare than the chosen path $\{C(\rho, t)\}$.

<u>Lemma 2</u>: Let $g(t, \delta)$ be a continuously differentiable function satisfying for all $\delta \geqq 0$: $g(0, \delta) = 0$, $0 \leqq \frac{\partial g(t, \delta)}{\partial t} \leqq 1$, $g(t, 0) = t$. Then

$$\int_0^\infty \dot{U}(\rho, t) \left[\frac{\partial g(t, \delta)}{\partial \delta}\right]_{\delta=0} e^{-\rho t}dt \leqq 0 .$$

<u>Proof</u>: From lemma 1

$$\int_0^\infty U(\rho, g(t, \delta))e^{-\rho t}dt \leqq \int_0^\infty U(\rho, t)e^{-\rho t}dt$$

for any $\delta \geqq 0$. This inequality plus the condition $g(t,0) = t$ implies that the right hand side derivative of $\int_0^\infty U(\rho, g(t,\delta))e^{-\rho t}dt$ evaluated at $\delta = 0$ is non-positive. Thus,

$$\left[\frac{\partial}{\partial\delta}\left(\int_0^\infty U(\rho, g(t,\delta))e^{-\rho t}dt\right)\right]_{\delta=0}$$

$$= \left[\int_0^\infty U_2(\rho, g(t,\delta))\frac{\partial g(t,\delta)}{\partial\delta}\, e^{-\rho t}dt\right]_{\delta=0}$$

$$= \int_0^\infty \dot{U}(\rho, t)\left[\frac{\partial g(t,\delta)}{\partial\delta}\right]_{\delta=0} e^{-\rho t}dt \leqq 0$$

which completes the proof.

By verifying that the hypothesis of lemma 2 holds for

$$g(t,\delta) \equiv te^{(e^{-\delta t}-1)}$$

we conclude that

$$-\int_0^\infty \dot{U}(\rho, t)t^2 e^{-\rho t}dt \leqq 0$$

From (39), this implies that

$$\frac{d\bar{V}}{d\rho} \leqq 0 \ .$$

Using (37) we obtain

$$F''(\bar{V}) \leqq 0$$

which proves (31). With this result the basic features of Figure 1 have all been justified.

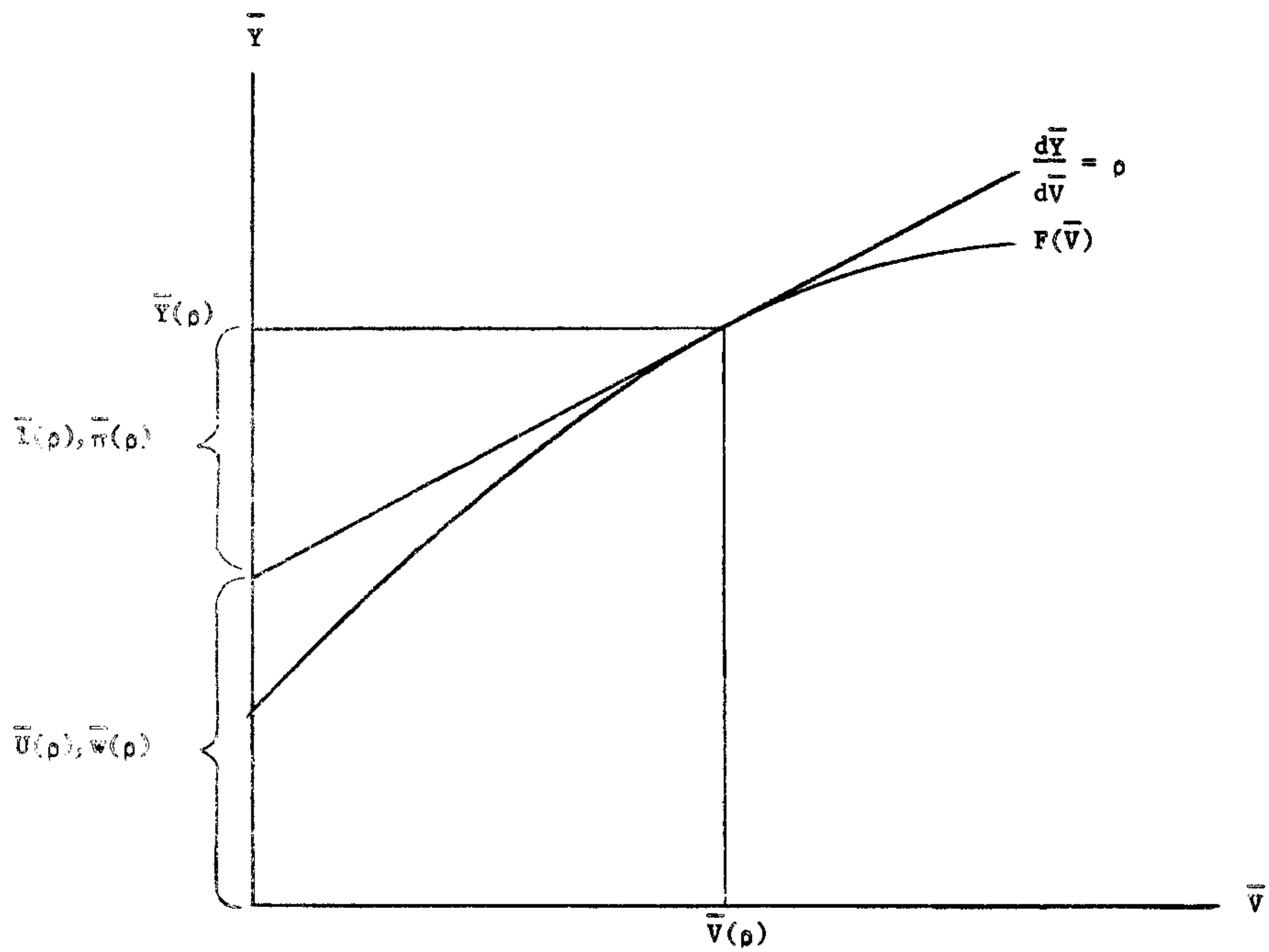

FIGURE 1

A NEW PARABLE

VII. A New Parable

How are the economic relations portrayed in Figure 1 to be interpreted? In this section a story is suggested which draws heavily on the concepts of classical interest theory developed by Irving Fisher.

Consider an infinitely long lived representative economic man. He is assumed to dwell in a perfectly competitive economy with interest rate ρ equal to his own rate of pure time preference. The man owns an unchanging labor endowment $\overline{L}$ meant to stand for his holdings of original inherited factors. Without loss of generality, labor units are selected so that $\overline{L} = 1$. An all purpose malleable capital symbolizes the man's accumulated factors. At any time labor $\overline{L}$ can be combined with amount $\overline{V}$ of acquired capital to produce output $\overline{Y}$ according to a given constant returns to scale production function

$$\overline{Y} = f(\overline{V}, \overline{L}) = F(\overline{V}).$$

Capital, considered infinitely durable, is raised by borrowing at the competitive rental rate ρ. If $\overline{V}$ is the amount of capital rented and $\overline{I}$ is the corresponding rental or interest cost, then $\overline{I} = \rho\overline{V}$. Given the unlimited opportunity to lend or borrow at rate ρ, any individual will at all times choose $\overline{V}$ to maximize net output $F(\overline{V}) - \overline{I}$. The value of $\overline{V}$ which maximizes net output, denoted $\overline{V}(\rho)$, will satisfy $F'(\overline{V}(\rho)) = \rho$. With a rate of pure time preference equal to ρ, the representative economic man will maximize his well being by always consuming at the level $\overline{W}(\rho) \equiv F(\overline{V}(\rho)) - \rho\overline{V}(\rho)$.

In a very rough sense the story of the one man economy is at least suggestive of certain aspects of the process of optimal economic growth. Like the representative economic man, an economy as a whole starts off with an inheritance which it can augment at a cost. The cost of accumulating capital in a closed economy is the current consumption which must be foregone to make way for investment. The benefit of capital accumulation is the eventual increased consumption to which it gives rise. The optimal amount of capital to be accumulated is a compromise between costs and benefits which depends in a crucial way on the rate of return or rate of time preference being imposed on the economy.

The kind of optimal balance which must be struck between the costs and benefits of capital accumulation is the basic message of the one man economy parable. In this simple model the role of investment cost is symbolized by $\bar{I} = \rho\bar{V}$, the service on the capital stock debt of $\bar{V}$. The benefit of more capital is reflected by the fact that output $F(\bar{V})$ increases with $\bar{V}$. The higher the interest rate the higher the opportunity cost (in terms of foregone consumption) of the debt payment required to raise a given amount of capital, and the lower the warranted stock of the accumulated factor.

The one man economy symbolically maximizes the benefits over costs of getting to and staying in a stationary state starting from the historical initial condition of no capital. With this new parable we are trying to force into a static model a story of the entire process of capital accumulation. The emphasis is quite different from that of the more usual neoclassical modified golden rule approach which comes closer to maximizing the benefits over costs of being in and staying in a stationary state independent of initial capital.

Given any value of $\bar{V}(\rho)$ with $F'(\bar{V}(\rho)) = \rho$, it is natural to ask about the associated imputed competitive returns to capital and labor. The imputed return to capital (profits) is $\bar{\pi}(\rho) = F'(\bar{V}(\rho))\bar{V}(\rho) = \rho\bar{V}(\rho)$. Let $\bar{Y}(\rho) \equiv F(\bar{V}(\rho))$ stand for total output. With constant returns to scale in production, the imputed return to labor (wages) is $\bar{w}(\rho) = \bar{Y}(\rho) - \rho\bar{V}(\rho)$. In the parable being presented it thus turns out that $\overline{I}(\rho) = \overline{\pi}(\rho)$ and $\overline{U}(\rho) = \overline{w}(\rho)$.

These basic relations are depicted in Figure 1. The comparative statics of the system as ρ changes can be read out of the diagram. As ρ declines, $\overline{U}$, $\overline{w}$, $\overline{V}$, and $\overline{Y}$ increase, while changes in $\overline{I}$ and $\overline{\pi}$ are indeterminate and depend on the elasticity of substitution between $\overline{V}$ and $\overline{L}$.

The main task of this paper has been to demonstrate a context in which the new parable depicted in Figure 1 applies to the process of optimal growth in a general multi-sector model. The parable has been shown to hold for appropriately defined averages of the corresponding variables.

REFERENCES

1. L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mischenko, The Mathematical Theory of Optimal Processes, New York and London: Interscience Publishers, 1962.

2. J. Robinson, The Accumulation of Capital, London: Macmillan, 1956.

3. P.A. Samuelson, "The Evaluation of 'Social Income': Capital Formation and Wealth", in The Theory of Capital, Proceedings of an IEA Conference, eds. Lutz and Hague, New York: St. Martin's Press, 1961.

4. ________, "A Summing Up", Quart. J. Econ., Nov. 1966, Vol. 70, No. 4, pp. 568-83.

5. R.M. Solow, "The Interest Rate and Transition between Techniques", in Socialism, Capitalism and Economic Growth. Essays presented to Maurice Dobb, ed. C.H. Feinstein, Cambridge: Cambridge University Press, 1967.