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PRICE DUOPOLY AND CAPACITY CONSTRAINTS

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PRICE DUOPOLY AND CAPACITY CONSTRAINTS*

by

Richard Levitan and Martin Shubik

I. Introduction

In this paper we examine an extremely simple model of a duopoly situation in which the two firms compete with price as the strategic variable and in which the firms are limited by capacity constraints. In terms of this model we shall review some of the important developments of duopoly theory concerned with the existence of equilibrium. Such a market, as Edgeworth [1] showed, does not in general have an equilibrium. We have found that our model, however, has a rather simply described equilibrium in mixed strategies.

Beckmann [2] in 1965 gave a mixed strategy equilibrium, as a solution of an integral equation for a similar model suggested by Shubik [3]. The difference between the two models is the mode by which demand gets redistributed in case shortages occur. Our model has a much simpler solution and has the appealing property that, as we let the capacities vary, we get at one end of an interval as a limiting case the Cournot quantity strategy equilibrium, and, at the other, the Bertrand price strategy equilibrium.

While we do not claim any particular realism for this model we present it both as a useful exposition of a game theoretic attack on the duopoly

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problem and as a valid point of departure for the study of model with more realism. These issues are discussed in the concluding section.

II. A Small Duopoly Model and a Historical Introduction

Our model of duopoly is starkly simplified but it will serve to illustrate the issues we are considering. Let two firms be selling identical goods in a market where demand is a linear function of price. We shall assume, for simplicity that production cost is zero.*

We shall assume that the firms have identical capacities. In the Appendix we shall analyze the extreme asymmetric case where only one firm is capacity-limited.

The firms compete by independently selecting a price to charge. In order to make the model complete we shall have to include an assumption about what happens when the low price firm does not have enough capacity to satisfy the whole market at his price. We assume that the market behaves as if the buyers all have uniform access to the sellers and further, that if their purchases at the low price is restricted, then their total demand is the maximum of the demand at the high price and that available amount at the low price. This amounts to the assumption of zero income effect on consumption.

Our last assumption is that in the event of equal prices the market is split evenly.

*However, all results of this analysis can be generalized to the case of constant unit rates by considering the price in the analysis as the unit margin over cost.

Let the market demand be:

$$(1) \quad q = a - p .$$

Capacities are given by k_1 and k_2 . To start with, we assume $k_1 = k_2 = k$; i.e., the firms have equal capacity.

We assume that $k \leq a$; i.e., a firm has no more capacity than enough to supply the whole market.

Suppose the firms charge prices p_i and p_j . We assume that demand is as follows:

$$(2) \quad q_i = \left\{ \begin{array}{ll} a - p_i & \text{if } p_i < p_j \\ \frac{1}{2} (a - p_i) & \text{if } p_i = p_j \\ a - k - p_i & \text{if } p_i > p_j \end{array} \right\} .$$

Three historical analyses of duopoly can now be illustrated quickly. The first is that of Cournot [4]. Suppose that each firm had $k \geq a/3$ and

named quantity not price as its strategic variable. The relation determining price is given by:

$$(3) \quad p = a - (q_1 + q_2) \quad \text{or} \quad p = a - q \quad \text{where} \quad q = q_1 + q_2 ,$$

hence the payoff to Firm 1 is:

$$(4) \quad \Pi_1 = q_1(a - q_1 + q_2) .$$

Taking derivatives of (4) and a like expression for Π_2 and setting them equal to zero we obtain

$$(5) \quad q_1 = q_2 = a/3 ; \quad p = a/3 \quad \text{and} \quad \Pi_1 = \Pi_2 = a^2/9 .$$

These describe the Cournot noncooperative equilibrium.

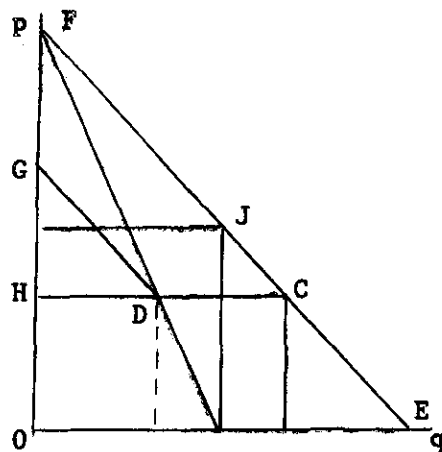


Figure 1

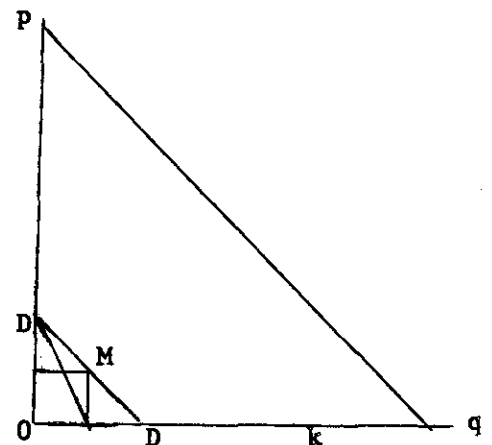


Figure 2

The price and total output in the Cournot market are shown at point C in Figure 1 . Point J gives the price and output if the firms act together as a monopolist.

$$(6) \quad q_1 = q_2 = a/4 ; \quad p = a/2 \quad \text{and} \quad \Pi_1 = \Pi_2 = a^2/8 .$$

Suppose that the firms were competing via price. Furthermore suppose that each had enough capacity to satisfy the whole market at any price. Bertrand [5] argued that the noncooperative equilibrium would fall to $p = 0$ as the firms would keep undercutting each other. This is shown at the point E in Figure 1.

$$(7) \quad q_1 = q_2 = a/2 ; \quad p = 0 \quad \text{and} \quad \Pi_1 = \Pi_2 = 0 .$$

Edgeworth introduced the possibility of limited capacity. The case for $k = 3a/4$ is illustrated in Figure 2. Suppose that one firm were charging $p_i = 0$ it can only sell up to capacity k . This would leave the other firm a demand given by DD' . It would pay to seek monopoly profit against this contingent demand by raising price to M as shown in Figure 2. Subsequently, if we are willing to follow a loose dynamic argument the other firm may raise its next price to just under that at M and a period of price-cutting may follow. Thus Edgeworth suggested that there was a range over which price might be expected to fluctuate. This range will depend upon the capacity k .

It is relatively simple to solve for the upper and lower bounds of this range as follows. Let the price at the bottom of the range be \bar{p} . Suppose that the price of one firm is indefinitely close to the bottom of the range and that the other firm has the choice of picking the price at the bottom of the range or raising its price to \hat{p} . It will be indifferent if profits are equal, i.e. if:

$$(8) \quad k\bar{p} = (a - k - \hat{p})\hat{p}$$

Fortunately in this simple example the profit to the high-priced firm depends only on its price and the capacity of the other hence we know that:

$$(9) \quad \hat{p} = \frac{a - k}{2} ,$$

(this is illustrated by the point M when $k = \frac{3a}{4}$ in Figure 2). Thus:

$$(10) \quad \bar{p} = \frac{1}{k} \left(\frac{a - k}{2} \right)^2 , \text{ and}$$

$$(11) \quad \bar{\pi} = \hat{\pi} = \left(\frac{a - k}{2} \right)^2 .$$

We note if (as would be the case in a competitive industry) there was no excess or shortage of capacity, then $k_1 = k_2 = a/2$ and the range and profits would be:

$$(12) \quad \hat{p} = \frac{a}{8} , \quad \bar{p} = \frac{a}{4} \quad \text{and} \quad \bar{\pi} = \hat{\pi} = \frac{a^2}{16}$$

We now come to a result that at first is surprising, then obvious. Suppose that we continue to shrink capacity in this market. When $k_1 = k_2 = a/3$ we see from (9) and (10) that $\bar{p} = \hat{p} = a/3$, in other words the range of fluctuations shrinks to a point and a pure strategy noncooperative equilibrium reappears. This occurs precisely at the value of the Cournot equilibrium point for the following reason. If the lower priced firm is at capacity at $a/3$ this leaves $2a/3$ of the market for the other who will act as a monopolist and produce $a/3$.

In Figures 3 and 4 the Edgeworth range and the reason why it disappears at $k = a/3$ are illustrated. We first consider $k = a/2$. This case is shown in Figure 3. Each firm has just enough capacity to satisfy the whole market at monopoly price $p = a/2$. If the firms had a capacity of $k = a$ then the curve OPM would show the growth of revenues to the lower priced firm as it raises price, always being able to satisfy total market demand. Because for prices $p < a/2$ a capacity of $k = a/2$ is not sufficient to satisfy the market, the growth of revenue is given by the line OM which intersects the curve OPM at M.

The curve OMMS shows the change in revenues as the higher priced firm increases its price. It has more than enough capacity to satisfy any demand that is left for it. Its revenues reach a maximum at the point MM and decline to 0 if it continues to increase its price to $p = a/2$.

Consider the lower priced firm charging $p = a/8$ and the higher priced firm charging a price a shade higher. The former will make a profit indicated by B and the latter will make a profit shown at D. Suppose that the firm with the higher price has an opportunity to change his price. If he cuts his price to just below the other his profit will be approximately B. If he raises his price to $p = a/4$ his profit, which is shown at MM, will be as high as at B.

The Edgeworth range is given by B and MM. Furthermore if a firm pessimistically assumes that it will be undercut it should set its price $p = a/4$. The point MM is also the maximum P_j . It is the security level P_j P_i

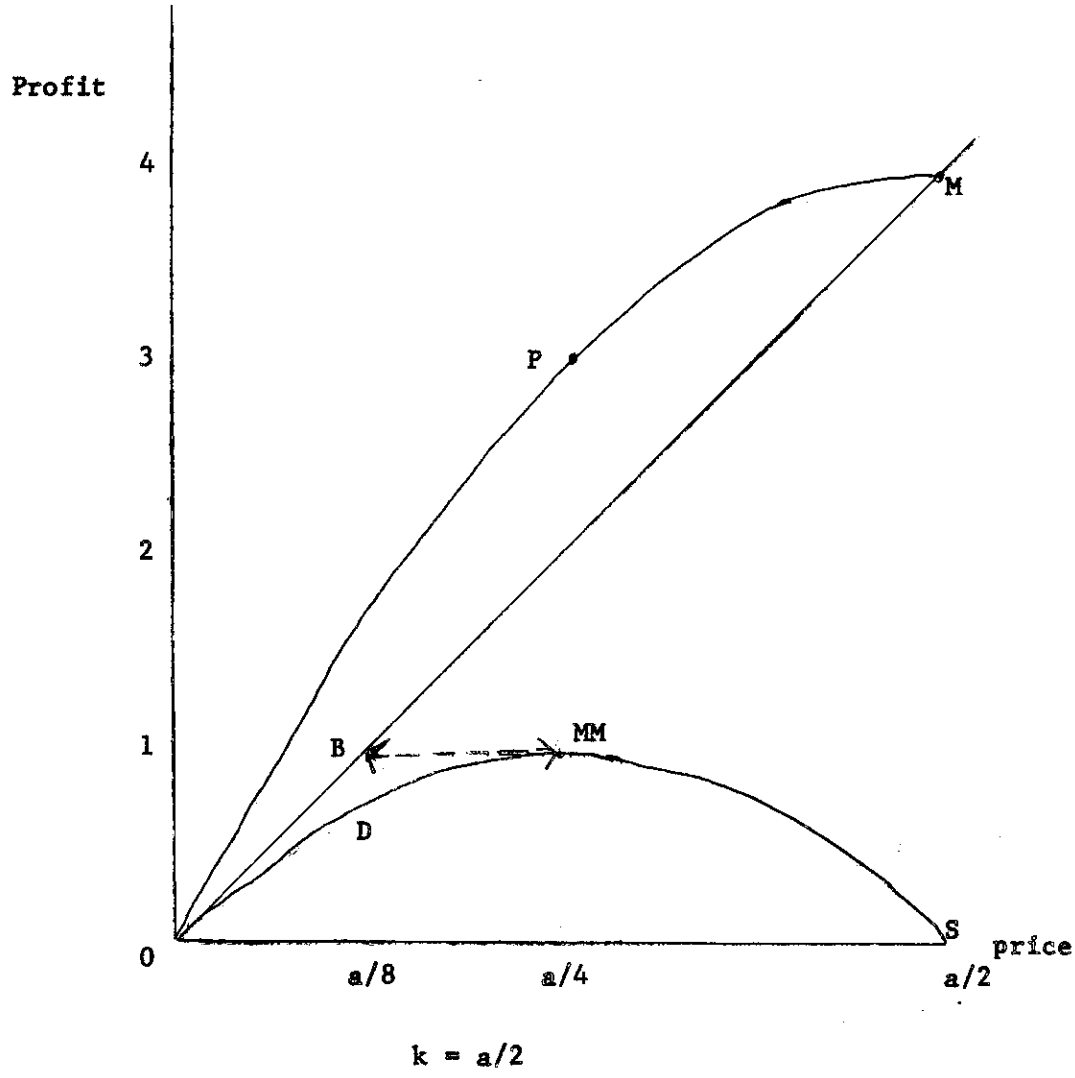


Figure 3

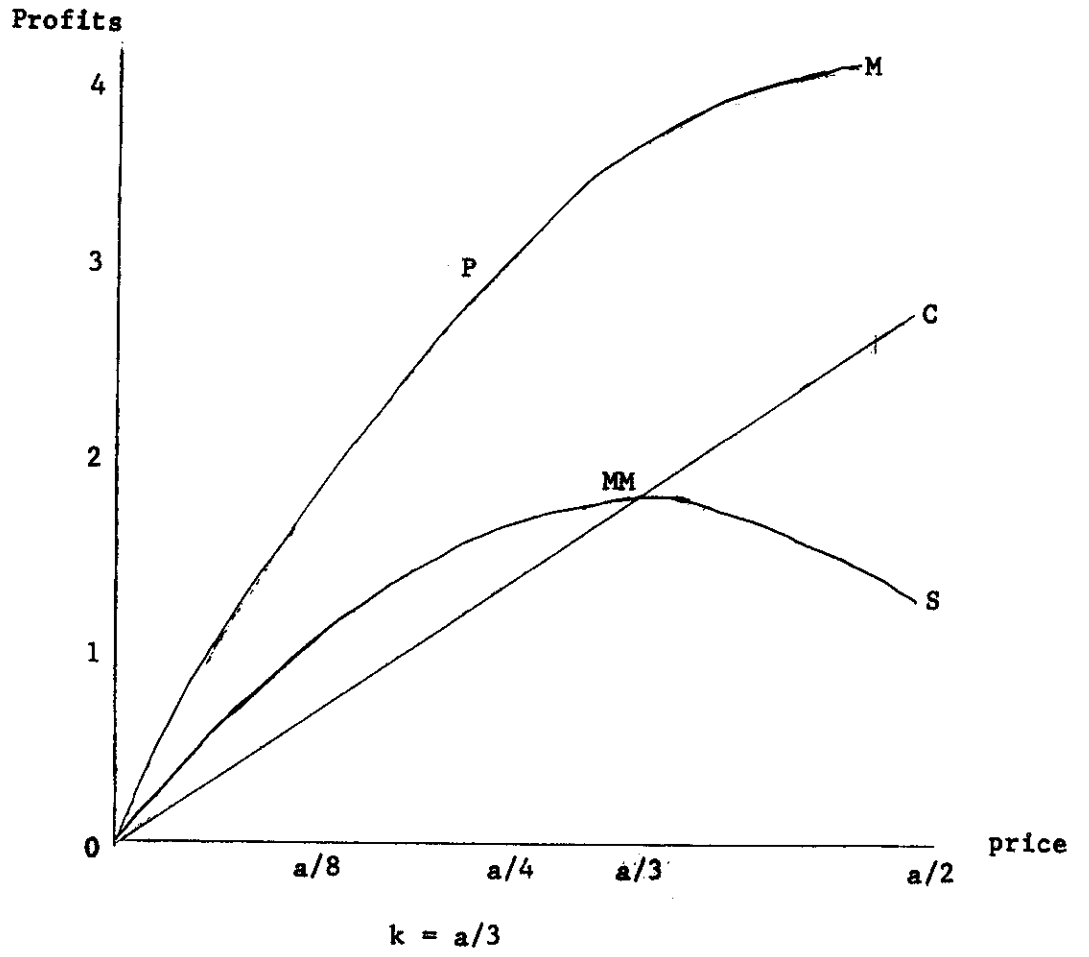


Figure 4

for either firm.¹

Turning to Figure 4, The curve OPM has the same meaning as in Figure 3. The line DMMC is related to the line OM, but here we observe that the individual firm no longer has enough capacity to satisfy the market at the monopoly price without capacity limits. At that price revenue is shown at C. We now note that this line goes through MM. But the Edgeworth cycle is determined by the horizontal distance from MM to oc which is now zero.

If one firm adopts its maximum strategy the optimum reply for the other is to also adopt its maximum strategy hence they are in equilibrium. Thus when $k = a/3$ the price and quantity noncooperative equilibria are the same. This holds true in the range $0 \leq k \leq a/3$ where both firms will produce to capacity and price will be $\bar{p} = a - 2k$.

When the firms together have a total capacity less than a , the efficient point or competitive equilibrium² is no longer at $p = 0$ but becomes $p = a - 2k$. This price can be interpreted as the shadow price for the worth of an increment of new capacity.

Our results can be summed up in Table 1

¹ In a more general model this is not the case. The relationships are somewhat more complicated. For further discussion and an example see Shubik [6].

² Without entry it is not quite correct to refer to the solution of Bertrand as the competitive equilibrium. It is better described as the efficient solution which assigns a shadow price to the value of capacity.

	Cournot	Edgeworth -Bartrand	Efficient Point
$k \geq a$	$a/3$	0	0
$a/2 < k < a$	$a/3$	fluctuation	0
$a/3 < k \leq a/2$	$a/3$	fluctuation	$a - 2k$
$0 < k \leq a/3$	$a - 2k$	$a - 2k$	$a - 2k$

Table 1

The entries in the table are prices for the appropriate solution. Instability in price competition is bounded from below by extreme excess capacity (if we regard $k_1 + k_2 = a$ as the "correct" amount of capacity then there is 100% excess). It is also bounded above by a 1/3 shortage of capacity. Beyond this point capacity is so tight that both firms need not worry about undercutting.

III. Mixed Strategy Solution to the Price Game

We now take the range $a/3 < k < a$ and investigate the nature of the mixed strategy in the price game. The mixed strategy could be interpreted as providing an indication of the distribution of prices in an unstable market.

A mixed strategy equilibrium of a two-person game consists of a pair of probability distributions over the respective strategy spaces with the property that for each player any strategy chosen with positive probability must be optimal against the other player's probability mixture.

In the present example we will avoid a complicated argument and merely assume that the equilibrium strategy involves a single interval, $[p_l, p_h]$ with positive probability density and no lumping of probabilities at any

point except perhaps at p_h .³

Since we have assumed no lumps except at the highest price and except in the case where both players sell capacity when they both charge their highest price, which is ruled out here, the probability of a tie is zero and we can write the sales of Firm i as a function of both prices as

$$x_i = \begin{cases} \min(k, a - p_i) & \text{if } p_i < p_j \\ \max(0, \min(k, a - k - p_i)) & \text{if } p_i > p_j \end{cases},$$

and expected sales as

$$(14) \quad E(x_i) = (1 - \phi_j(p_i))\min(k, a - p_i) + \phi_j(p_i)\max(0, \min(k, a - k - p_i)).$$

Expected profit, of course, is

$$(15) \quad \Pi_i = p_i E(x_i).$$

The reason why our model is solved with such facility is the fact that sales are a step function of the competitor's price and the functional equations (14) and (15) involve no integrals.

We now assert that $p_h < a - k$. Suppose $p_h \geq a - k$. Then

$$(16) \quad \Pi_i(p_h) = p_h(1 - \phi_j(p_h))(a - p_h)$$

which is equal to zero unless j has a lumped probability at p_h . But both firms cannot sell k at $a - k$ so this is ruled out.

³The interested reader is referred to Reference [7] for a rigorous argument for these assumptions in the case of a sealed bid auction.

Further, from equation (10) and its associated argument, it will never pay a firm to charge less than $\bar{p} = \frac{1}{k} \left(\frac{a-k}{2} \right)^2 \geq a - 2k$. Thus we can rewrite (14) as

$$(17) \quad \Pi_i(p) = p[(1 - \phi_j(p))k - \phi_j(p)(a - k - p)] , \text{ or}$$

$$(18) \quad \frac{\Pi_i(p)}{p} = k - \phi_j(p)[p + 2k - a] .$$

Solving (18) for ϕ_j , one obtains

$$(19) \quad \phi_j(p) = \frac{k - \Pi_i/p}{p + 2k - a} .$$

Since Π_i is constant on $[p_\ell, p_h]$, (19) gives the mixed strategy equilibrium once Π_i , p_ℓ , and p_h are evaluated.

It is simple to show that $p_h = \frac{a-k}{2}$. If a firm is charged p_h he knows that he will be surely undercut, and charging other than $\frac{a-k}{2}$ will not be optimal. Thus $\Pi_i = \Pi_i(p_h) = \left(\frac{a-k}{2} \right)^2$, and further, since $\Pi_i = \Pi_i(p_\ell) = k \cdot p_\ell$ we have $p_\ell = \frac{1}{k} \left(\frac{a-k}{2} \right)^2$, and the upper and lower bounds for the mixed strategy coincide with those for the Edgeworth fluctuation or cycle (this is not generally true).

Substituting the value of $\Pi_i = \left(\frac{a-k}{2} \right)^2$ into (19) we obtain the cumulative probability function:

$$(20) \quad \phi_j(p) = \frac{kp - \left(\frac{a-k}{2} \right)^2}{p(p + 2k - a)} .$$

The graph in Figure 5 shows how the distribution changes for the values $k = .9a$, $.5a$ and $.4a$. It is easy to check the limiting values of $k = a$ and $a/3$ in equation (19).

What happens when capacities are different? If $k_1 \geq a$ and $k_2 \geq a$ or $k_1 \leq a/3$ and $k_2 \leq a/3$ a pure strategy exists. When the capacities are unequal but not in the ranges noted there is a mixed strategy solution.

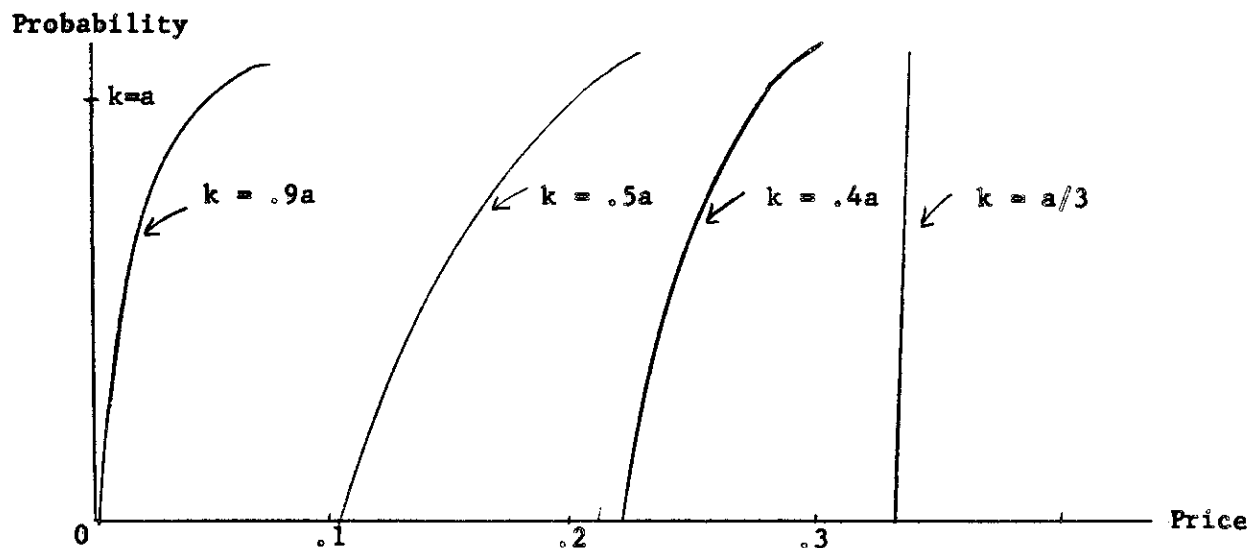


Figure 5

The mixed strategy is no longer continuous when the firms have unequal capacities but the firm with the larger capacity selects the upper point in its bidding range with a finite probability. The solution appears to be more of a mathematical curiosity than of economic interest. An example with $k_1 = a$ and $k_2 < a$ is given in the Appendix.

4. Concluding Remarks: Realism and Generality

In Section 2 we encountered the surprising result that as capacity was shrunk the pure strategy equilibrium reappears at the Cournot equilibrium point. This is not general; it will be determined by the type of contingent demand structure that is postulated. For example, Beckmann using a contingent demand method originally suggested by Shubik [8], does not obtain a pure strategy equilibrium at the Cournot point [2].

The determination of the reappearance of the pure strategy equilibrium depends upon the value of $\partial \Pi_1 / \partial p_1$ at the point of potential equilibrium. The effect of moving capacity into the range $k_1 + k_2 \leq \frac{2a}{3}$ puts a constraint on the derivative in the price-cutting direction. If a pure strategy equilibrium exists then both firms will be producing at capacity.

The test to see if capacity production is in equilibrium comes in the direction of raising price. In Figure 1 the contingent demand at the point D is GD which has an elasticity of 1 at D hence there is no motivation to move price up.

Depending upon the method used for the calculation of contingent demand, as capacity is varied the slope of the contingent demand for each firm (when both are at capacity and the price is such that the market just clears) may change. This point will be an equilibrium point if the elasticity of demand along the contingent demand curve is 1.

We have picked the most pessimistic method of calculating contingent demand. The high priced part of the demand curve is satisfied first. This gives a parallel translation of the demand inward as the shape of the contingent demand. In this case the Cournot point must be the point of demarcation for the reappearance of the pure strategy.

The actual shape of contingent demand cannot be specified generally from a priori reasoning. It will depend upon priorities in service of customers and details of the summation of individual demand in specific markets.

It is an important and complicated marketing problem which needs specific empirical investigation and model building.

We have discussed capacity constraints in this paper. Inventories are often used as a means for avoiding the short term effect of capacity constraints. We deal with the introduction of inventories elsewhere [9].

Unequal capacities complicate the analysis but do not appear to introduce any particularly interesting new phenomena.

Is price the right variable? Furthermore is the simultaneous move noncooperative game the right model? In general we would argue that price may not be the most important variable. Furthermore the one period noncooperative game is a gross oversimplification of oligopolistic competition. However this model does represent an extreme case and as such merits investigation. What strategic variable is important depends upon the specifics of the market. Furthermore the manner in which an economic weapon can be used also depends upon the market. In some markets price can be moved almost instantaneously; in others it may be highly inflexible.

A dynamic theory of oligopolistic competition needs to take into account both technological and institutional detail which enable us to give structure to the strategic possibilities for each firm and to the nature of threats.

The introduction of product differentiation does not appear to add any new qualitative results. The capacity limits on the conditions for the existence of a pure strategy equilibrium will change but beyond that the phenomena encountered will be qualitatively the same as without product differentiation.

It has been shown elsewhere that as the number of firms is

increased in the market, if $\sum_{i=1}^n k_i \geq a$ and $k_i \geq a/n$ then the value of

the game approaches that of the competitive equilibrium and the probability distribution on prices shifts towards the lower end of the range as n increases [10].

APPENDIX

A Price Game with Unequal Capacities

The payoff to Firm i may be expressed as:

$$(20) \quad \Pi_i = p \left\{ (1-\phi_j) \min(k_i, a-p) + \phi_j \max(0, a-k_j-p) \right\}$$

which may be written as

$$(21) \quad \Pi_i = p \left\{ \min(k_i, a-p) - \phi_j \min(k_i, a-p, k_i+k_j-a+p, k_j) \right\} .$$

We solve for the special case: $k_1 = a$ and $k_2 = k < a$. The payoffs to the two firms are:

$$(22) \quad \Pi_1 = p[a-p - \phi_2 \min(a-p, k)]$$

$$(23) \quad \Pi_2 = p[\min(a-p, k)] (1 - \phi_1) .$$

We see immediately from (23) that ϕ_1 cannot take the value 1 for an active strategy in the continuous range, as the value of Π_2 would then be zero.

At the lower end of the range of active strategies we may assume that the capacity constraint is effective on the firm with limited capacity, hence $a-p \geq k$ and

(24) $\Pi_2 = p_\ell k$ where p_ℓ is the lowest active price. This follows immediately from (23).

At the highest price in the range, p_h the condition $a - p_h \geq k$ must hold, or $p_h \leq a - k$.

From (22) we have:

$$(25) \quad \Pi_1 = p_h [a - p_h - k] .$$

At p_h the derivative $\partial \Pi_1 / \partial p$ must be nonpositive, as this is the end of the range of active strategies.

$$(26) \quad \frac{\partial \Pi_1}{\partial p} = a - p_h - k + p_h (-1) \leq 0 , \text{ hence}$$

$$(27) \quad p_h \geq \frac{a - k}{2} .$$

From (23) we may write:

$$(28) \quad \phi_2^1 = \frac{a - p - \Pi_1 / p}{\min(a - p, k)} = \frac{a - p - \Pi_1 / p}{k} ,$$

this must have a positive derivative.

$$(29) \quad \phi_2^1 = \frac{1}{k} \left\{ -1 + \frac{\Pi_1}{p} \right\} \quad \text{hence}$$

$$\frac{\Pi_1}{p} \geq 1 \quad \text{and from (25)} \quad p_h^2 \leq \Pi_1 = p_h (a - p_h - k) ;$$

$$\text{hence} \quad p_h \leq \frac{a - k}{2} \quad \text{and from (27)}$$

$$(30) \quad p_h = \frac{a - k}{2} .$$

Given the top of the range the values and the cumulative density functions Φ_1 and Φ_2 become easy to calculate. We modify our notation replacing k by $k = \theta a$ where $0 < \theta < 1$.

From (30) and (22):

$$(31) \quad \Pi_1 = \left(\frac{a-k}{2} \right)^2 = \left(\frac{a(1-\theta)}{2} \right)^2 = F_\ell(a-p_\ell), \text{ from which:}$$

$$(32) \quad p_\ell = \frac{a}{2} \{1 - \sqrt{\theta(2-\theta)}\}.$$

We check to verify that $p_h - p_\ell > 0$

$$p_h - p_\ell = \frac{a}{2} \{ \sqrt{\theta(2-\theta)} - \theta \} > 0 \quad \text{for} \quad \theta < 1.$$

From (23) we have:

$$(33) \quad \Phi_1 = 1 - \frac{a(1 - \sqrt{\theta(2-\theta)})}{2p}$$

and from (28)

$$(34) \quad \Phi_2 = \frac{a - p - \frac{a^2(1-\theta)^2}{4p}}{\theta a}.$$

In Figure 6 we illustrate the distributions for the case where $k = a/2$.

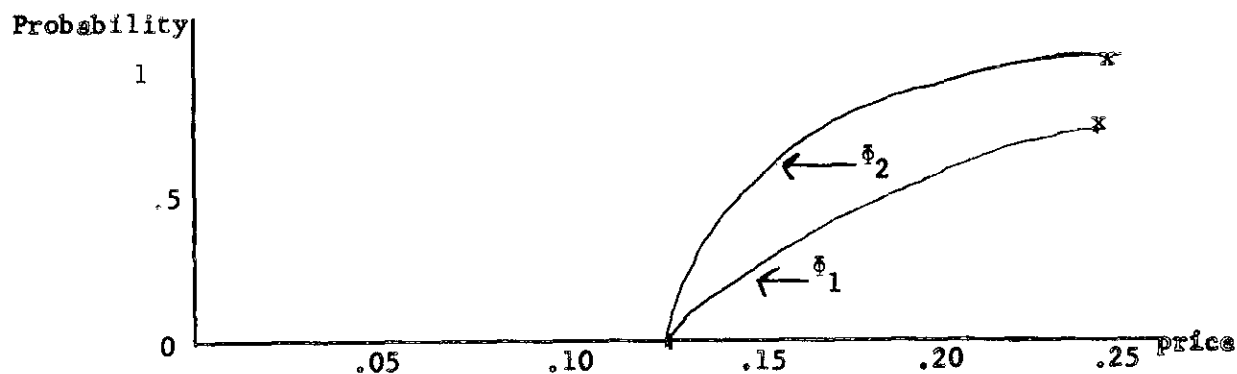


Figure 6

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