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INVESTMENT, SAVINGS AND EMPLOYMENT IN THE LONG-RUN

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INVESTMENT, SAVINGS AND EMPLOYMENT IN THE LONG-RUN*

by

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Most studies in the theory of growth assume that goods and factor

markets work sufficiently well, or clearly sufficiently fast, so that they

can always be considered in equilibrium. With the exception of the few

articles surveyed in the last part of this paper most of the debate has

centered on the determinants of the rate of capital accumulation--as to

whether it is the 'animal spirits' of the investors or the savings behavior

of individual consumers which "determines" the rate of investment in fixed

capital. To our minds this seems to be only a partially formed question

--the real problem is to discover how the aims and behavioral patterns of

the different economic agents interact in the various markets, and what

the consequences of such interactions are.

The aim of this paper is to examine various possible specifications

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of market behavior and see what differences they make to the normal picture
of the long run growth of the economy. It will be remembered that under
appropriate assumptions most growth models discussed in the literature possess
a "steady-state" growth path which the economy will follow if it starts with
the appropriate capital endowment. If it starts from some different initial
position it will, in general, follow an "equilibrium path" (full employment,
with equilibrium in the goods market) which will asymptotically converge
to the steady-state path.[1] We are interested in models which are not always
in equilibrium, and we shall be interested in the characteristics of the
'disequilibrium' path, in particular, whether it converges to a steady-state
path. If so, then we shall consider the model to be stable. This differs
from the conventional (neo-classical) view of stability, which merely re-
quires convergence of the equilibrium path to the steady-state path; in our
view a rather misleading concept which causes confusion when discussing the
relationship between short-run Keynesian behavior and long run growth.

If a model is to handle possible disequilibrium in the goods and
factor markets it should contain the following component parts:

A production function which relates output to the factors employed. In this
paper we shall assume a malleable capital neoclassical production function
with constant returns to scale and Harrod-neutral technical progress;

$y = e^{gt}f(ke^{-gt})$, where y is per capita output in homogenous physical
units, k is capital/man in the same units.

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[1]Definitions of steady-state path, etc. correspond to those used by Hahn
and Matthews [1].

<u>The demand for goods</u>, or more generally a specification of the behavior of the product market. The supply of goods is determined by production and drawing down stocks, if necessary, whilst demand is generated by consumption and investment. Insufficient demand will lead to unintended stock-piling. We assume that prices will vary with excess demand in a Walrasian manner:

$$\hat{p} = uZ$$

where $\hat{p}$ is the proportional rate of change of prices $\frac{1}{p}\frac{dp}{dt}$, and Z is relative excess demand for goods, $Z = (C + I - Y)/Y$.

$$\text{Thus} \quad \hat{p} = u\left\{\frac{C + I - Y}{Y}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

<u>A consumption function</u>, or a savings function. In general this will depend on the distribution of income:

$$C = \{1 - s(W)\}Y \qquad s' \leqq 0 \quad \dots\dots\dots\dots\dots\dots (2)$$

where s is the aggregate savings ratio and W is the share of wages. It is possible that capital gains or losses may also affect consumption, in which case the consumption function could be written as

$$C = (1 - s)(C + I) + c\hat{p}K \quad \dots\dots\dots\dots\dots\dots (3)$$

where $C + I$ is total cash income for the economy (in real terms), and c is the propensity to consume out of capital gains. Alternatively, if we wish to emphasize the class differences in saving,

$$C = (1 - s_w)\frac{wL}{p} + (1 - s_p)(f' + \hat{p})K \quad \dots\dots\dots\dots (4)$$

where w is the money wage,   L is total employment, and $f' + \hat{p}$ is the total real rate of profit.

If, on the other hand, all saving done by workers earn a rate of interest r (paid by the financial intermediaries), whilst the capitalists (or the firms) have a different rate of saving, then:

$$C = (1 - s_w)(\frac{wL}{p} + rK) + (1 - s_p)\{f' + \hat{p} - r\}K \ldots\ldots\ldots\ldots (5)$$

An investment function, which is usually the most difficult aspect of any growth model to make plausible. This is where the malleable capital assumption makes a considerable difference, and where one would expect expectations to be important. We certainly expect the rate of investment $\hat{K}$ $(= \frac{1}{K}\frac{dK}{dt})$ to depend on expected profitability, and if investment is financed by bonds bearing a rate of interest r , the current real rate of net profit will be $f' + \hat{p} - r$. Thus we could write

$$\hat{K} = \emptyset(f' + \hat{p} - r) \, , \quad \emptyset' > 0$$

indicating that investors increase the rate of investment as the rate of profit increases. The next problem is to determine the "normal" rate of investment, $\emptyset(0)$. Our initial idea was to take $\emptyset(0)$ as N , the natural rate of growth of the economy, but this might seem to presuppose the convergence of the economy to balanced growth. We experimented with $\emptyset(0) = M \neq N$, and found that this makes little substantial difference, except that there will be long-run inflation or deflation as $M \gtrless N$. Professor Vogt, of Regensburg University, has described in correspondence the use of the function

$$\hat{K} = sa + \emptyset(f' + \hat{p} - r) , \quad \emptyset(0) = 0 \ldots\ldots\ldots\ldots (6)$$

where $a = y/k$, the output-capital ratio. Thus $\emptyset(0)$ corresponds to equilibrium in the product market, a neat and plausible assumption. We wish to examine the contribution each market makes separately, and so for the moment we will adopt the linearized form

$$\hat{K} = N + v(f' + \hat{p} - r) , \quad v > 0 \ldots\ldots\ldots\ldots (7)$$

Since the equilibrium value of the bracket is zero the local stability properties of the model are unaffected by the linearization.

A money equation, since the model has prices, inflation, and financial intermediaries which stand ready to finance all investment with short term loans at the current rate of interest, r . The easiest assumption is that the Banks' supply funds elastically at r , and the monetary authorities expand the money supply as required. r may or may not itself adjust, but to simplify the model we shall assume it is held constant at its equilibrium value $r*$.

A wage-employment relation relating the demand for labour, its cost, and the rate of change of wages to the demand for labour. For the latter, we take the simple Phillips assumption

$$\hat{w} = h(z) , \quad h' \geq 0 , \quad h(1) = g \ldots\ldots\ldots\ldots (8)$$

where w is the money wage, and z is $L/L*$, the fullness of employment --$L*$ is conventional full-employment, L is actual employment.

The demand for labour is analagous to the demand for investment:

$$\hat{L} = n + \psi\left(\frac{\partial Y}{\partial L} - \frac{w}{p}\right) \qquad \psi' > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

where n is the rate of increase of the labour force, and $\partial Y/\partial L$ is the real marginal product of labour, $= y - kf'$.

It is in fact more convenient to work with z , and we replace $\hat{L}$ by $\hat{z} + n$ in (9) to obtain

$$\hat{z} = \psi\left(1 - \frac{f'}{a} - \frac{w}{py}\right) \cdot y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since the equilibrium value $\psi(0) = 0$, this can be linearized to

$$\hat{z} = m\left(1 - \frac{f'}{a} - W\right) \quad , \quad m > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

where W is the share of wages.

The main question we are asking is how sensitive is the behavior of the assembled model to the specification of its component parts? Until this question is answered it is a poor guide to the plausibility of a model merely to find conditions under which it gives "realistic" results. To answer this question we procede in two steps. First of all we will examine the competing assumptions in the simplest possible model, then we will generalize one formulation to a more complex model and examine the significant changes in its behavior.

Model A

Most of the variation in the specification has to do with the product

market, so in this simple model we will assume away any adjustment problems in the factor market. Thus the real wage w/p will always be equal to its real, full employment marginal product, $\partial Y/\partial L$, (L = L*, z = 1, $\hat{L} = n$, m = 0).

Model A1

We take equations (1), (2), and (7), and derive:

$$\hat{K} = \frac{(N + v(f' - r - us))}{1 - \frac{uv}{a}}$$

We assume that uv < a* for plausible behavior, where starred values are the equilibrium values of the variables. Define $x = k/k*$, so that $\hat{x} = \hat{k} - g = \hat{K} - N$. (Again using the convention that $\hat{x} = \frac{1}{x}\frac{dx}{dt}$.) Substituting above we have:

$$\hat{x} = \frac{v}{1 - \frac{uv}{a}}\left\{ (f' - r) + u\left(\frac{N}{a} - s\right)\right\} \quad \ldots\ldots\ldots\ldots\ldots(11)$$

The differential equation (11) determines the behavior of the system. The terms in the right-hand bracket have been grouped to show the two factors at work in the model. The first term, $f' - r$, is the return to investment before allowing for capital gains, whilst the second is the capital gain (or loss). To see how this works let us suppose that the capital-labour ratio exceeds its equilibrium value (x > 1). The marginal product of capital is reduced below its equilibrium value, and for this reason investment is reduced below its natural rate--see Figure 1, which shows the supply and demand functions of investible funds. But what happens to the price level? This depends on whether $I \gtrless S$.

In the Cobb-Douglas case it is readily shown that if $vb > s$ [1] the investment schedule is steeper than the savings schedule, so that S is above I for $x > 1$, prices will then fall, and the consequent capital losses will further depress investment (the dashed line shows investment before taking acount of capital gains, the solid line includes capital gains in Figure 1). In this case, introducing capital gains makes the model more rather than less stable; and it is clear that it will tend to return to equilibrium more quickly than in the Solow case where the rate of capital formation is governed by savings.

Where $bv < s$, we have the situation shown in Figure 2: when $x > 1$ the price level rises, thus encouraging investment. The system may be stable (as $I'I'$) or it may be unstable $(I"I")$ at the point $x = 1$. In the Cobb-Douglas case we can completely characterize the behavior as follows: substituting for $f' = ab$, $r = Nb/s$, in (11) we have

$$\hat{x} = v(a - N/s)(ba - su)/(a - uv)$$

or since $\hat{a} = (b - 1)\hat{x}$

$$\dot{a} = \{b(1 - b)va(a - N/s)(su/b - a)\}/(a - uv) \ldots \ldots \ldots (12)$$

We can distinguish three possible configurations depending on the magnitudes of the parameters (see Figures 3a, b, c). The curious one is 3c, where the steady-state is unstable, but there exists a stable path with a higher output-capital ratio. If the economy is displaced to a point above the steady-state value of a it will converge to this 'pseudo steady-state', in which prices fall steadily and stocks accumulate (although the ratio of stocks to output remains constant). The condition for local stability of the steady-

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[1] b is the share of profits; $y = k^b e^{g(1-b)t}$.

state is ($u < bN/s^2) , a special case of (14) below, though it does not completely characterize the equilibria. To elaborate, suppose $\frac{us^2}{bN} = \theta$, and $bv < s$. The stable value of a , which we denote $\bar{a}$, is then:

$$\bar{a} = a^* \qquad \theta \leq 1 \quad,$$

$$\bar{a} = \theta a^* \qquad \theta \geq 1 \quad.$$

For the case $\theta > 1$ (i.e. the unstable case) we have the following results:

$$\bar{y} = \theta^{b/(b-1)} y^* \quad, \quad \text{i.e.} \quad \bar{y} < y^* \quad.$$

The ratio of stocks to output is:

$$\frac{\text{Stocks}}{\text{Output}} = \frac{\theta - 1}{\theta} (s - bv) \quad.$$

Thus for values of θ near unity actual behavior in the neighborhood of the <u>stable</u> 'steady-state' path does not depend critically on the exact value of θ . In other words the boundary between local stability and instability is blurred. Figure 4 indicates the range of stability for values of θ .

This analysis brings out rather sharply the need to investigate global stability, rather than just confining attention to local stability. Although linear approximations become suspect for significant deviations from the equilibrium values, and make the global analysis very tentative, nonetheless it is insufficient to deduce local instability without enquiring into behavior in a more extensive neighborhood of the instability.[1]

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[1]Notice the similarity with the treatment of Harrod instability--which leads directly to Trade cycle analysis.

The Cobb-Douglas ease, though mathematically very easy to handle, has a number of special properties which might lead to rather special behavior. We return to the more general form of equation (11). This can be written in the form:

$$\frac{dx}{dt} = \frac{v}{\left(1 - \frac{uv}{a}\right)} F(x) , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

where $F(x) = f' - r + u[N/a - s(W)]$ and $F(1) = 0$. The path corresponding to $x = 1$ will be locally stable if and only if $F'(x^*) < 0$ for $x^* = 1$.

[The set of singularities is given by the roots of $F(x) = 0$, and the root x^* is stable if $F'(x^*) < 0$, unstable if $F'(x^*) > 0$. This is readily seen in the phase diagram (Figure 6a), as is the fact that stable and unstable 'steady-state' paths will alternate.

Whilst we are confronted with the phase diagram we can return to the problem of the consistency of savings plans and investment intentions. Suppose the investment function is

$$\hat{K} = M + v(f' + \hat{p} - r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

instead of (7). The differential equation then becomes

$$\frac{dx}{dt} = \frac{v}{\left(1 - \frac{uv}{a}\right)} \left[F(x) + \frac{x(M - N)}{v} \right] \quad (13a)$$

Depending whether $M \gtrless N$ the phase diagram will be modified as in Figure 6a, but clearly, the character of the model is not violently altered—some inflationary or deflationary price changes will ensue.]

Since

$$F(x) = f' - r^* + u\left[\frac{N}{a} - s(W)\right]$$

$\therefore$ $F'(x) = k^*e^{-gt}f'' - \frac{uN}{a^2x} \cdot \frac{ye^{-gt}f''}{f'} \cdot \sigma - us \cdot \frac{k^*}{y}(\sigma-1)ke^{-gt}f''$ [1]

$F'(x) < 0$ for $x = 1$ if and only if

$$1 - \frac{uN\sigma}{af'} - \frac{u}{a}(1 - \sigma)s' > 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (14)$$

As the model shows there are two mechanisms affecting stability--factor substitution, which will depend on σ, the elasticity of substitution, and changes in income distribution affecting the savings rate (if $s' \neq 0$). We consider these in turn, first ignoring the "Kaldor-effect."

Stability condition: $af' > uN\sigma$

There is a problem of interpretation here, since all the variables except u are interdependent. If we express the stability condition as a condition on u, and deal with _observable_ statistics (N, s, W) and σ, we have

$$f' = a(1 - W)$$

$$u < \frac{a^{*2}(1 - W)}{N\sigma}$$

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[1]

$$\frac{d}{dx} = k^*\frac{d}{dk}, \quad \frac{da}{dx} = \frac{f' - a}{x}$$

$$\frac{dW}{dx} = \frac{k^*}{y}\{\sigma - 1\}kf''e^{-gt}$$

where $\sigma = \frac{f'(f'-a)}{ye^{-gt}f''}$, the elasticity of substitution.

or $$u < \frac{N(1-W)}{s^2 \sigma} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

This is the condition, together with $v < \frac{N}{su}$, which ensures that $a = \frac{N}{s}$ is a locally stable steady growth path. Comparing economies with the same N , s , W , but differing σ , we can see that such stability increases for lower σ , a strange anti-Harrodian conclusion. However, we have a very special distribution theory built with this model, quite apart from the fact that we are dealing with malleable capital (see Appendix I).

For $W = 65\%$, $N = 4\%$, $s = 12\%$, we have $u < \frac{1}{\sigma}$, $uv < .33$.

The "Kaldor-effect"

We expect $s' \leq 0$ in general, and the effect on stability will depend on σ . For $\sigma < 1$ stability will be improved and conversely for $\sigma > 1$.

i.e. $u < \frac{N}{s} \left\{ \dfrac{1}{\sigma s/(1-W) + (\sigma-1)(-s')} \right\}$

Suppose s_π (savings out of profit income) $= 30\%$, $s_w = 5\%$, $W = 65\%$; $s = 12.75\%$, $-s' = s_\pi - s_w = 0.25$. $\dfrac{(\sigma-1)}{\sigma}(1-W)\left[\dfrac{-s'}{s}\right]$ gives the relative influence of the Kaldor effect, in this case about $\dfrac{2}{3}\dfrac{(\sigma-1)}{\sigma}$. Most estimates of σ place it between 0.6 and 1.2, making, on these assumptions, the Kaldor effect have not more than 1/3 the influence of the straight substitution effect.

Other stabilizing influences

An examination of the literature suggests that there are two prin-
cipal means by which investment is equated to full employment savings:
(i) by the effect of prices on savings—Real Balance or "Pigou" Effect
—(e.g. Kahn, [13]).

(ii) by the deliberate adjustment of the interest rate by the Mone-
tary Authorities (e.g. Meade [14]).

Let us consider the Pigou effect first.[1] In this case savings are
assumed to be an increasing function of the price level (since if the price
level rises, real balances will decrease, and so consumption will fall).
We need to modify (1) to

$$\hat{p} = u\left(\frac{N + \hat{x}}{a} - s(p)\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1a)$$

Now it can be shown (see Appendix III) that the steady state will be locally
stable in the Cobb-Douglas case for

$$\theta < \frac{1}{1 - \frac{\epsilon}{v(1-b)}}$$

where ϵ is the "elasticity" of the Pigou effect $\left(\frac{pds}{sdp}\right)$. Thus the Pigou

effect is stabilizing, although crude estimates of the order of magnitude

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[1]While the real world effect of the Pigou effect is probably small (as is
also the effect in this model), this also covers the case of an economy
open to foreign trade, where net exports fall as the domestic price level
rises, an effect that may be of considerable importance.

of its effect suggest that it will contribute less than 10% of the total

stabilizing force.   (See Appendix III.)

An interesting feature which is illustrated in the phase diagram

(Diagram 7), and which is general to models with more than one stabilizing

force, is that if the model is on an equilibrium path with savings equal

to investment ($\dot{p} = 0$) ,   this is in general not a solution path of the

model. The model will thus diverge from the equilibrium path, and in this

case it will cycle around it.

The second mechanism by which investment and full employment savings

may be brought into equality has been discussed by Hahn, [15], who proposed

the following simple adjustment process:

$$\dot{r} = c\left(\frac{\dot{K}}{a} - s\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (16)$$

As he shows, the model will then be stable in the Cobb-Douglas case when

$$bN^2/s - Nu + c/(1-b) > 0$$

and a sufficient condition for this to be satisfied is that

$$c > Nu$$

Variation of the rate of interest (or, more generally, the financial attrac-

tiveness of investment) is a powerful stabilizing influence. Again, we

note from the phase diagram (Diagram 8) that we have the same problem as

with the Pigou effect--the equilibrium path is not a solution path of the

system. For it to be so a particular time path of the rate of interest

would be presupposed that is unlikely to be followed by any simple rule

such as given by (16).

This need not worry us, since there is nothing optimal about the equilibrium path, but it should make us sceptical that the equilibrium path is a good approximation to the actual path followed by an economy.

__Model A2.__ We replace equation (2) by (3), which can be rewritten thus

$$C = \left(\frac{1-s}{s} \right) I + \frac{c}{s} \hat{p}K \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Then (1) becomes

$$\hat{p} = \frac{u(\hat{\hat{K}} - as)}{as - uc}$$

We can substitute $\dfrac{au}{as - uc}$ for u wherever it occurs and thus obtain

$$\hat{x} = \left(\frac{v}{1 - \dfrac{uv}{as - uc}} \right) \left\{ (f' - r) + \frac{u(N - as)}{as - uc} \right\}$$

Suppose, for simplicity, that $\dfrac{ds}{dW} = 0$. Then the stability condition reduces to (dropping e^{-gt} now wherever it occurs):

$$k*f'' - \frac{us}{N - uc} \frac{da}{dx} < 0$$

or

$$k*f'' - \frac{usyf''\sigma}{(N - uc)f'} < 0$$

or

$$u < \frac{f'}{\sigma + \dfrac{cf'}{N}}$$

or $\qquad$ $u < \dfrac{N\Pi}{s\sigma + c\Pi}$ $\qquad$ where Π is the share of profits.

For $N = 4\%$, $s = 12\%$, $\sigma = 1$, $\Pi = 35\%$, $c = 6\%$, $u < 0.1$, $\dfrac{au}{as + uc} < 1.0$

Model A3. This time we replace (2) by (4) to derive:

$$\hat{p} = \frac{u}{1 - \dfrac{c_p u}{a}}\left\{\frac{\hat{K}}{a} - s_w - (s_p - s_w)\frac{f'}{a}\right\}$$

where $c_p = 1 - s_p$.

Again we replace u by $u/\left(1 - \dfrac{c_p u}{a}\right)$ and s by $s_w\Pi + s_p\Pi$:

$$\hat{x} = \frac{v}{1 - \dfrac{uv}{a - c_p w}}\left\{(f' - r) + \frac{u}{1 - \dfrac{uc_p}{a}}\left(\frac{N}{a} - s\right)\right\}$$

replace $u\left(1 - \dfrac{uc_p}{a}\right)$ by u^* and the stability condition is identical to IA :

$$1 - \frac{u^* s^2 \sigma}{N\Pi} - \frac{u^* s(1-\sigma)(s_p - s_w)}{N} > 0 \quad \ldots\ldots\ldots\ldots \text{(17)}$$

Model A4. This time we use assumption (5) in place of (2), which differs from A3 in having an extra term $(s_p - s_w)rK$. Thus we have

$$\hat{p} = \frac{u}{1 - \dfrac{uc_p}{a}}\left\{\frac{\hat{K}}{a} - s_w - (s_p - s_w)\left(\frac{f' - r}{a}\right)\right\}$$

where

$$\hat{x} = \frac{v}{1 - \frac{u^*v}{a}} \left\{ \left(1 - \frac{u^*(s_p - s_w)}{a} \right)(f' - r) + u^*\left(\frac{N}{a} - s_w \right) \right\}$$

$$F'(x^*) = \left(1 - \frac{u^*(s_p - s_w)}{a} \right)k^*f'' - \frac{Yu^*Nf''\sigma}{a^2 f'}$$

Stability if

$$1 - \frac{u^*(s_p - s_w)s}{N} - \frac{u^*s_w^2\sigma}{N\Pi} > 0 \quad \dots\dots\dots\dots\dots\dots (18)$$

which bears a strong similarity to (17) except for the ommission of a $(1-\sigma)$ term. Put another way, the Kaldor effect will operate in this model even if the production function is Cobb-Douglas, whilst in the other three models it will not.

Model A5. This time we vary the investment function, and take equations (1), (2) and (6) linearisec to

$$\hat{K} = sa + v(f' + \hat{p} - r) \quad \dots\dots\dots\dots\dots\dots\dots (6a)$$

This gives

$$\hat{x} = \frac{v}{1 - \frac{uv}{a}}(f' - r) + (as - N)$$

which will be stable if

$$\left(\frac{v}{1 - \frac{uv}{a}} \right)k^*f'' + \frac{syf''\sigma}{f'} + s'k^*(\sigma-1)f'' < 0$$

or

$$\frac{v}{1 - \frac{uv}{a}} + \frac{N^2\sigma}{s\Pi} + s'(\sigma - 1) > 0$$

with $s' = 0$, the constraint on u is

$$u < \frac{v\Pi + \frac{N^2\sigma}{s}}{N\sigma v}$$

Take the same numerical values as in A1, also put $\sigma = 1$, $v = .2$

$$u < 10.4$$

a much looser condition than A1.

Conclusions

So far they are encouraging. The qualitative behavior of all models is similar, and does not seem to depend too critically on the particular assumptions. The quantitative restrictions differ, though it is difficult to say how important this is (see Appendix I), as the estimating procedure will differ for each model.

Thus encouraged, we will consider one of the above models (A1) and relax the assumptions we made about equilibrium in the remaining markets to see what qualitative difference this makes.

The earlier results can be derived directly from this model by imposing specific conditions, and by viewing the earlier model in a more general framework we may more readily judge the sensitivity of its results to the specific assumptions. We also discuss the effect on the stability of steady growth of a Phillips curve relation between the rate of change of money wages and the level of employment, and of the introduction of an element of cost-push, as opposed to demand-pull, inflation.

The most important difference between this model and model A1 lies in the wage determination equation (8). This simple assumption has come to be known as 'the Phillips curve,' its general form is as shown in Figure 9,[1] although it should be remembered that both Phillips and subsequent writers have suggested that there may be other important explanatory variables (such as the rate of change of employment). In addition, we have decided not to restrict z to be less than unity: L* is taken to mean the total man-hours supplied at normal participation rates and with a normal working week, but it may be exceeded (at the cost of more rapidly rising money wages).[2]

The three dependent variables in this model will be z, x and W, and the remaining assumptions are as in A1. We summarize them here for convenience, writing $\hat{z} + n$ for $\hat{L}$.

$$\hat{p} = u\left(\frac{\hat{K}}{a} - s\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

$$\hat{K} = N + v(f' + \hat{p} - r) \quad \dots\dots\dots\dots\dots\dots\dots\dots(7)$$

$$\hat{z} = m(1 - f''/a - W) \quad \dots\dots\dots\dots\dots\dots\dots\dots(10)$$

This last equation is the demand for labor equation discussed earlier.

Whence,

$$\hat{x} = \frac{v}{(1 - uv/a)} \cdot [(f' - r) + u(N/a - s)] - m(1 - f'/a - W) \quad \dots(19)$$

$$\hat{W} = h(z) - g - u(N/a - s) - \frac{(u + f')}{a} \cdot \hat{x} - \frac{u}{a} \cdot \hat{z} \quad \dots\dots\dots(20)$$

[1] See Phillips [12].

[2] For an empirical discussion see Brechling [5].

These three equations (10, 19, 20) in three variables (x, W, z) have a singularity (equilibrium) where $x = 1$, $z = 1$ (or more generally, $z = h^{-1}(g))$, and $W = 1 - f'/a$. We are interested in the stability of this equilibrium, and for this we require that all the roots of the equation

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0$$

(given in Appendix IV) be negative.

The necessary and sufficient conditions for stability are:

$$R > 0, \quad Q > 0, \quad \text{and} \quad PQ > R. \quad \text{(See Appendix II)}.$$

From the form of R given in Appendix IV, it is clear that this condition is identical to the stability condition of model A1 (see section 9 above). The second condition is more complex, and includes h'. Essentially, though, if we ignore the Kaldor effect, consider the Cobb-Douglas case, and put $h' = 0$ for the moment, then the $Q > 0$ condition for stability becomes

$$bv < s$$

As we have shown in dealing with A1, this is also the condition for investment to exceed saving when x rises above its equilibrium value (not taking into account capital gains) and hence the price level to rise and encourage investment still further. In the case of model A1 we showed that $bv > s$ was a sufficient condition for stability; here it is a sufficient condition for instability.

If we turn to the first condition, $P > 0$, the second term has exactly

the same form as the stability condition obtained for model A; moreover the other terms are unambiguously positive. It follows that the stability of model Al is a sufficient but not necessary condition for the stability of this simplified model B. If we now let $h' > 0$, then the condition on s is weakened.

Figure 10 summarizes the different stability conditions for each model. It will be seen that the stable region of model B is contained within model A, so relaxing the original assumptions has reduced the stability of the system.

If we take crude values for the parameters and look at Q, for $u = 1$, $v = 0.25$, $a* = 0.35$, $b = 30\%$, $h' = 1$, the Phillips factor has about four times the effect of the other factors--it may be a major stabilizing force, in other words. (Note--it is difficult to find a plausible value of h' --it is the rise in percentage points in the rate of increase of money wages resulting from a 1% increase in the level of employment, so a figure of between 1 and 3 is plausible.)

Alternative Specifications of the Wage Equation

As we have already noted, the simple assumption (8) does not do justice to the empirical work which has been done in this field. We now investigate the implications for stability of some alternative formulations of the wage equation, assuming $s' = 0$ throughout for convenience.

Several writers, including Phillips and Lipsey [8] have suggested that wage claims may be influenced by changes in the cost of living index. This can easily be incorporated into the model:

$$\hat{w} = h(z) + f_1(\hat{p}) \ , \quad \text{where} \quad f_1^0 > 0 \ . \quad \ldots\ldots\ldots\ldots\ldots \quad (8a)$$

Expressing $\hat{p}$ as a function of x , we have:

$$\hat{p} = \frac{u}{(1 - uv/a)} \cdot \left(\frac{N}{a} - s + v(f^0 - r) \right)$$

Whence:

$$\frac{d\hat{p}}{dx} = \frac{u}{(1 - uv/a)} \left\{ v \frac{df^0}{dx} - \frac{N}{a^2} \frac{da}{dx} \right\}$$

The only term affected is a_{32} (see Appendix IV), and this term only appears

in Q , having the effect of increasing h^0 by a factor $1/(1 - f_1^0)$.

Thus the effect is to increase the range of stability of the model, which

is not surprising, since the introduction of the cost-of-living term will

reduce the responsiveness of the wage share to changes in x . With the

values obtained in empirical estimates of equation (8a), this factor may

be quite significant: e.g. Perry (REStud 1964) estimates f_1^0 as 0.367 for

the U.S., which implies raising h^0 by a factor of 1.58.

The second possible modification is to introduce the actual wage

share as a determinant of the rate of increase of wage rates: if the profit

share is large, then wage claims increase and can more readily be met.

Although Lipsey and Steuer found little evidence of such a relationship

in the U.K., Eckstein and Wilson[1] concluded that profits played a signifi-

cant role in explaining wage changes in the U.S.

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[1] Lipsey and Steuer [9], Eckstein and Wilson [7].

If we then write:

$$\hat{w} = h(z) + f_2(W) , \quad \text{where} \quad f_2^! < 0 \ldots\ldots\ldots\ldots\ldots \quad (8b)$$

This has the effect of reducing a_{33}, and since a_{22} is negative, of raising Q . R is left unaltered, and so stability is increased.

Finally, we can consider the effect of making wage changes depend on the rate of change of employment: money wage rates rise more rapidly when employment is rising than when it is falling. Lipsey[1] found that this variable was significant for the period 1862-1913, and for 1929-39, but for the post-war period it had the 'wrong' sign. Let

$$\hat{w} = h(z) + f_3(\hat{z}) , \quad f_3^! > 0 \ldots\ldots\ldots\ldots\ldots\ldots \quad (8c)$$

Again, this has the effect of reducing a_{33} (by $mW^*f_3^!$), and so has the same effect as introducing $f_2(W)$ --the model becomes more stable.

Cost-Push Inflation

Up to now, we have assumed that price increases result solely from excess demand in the goods market. We now investigate the effect of intro-ducing an element of cost-push in the determination of the price level. Let us assume that firms pass on a certain proportion of the increase in labor costs. The price equation now becomes:

$$\hat{p} = u\left(\frac{\hat{K}}{a} - s\right) + \mu\{h(z) - g\} \ldots\ldots\ldots\ldots\ldots \quad (2a)$$

---

[1]Lipsey [8].

This has the effect of adding an extra term $\mu(h - g)$ inside the square bracket of equation (19) and of subtracting $\mu(h - g)$ from equation (20). As shown in Appendix IV, this reduces both Q and R and thus reduces the stability of the model at both ends of the range. For values of μ as high as 0.5 (which have been suggested) the destabilizing effect is very powerful, $\bar{s}$ falls dramatically, and there is no longer any guarantee that the range of stability is non-empty.

The Next Step

This brings us to the end of our investigation of a fairly simple malleable capital disequilibrium model. Within its framework we have attempted to assess the importance of the various assumptions and the relative significance of the different stabilizing forces. Sometimes it has been difficult to find appropriate numerical estimates for the parameters on which stability depends, and we have not followed the advice given in Appendix I(ii). Ideally we should now simulate the behavior of the model and estimate the parameters in a comparable way to the econometric estimates on which we have relied, or, better, we should use our model to interpret the econometric data--this is more difficult, since often it is impossible to find data for some of the important endogenous variables-- especially depreciation and capital stock.

As we commented earlier, probably the least reliable part of the model is the investment function, together with the fact that the models are malleable capital models. We have made preliminary attempts to investigate disequilibrium behavior in a vintage model, and it is clear that if

we retain a choice of techniques in the investment function (i.e. have capital
malleable ex-ante, as in the model discussed, for instance, by Bliss (REStud
1968)) it is impossible to investigate behavior analytically. The only solu-
tion seems to lie with computer simulation, and we have made some forays in
this direction. Anyone who has dealt with simulating systems of unknown
stability by numerical methods will appreciate the type of problems that we
have encountered, and it is too soon to make any definite conclusions.

The choice of an investment function is even more difficult with a
vintage model, since it is necessary to bring expectations more fully into
the model--the choice of factor proportions will affect future profitability.
One model we have been experimenting with makes the following assumptions
about entrepreneurs' behavior. It assumes perfect clearing in the labor
market, and the real wage equal to the average product on the marginal vin-
tage.

The entrepreneur invests to provide a desired increment of output
at maximum expected PDV of profit, making the assumption that the wage rate
will rise at the equilibrium rate, and that the investment will be scrapped
at zero gain when its quasirent falls to zero. The desired increment of
output is composed of three elements:

(i) the expected rise in total demand (a markup on current demand
or previous period's demand) plus existing demand supplied by equipment now
being scrapped,

(ii) some proportion of observed excess demand (currently being met
from stocks),

(iii) an increase in the firm's market share if the investment looks more profitable than normal (zero net profits).

It is very easy to build lags into any simulation model. We feed into or generate inside the computer a past history (usually a steady state history) so that the computer has in its memory output, capacity, and employment fi- gures for a sufficient number of vintages. Given the present state of the economy--demand, savings, wage rate, rate of discount, etc., the computer chooses simultaneously the amount and type of new investment. This in term determines the distribution of employment over past vintages, hence the wage rate and total output. The computer then traces out a path from the parti- cular initial history, and can be subjected to shocks or displacements to investigate stability.

One basic problem with such a putty-clay model is that we are not even sure of the stability (in the sense of convergence) of the Solow path, at least not in the general case of elastic wage expectations. Sheshinski (REStud 1967) has proved convergence for the case in which the current wage is expected to rule indefinitely, but this is clearly unsatisfactory.

Conclusions

We have found that under a wide range of assumptions the economy will return to a steady-state growth path in which there is a tendency to full- employment and equilibrium in the goods markets. Allowing possible unemploy- ment has a marked effect on the stability of the model, and it should be pointed out that "full-employment" is defined as the level of employment

at which money wages rise at their steady-state rate; the point at which there is no inflation. This may not be a very satisfactory level of employment, in which case one solution would seem to be to increase the equilibrium rate of investment--a possibility discussed earlier. Perhaps our strongest conclusion is that the equilibrium non-steady state paths have very little appeal, and it is only in the long run that disequilibrium paths and equilibrium paths converge - both to the steady-state path.

Bibliography and Survey of Disequilibrium Growth Models

A. **General Bibliography**

1. Hahn and Matthews: The Theory of Economic Growth: A Survey, _EJ_, 1964 (p. 2).

2. Bergstrom: The Construction and Use of Economic Models, 1967.

3. Gantmacher: The Theory of Matrices, 1959.

4. Ball and St. Cyr: Short-term Employment Functions in British Mfg., _REStud_, 1965.

5. Brechling: Relationship between Output and Employment in British Manfuacturing Industry, _REStud_, 1965.

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7. Eckstein and Wilson: The Determinants of Money Wages in American Industry, _QJE_, August 1962 (p. 31).

8. Lipsey: The Relationship between the Rate of Change of Money Wage Rates and the Level of Unemployment in the U.K. 1862-1957: A Further Analysis, _Economica_, 1960 (p. 31).

9. Lipsey and Steuer: The Relation between Profits and Wage Rates, _Economica_, 1961.

10. Mayer: The Empirical Significance of the Real Balance Effect, _QJE_, 1959.

11. Perry: The Determination of Wage Rate Changes and the Inflation-Unemployment Trade-off for the U.S., _REStud_, 1964.

12. Phillips: The Relationship between the Rate of Change of Money Wage Rates and the Level of Unemployment in the U.K. 1862-1957, _Economica_, 1958.

13. Kahn: Exercises in the Analysis of Growth, _OEP_, 1959 (p. 20).

14. Meade: A Neo-Classical Theory of Economic Growth, (p. 20).

B. <u>Short Survey of Long-run Disequilibrium Growth Models</u>

15. Hahn: The Stability of Growth Equilibrium, <u>QJE</u>, 1960.

This article was the starting point for the present paper, and Hahn's model A is almost identical to Al. Some of his conclusions (especially the numerical ones) need correction, and the appropriate modifications are readily seen. Hahn's model B can be derived from our model B by letting $h' = 0$. It possesses peculiar properties, in particular it is only neutrally stable, and there is no guarantee that if displaced it will return to full employment.

16. Sargent: The Stability of Growth Equilibrium: Comment, <u>QJE</u>, 1962.

This is a comment on the Hahn paper, and as we have not explicitly referred to it in the text, we do so here. Sargent draws attention to the assumption that producers get the capital they desire--so that the whole burden of adjustment is placed on the consumers. There is in principle no difficulty in altering this assumption, as indeed, Sargent suggests, but the only effect is to weaken the stability conditions. As Hahn points out in his reply (same journal), this is not really a substantive alteration.

Bergstrom, A.: 'The Construction and Use of Economic Models' 1967.

We mention this book, as it is the most convenient summary of the work by Phillips and Bergstrom on disequilibrium growth models (Phillips Ec 1961, Bergstrom Ec 1962). By and large, the focus is rather different, the models have fixed coefficients, and the monetary side is more highly developed--they are more accurately described as Keynesian growth models. Unfortunately, the most articulated model by Bergstrom rests upon very strange investment and scrapping assumptions. The Control Group in the Engineering Faculty at Cambridge have experimented with a 12 state equation 3 control variable version of this model in an attempt to develop an optimal dynamic control system for the model (unpublished paper by Noton and Lively).

17. Rose, H.: 'Expectations and Stability in Neo-Keynesian Growth Theory' <u>QJE</u>, 1963.

This is a model to show that the requirement that savers react more strongly than investors is not a necessary condition for the stability of Keynesian growth models. Rose assumes that labor is paid its marginal product, though full employment is not guaranteed. The goods market clears instantly, savings are Kaldorian, and Investment reacts with a lag to expectations which can be either flexible or elastic. The article derives stability conditions, but it is difficult to judge the robustness of the formulation, and the sensitivity of the conclusions to particular assumptions.

18. Rose: Unemployment in a Theory of Growth, IER, September 1966.

In this article Rose is concerned less with stability than comparative statics when account is taken of unemployment. Factors earn their marginal products, saving varies with the profits, wages and the rate of interest. The rate of investment depends on the relation between the cost and return of funds, the price level depends on the excess demand for goods, and money is supplied to maintain a constant relationship with real N.I. The rate of interest depends on liquidity preference. The model is akin to the London School models in its complexity and the articulation of the monetary side, but it has flexible coefficients. It is rather difficult to see what is going on, given the complexity of the formulation.

19. Rose: On the Non-Linear Theory of the Employment Cycle, REStud, 1967.

This is a rather strange model in which savings plans are always realized, and determine the rate of 'productive' investment, whilst stocks (unplanned investment) enter the production function. The model gives rise to cyclical unemployment--this effect seems to depend on the peculiar shape of the Phillips curve, which looks like an S on its side. The mathematics is quite sophisticated--the economics is accepted rather uncritically.

20. Meade, J.: The Adjustment of Savings to Investment in a Growing Economy, REStud, 1963.

This is a study not of the dynamic behavior of a fully articulated growth model, but of the way in which savings may be equated to some exogenously given level of investment--in fact the ratio of investment to N.I. is assumed fixed; which is not quite the same thing, as investment plans can now be influenced in the model via changes in Y . Whilst it is difficult to formulate an adequate theory of investment, in practice almost any model makes some implicit or explicit assumptions about investment behavior, and it is important to examine the effect of these particular assumptions have on the behavior of the model. Consumption and wages respond with a lag to their appropriate levels, and the model studies the relative size of the Kaldor effect and the various lags in consumption response. The article goes on to introduce imperfections in the labor and product markets, but the analysis is concerned with comparative statics (or dynamics) rather than with the stability of the system.

21. Conlisk: Unemployment in a Neo-Classical Growth Model, EJ, 1966.

The model is neo-classical in that it has an aggregate production function (Cobb-Douglas with increasing returns), and the rate of return on capital is related to the marginal productivity of capital, although in a somewhat odd way, since it is not made clear whether the model is a straight malleable capital model or whether it has elements of a vintage

model (see e.g. p. 555). The model has grafted on to it a 'Keynesian' demand relation which permits unemployment, although unemployment adjusts via the savings proportion—claimed to reflect a life-cycle savings theory, although the same effect would occur with Kaldorian savings or a lagged response of consumption. Thus the model is to be contrasted with monetary formulations where inflation plays a significant role in balancing savings and investment demands.

22. Bruno: Estimation of Factor Contribution to Growth under Structural Disequilibrium, _IER_, February 1968.

This paper is interesting, not so much for the disequilibrium model advanced, as in the recognition that it is necessary to take account of disequilibrium in estimating the underlying parameters. The postulated wage determination is compatible with a production function of the form:

$$y = Ae^{g(1-b)t}k^b - m , \quad \text{(contrast A1, p. 10)}$$

There is imperfection in the labor market, and capital rationing—and the model is used to estimate parameters for the Israeli economy

23. Solow and Stiglitz 'Output, Employment, and Wages in the Short Run,' _OJE_, November 1968.

This is a short run model in that investment is exogenous, and employment adjusts towards a desired level, determined by its price and demand. Prices depend on total costs and excess demand, money wages on unemployment and prices. The demand for labor is slightly strange, and the model is analyzed for stability and its comparative statics are explored.

24. Akerlof and Stiglitz: 'Capital, Wages, and Structural Unemployment,' _EJ_, June 1969.

The purpose of the paper is to examine the role of wages in determining the long-run behavior of the economy, and in particular to see if the Phillips curve will lead to cycles. The product market is always in equilibrium, though the labor market need not be. The authors treat a putty-clay and a malleable capital model, with no technical progress, and use a Kaldorian savings function. In general the balanced growth path is stable, though cycles are possible. There is an interesting "justification" of a more general Phillips curve $w = w(u, k)$, $\partial w/\partial u < 0$, $\partial w/\partial K > 0$ (where w is the real wage, u is unemployment, K is the capital-labor ratio), and an examination of the leverage this specifi-

cation gives the government in influencing the long-run rate of unemploy-
ment. However, apart from this, it is misleading to suggest that the
model explains "structural unemployment"--since this is built so firmly
into the assumed Phillips curve.

This survey has ignored all short-term Keynesian disequilibrium models,
and also those predominantly concerned with explaining the Trade Cycle--
of which the nearest to our present work is Goodwin: 'A Growth Cycle'
in 'Socialism, Capitalism and Economic Growth--Essays presented to Maurice
Dobb, ed. Feinstein.

APPENDIX I

OBSERVABILITY AND ESTIMATES OF THE SPEEDS OF ADJUSTMENT

This is not the place to discuss fully the problem of statistical inference and econometric estimation when the subject of our observation (the economy) is a complex inter-related simultaneous system. We just make two points:

(i) The estimate of most parameters (u , v , etc.) will be specific to the particular model. There is no a priori reason to believe that a value of v which is plausible for a malleable capital model will be plausible for a vintage model, for example.

(ii) We should be able to estimate parameters from the model which are similar to similarly estimated parameters for the economy.

Unfortunately this is computer-intensive and time-consuming, so we have not made much progress in this direction. Nonetheless, drawing on empirical results and a certain amount of introspection, we can make the following 'order-of-magnitude' estimates.

Beginning with v , it may be useful to consider this as giving the absolute increase in the annual rate of growth of the capital stock when the interest rate is lowered 1%. From this it seems clear that however optimistic one may be about the effect of the interest rate on investment, v must be considerably less than unity, and that it is probably more of the order

of .25: a 1% fall in the interest rate causes the rate of capital forma-
tion to rise from, say, 3.5% to 3.75% per annum. Some support of this view
is provided by the estimates of Dhrymes (International Economic Review,
1967) for the U.S. two-digit industry primary metals. He estimates

$$K_t - K_{t-1} = \delta(K^* - K_{t-1}) \quad \text{where} \quad \delta = .05$$

Assuming a CES production function (as Dhrymes), and taking a linear approxi-
mation, we obtain the following estimate of $v = .05/2r$ so that when $r =$
10% we obtain $v = .25$, when $r = 15\%$ we have $v = .166$ and when $r = 5\%$,
$v = 0.5$. While we have no reason to expect these results to carry over
to other industries, they do provide some support for the presumption that
v will be significantly less than unity.

Dhrymes also estimates a labor adjustment equation for the same in-
dustry group, from which we can deduce that $m = 1.5/(1-b)$, so that for
$b = .3$, $m = 2.1$. On the other hand, we have estimates of the equation

$$dL/dt = \gamma(L^* - L)$$

for the U.K. made by Brechling (REStud, 1965) and Ball and St. Cyr (REStud,
1966). In the Cobb-Douglas case,

$$m = \gamma/(1-b) \quad . \quad L/L^* \simeq \gamma/(1-b)$$

With $b = .3$, their estimates imply a value of m of the order 1.2-2.0.

Finally, we come to $\underline{u}$. Here we have no more to offer than the
view that Hahn's implicit assumption[1] that $u = 1$ may be a reasonable

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[1] But when he makes numerical estimates, he also assumes the timescale to
be measured in years.

one, but that $u = .5$ or $u = 2.0$ would be equally convincing. The inter-
pretation of $\underline{u}$ is clear: the rate of price increase caused by 1% excess
demand. Very roughly, if one assumes that 2% of the 3% average inflation in
the past decade is caused this way, and that there has been on average 1-4%
excess demand, then this gives the range we mentioned.

This is a convenient point at which to discuss the assumption that
$a^* > uv$, which Hahn makes without comment. Since the steady state output-
capital ratio is given by N/s, this reduces to $s < N/uv$. Taking u
at the (relatively high) value of 2.0, and $N = .05$, we get $s < 2.5/v\%$,
so that with $v = .25$ this quite restrictive condition $(s < 10\%)$. On
the other hand, with $u = .5$, we have $s < 40\%$, which is very unlikely
to be binding. So that while we have assumed that the condition is always
satisfied, the first example shows that there are situations in which it
may not be; this then raises the question as to what happens when $a^* < uv$.
It is immediately clear that this reverses the sign of B, and of the
first term of A, so that it is very important for the stability condi-
tions. On the other hand, if we look at the investment function, we can
see (Equation 2) that the rate of growth of the capital stock will then
respond negatively to an increase in the return (before capital gains) and
to a fall in the interest rate--a result contrary to both casual and more
careful empiricism.[1] This suggests that the condition $s < N/uv$ is in
fact satisfied, and that $\underline{u}$ and $\underline{v}$ are the lower rather than the higher
end of the ranges suggested earlier. We shall now assume that this inter-
pretation is in fact correct, although with more misgivings than Hahn ap-
parently felt.

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[1] Even though in many cases investment functions incorporating the interest
rate do not have significant coefficients for this variable, in general it
is negative.

APPENDIX II

CONDITIONS FOR STABILITY

Let y be a vector of deviations from equilibrium i.e. $y^* = \underline{0}$. Suppose we have a differential equation system

$$Dy = \underline{F}(y) \quad \text{where} \quad \underline{F}(y) = (f_1(y),\ f_2(y),\ \ldots)$$

To examine stability in the small (local stability)

$$Dy = \left.\frac{\partial F}{\partial y}\right|_{y=0} \cdot y$$

where $\dfrac{\partial F}{\partial y}$ is a matrix A whose element $a_{ij} = \dfrac{\partial f_i}{\partial y_j}$. From the theory of linear algebra we can transform the linearized equation to new bases so that $Dy = Ay$ becomes $Dz = Az$, where A is a diagonal matrix whose elements are the eigenvalues of A . The solution of this equation is

$$z = \sum_{i=0} z_i e^{\lambda_i t}$$

and so the equilibrium will be stable if the real parts of all the eigenvalues are negative. In general the Routh-Hurwitz conditions are the easiest means of establishing such conditions (see e.g. Gantmacher, F. "Theory of Matrices," 1959; or Bergstrom, A. "The Construction and Use of Economic Models" 1967, p. 33).

In simple cases solve $|A - \lambda I| = 0$ to give an equation

$$A_0 \lambda^n + A_1 \lambda^{n-1} \cdots + A_n = 0$$

and set $A_0 > 0$.

For a second order equation we have <u>stability</u> if $A_0 > 0$, $A_1 > 0$, $A_2 > 0$. For a third order equation, stability if $A_0 > 0$, $A_3 > 0$, $A_2 > 0$, $A_1 A_2 > A_0 A_3$.

APPENDIX III

THE PIGOU EFFECT IN THE COBB-DOUGLAS CASE

The stability conditions can be readily calculated from the following two equations:

$$\dot{x} = vx(ab + \hat{p} - r*) \quad \text{from (7), p. 5, and p. 8.}$$

$$\dot{p} = up\left(\frac{\hat{x} + N}{a}\right) - s(p) \quad \text{from (1a), p. 13}$$

Let $a_{11} = \left.\dfrac{\partial \dot{x}}{\partial x}\right|_{x=1}$, $a_{12} = \left.\dfrac{\partial \dot{x}}{\partial p}\right|$ etc.

Then

$$a_{11} = \frac{v}{(1 - uv/a)} \cdot [b - Nu/a^2] \cdot \frac{da}{dx} , \qquad a_{12} = \frac{uv}{(1 - uv/a)} \cdot \left(- \frac{ds}{dp}\right)$$

$$a_{21} = \frac{up}{a} \cdot (vb - s) \cdot \frac{1}{(1 - uv/a)} \cdot \frac{da}{dx} , \qquad a_{22} = - \frac{up}{(1 - uv/a)} \cdot \frac{ds}{dp}$$

The model will be stable at $a*$ if:

(i) $a_{11} + a_{22} < 0$

(ii) $a_{11}a_{22} - a_{21}a_{12} > 0$

(i) $\qquad v(1 - b) \dfrac{N}{s} (b - us^2/N) + us\left(\dfrac{pds}{sdp}\right) > 0$

i.e. $\qquad v(1 - b)Nb/s + u[s\epsilon - uv(1 - b)s] > 0$, where $\epsilon = \dfrac{pds}{sdp}$

$$\text{or} \qquad u < \frac{bN/s^2}{(1 - \epsilon/v(1 - b))}$$

The larger ϵ is the more likely this is to be satisfied. An estimate of $1/v(1-b)$ would put it between 5 and 10, so that the constraint is not binding for ϵ, the "elasticity" of the Pigou Effect $\left(\frac{pds}{sdp}\right)$ greater than about 0.2. Using the data given in the article by Mayer (QJE, 1959, p. 281) a value of ϵ of 0.02 would be quite high.

(ii) This reduces $v > 0$, which is trivially satisfied.

APPENDIX IV

STABILITY CONDITIONS FOR MODEL B

Let $a_{11} = \dfrac{\partial \dot{z}}{\partial z}\bigg|_{z=1}$; $a_{12} = \dfrac{\partial \dot{z}}{\partial x}\bigg|_*$ etc. Then from equations 10, 19, 20,

$$a_{11} = 0 ; \qquad a_{12} = -m\,\frac{d}{dx}\left(\frac{f'}{a}\right) ; \qquad a_{13} = -m ;$$

$$a_{21} = 0 ; \qquad a_{22} = \frac{v}{(1 - uv/a)}\left\{\frac{df'}{dx} - u\,\frac{N}{a^2}\,\frac{da}{dx}\right\} + m\,\frac{d}{dx}\left(\frac{f'}{a}\right) ;$$

$$a_{23} = m - uvs'/(1 - uv/a) ;$$

$$a_{31} = W^*h' ; \qquad a_{32} = W^*\left(\frac{uN}{a^2}\,\frac{da}{dx} - \left(\frac{u + f'}{a}\right)a_{22} - \frac{u}{a} \cdot a_{12}\right)$$

$$a_{33} = W^*\left(us' - \left(\frac{u + f'}{a}\right)a_{23} + um/a\right)$$

If we write the characteristic equation as $\lambda^3 + P\lambda^2 + Q\lambda + R = 0$, then

$$R = a_{31}(a_{22}a_{13} - a_{23}a_{12})$$

$$R = W^*h'm = \left\{\frac{v}{(1 - uv/a)}\left[\left(-\frac{da}{dx}\right)\left(\frac{df'}{ds} - \frac{uN}{a^2}\right)\right] + (-s')\,\frac{uv}{(1 - uv/a)} \cdot \frac{d}{dx}\left(\frac{f'}{a}\right)\right\}$$

The bracket is the same as the expression at the top of page 11.

$$P = (-a_{22} - a_{33})$$

If we ignore the Kaldor effect, and substitute $W* = 1 - f'/a$, we have:

$$-P = A + m \frac{d}{da}\left(\frac{f'}{a}\right) \cdot \frac{da}{dx} + \frac{mf'}{a}\left(\frac{f'}{a} - 1\right)$$

$$P = -mk*f''/a - \frac{v}{(1 - uv/a)}\left\{\frac{df'}{dx} - \frac{uN}{a^2}\frac{da}{dx}\right\}$$

$$Q = (a_{22}a_{23} - a_{32}a_{23} - a_{13}a_{31})$$

$$Q = mW* \left(a_{22}\frac{u}{a} - \left(\frac{uN}{a^2}\frac{da}{dx} + \frac{mu}{a}\frac{d}{dx}\left(\frac{f'}{a}\right)\right) + h'\right) .$$

$$Q = \left[\frac{uv}{(a - uv)}\left\{k*f'' + \frac{N}{av}(a - f')\right\} + h'\right] mW*$$

In the Cobb-Douglas case this reduces to a positive factor $(1 - bv/s)$ plus

the h' term. Thus we have weakened the stability condition of the $h' = 0$ case—

—the larger h' is the weaker in the constraint on the lower bound of s .

There remains the condition $PQ > R$.

Let $R* = R/W*m$; $Q* = Q/W*m$, then the condition reduces to:

$$R*(Q* - h') - Q*mk*f''/a > 0$$

The second term is clearly positive, the first term is a more restrictive

condition on Q than those presumed already satisfied. In fact, it reduces

to the condition $bv < s$ in the Cobb-Douglas case so that the whole expression is

again a less severe constraint than that found when $L' = 0$; thus

the larger is h' the more stable the model will be.

Assumption 8a, p. 22.

a_{32} has added to it $W*\left\{uvf_1'/(1 - uv/a)[k*f'' + \frac{N}{av}(a - f')]\right\}$ and so Q has the first term reduced to $(1 - f_1')$ of its original value.

Assumption 2a of p. 23.

a_{21} becomes $\mu h'v/(1 - uv/a)$; $a_{31} = W*(1 - \mu)h'$ R is reduced to $(1 - \mu)\bar{R} - W*m^2 bv\mu h'/(1 - uv/a)$ in the Cobb-Douglas case where $\bar{R}$ is the old value of R .

Q falls by $m\mu h'W*$.

If $b = 30\%$, $m = 1$, $u = 1$, $h' = 1$, $\mu = 0.5$, $a* = 0.3$, then $\bar{s}$ falls to zero—there is no stable range of s . (R negative for all positive s .)

LIST OF DIAGRAMS

A note on Figures 1, 2 and 3

These curves are readily plotted for the Cobb-Douglas case. At a moment of time (i.e. ignoring technical progress) we have

$$y = Ak^b = Ak^* x^b$$

Thus y is an increasing function of x

$$a = y/k = Ak^{b-1} = Ak^{*b-1} x^{b-1}$$

$$i.e. \quad a = a^* x^{b-1}$$

a is a decreasing function of x .

The "savings schedule" is $\frac{sY}{K}$, or, sa , in this case Nx^{b-1} . The "investment schedule" before allowing for capital gains is

$$\hat{K} = N + v(f' - r^*)$$

$$\hat{K} = N \left(1 - \frac{vb}{s} \right) + \frac{vb}{s} Nx^{b-1}$$

If $vb > s$ this is steeper than the savings schedule, so that there is deficient demand and thus prices fall for $x > 1$.

The schedule allowing for capital gains is

$$\hat{K} = \frac{N \left(1 - \frac{vb}{s} \right) avb - uvs}{1 - \frac{uv}{a}}$$

Clearly the only steady growth equilibria are $\hat{K} = N$ and these will be stable if the investment schedule is downward sloping at this point. We can read the equilibrium values of x from the graphs and compute a from it.
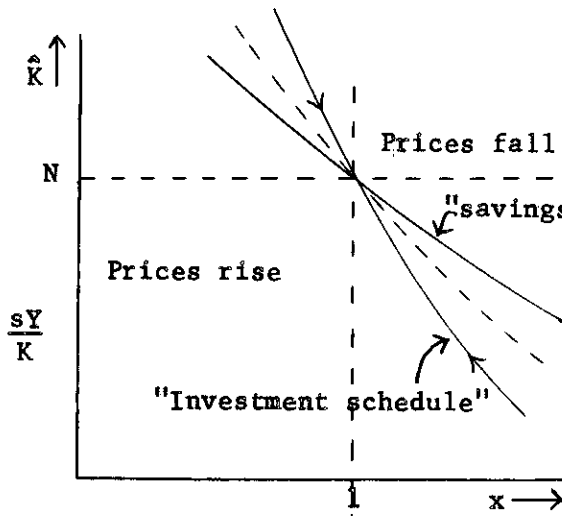
Figure 1   bv > s
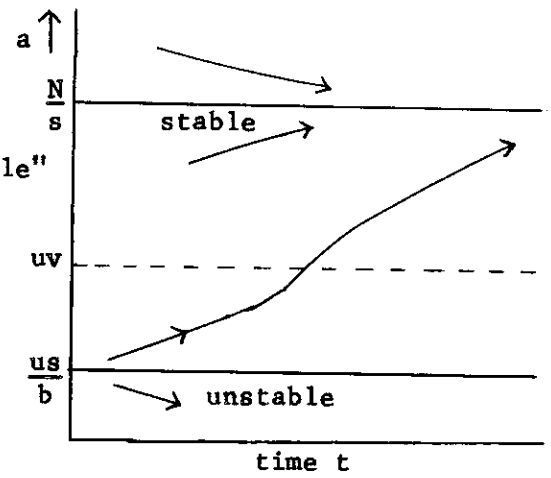
Figure 3a
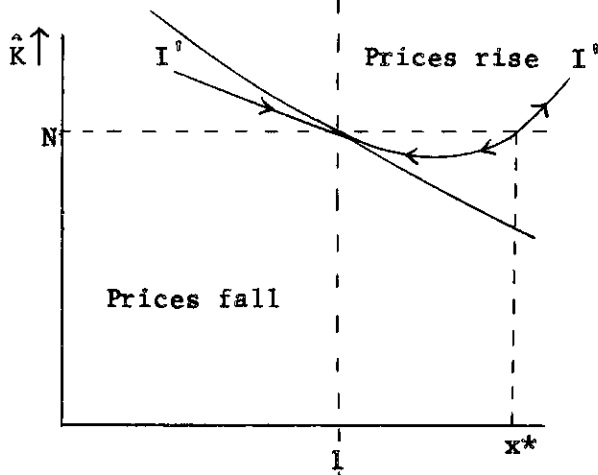
Figure 2a   bv < s
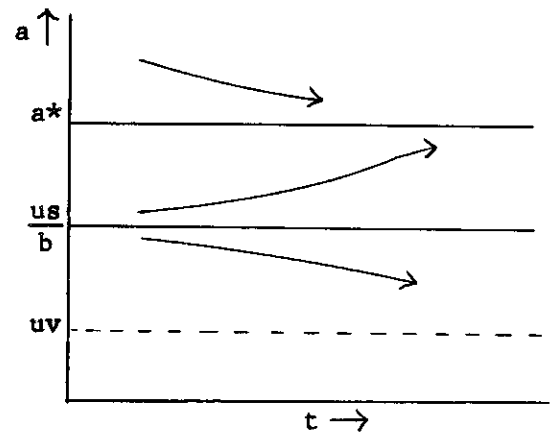
Figure 3b
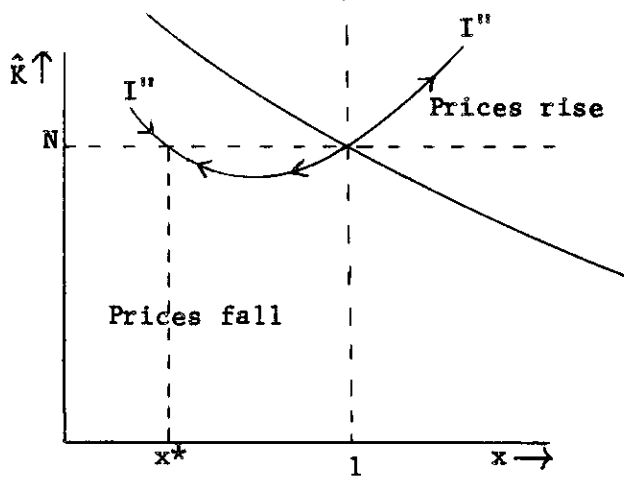
Figure 2b
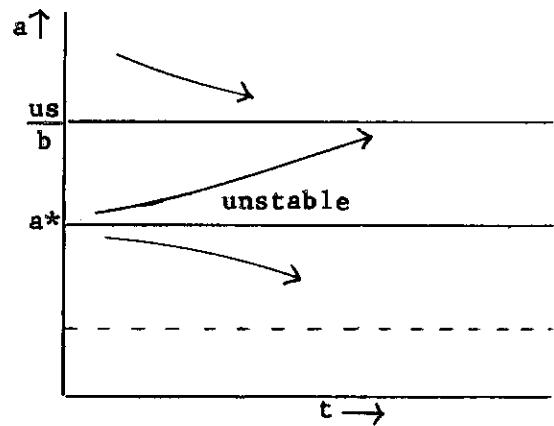
Figure 3c
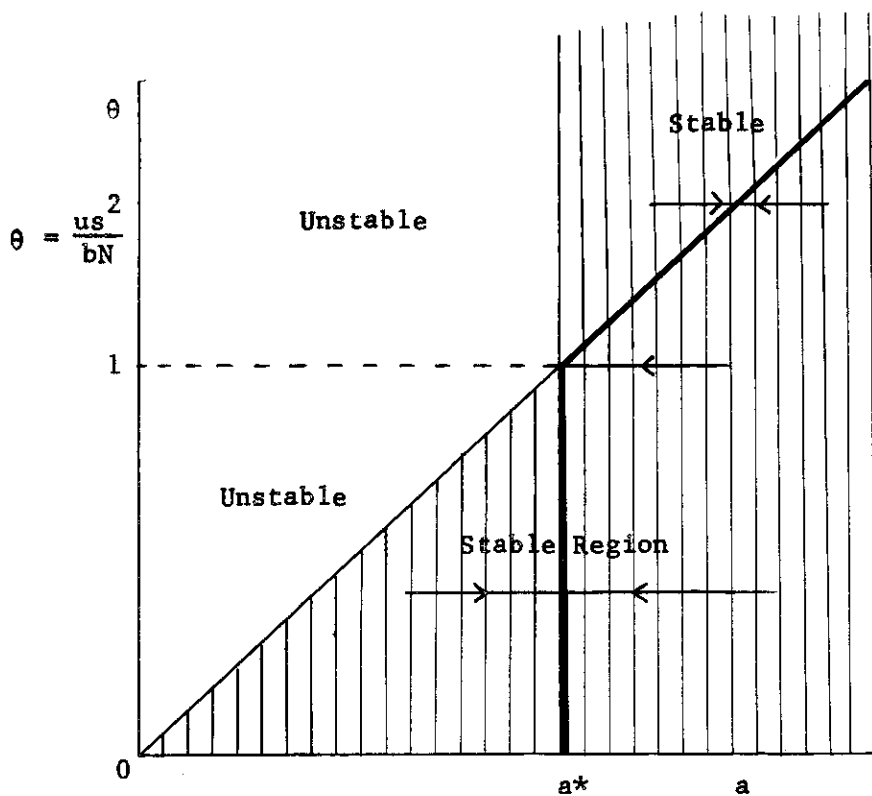
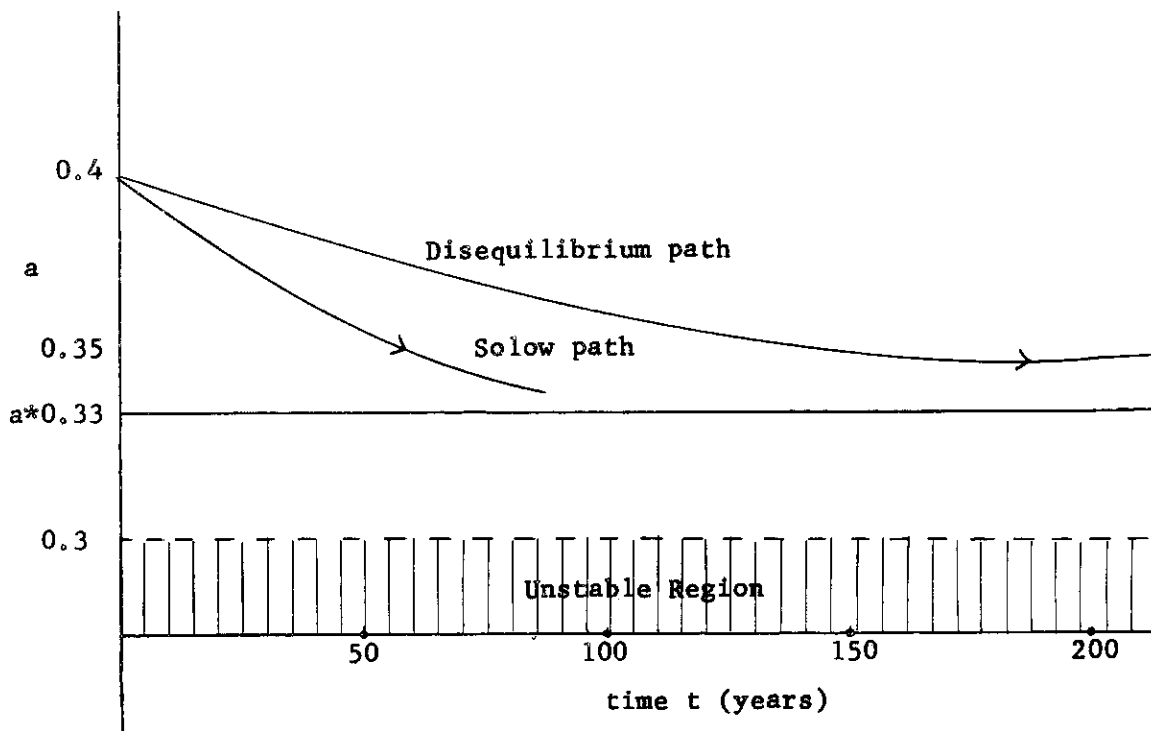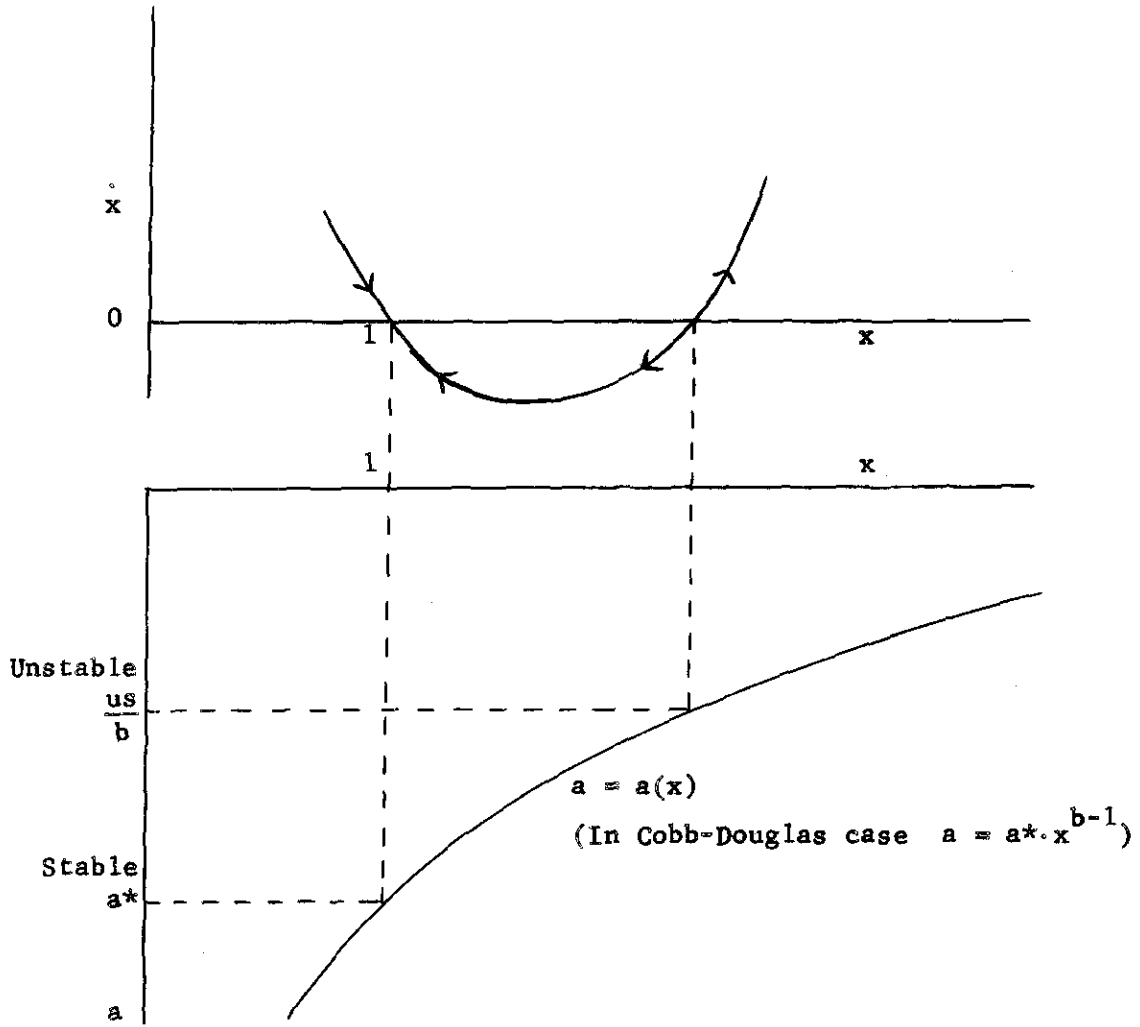Figure 4 Heavy line indicates stable steady-state

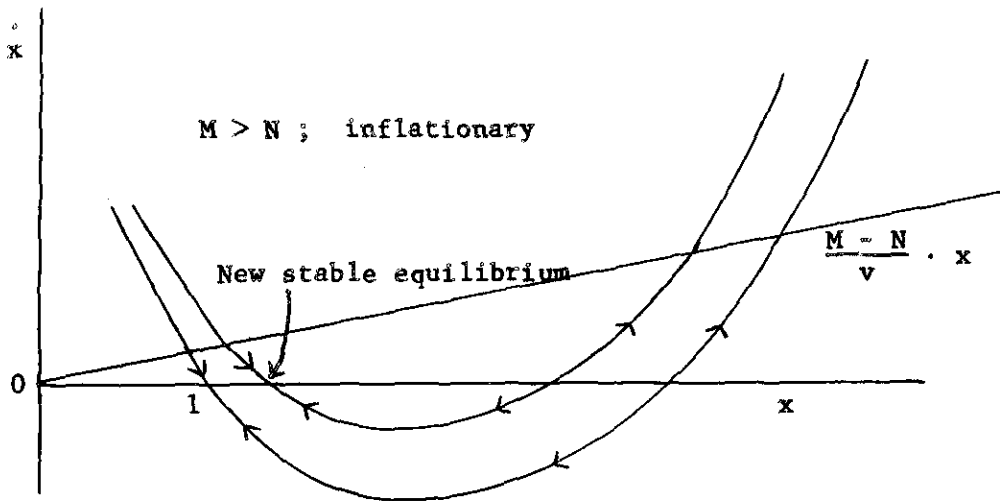Figure 5 Time Paths

Figure 6a   Phase Diagram

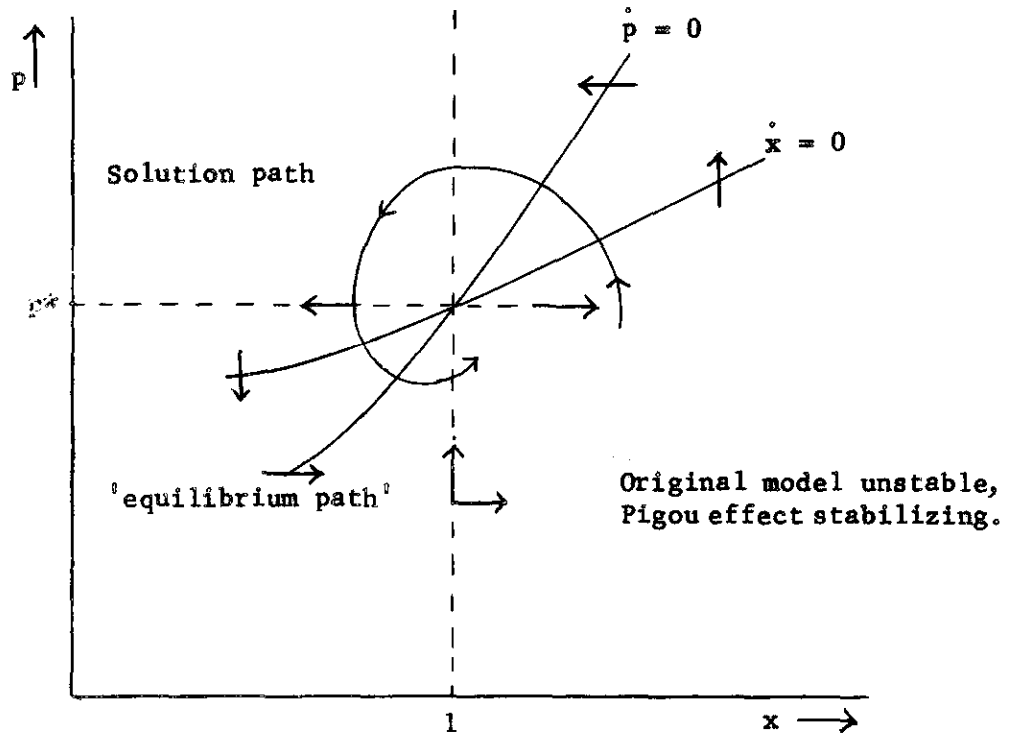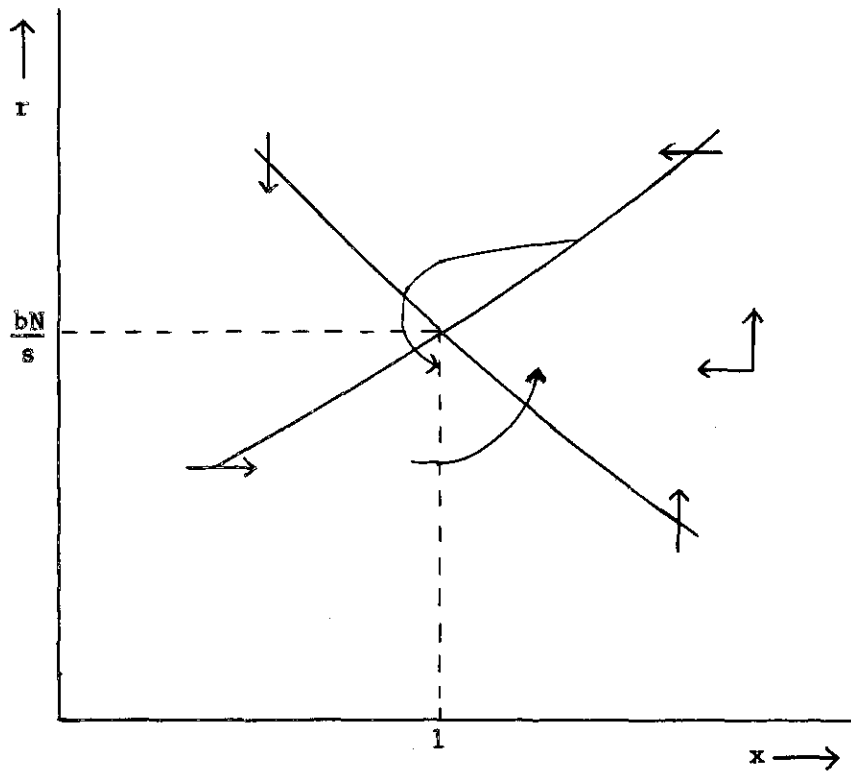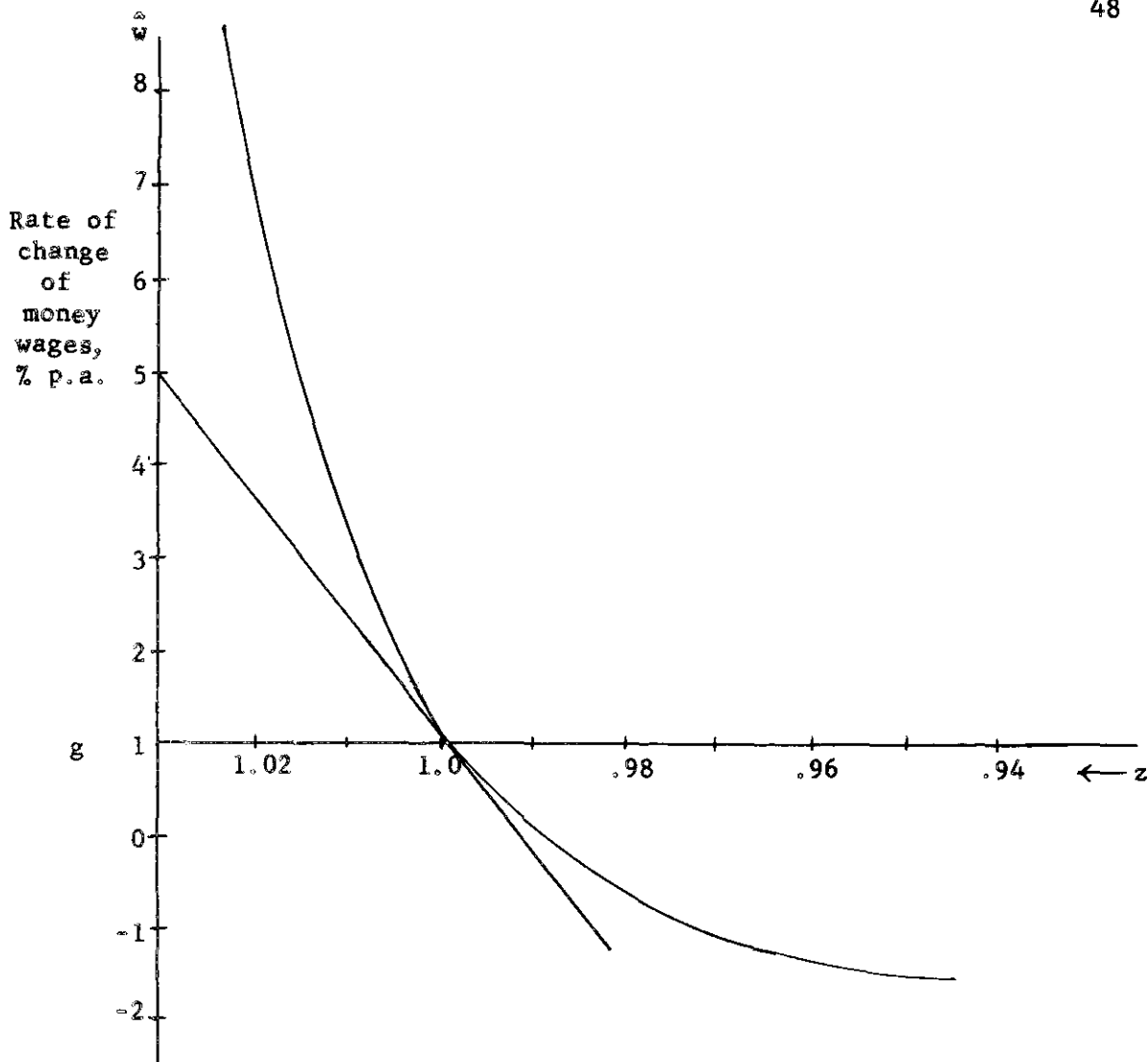Figure 6b   Phase Diagram for Inconsistent Plans
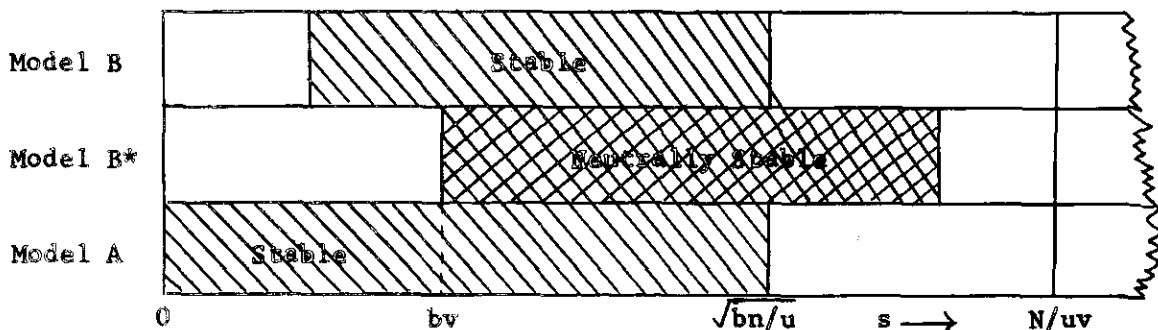
Figure 7   Phase Diagram for Pigou Effect

The curve above is drawn for a period 1862-1913, and is centred on roughly the natural rate of growth. z = 1 would then correspond to 3% unemployed. The curve for later periods shifts up (g increases), moves to the left and becomes more rectangular.

Figure 9   The Phillips Curve

Figure 10   The Three Models Compared

B* is Model B with h' = 0