COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 278

THE CHOICE OF TECHNIQUES IN A DUAL ECONOMY

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August 5, 1969

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by

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In the last decade planning offices in a number of underdeveloped
countries have become increasingly aware of the need for social cost-benefit
studies of potential investment possibilities, and also of the serious gaps
in the theoretical treatment of cost-benefit analysis, which has in large
part been created to handle project selection in developed full-employment
economies. Unemployment in many underdeveloped countries has been rising,
inflows of investible funds have stagnated, and the initial hopes of the
Development Decade have faded. The problem of correctly allocating scarce
investment resources and deciding on employment levels has never seemed more
acute, and at the same time many of these countries have trained and have
working in the government economists who see these problems in economic
terms and are uncertain of the appropriate method of solving them. The
fundamental problem is that the observable prices probably give a very poor
guide to the opportunity costs, and in particular, it is often argued that

in the face of extensive unemployment the wage paid seriously overstates the

cost of using labour. Fortunately, the last few years has seen a steady

flow of manuals and books on social cost-benefit analysis, of which the

most recent and one of the best is "Social Cost Benefit Analysis" by

I.M.D. Little and J.A. Mirrlees, published as Volume II of the "Manual of

Industrial Project Analysis in Developing Countries", OECD Paris, 1969.

The economic aspects of the problem are fairly clear, and the correct

accounting prices to use in project selection for most inputs can be estab-

lished without too much difficulty. Practically and conceptually the most

difficult price to calculate is the accounting price of labour, which in

turn determines the capital-intensity of investment, and more importantly,

the level of employment in the modern sector. This paper is directed to

solving that problem, and I shall assume that once we have found this price

all other prices can be calculated (for the details, see Little and Mirrlees,

op.cit.).

Before I turn to the details of the model a few words of caution

are needed in interpreting the results. Choosing the right investment (which

usually amounts to choosing the industrial mix) in only one of the decisions

to be made by governments bent on economic development, and it is difficult

to judge how important it is. It is possible to argue that there is little

room for significant choice, and that the pattern of development is similar

in similarly placed economies no matter what the government does. Cross

country studies (e.g. the work by Chenery, in particular "Patterns of

Industrial Growth" AER, September, 1960, and Kuznets) provide some support

for this view, though it is not difficult to find detailed case studies where an erroneous investment strategy has proved costly. Perhaps a more fundamental criticism is that economic development is a complex process involving social and political change as well as economic change, and subtle differences in economic policies may be of minor importance, at least at very low levels of development. This view is suggestively illustrated in the recent book by I. Adelman and C. Taft Morris, "Society, Politics, and Economic Development," which leads one to suspect that it is precisely those consequences of economic policies which are usually left out of investment appraisals that may have the most significant effect in inducing social change. It could be argued that it is the social structure, the presence or absence of enterprise and purpose, the existence of a sufficient body of men with a rational, scientific or material approach to the world, rather than the size of the capital stock which is the main determinant of economic progress, and these may be influenced in indirect and subtle ways by the consequences of economic decisions. This may well be true, and if it is, it suggests that as development economists, we should broaden our study to embrace these factors. My defense for appearing to ignore them is not that they are unimportant, but that I want to have done with the theoretical difficulties which have been associated with the problem of the choice of techniques. The problem ought to be despatched reasonably satisfactorily in order to free economists' concern for more significant problems. It may even, in passing, be of some practical usefulness when other things are deemed to be equal.

The economy I have in mind during the remainder of this paper is dual in the sense that much of the population lives a traditional way of life relatively beyond the reach of the government. The modern sector, which is entirely state-controlled, can almost be imagined as another country, and whilst labour and goods can pass between the two sectors, the government has severly limited influence over the production and consumption of the traditional sector, and consequently, very limited ability to raise investible tax revenue from this sector.

The traditional sector is defined more by the form of economic organization there than by its production technology or range of goods produced and traded. It consists typically of peasants working on predominantly subsistence plots, often on collectively owned land, and working for the collective benefit of the relevant kinship group. Its typical product is some kind of agricultural produce, which I shall typify as rice. Other goods are produced, but in their nature can only be locally consumed--such items as hut-building, beer brewing and so on can be ignored for our purposes. Any agriculture which is commercial in the sense of being profit motivated and which hires labour I deem to be part of the modern sector, even though it may be producing identical products. Indeed, typically we have plantation agriculture alongside peasant agriculture, and although some important interactions will be slurred over in separating them into these two simple sectors, nonetheless this is a more plausible procedure than the alternative, which frequently implies that the modern sector is incapable of producing (or even importing) food itself. (See Bose (2), Dixit (5) and (6).

I shall first set out the simplest model of a dual economy which is able to handle those economic factors that we feel are likely to be important in determining the choice of technique and the level of employment in the modern sector. In particular, I wish to take into account the constraint on saving that the governments' severely limited power of taxing the traditional sector implies, the fact that the supply price of labour cannot fall below a certain level, and I do not wish to impose any prior restrictions on the marginal product of labour in the traditional sector.

Perhaps this last point needs stressing, as considerable and largely unproductive effort has been directed to the question of what "surplus labour" is, and whether it exists.[1] Most careful studies show that the marginal product of a manhour of labour is positive at least during considerable portions of the year, and certainly averaged over the year. This is in itself not enough to ensure that the total product of the traditional sector will decline if a labourer is withdrawn, especially if the economic organization of the sector is at all complex,[2] but it should at least make us sceptical that the marginal product is zero. More importantly, the significant feature of dualism is the difference in economic organization in the two sectors and particularly in the determinants of labour use and the utilization of the surplus, rather than the exact value of the marginal product of labour in the traditional sector.

In this model, I shall make very simple assumptions about the structures of the two sectors. The modern sector will be state-owned, and the planners have complete control over the profits of the enterprises, and

over the choice of the type and scale of new investments and the hiring of wage labourers. Production in this sector enjoys constant returns to scale, and there is perfect world trade. This last assumption enables us to take the accounting prices for traded goods to be proportional to world market prices, and determined exogenously.[3]

It also permits us to aggregate over inputs and outputs and use a one-good model to represent the production process. The output of the modern sector may be consumed, invested, or traded abroad to obtain the necessary imports for consumption or investment. Since one of the most interesting questions to ask is what accounting wage rate to use in investment decisions I shall assume that the marginal product of labour varies with the choice of techniques--in fact, I shall assume that production in the modern sector is represented by a neoclassical production function with constant returns to scale.

I also assume that there are no gestation lags between investment and ensuing production. This is a strong assumption, as is the malliability assumption, and both are least appropriate to one of the most important factors of development, namely the provision and use of skills, education, enterprise, creativity and organizational talent--factors often mechanistically dismissed as "human capital".

All of these factors can only be produced after a significant gestation period, and some of them (enterprise, etc.) may only be induced by long term changes in the ideologies of the society, which change only slowly.

The only way I can think to remedy this serious shortcoming has yet to be tried, but I think it is sufficiently important to be worth quickly sketching here. The model I shall describe is an optimal growth model, and as such it must be mathematically simple if it is to yield analytically intuitive insights. It is clear that the real world cannot be completely described by simple relations, but we hope that by careful selection and elimination it can be approximately so described. The criterion of success for our model building is that if we then go on to build a more complex model (paradigmatically, a large simulation model too complex to handle except on a computer) then the solution of the simple model should provide the first approximation to the solution of the complex model. Typically it will be either analytically impossible or computationally too expensive to solve the more realistic model, but it will be fairly simple to use it to test the robustness of the conclusions of the simple model. My aim here is to build the simple model, show how it can be solved, and illustrate typical solutions. Later in this paper, I will indicate extensions of the simple model, and in due course I hope to build the more complex simulation model.

To summarize, the output of the modern sector, O , is given by the function

$$O = f(K, L)$$

where L is modern sector employment, K is modern sector capital. Any modern sector infrastructure (roads, ports, urban capital, etc.) is subsumed into K , and any costs which are entailed by employment not borne by the firm or labour directly (health services, housing, transport) can also be included in f as follows.

Let this cost be m per worker, and the manufacturing production function be F(K, L) . Then

$$f(K, L) = F(K, L) - mL$$

which is also homogenous of degree one in K and L .

The population not employed in the modern sector, N - L in number, works in the traditional sector to produce a gross output of X , or per capita output x . [x = X/(N-L)]. This may be traded with the modern sector (or the outside world) but at this stage I assume that the trade is not taxed--clearly an unreasonable assumption, but one that will be retaxed later. There is no difficulty in handling tax revenue which is simply related to L , and to illustrate this I shall assume that the rural surplus available to the government is R and redefine X to be the net product after taxation.

The government cannot influence X except indirectly by inducing labourers to work in the modern sector, thus reducing the rural labour force. I shall assume that as a result of either the availability of free land, or more plausibly, the endeavours of agonomists and the improvements in technical knowledge, that output per worker is maintained in the face of rising population.[4] As an illustration consider the following Cobb-Douglas production function with as its two arguments land and labour (I intend to slur over the distinction between men and manhours). Land is fixed and can be subsumed in the constant, technical progress is exogenous at rate g .

$$X = Be^{gt}(N-L)^b , \qquad 0 < b < 1$$

Suppose $N = N_0 e^{nt}$, and define $\ell = L/N$,

$$x = D(1-\ell)^{b-1} e^{[g-(1-b)n]t}$$

Then if $g = (1-b)n$, we have $x = x(\ell)$, $x' > 0$

The goal of the government is to raise the standard of living of the population as a whole, and it can do this in two ways--either by offering employment at a higher standard of living in the modern sector, or by improving the quality of life in the traditional sector. This two-fold choice comes out very clearly in the country with which I am most familiar, Tanzania, where the President has alternated his development strategy from an initial concentration on modern commercial agriculture conducted on capital intensive settlement schemes, to a subsequent programme of channeling assistance to promising developments originating in the traditional sector, and providing both developmental and social services to the traditional sector. In this model I shall try to capture this two-fold choice by allowing the government to make transfers to the traditional sector which will affect the well-being of the peasantry, though not their production. Per capita consumption in the traditional sector will be y , greater than or equal to x . We shall be interested in whether the government's development strategy should allow y to rise above x before the traditional sector is fully absorbed; whether, in other words, the modern sector should subsidise the subsistence sector.

The consumption of modern sector employees is z , (which will be closely related to the wage paid).

Workers are only prepared to migrate to the urban sector when their expected wage there is at least as high as the consumption they would have enjoyed in the traditional sector, which, given the collectivist ethic prevailing there, is equal to the _average_ consumption per head. Thus $z \geq y$.[5] Also the modern sector will find it necessary either to pay at least the legal minimum wage, or a wage adequate to ensure reasonable nutrition standards. Thus $z \geq \gamma$ also.

We can summarize these constraints as follows:

$$z \geq \text{Max} \ (\gamma, \ y)$$

$$y \geq x(\ell)$$

Diagram I shows the relation between the price of labour and the level of modern employment.

The diagram also defines ℓ^* , the fraction of the labour-force employed for which $\gamma = x(\ell)$. For $\ell < \ell^*$, the industrial wage is not related to rural consumption, and consequently, changes in rural population do not directly affect the price of labour. For $\ell > \ell^*$ this is no longer true and the ability of the government to act as a monopsonist in the labour market becomes important. This point will be elaborated later.

All investment is allocated to the modern sector, and the equation governing capital formation is

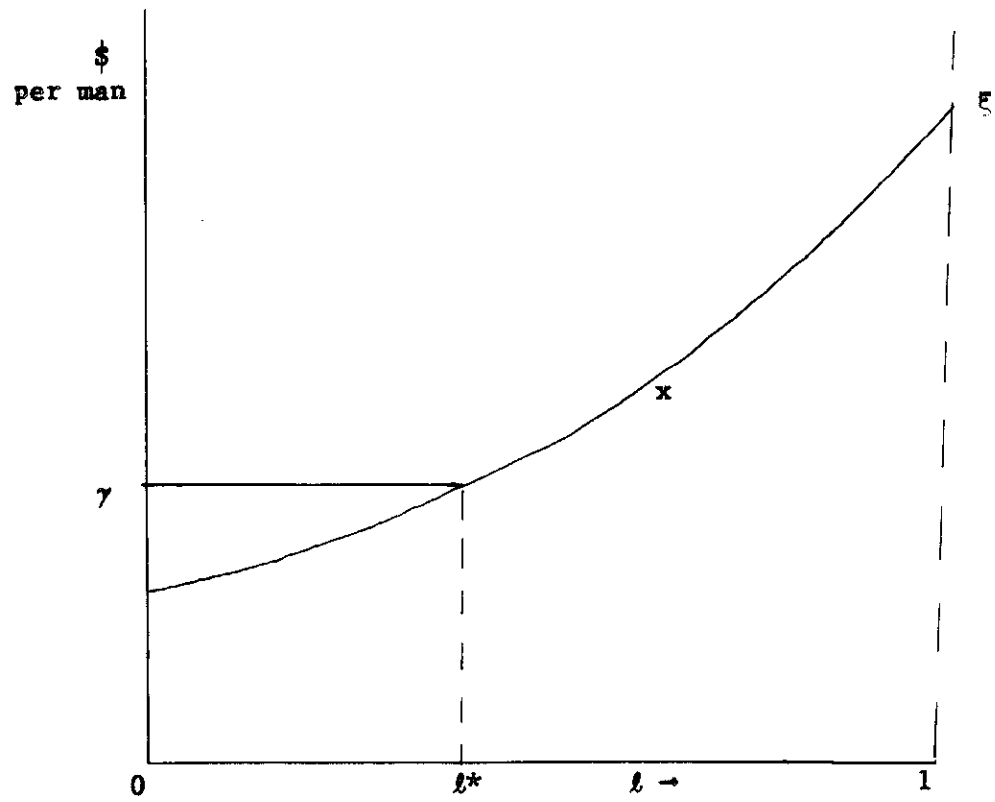$$\dot{K} = f(K, \ L) + R - \alpha K - zL - (N - L)(y - x)$$

Diagram I.   **Supply price of labour**

where α is the rate of depreciation of capital, zL is the industrial
wage bill, and $(N - L)(y - x)$ is the transfer to the rural sector. Divide
by $N = N_o e^{nt}$, and put $\ell = L/N$, $k = K/N$.

$$\dot{k} = f(k, \ell) - \alpha k - nk - z\ell - (1 - \ell)(y - x) + R/N .$$

As a first approximation suppose that R/N depends only on ℓ , and
$R/N = r(\ell)$.

The problem can now be stated formally. The government wishes to
choose values for ℓ , z and y to maximize the present discounted value
of total utility:

$$\text{Max} \int_0^\infty \{Lu(z) + (N-L)u(y)\}e^{-\rho t}dt .$$

$u(.)$ is a concave function (to reflect the diminishing marginal utility of
consumption imputed by the planner,) and ρ , the rate of pure time preference,
reflects the preference the planner has for the welfare of the present as
opposed to future generations. It is clearly appropriate to take account of
the traditional sector, in evaluating alternative programmes, although there
may be some argument as to the appropriateness of assuming the same utility
function for both sectors. Whether we should take total utility as opposed
to the utility of the representative man is essentially a decision about the
rate of pure time preference, ρ , and since I am avoiding any discussion
of the political process which would select the form of u and ρ , I shall
not take any stand on this issue.[6]

This can be transformed and more fully stated as follows:

$$\text{Max } {}_0\!\int^\infty \{\ell u(z) + (1 - \ell)u(y)\}e^{-\delta t}dt$$

Subject to

$$z \geqq \text{Max } (\gamma, y) \qquad \delta = \rho - n$$

$$y \geqq x$$

$$\dot{k} = f(k, \ell) - (\alpha + n)k - z\ell - (1-\ell)(y-x) - r(\ell) \quad .$$

The most intuitively appealing way to handle this maximization is to find prices for consumption and investment at each instant that allow us to take nyopic maximizing decisions. Let us define the present value (in utility units) of G.N.P. at time t to be H

$$H = \{\ell u(z) + (1-\ell)u(y) + \lambda\dot{k}\}e^{-\delta t}$$

where λ is to be interpreted as the price of capital/head. For convenience define $[\ell u(z) + (1-\ell)u(y)]e^{-\delta t}$ to be W, and $\lambda e^{-\delta t}$ to be p, a transformation which removes the explicit time dependence from H, which is now just equal to $W + p\dot{k}$. We will now find the optimal path (under conditions to be made more precise in the theorem below) if we choose values for the control variables ℓ, z and y (which from now on we will signify by a vector $\underline{u}$) to maximize H at the price p, and we can find the values of p from the arbitrage equation $\dot{p} + \frac{\partial H}{\partial k} = 0$. This in effect says that if we allow for capital gains correctly, "National Income" cannot be increased by varying k, for N.I. $= H + \dot{p}k$, where $\dot{p}k$ is the instantaneous capital gains.

A more mathematically oriented derivation can be constructed using

the techniques of dynamic programming, but it should be sufficient for our

purposes to verify that the proposed approach will yield the desired solution.

This is the content of the theorem, which also specifies the required

assumptions. The proof is essentially based on the concept of a separating

hyperplane, and is standard.[7]

Theorem

Suppose $W + pF = H(k, \underline{u})$ is a concave function of k, and $\underline{u}$,

then if $k*(t)$, $u*(t)$ and p_t are such that:

(i) $\dot{k}* = F(k*, \underline{u}*)$

(ii) $\dot{p}(t) + \frac{\partial H}{\partial k} (k*, \underline{u}*) = 0$

(iii) $H(k*, \underline{u})$ is maximized at $\underline{u}*$

(iv) $\lim_{t \to \infty} p_t k_t = 0$

Then $(k*, u*)$ is an optimal programme in the sense that there

exists a T_o, such that $\int_0^{T_o} [W(k*, \underline{u}*) - W(k, \underline{u})]dt \geqq 0$ for $T > T_o$.

Proof

Let $\underline{\xi}$ be $(k, \underline{u})$. Then, since H is concave in $\underline{\xi}$ we have

$$H(\underline{\xi}^*) - H(\underline{\xi}) > \nabla \underline{\xi}* \cdot (\underline{\xi}* - \underline{\xi})$$

But $\nabla \underline{\xi}* \cdot (\underline{\xi}* - \underline{\xi}) = - \dot{p} (k* - k) + \frac{\partial H}{\partial u} (k*, \underline{u}*) \cdot (\underline{u}* - \underline{u})$

and by the maximization of H wrt $\underline{u}$: $\frac{\partial H^*}{\partial u} \cdot (\underline{u}* - \underline{u}) \geqq 0$.

$$\therefore \quad H(\underline{\xi}^*) - H(\underline{\xi}) > - \dot{p}(k^* - k)$$

or $\quad W(k^*, \underline{u}^*) + p\ F(k^*, \underline{u}^*) + p\ k^* > W(k, u) + p\ F(k, u) + \dot{p}\ k$.

Integrate by parts

$$\int_0^T [W(k^*, \underline{u}^*) - W(k, \underline{u})]dt > p_T(k_T - k^*_T)$$

$$\therefore \quad \underset{T\to\infty}{\text{Lim}} \int_0^T [W(k^*, \underline{u}^*) - W(k, \underline{u})]dt > 0$$

Thus (k^*, u^*) is optimal QED.

We replace p in the theorem by $\lambda e^{-\delta t}$, λ must satisfy the equation derived from conditions (ii) and (iv):

$$\lambda(f_k - n - \alpha) = -\dot{\lambda} + \delta\lambda$$

or $$\dot{\lambda} = \lambda(\rho + \alpha - f_k) \quad \dots\dots\dots\dots\dots\dots(1)$$

Also $$\underset{t\to\infty}{\text{Lim}} (\lambda k e^{-\delta t}) = 0 \quad \dots\dots\dots\dots\dots\dots(1a)$$

k must satisfy the equation

$$\dot{k} = f(k, \ell) - (n + \alpha)k - \ell z - (1-\ell)(y - x) + r(\ell) \quad \dots\dots\dots(2)$$

The strategy is now to show that the problem satisfies the conditions of the theorem, and to construct a phase diagram to show the relationship between k and λ. This will enable us to calculate the momentary GNP for any value of k, and also the correct choice of the policy instruments ℓ, z and y. The constraints may or may not be binding, and they can be handled by Lagrange multipliers. It is convenient to partition the phase diagram into six zones according to which constraints are binding, and accordingly I define a Lagrangian, L :

$$L = He^{\delta t} + \mu(y - x) + \nu(z - \gamma) + \omega(z - y) + \rho(1 - \ell) + \theta(\ell^* - \ell) \quad ..(3)$$

Each of the langrangian multipliers will obey the following inequalities:

$$\mu \geq 0 \qquad \mu(y - x) = 0 \ , \ \text{etc.}$$

The first three multipliers, μ , ν , ω are straightforward, and correspond to the savings constraints. The last two, ρ and θ , require some further comment. Clearly ℓ cannot exceed unity, and when $\ell = 1$ the traditional sector vanishes. We require that it then cease to influence policy--hence at this point $\rho \geq 0$. At the point $\ell = \ell^*$ there is a significant discontinuity in the determination of the price of labour, and in this model the government can exercise monopsony power only for $\ell \geq \ell^*$. The magnitude of θ at $\ell = \ell^*$ gives a measure of the extent of this monopsony power. I shall now describe the zones, or stages of development, more closely. In each case the proof that $H + \rho\dot{k}$ is concave in $\underline{u}$ is left to the appendix; it is clearly concave in k .

Stage 1. **Initial Austerity**

The government maximizes the rate of reinvestment by holding y and z to their minimum values. The urban sector is small (i.e. $\ell < \ell^*$).

$$y = x \ , \ z = \gamma \ , \ \mu \geq 0 \ , \ \nu \geq 0 \ , \ \omega = \rho = \theta = 0 \ .$$

Condition (iii) of the theorem implies:

$$\frac{\partial L}{\partial y} = (1-\ell)\{u'(y) - \lambda\} + \mu = 0$$

or, since $\mu \geq 0$, $y = x$:

$$\lambda \geq u'(x)$$

Also: $\frac{\partial L}{\partial \ell} = \lambda(f_\ell - \gamma) + u(\gamma) - u(x) + (1-\ell)\frac{dx}{d\ell} u'(x) = 0$.

Denote the marginal product of labour in the traditional sector as m , then

$(1-\ell)\frac{dx}{d\ell} = x - m > 0$.

Thus $f_\ell = \gamma - [\frac{u(\gamma) - u(x) + (x-m)u'(x)}{\lambda}]$ ....(4)

This is the accounting wage rate to use in the choice of the capital
intensity of industrial investment, or equivalently, the level of employment
ℓ . If we know λ as a function of k , then $\ell = \ell(k, \lambda)$ is also
specified by this relation.

As an illustration of the technique used in the actual computation
of ℓ , graph the left and right hand side of (2) as functions of ℓ for
given k , λ :
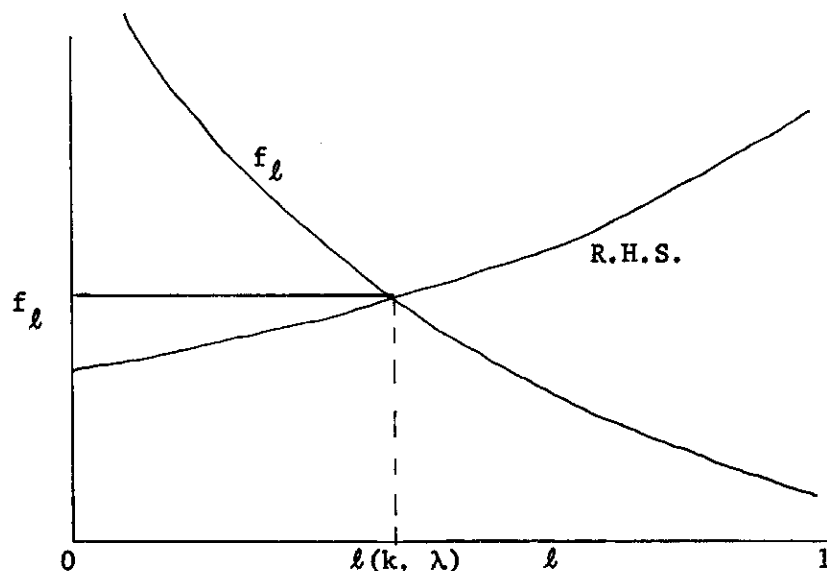
It is easy to see the effect on ℓ of varying k or λ .

Stage 2. Rural Socialism

Since we have assumed the same utility function for both sectors (benevolent indivudualistic socialism) and since the urban wage is constrained to be higher than the rural income, or at least equal to it, if the government reaches the point at which savings cease to be constrained whilst the modern sector is still small, then it may be desirable to distribute any of the surplus left over after meeting optimal investment requirements to the peasantry first, and only later to allow urban wages to rise. This is a council of idealism and political naivity, since invariably the urban proletariate has greater political power than the peasantry, but in the normative world of ideal planning it is nonetheless interesting to make it. The form in which this consumable surplus is distributed to the peasants is left unspecified--we can imagine it distributed in the form of education, health services, and community development expenditures, higher prices for the marketed surplus, or subsidised manufactureds. The only stipulation is that it should not affect rural production significantly, or at least should not be perceived by the government to do so.

$$z = \gamma \geq y \geq x \ , \ \ell \leq \ell* \ , \ \nu \geq 0 \ , \ \theta \geq 0 \ , \ \mu = \omega = \rho = 0 \ .$$

As before $\dfrac{\partial L}{\partial y} = (1 - \ell)[u'(y) - \lambda] \ ; \ \dfrac{\partial^2 L}{\partial y^2} \leq 0$

$$\therefore \qquad\qquad\qquad \lambda = u'(y)$$

$$\frac{\partial L}{\partial \ell} = u(\gamma) - u(y) + \lambda[f_\ell - \gamma + y - x + (1 - \ell)\frac{dx}{d\ell}] - \theta = 0$$

$$\therefore \qquad f_\ell = \gamma - y + m - [\frac{u(\gamma) - u(y) + \theta}{u'(y)}] \quad \dots\dots\dots\dots\dots\dots\dots\dots(5)$$

Stage 3. <u>Monopsony power exerted</u>.

If stage 2 is not reached before $\ell = \ell^*$, then the government will hold employment at this level until the now suddenly increased marginal cost of allowing <u>all</u> wages to rise has been exceeded by the gradually rising marginal product of labour.

$$z = x = y = \gamma \, , \quad \ell = \ell^* \, , \quad \mu \geqq 0 \, , \quad \omega \geqq 0 \, , \quad \theta \geqq 0 \, , \quad \nu = \rho = 0 \, .$$

Again

$$\frac{\partial L}{\partial y} = (1-\ell)\{u'(\gamma) - \lambda\} + \mu - \omega \geqq 0$$

$$\frac{\partial L}{\partial z} = \ell(u'(\gamma) - \lambda) + \omega \geqq 0$$

Adding equations eliminates ω, and since $\mu \geqq 0$,

$$\lambda \geqq u'(\gamma) \, .$$

Similarly

$$f_\ell = \gamma - \frac{(\gamma - m^*)u'(\gamma) + \theta}{\lambda} \quad \ldots \ldots \ldots \ldots \ldots (6)$$

Stage 4. Urban austerity

If the savings constraints are still binding for $\ell > \ell*$, the situation is as follows:

$$y = x = z \, , \, \ell > \ell* \, , \, \mu \geq 0 \, , \, \omega \geq 0 \, , \, \rho \geq 0 \, , \, \nu = \theta = 0$$

$$\frac{\partial L}{\partial z} = \ell \{u'(z) - \lambda) + \omega = 0$$

$$\therefore \qquad\qquad \lambda \geq u'(x)$$

$$f_\ell = m + \{1 - \frac{u'(x)}{\lambda}\}\frac{dx}{d\ell} + \frac{\rho}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots(7)$$

when θ is such that f_ℓ in Stage 3 has risen to the value, (with $\rho = 0$), the economy enters stage 4, i.e. when

$$\theta = \ell* \frac{dx}{d\ell*} (\lambda - u'(\gamma))$$

Stage 5. Unconstrained investment

Eventually the surplus is sufficiently large for the government to permit some of it to be consumed. This may well happen before the traditional sector has vanished, i.e. for $\ell < 1$. Then:

$$z = y > x \, , \, \ell < 1 \, , \, \omega \geq 0 \, , \, \mu = \nu = \rho = \theta = 0$$

[Note: the θ constraint ceases to be relevant, since the government chooses not to exert monopsony power.]

Differentiating L with respect to z and y shows $\omega = 0$ and $\lambda = u'(y)$. As we would expect the price of capital has now fallen to the price of consumption and we need not distinguish between the alternative uses of output.

Also $\qquad\qquad\qquad\qquad f_\ell = m$ ....(8)

Thus the marginal products in the two sectors are equated, as we would expect.

Stage 6. <u>Complete moderization</u>

Finally, capital has accumulated sufficiently to permit the entire labour force to be employed in the modern sector at a per capita consumption greater than ξ ($= x(1)$).

Then: $z = y \geq \xi$, $\mu = 0$, $\nu = 0$, $\omega \geq 0$, $\rho \geq 0$. For maxima, since L is concave in y , z:

$$\frac{\partial L}{\partial y} = (1-\ell)[u'(y) - \lambda] - \omega' = 0$$

$$\frac{\partial L}{\partial z} = \ell[u'(z) - \lambda] + \omega = 0$$

$$\therefore \qquad\qquad \omega = 0$$

$$\underline{\lambda = u'(y)}$$

Also $\qquad\qquad\qquad f_\ell = m(1) + \dfrac{\rho}{\lambda}$ ....(9)

It is possible for the economy to pass through various sequences, such as 1, 3, 4, 6 or 1, 2, 5, 6, but we now have adequate information to map the zones on the phase diagram. The optimal path in the phase diagram

will be described as a trajectory satisfying the differential equations for $\dot{k}$ and $\dot{\lambda}$, and the terminal conditions. In order to study the distinctive nature of this trajectory we examine the stationaries, piecing the various zones together. Their form is justified in the appendix.

The stationaries will have the form illustrated in the phase diagram below, and the signs of $\dot{k}$ and $\dot{\lambda}$ in each of the four parts of the phase diagram that they define are shown by arrows. It can then be seen that E is a saddle point,
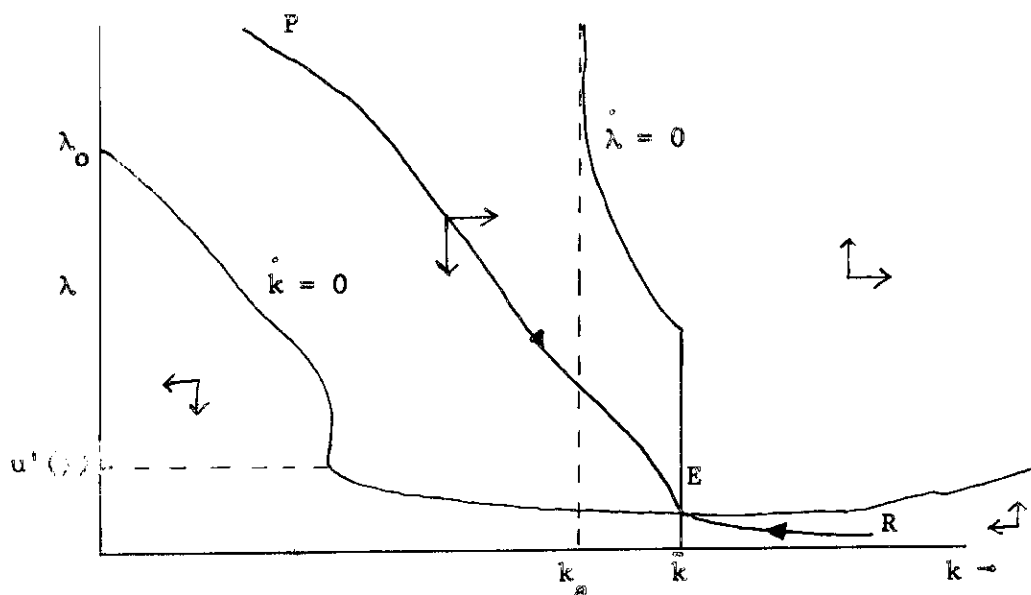
PER is the uniquely defined path satisfying all the equations, and at E both λ and k have finite values, so that $\lim_{t \to \infty} (\lambda k e^{-\delta t}) = 0$, as required.

Thus PER describes the optimal strategy. At time zero, K_o and N_o are known, so therefore is k_o, and λ_o can be read off the phase diagram. The solution of equation (2) gives both the level of employment in the modern sector, and also the accounting wage rate (f_ℓ). The accounting

discount rate is derived either as $f_k(k, \ell)$, or, equivalently, as

$[f(k, \ell) - \ell \, f_\ell(k, \ell)]/k$. A few comparitively simple extensions can be

made to the basic model without affecting its structure significantly, and

some suggestions are illustrated below.

The distribution of income.

The assumptions made above imply that the government has such

power over the wage rates in the modern sector that the consumption of all

employees can be held to the supply price if desired. Of course, this does

not mean that all qualities of labour will have the same accounting price,

but it must be admitted that this assumption has an unrealistic utopian

ring to it. It is, however, a simple matter to modify the model to take

account of a plausibly inegalitarian modern sector after-tax income dis-

tribution. Suppose the income distribution is described by the density

function $\Phi(y)$; when $\Phi(y)\delta y$ is the number of income recipients whose

income lies between y and $y + \delta y$. Suppose the minimum industrial wage

is z (determind as before) and all incomes are proportional to z . Pro-

vided the utility function $u(y)$ is homogenous, (e.g., isoelastic,

$u(y) = -y^{-\nu}$) , and the density function is homogenous of degree -1 in

y , then two constants β_1 and β_2 can be found so that total industrial

utility is $\ell\beta_1 \, u(z)$, and total consumption is $\ell\beta_2 z$, where $\beta_2 > \beta_1 > 1$.[8]

The equation for $\dot{k}$ and f_ℓ are readily modified, for example, in stage 3

the conditions become

$$u'(\gamma) \lesseqgtr \left(\frac{1+(\beta_2 - 1)\ell*}{1+(\beta_1 - 1)\ell*}\right) \lambda$$

$$f_\ell = \beta_2 \gamma + \left[\frac{(\beta_1-1)u(\gamma) + (\gamma-m*)u'(\gamma) + \theta}{\lambda}\right]$$

and in the last stages $\lambda = \dfrac{\beta_1}{\beta_2} u'(y)$, which we can take as the "representative"

price of consumption.

Technical progress.

Provided that the technical progress is of a simple kind it can readily be accommodated. If it is Harrod-neutral at rate g , and if γ , $r(\ell)$, and $x(\ell)$ also increase at rate g , and if the utility functions are isoelastic of degree $\nu(>0)$, then all the equations can be transformed to starred values thus:

$$k* \qquad = ke^{-gt}$$

$$z* \text{ etc.} = ze^{-gt}$$

$$\lambda* \qquad = \lambda e^{-(\nu+1)gt}$$

The equations for the trajectory then becomes:

$$\dot{\lambda}* = \lambda*\{\rho + \nu g + \alpha - f_{k*}(k*,\ \ell)\}$$

$$\dot{k}* = f(k*,\ \ell) - (n + \alpha + g)k* - \ell z* - (1-\ell)(y* - x*) + r*(\ell)$$

Thus the effects are formally identical to changing the values of n to $n + g$, and ρ to $\rho + \nu g$. The terminal conditions are weakend as well.

Taxation

The original model has provision for the only rational taxation
--taxing the traditional sector. The government is already presumed to
control both wages and profits in the modern sector, and so additional tax
powers there are meaningless. I shall show first of all how general the
original formulation is, and then point to some further generalisations.
Suppose the government can only tax goods which pass between the two sec-
tors, and cannot tax total production--a situation common in Africa where
land tenure is rare and shifting agriculture is practiced. It is easiest
to imagine this taking place through government organized marketing boards,
and their capacity to tax has long been recognized. (See Helleiner [7]
for a particularly good illustration of the principles and practice.) Al-
ternatively, if the government levies indirect taxes on either rice, or
the products of the modern sector, and also manages the exchange rate, the
same effect can be achieved.

Suppose also that gross output is insensitive to the tax rate,
which only affects the proportion of output traded. This is plausible where
only a small fraction of production ever reaches the market. I am interested
in finding an expression for the maximum tax revenue that can be derived
from the traditional sector. Let p be the fraction of the accounting
price paid to farmers for their product, so that the tax rate is 1-p .
Define m as sales of rice per farmer, then total revenue, R , will be

$$R = (N - L)(1 - p)m(p, x)$$

and
$$r = \frac{R}{N} = (1-\ell)(1 - p)m(p, x) \; .$$

At any moment the value of p which maximises R , p* , is a function
of x alone.[9] Thus r can be rewritten as

$$r = (1-\ell)(1 - p*(x))m(p*(x), \ x)$$

or $r = r(\ell)$, as before. We should also redefine $\bar{x}$ to be the equivalent
consumption as follows:

$$u = u(\bar{x}) = \psi(p, \ px) = \psi(1, \ \bar{x})$$

where u is utility, $u(\cdot)$ is the direct utility function, and $\psi(\cdot, \ \cdot)$
is the indirect utility function.

Naturally the government will not wish to exercise continuing
complete monopoly power over the peasantry, but we have allowed for that
by permitting consumption to rise above $\bar{x}$ as development proceeds.

Poll taxation (an important source of revenue in some countries)
can obviously be handled very simply, as can conventional tithes on gross
output.

Taxation of the private sector

So far I have assumed that the modern sector is completely state-
owned, but many countries have significant private owned manufacturing capa-
city. It is probably true that these entrepreneurs do not reinvest the
entire surplus, and that it is not possible to entirely tax away the non-
reinvested surplus -- if it were true there would.be no economic difference
between the mixed economy and a socialist economy.

Presumably, then, the government's power to tax profits is limited by its fear that the entrepreneur will reduce his rate of reinvestment. What then is the optimal tax strategy?

We assume that entrepreneurs take the supply price as the wage rate, w, and maximise profits at this wage, that they both invest and consume the surplus, and that there is some taxable capacity which can be taken as a function of profits. Their presence may have political implications, felt economically on the level of taxable surplus that can be raised, and the direction of investment, and possibly even its level, but these considerations will be ignored here.

$Y = F(K, L)$, $F_L = w$, therefore $K/L = k = k(w)$, and $y = y(k) = y(w)$

$P = Y - wL$ profits

$I = \overset{o}{K} = s(t)P$ private investment

$T = t \cdot P$

where t is the profits tax rate, T is government revenue. Utility produced by the private sector is taken as $Lu(w)$ to a first approximation, ignoring the consumption of the capitalists on the principle that this is offset by the political power they weild. They are tolerated solely for their ability to create employment and yeild government surplus.

I shall make a number of simplifying assumptions, which can be heuristically justified by the insensitivity of the numerical results to similar simplifications. Suppose that the wage rate is static for the period in question, and that supply of labour is elastic at wage w. Then

the profit maximising choice of technique for the private sector is constant, say with capital-labour ratio $k*$. Suppose that the government must make a once and for all choice of t to maximise the Present Discounted Value of tax receipts.

Now $I = s(t)(y* - w)L$, and $\dot{K} = k*\cdot\dot{L} = I$, thus K , L and P , profits all grow at rate $g = s(t)(y* - w)/k*$.

Discounting at an interest rate r , the PDV of tax receipts is $tP_0/(r - g)$, which is maximised for

$$r + \frac{t(y* - w)}{k*} \frac{ds}{dt} = 0$$

We can go slightly further in constructing the savings function $s(t)$, if we make the assumption that entrepreneurs maximise the discounted utility of consumption:

$$W = \int_0^\infty u(C)e^{-\delta v}dv \ .$$

Let $u(x) = -x^{-\nu}$, $C_v = (1 - s - t)\cdot P_0 e^{gv}$. Thus $W = \dfrac{[(1 - s - t)P_0]^{-\nu}}{\nu g + \delta}$.

If the entrepreneur believes that he cannot affect the tax rate by his savings behavior (essentially the competitive assumption), then if he makes a once and for all decision about the savings rate this utility W is maximised for

$$0 = \frac{\nu}{1 - s - t} - \frac{\beta}{\nu s \beta + \delta} \quad \text{where} \quad \beta = (y* - w)/k*$$

i.e.
$$s(\nu^2\beta + \beta) + \delta\nu - \beta(1 - t) = 0$$

or
$$s = \frac{\beta(1 - t) - \delta\nu}{\beta(\nu^2 + 1)}$$

$\therefore$
$$\frac{ds}{dt} = - \frac{1}{\nu^2 + 1}$$

and
$$\frac{t\beta}{\nu^2 + 1} = r \quad \text{for optimal} \quad t$$

i.e.
$$t = \frac{(\nu^2 + 1)r}{\beta}$$

$$s = \frac{1 - \frac{\delta\nu}{\beta}}{1 + \nu^2} - \frac{r}{\beta}$$

Foreign Borrowing

The present value of \$1 of investment at time t is $\lambda_t e^{-\delta t}$, and for small amounts of foreign borrowing its desirability can be assessed by evaluating the present discounted cost of the loan amortisation using the accounting prices, i.e. if repayment in year t is A_t, size of loan is L, period is T then borrow if

$$\lambda_0 L > \int_0^T A_t \lambda_t e^{-\delta t} dt .$$

If the cost of borrowing depends on the size of the contracted debt, then A_t is the marginal cost of amortizing the loan.

If loans are a significant fraction of total investment (and recently the net flow of resources from developed to underdeveloped countries

was estimated at $9 billion per year, or one quarter of their gross invest-
ment, see Chenery and Strout [4]), then they should be explicitly included
in the analysis. The difficulty then is to decide what factor limits the
rate of borrowing. One plausible hypothesis is to suppose that gross bor-
rowing is proportional to taxable capacity, and that loans are amortized
at a constant rate. Let taxable capacity (per capita) be s . Then the
equation $\dot{k} = s$ is modified to read $\dot{k} = (1 + \mu)s - \omega k$, where μ is
the ratio of borrowing to taxation, and $\omega = a\mu/1+\mu$, where a is the
annual amortization per $ of loan. This equation will hold until the P.D.V.
of a loan falls to zero -- a point in time readily found by a few iterations
of the time path of λ .

Foreign investment can be treated in exactly the same way.

The Marketed Surplus Problem

In several recent papers (particularly, Bose [2], Dixit [6], and,
in a geometric spirit, Findlay's early paper [7]), considerable attention
has been devoted to the problem of extracting enough food from the agri-
cultural sector to feed the urban workers. The models used are autarkic,
in contrast to the model of this paper which assumes perfect world trade,
and some writers permit the government to influence the supply of food by
rural investment. How should this work be reconciled with the approach
adopted in this paper?

The first point to make is that the foreign trade situation as-
sumed is crucial. It seems reasonable to assume that _food_ crops are sold

on a competitive market, though some of the other cash crops may have a marginal revenue well below the average revenue. This is not serious, the accounting price to use is the M.R., although it will modify the function giving the marketed surplus. [It will still be a function of x and ℓ , though.]

Transport costs will only be important if the country alternately exports and imports food, and this may happen with fluctuating weather. In this case an optimal stockholding programme must be worked out, and adequate reserves of food or foreign exchange held. As we are allowing ourselves to extract the maximum (food) surplus from the traditional sector, the pure terms of trade problem can be deemed solved. The remaining difference in treatment lies in the possibility of influencing rural output by allocating investment funds to the rural sector.

The distinctive feature here is that presumably the government has incomplete control over the increase in output engendered by this investment--which we can picture as infrastructural investment--otherwise we could consider it to be in the modern sector. The government is interested in the increased surplus accruing to itself as a result of the investment, and it is not too difficult to solve for the allocation of investment between sectors.

I will conclude by commenting that the marketed surplus problem is really a catch-all category for a variety of other problems--the modern sector failing to produce exportable goods, failure on the part of the government to provide contingency stocks of foreign exchange, long gesta-

tion periods for agricultural investment, failure to search for profitable agricultural projects, and a variety of others. Conceptually it is not clear that it constitutes a separate problem, and the tacit assumption of autarky is dangerous in a world where governments need little encouragement to be isolationist.

Numerical sumulation of the optimal path and accounting prices.

It is relatively simple to integrate the pair of equations which govern the motion of k along the optimal trajectory. For this purpose a modified Runge-Kutta-Gill 4th-order integration routine was used, and the equations were integrated from a point very close to E , whose co-ordinates can be derived from the stationaires: $f_k(\bar{k}, 1) = \rho + \alpha$,

$\bar{\lambda} = u'(\bar{y})$ where $\bar{y} = f(\bar{k}, 1) - (n + \alpha)\bar{k}$. The start of the integration was then to be $(\bar{k}(1-\epsilon) , \bar{\lambda}(1+\epsilon))$ where ϵ was a small number (0.001, usually), and the differential equations (1) and (2) were reversed in sign to trace out the trajectory in the reverse direction back to k = 0 . The only complication was to specify the form of the equations depending on the stage, and this was done by switching to different sub-programmes when values of ℓ , λ or k reached bounding values, e.g., $\lambda > u'(y)$, $\ell < 1$, etc.

Values for ℓ are found from the equations giving the marginal product of labour, (which, given k , is a function of ℓ alone) as described in the text. A simple iteration converges rapidly to the value of ℓ .

As the programme is set up at the moment it can handle a perfectly general specification of the production functions of the modern and traditional sectors--analytical convenience is not necessary, provided that numerical expressions for the total and marginal products in the two sectors can be expressed. The algorithm used is described in detail in Ralston and Wilf, "Mathematical Methods for Digital Computers", New York, Wiley 1962; Ch. 9, and it takes about 30 seconds of computation time on the IBM 7094 to sketch out the whole phase diagram and compute all the interesting parameters--taking about 300 integration steps to do so.

Details of the numerical calculations

Several specifications of the advanced sector were tried, although for all experiments the traditional sector had the quadratic form:

$$x = a_1 + a_2 \ell + a_3 \ell^2$$

and the utility function chosen was isoelastic:

$$u(c) = -c^{-\nu} \ , \ \nu > 0 \quad .$$

The results do not depend very sensitively on the parameters a_i , which is a sufficient justification for using such a simple formulation for the function $x(\ell)$. The simplest specification for the modern sector was a modified Cobb-Douglas production function:

$$f = (k + k_o \cdot \ell)^b \cdot \ell^{1-b} \quad .$$

This has no technical progress or depreciation built into it explicitly (although the parameters can be readily reinterpreted), and no taxation of the traditional sector was envisioned.

The graphs appended specify the values of the parameters b , k_o , r , n , ν , a_i and ℓ^* . They give the phase diagram, and the time paths of the level of employment, the shadow price of capital in terms of consumption in the modern sector (it is assumed that the workers save nothing.) It will readily be noticed that especially in the earliest stages, the shadow price of capital changes very rapidly, whilst the shadow wage ratio is far more stable, confirming the hypothesis of Little and Mirrlees [12].

A more ambitious experiment with the following features was then run:

$$f(k, \ell, t) = e^{gt} \left\{ a(ke^{-gt})^{-b} + (1-a)\ell^{-b} \right\}^{-1/b} - m\ell e^{gt}$$

This is a C.E.S. production function[9] with elasticity $\sigma = 1/1-b$, and Harrod-neutral technical progress at rate g . me^{gt} is the social cost per worker, and exponential depreciate at rate α was assumed.

$$x(\ell, t) = e^{gt} x ; \quad x \text{ specified above}$$

$$r(\ell, t) = \tau(1-\ell) x(\ell, t)$$

where τ is the tax rate on gross agricultural output.

Values for σ , π (the asymptotic share of profits), ρ , n , g , α , τ , ν and ℓ^* are appended with the graphs.

FOOTNOTES

[1]The literature is extensive, and often unhelpful. See Sen [14] and its bibliography, and the discussion on surplus labour in Adelman and Thorbecke [1] and its references. L.G. Reynold's in an article in Oxford Economic Papers, March, 1969 gives further references.

Several writers have tackled aspects of the problem in the same spirit as this paper, particularly Bose [2], Dixit [6], [5], Hornby [10], Marglin [13] and Stern [15].

[2]If the traditional sector is composed of land-owning profit maximizing land-lords hiring labour whose alternative employment is on communally-held land, then induced changes in the internal structure of the sector can have repercussions on the total marginal product of labour.

The relationship of the peasant economy to the modern economy may be further complicated if modern sector products are substitutes for non-traded peasant produced goods such as cloth, roofing materials for huts, etc. For an illuminating discussion of the internal organization of the peasant economy and its implications see Hymer and Resnick [9].

[3]For a more extensive discussion of the appropriate accounting prices to use refer to Little and Mirrlees [12].

[4]The relationship between output, agriculture technology and population pressure is handled in an illuminating way by Boserup [3]. Increases in agriculture productivity typically occur at the same time as the drive for industrialisation, and all I require is that the two trends of technical progress and population growth approximately offset each other.

[5]Several comments are needed here. One of the least satisfactory features of the theory of underdeveloped economies is to know what it is that determines the supply price of labour, and it is fair to say that despite the voluminous literature on the subject, not much advance has been made since Lewis' suggestion (Lewis [11]). Next, the relevant figure is that of urban consumption per income unit, measured at accounting prices. This may be calculated from budget studies, observations on tariffs and taxes and so on. In the Manual [12] a numerical example is cited in which the consumption of Pakistani workers evaluated at the official exchange rate is 62 1/2% of the rupee wage paid. Lastly, this is a normative assumption, which is clearly not in force in many countries -- and the economic consequences of rising urban unemployment are clear to see. This unemployment would seem to be the consequence of the disparity in the standards of living and opportunity between the two sectors, and an equilibrium (of sorts) is reached when the expected returns to either alternative are the same, the expectation in the urban sector including a substantial period of

unemployment. It also seems to be a consequence of independence, and in some countries is linked to the various trade unions and political pressure groups. Whether it is possible to reduce the disparities between the sectors is an open question, and clearly needs careful examination.

[6]This concept of evaluating the integral of total utility (as opposed to the utility of the representative man, or the utility of the priviledged modern sector labourer) is not fully accepted, although the arguments advanced by Meade (in "Trade and Welfare") and Mirrlees (in various papers) seem to carry more weight than the alternatives, at least when population is exogenous. N. Stern has given this choice of utility integral more extensive defense in [15].

[7]Numerous references could be given, I will choose two. A fairly non-technical discussion is provided in Koopmans "Allocation of Resources and the Price System," in "Three Essays on the State of Economic Science," McGraw Hill, 1957. A simple mathematical exposition is given in Hadley "Linear Algebra," Addison Wesley, 1961, Ch. 6, Section 6.

[8]For a treatment of the theory of income distribution, see D. Champernowne "A Model of Income Distribution," E.J., 1953, p. 318. He derived and fits the density functions

$$\emptyset(y) = \frac{\alpha N \sin \theta}{\theta y \left(\left(\frac{y}{y_0} \right)^{\alpha} + 2 \cos \theta + \left(\frac{y_0}{y} \right)^{\alpha} \right)}$$

which is homogenous of degree -1. α, N, θ are parameters, y_0 is median income, and β_1, β_2 are given for the isoelastic case by

$$(\beta_1)^{-\nu} = \int_0^{\infty} z \left(\frac{y}{z} \right)^{-\nu} \emptyset \left(\frac{y}{z} \right) dy$$

$$\beta_2 = \int_0^{\infty} \frac{y}{z} \emptyset \left(\frac{y}{z} \right) dy$$

[9]This is readily seen if we work with indirect utility functions. Let $\psi(p, y)$ be the utility of the farmer whose income is y ($= px$), for whom the price of rice $= p$, and manufacturers $= 1$.

Then $m = \left\{ x + \frac{\frac{\partial \psi}{\partial p}}{\frac{\partial \psi}{\partial y}} \right\} / p$, which is clearly a function of x and p alone.

(See Roy, R., "De l'Utilite, Contribution a la Theorie des Choix," Paris, 1943 and Houthakker, H.S., "Additive Preferences," Econometrica, April, 1960.)

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APPENDIX I CONCAVITY OF $H + p\dot{k}$

$$\frac{\partial^2 L}{\partial y^2} = (1-\ell)u''(y) \leqq 0$$

$$\frac{\partial^2 L}{\partial z^2} = \ell u''(z) \leqq 0 \ .$$

For ℓ the situation is more complex. In stage I, for example

$$\frac{\partial^2 L}{\partial \ell^2} = \lambda f_{\ell\ell} - u'(x) \frac{dm}{d\ell} + u''(x)(x-m)$$

Since $\frac{dm}{d\ell} > 0$, $x-m > 0$, all terms are separately negative,

thus $\frac{\partial^2 L}{\partial \ell^2} < 0$. Essentially the same equation applies to each stage.

APPENDIX II THE STATIONARIES AND THE PHASE DIAGRAM

The λ Stationary

This is readily calculated. Equation (1), we recall, is $\dot{\lambda} = \lambda(\rho + \alpha - f_k)$. Thus $\dot{\lambda} = 0$ when $f_k = \rho + \alpha$. As we have assumed no technical progress, this equation uniquely specifies the ratio $k/\ell = \bar{k}$. This ratio in turn uniquely specifies $f_\ell = \bar{w}$, say. If we now return to the equations for f_ℓ , and substitute for $\ell = k/\bar{k}$, we can derive an equation for $k = k(\lambda)$. Consider stages 5 and 6

(a)
$$\bar{w} = x - (1-\ell) \frac{\partial x}{\partial \ell} \quad ; \quad \ell < 1 \quad \ldots \ldots \ldots \ldots \quad (8)$$

or

(b)
$$\bar{w} = \xi + \frac{\rho}{u'(y)} \quad \ell = 1 \quad \ldots \ldots \ldots \ldots \quad (9)$$

I shall in fact assume that $\bar{w} \geq \xi$, so that the economy ultimately does wish to absorb the entire traditional sector.

Equation (9) gives a relation between ρ and y, and the λ stationary in stage 6 occurs at $\ell = 1$, $k = \bar{k}$.

The next stage to consider is stage 4:

$$f_\ell = m + \left\{ 1 - \frac{u'(x)}{\lambda} \right\} \frac{\partial x}{\partial \ell} + \frac{\rho}{\lambda} \quad \ldots \ldots \ldots \quad (7)$$

Clearly f_ℓ is continuous over the boundary between stage 4 and 6, consequently the λ stationary is also continuous. At the boundary $\rho = (\bar{w} - \xi)u'(\xi)$, $\ell = 1$. At some value of λ, with $k = \bar{k}$, $\ell = 1$, ρ falls to zero:

Then λ satisfies: $\quad \bar{w} = \xi + \left(1 - \frac{u'(\xi)}{\lambda} \right) \frac{\partial x}{\partial \ell} \bigg|_{\ell=1}$

i.e.
$$\lambda_a = \frac{u'(\xi) \frac{\partial \xi}{\partial \ell}}{\left(\frac{\partial \xi}{\partial \ell} - \bar{w} + \xi \right)} > u'(\xi)$$

where $\frac{\partial \xi}{\partial \ell}$ is shorthand for $\frac{\partial x}{\partial \ell} \bigg|_{\ell=1}$.

Thereafter ρ remains zero, ℓ is less than 1, and equation

(7) traces out the λ, k relationship. We are also interested in $\underset{\lambda \to \infty}{\text{Lim}} k(\lambda)$. This is derived from equation (8) above. Let this asymptotic value of k be $k_a \leq \bar{k}$. We note that it is very unlikely that $\ell_a = \dfrac{k_a}{\bar{k}} < \ell^*$ (for plausible values of γ, ξ) and so the λ stationary will only pass through stages 4 and 6.

We can now sketch the λ stationaries:
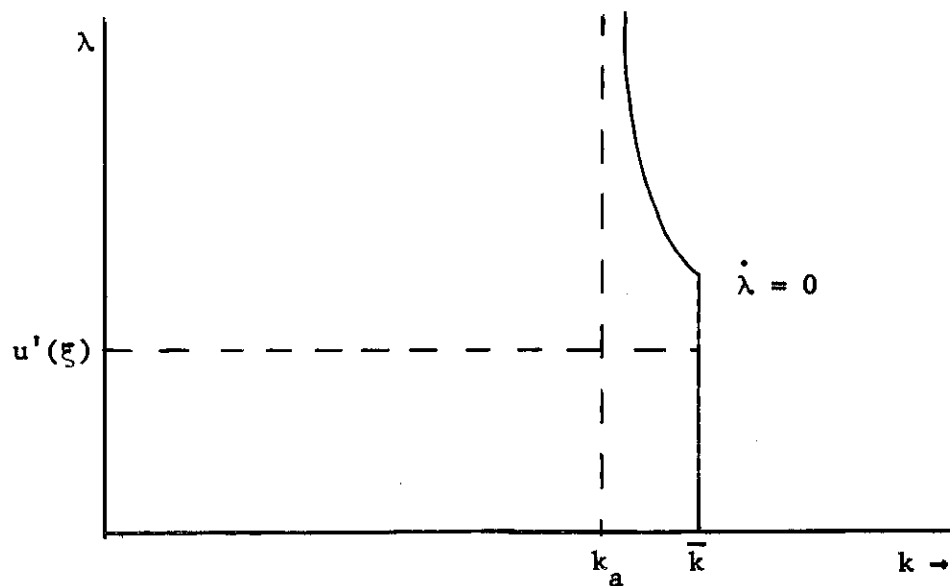
The k stationary

We look first at phase 1:

$$\dot{k} = f(k, \ell) - (\alpha+n)k - \gamma\ell + r(\ell) .$$

The argument is simplified if we temporarily assume $r(\ell) = 0$. Then

$$\dot{k} = f(k, \ell) - (\alpha+n) - \gamma\ell \quad \dots\dots\dots\dots\dots (10)$$

$$f_\ell = \gamma - \left[\frac{u(\gamma) - u(x) + (x-m)u'(x)}{\lambda} \right] \quad \dots\dots\dots (4)$$
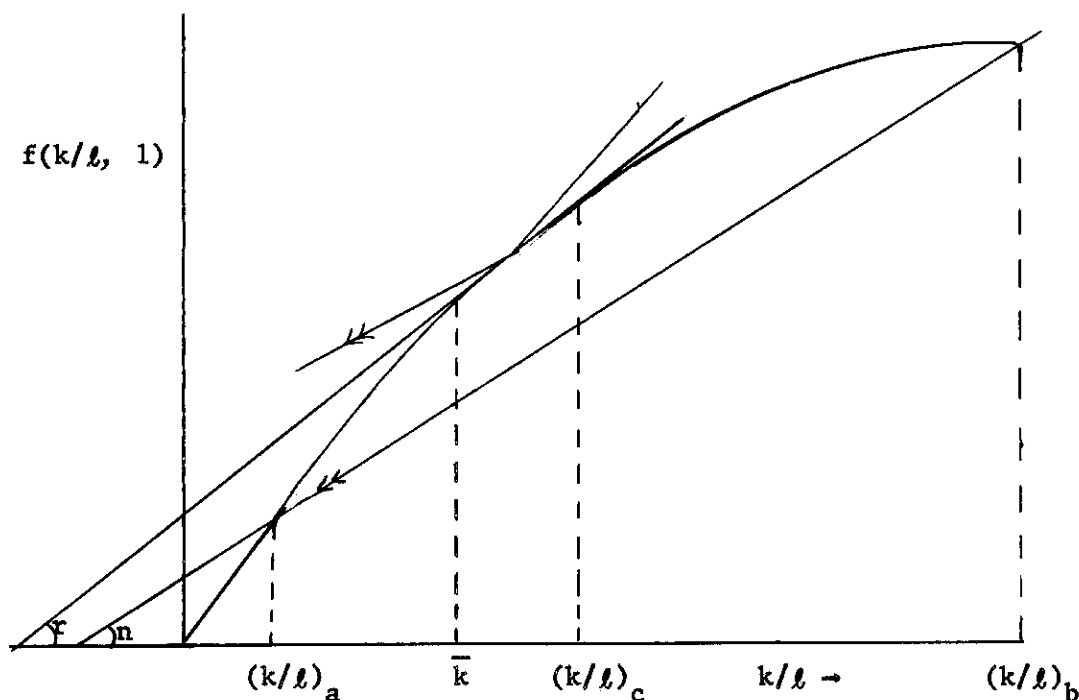
Equation (10) is homogenous of degree one in k and ℓ and can be rewritten as a function of k/ℓ. As can be seen in the diagram above, in general it will have two solutions for k/ℓ, one less than $\bar{k}$, one greater, respectively $(k/\ell)_a$ and $(k/\ell)_b$. For simplicity denote $(k/\ell)_a$ as κ, and observe that on the k stationary $k/\ell = \kappa$, constant, and so $f_\ell(\kappa, 1)$ is also constant. Further it is clear from the diagram below that $f_\ell < \gamma$. The k stationary will hit the axis when both k and ℓ are zero, and the value of λ will be:

$$\lambda_0 = \frac{u(\gamma) - u(x_0) + u'(x_0) \cdot x_0'}{\gamma - f_\ell(\kappa, 1)}$$

In general the value of λ is given by:

$$\lambda = \frac{u(\gamma) - u(x) + (x-m)u'(x)}{\gamma - f_\ell(\kappa, 1)}$$

and clearly as ℓ increases to ℓ^* the R.H.S. will fall to zero, and before this happens λ will have fallen to $u'(x)$. At this point the stationary enters stage 2.

$f(k/\ell, \ 1)$

n

$f_\ell(\kappa, \ 1)$

γ

κ

$k/\ell \rightarrow$
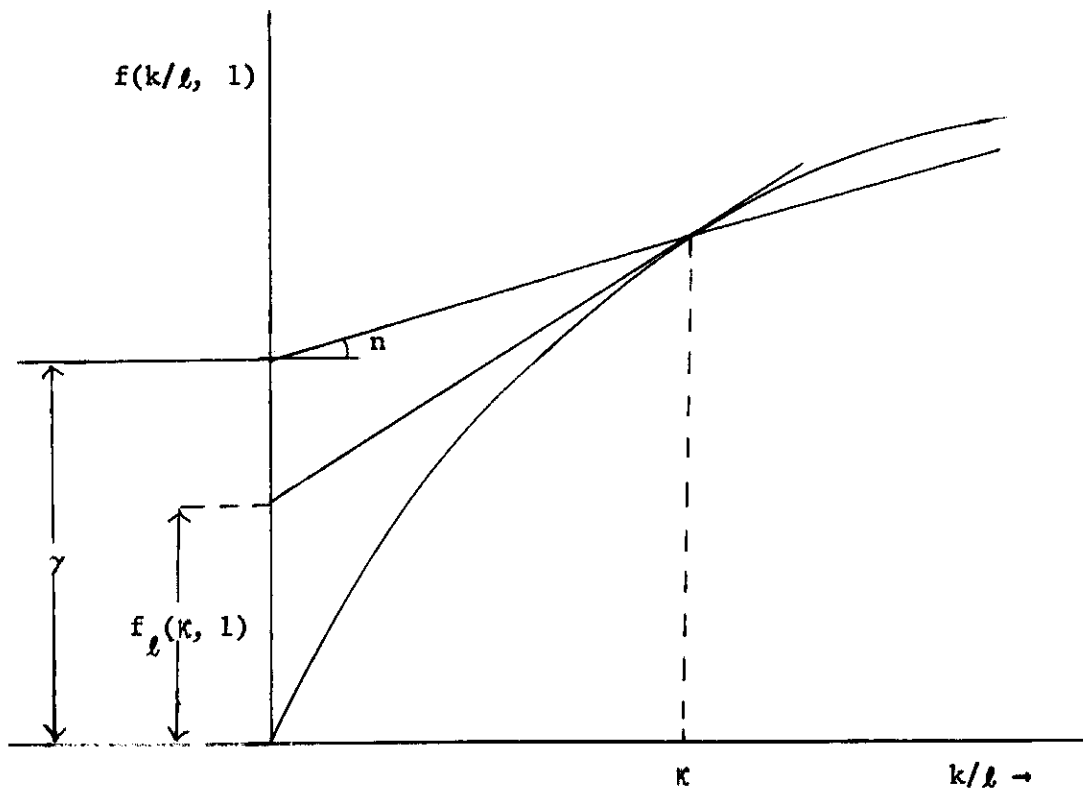
Returning to the diagram with its two values in k/ℓ we see that the k stationary lies to the left of the λ stationary, and this is true by the same argument as the above in each zone. It is also true that the k stationary has another trajectory to the right of the λ stationary (corresponding to the ratio $(k/\ell)_b$ in the above diagram. The same method can be used to describe the shape of the stationary in each zone. It is readily established that f_k and f_ℓ are continuous across all boundaries, as are the stationaries. Further, the k stationary has unambiguous slope in stage 5 and has its minimum there, at $k_c/\ell_c = f_k^{-1}(n + \alpha)$; for, we have differentiating with respect to λ on the k stationary:

$$(f_k - n - \alpha) \frac{dk}{d\lambda} + \left[f_\ell - \left(x + \ell \frac{dx}{d\ell} \right) \right] \frac{\partial \ell}{\partial \lambda} = 0$$

$$(f_k - n - \alpha) = \frac{u'(x)}{\lambda} \frac{dx}{d\ell} \frac{\partial \ell}{\partial \lambda} < 0$$

$$\frac{dk}{d\lambda} \leq 0 \quad \text{for} \quad f_k \leq n + \alpha$$

$$> 0 \quad \text{for} \quad f_k > n + \alpha$$

It is easy to see that $k_c > \bar{k}$. Simple continuity arguments show that the inclusion of $r(\ell)$ will make no substantial difference. The stationaries can now be drawn.

APPENDIX III Numerical Results

Graphs take up too much space for it to be practical to illustrate all the various computations and so I have tabulated the salient results in two tables and only   attached one set of graphs as an illustration. Table 1 identifies the parameters for each computation and Table 2 tabulates the key results. I have chosen to tabulate the duration of each stage, the boundary values for capital stock and its price in terms of consumption, and the maximum and minimum values of the ratio of the accounting price of labour to the urban consumption (the wage actually paid.)

TABLE 1

		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
Depreciation	$\delta\%$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
agricultural tax rate	$\tau\%$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
overhead costs	m/ξ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	$\ell*\%$	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30	30
relative productives	$\bar{w}/\xi$	1.5	1.5	1.5	1.5	1.5	1.5	1.5	3.7	4.6	1.55	1.62	1.89	1.5	1.5	1.5	1.5
Traditional sector ratios	a_3	1.0	1.0	1.0	1.0	1.0	2.8	0.1	1.0	1.0	1.0	0.1	1.0	1.0	1.0	1.0	1.0
	a_2	2.0	2.0	2.0	2.0	2.0	2.9	0.5	2.0	2.0	2.0	0.5	2.0	2.0	2.0	2.0	2.0
	a_1	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0	3.0
	k_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
asymptotic share of profits	b or $\Pi\%$	25	25	25	25	25	25	25	25	25	35	35	45	25	25	25	35
elasticity of substitution	σ	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.9	0.8	0.75	0.75
elasticity of utility	ν	1.0	2.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
time preference rate	$\rho\%$	3	3	3	3	4	3	3	3	3	3	3	3	3	3	3	3
rate of technical progress	$g\%$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
population growth rate	$n\%$	2	2	2.5	1.0	3.0	2	2	2	2	2	2	2	2	2	2	2
simulation number		A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P

	n%	g%	ρ%	ν	σ	b or Π%	k_o	a_1	a_2	a_3	w_ξ	ℓ*%	m/ξ	τ%	α%
Q	2	0	3	1.0	<u>0.5</u>	<u>17</u>	0	3.0	2.0	1.0	1.5	30	0	0	0
R	2	0	3	1.0	1.0	25	0	3.0	2.0	1.0	1.5	<u>10</u>	0	0	0
S	2	0	3	1.0	1.0	25	0	3.0	2.0	1.0	1.5	<u>50</u>	0	0	0
T	2	<u>2</u>	3	1.0	1.0	25	0	3.0	2.0	1.0	1.59	30	0	0	0
U	2	0	3	1.0	1.0	35	0	3.0	2.0	1.0	1.5	30	0.25	<u>2</u>	<u>2</u>
V	2	0	3	1.0	1.0	25	<u>0.05</u>	3.0	2.0	1.0	1.51	30	0	<u>0</u>	<u>0</u>

TABLE 1
(con't)

Notes:

$\overline{w}$ is the asymptotic wage rate, b and k_o refers only to the Cobb-Douglas case, Π is the asymptotic profits share in all cases, a_i are given as ratios, the absolute value can be determined from the ratio $\overline{w}/\xi$. When α , m , or τ are non zero there is some ambiguity about the definition of Π -- here it is taken to be the share which would have prevailed if all the parameters were zero.

Simulation Number	Phases						accounting wage / urban consumption / max w/c / min w/c	T years
	1	2	3	4	5	6		
	Time spent in phase (years) % terminal capital stock			price of capital				
A	6.6 0.04%		1.0 0.66% 1.0	17.5 1.12% 4.25		39.0 25.2% 1.0	1.06 0.93	25
B	6.4 0.04%		1.0 0.6% 5.7	16.0 1.0% 3.4		54.0 19.4% 1.0	1.02 .91	25
C	6.8 0.04%		1.25 0.7% 7.4	20.0 1.13% 4.5		36.5 28.3% 1.0	1.07 .93	28
D	5.2 0.4%		.08 0.6% 6.1	13.0 1.0% 3.9	0.5 18.3% 1.0	47.0 19.4% 1.0	1.03 .92	19.5
E	5.0 0.04%		1.05 0.7% 7.2	14.0 1.2% 4.2		30.0 26.2% 1.0	1.07 .93	20
F	3.0 0.02%		.85 0.3% 7.7	17.25 0.6% 4.1		39.5 25.2% 1.0	1.11 .91	21
G	17.5 0.3%		1.25 3.2% 4.4	18.5 3.75% 3.8		39.5 25.2% 1.0	1.06 .93	37
H	?		?	1.0 0.25% 1.9		51.0 1.5% 1.0	1.15 1.10	?

	1	2	3	4	5	6	w/c	T
I	?		?	0.5 0.13% 1.7		52.0 0.7% 1.0	1.11 .91	21
J	18.0 0.11%		2.2 1.6% 4.8	27.5 2.3% 3.3	2.5 25.4 1.0	55.0 28.8 1.0	1.0 .90	50
K	66+ 0.27		2.6 5% 4.2	45.25 5.64% 3.7		70.0 30.7 1.0	1.02 .97	113
L	34 0.17		2.8 2.5% 3.4	32.5 3.2% 2.6	15.2 21.1% 1.0	72.5 33.6 1.0	1.02 .86	84
M	9.8 0.08		1.25 1.3% 5.2	18.75 1.9% 3.7		35.0 28.3% 1.0	1.03 .91	27
N	13.6 0.14		1.45 2.4% 4.26	19.0 3% 3.14	0.8 30.3% 1.0	31.5 31.8% 1.0	1.1 .89	34
O	15.4 0.18		1.25 2.9% 3.8	18.75 3.6% 3.0	1.75 31% 1.0	30.0 34.5% 1.0	1.09 .87	37.2
P	?		?	26.0 5.38	12.75 29% 1.0	35.0 44.9% 1.0	1.1 0.91	?
Q	13.8 0.36		0.7 5.8% 2.7	11.25 6.7% 2.4	3.0 35% 1.0	17.0 45% 1.0	1.25 .82	29.0
R	2.8 0.02%		0.45 0.15% 18.4	21.5 0.2% 14.0		39.5 25.2% 1.0	1.06 .98	25
S	12.4 0.07%		2.25 1.85% 4.2	14.75 3.34% 2.4		39.5 25.2% 1.0	1.06 .91	29.4

Notes

        In each box under the phase number are three numbers. The first gives the time spent in the phase, the second (below the first) gives the % of the terminal capital stock per head present at the start of the phase, and the last gives the accounting price of capital in terms of consumption goods. The time in phase 6 is the time taken to reach 75% of terminal k . An empty box indicates that the phase was omitted, a query that the programme stopped prematurely.

        The eighth column gives the maximum and minimum values of f_ℓ/z attained in phases 1 - 5 , and the last column gives the time taken before the traditional sector disappears. The value of k/k_∞ in phase 1 gives some feel for the time taken in that phase. Clearly it would take infinite time if k started at zero and τ were also 0.

Conclusions

        The numerical results are encouraging in that they are fairly robust. It is comforting to see that the results are not particularly sensitive to discount rate ρ , the utility elasticity ν , and the specification of the traditional sector (a_i , ℓ^*) . As we would expect, the share of profits and the ratio $\bar{w}/\xi$ are influential, but almost without exception the accounting wage rate to use differs very little from the actual wage paid. The conclusion is that for some time at least the government ought to maximize the rate of re-investment and not make much correction to the accounting wage rate -- a result which concurs with Little and Mirrlees [12].

TIME PATHS OF ACCOUNTING PRICES FOR MODEL 0

A-15

TIME PATHS OF EMPLOYMENT AND CAPITAL STOCK FOR MODEL 0

k/k_∞

30%

$\ell\%$

100% — 20%

10%

Phase 1

Phase 3

Phase 4

k

$\ell\%$

100%

Phase 5   Phase 6

50%

ℓ^* 30%

10   20   30   40 time
         3   years

A-16