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THE "BRIDGE GAME" ECONOMY

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THE "BRIDGE GAME" ECONOMY*
(an example of indivisibilities)

by

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Most of the mathematics associated with the more general cases of production with indivisibilities than are dealt with here has been presented by Shapley and Shubik elsewhere[1] in an article dealing primarily with non-convex preference sets. That analysis however could have been applied to production.

We assume that there are n people in an economy with one unit of one producer good "time" and each with a utility function or a value for the one possible consumer good "the Bridge game."

For simplicity we assume that a Bridge game requires the participation of four players, each for one unit of time. The production function for Bridge games can be described as:

$$x = \left[\frac{t}{4} \right]$$

i.e. the output is the largest integer in the number $t/4$.

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[1] Shapley, L.S. and M. Shubik, "Quasi-Cores in a Monetary Economy with Non-convex Preferences," Econometrica, Vol. 34, No. 4, 1966, pp. 805-827.

For further simplicity we assume that the utility function for each individual is $U_i(x_i) = x_i$ where x_i is the amount of Bridge playing he obtains. In this trivial case x_i will be 0 or 1 as each individual has the time for only one game.

We do not need to assume that the individuals' values for a Bridge game are comparable or that there exists any monetary mechanism. If we did then we could state the characteristic function of the economy as below and illustrate it with a simple drawing as is shown in Figure 1.

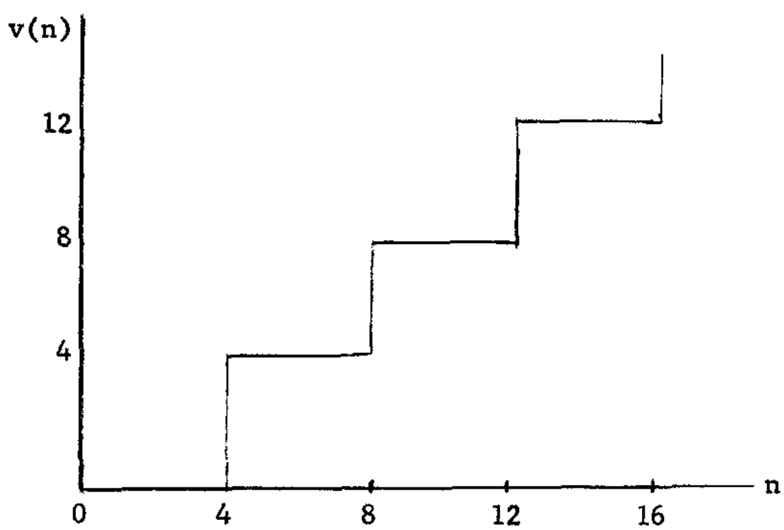

FIGURE 1

$$v(s) = 4\left[\frac{s}{4}\right] \quad \text{for all } s \text{ where } s = |S| \text{ and } S \subset N .$$

$v(s)$ may be read as "the value that can be obtained by a set of players S ." As the game is symmetric we may save ourselves a small amount of notation and use only 's' the number of players in the set 'S' as all sets of the same size have the same total value even though they may have different players.

If we do not wish to compare utilities we must use a "characterizing function" whose values must be described by a vector with a component for each member of the coalition. For clarity the characterizing function for the 5 person game is given. Here we use the following notation: $V(\overline{123})$ stands for the outcome achievable by the set consisting of players 1, 2, and 3.

$$V(\overline{i}) = (0) \qquad \text{for } i = 1, \ldots, 5$$

$$V(\overline{ij}) = (0,\ 0) \qquad \text{" all pairs } i,\ j.$$

$$V(\overline{ijk}) = (0,\ 0,\ 0) \qquad \text{" all triads } i,\ j,\ k.$$

$$V(ijkl) = (1,\ 1,\ 1,\ 1) \qquad \text{" all tetrads } i,\ j,\ k,\ l.$$

$$V(\overline{12345}) = \begin{cases} (1,\ 1,\ 1,\ 1,\ 0) \text{ or } (1,\ 1,\ 1,\ 0,\ 1) \text{ or } (1,\ 1,\ 0,\ 1,\ 1) \\ \text{or } (1,\ 0,\ 1,\ 1,\ 1) \text{ or } (0,\ 1,\ 1,\ 1,\ 1) \end{cases}$$

It is easy to observe that if we treat the Bridge game problem as an economy a price system only exists when the number of players is divisible by 4 (or trivially when the number of players is less than 4).

In the terms of game theory, the core[2] exists only for $n = 1, 2, 3, 4, 8, 12, \ldots, 4k$. For $n = 1, 2, 3$ the core is trivial, no

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[2] Shubik, M. "Edgeworth Market Games," in A.W. Tucker and R.D. Luce, Contributions to the Theory of Games, Vol. IV, pp. 267-278, 1959.

Debreu, G., and H. Scarf, "A Limit Theorem on the Core of an Economy," International Economic Review, Vol. 4, 1963, pp. 235-246.

group can obtain anything. For $n = 4$ it is every imputation[3] in the game with side-payments and the single point (1, 1, 1, 1) for the no side-payment game. For $n = 4k$ where $k > 1$ the core is a single point for all of these games.

In this model our economy suffers from the effects of the indivisibility or the integral aspects of the Bridge game. As the numbers increase

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[3]In the side-payment game an imputation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ will add to $\sum_{i=1}^{n} \alpha_i = 4\left[\frac{n}{4}\right]$.

For a core to exist the above condition must be satisfied together with the inequalities

$$\sum_{i \in S} \alpha_i \geq v(s) \quad \text{for all} \quad S \subset N .$$

These conditions are illustrated for $n = 4$, 5, and 8.

$n = 4$	$v(1) = 0$	$n = 5$	$v(1) = 0$	$n = 8$	$v(1) = 0$
	$v(2) = 0$		$v(2) = 0$		$v(2) = 0$
	$v(3) = 0$		$v(3) = 0$		$v(3) = 0$
	$v(4) = 4$		$v(4) = 4$		$v(4) = 4$
			$v(5) = 4$		$v(5) = 4$
					$v(6) = 4$
					$v(7) = 4$
					$v(8) = 8$

$n = 4$:

$$\alpha_i \geq 0$$
$$\alpha_i + \alpha_j \geq 0$$
$$\alpha_i + \alpha_j + \alpha_k \geq 0$$
$$\sum_{i=1}^{4} \alpha_i = 4$$

any imputation

$n = 5$:

$$\alpha_i \geq 0$$
$$\alpha_i + \alpha_j \geq 0$$
$$\alpha_i + \alpha_j + \alpha_k \geq 0$$
$$\alpha_i + \alpha_j + \alpha_k + \alpha_m \geq 4$$
$$\sum_{i=1}^{5} \alpha_i = 4$$

no imputation

$n = 8$:

only the imputation

$\alpha_i = 1$ for $i = 1, \ldots, 8$

the price system and the core appear and disappear periodically.[4]

Does it seem reasonable to have the price system disappear when the economy has 1,000,001 people instead of 1,000,000? A way to avoid this undesirable situation is to introduce a lottery ticket for the Bridge game. We assume that for any size n, each individual sells his time in exchange for a lottery ticket which carries the probability of $4/n[n/4]$ that he plays in a Bridge game and $1 - 4/n[n/4]$ that he is left out. As the number of players increases the probability of getting a Bridge game becomes arbitrarily close to 1.

The interpretation of the "lottery ticket" in this case is quite natural. If we leave out special social structure then everyone has the same chance to find a game. As the number increase in spite of the fact that the indivisibility may still cause an annoyance and 1, 2, or 3 people may fail to play, the odds become insignificantly small as the numbers increase. The price of the lottery ticket becomes approximately the same as the guaranteed game.

As long as the number of types of indivisible factors of production is finite and their capacities are finite the core will still appear and disappear in a periodic manner as the population mix of the different owners of resources is or fails to be in the correct ratio. As the size of the

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[4] It is of interest to note that this unsatisfactory state of affairs is not encountered with the value [Shapley, L.S. and M. Shubik, "Pure Competition, Coalitional Power and Fair Division," International Economic Review (Forthcoming).] of a game to a player. For the n-person game the value of the i^{th} player φ_i is given by:

$$\varphi_i = \frac{4}{n}\left[\frac{n}{4}\right]$$

which fluctuates between 1 and $1 - 3/n-3$ for $n = 4k + 3$ and $k \geq 1$.

economy grows sufficiently there will still remain the possibility of a mismatch of resources but relative to the whole economy the amount of the mismatch approaches zero.

The implications of this example for the economy as it is, are that when indivisibilities are small relative to the economy as a whole they do not matter very much and lottery tickets could be formally introduced and sold to preserve the price system; otherwise the lottery aspect will come about by a relatively minor amount of queuing, or social convention to correct for the minor aberration from the price system. In many instances the indivisibilities in society are large relative to the economy as a whole hence the approximation and limit argument will not apply.

The approach here appears to be related to the type of work on integer programming by Gomory[5] and is also related to the comments on nonconvexity of Farrell.[6]

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[5] Gomory, R.E., "Some Polyhedra Related to Combinatorial Problems," RC 2145 IBM Research, Yorktown Heights, July 25, 1968.

[6] Farrell, M.J., "The Convexity Assumption in the Theory of Competitive Markets," Journal of Political Economy, Vol. 67, 1959, pp. 377-391; "A Reply," ibid., Vol. 69, 1961, pp. 484-489; "Rejoinder," ibid., Vol. 69, 1961, p. 493.