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# A RE-EXAMINATION OF THE MODIGLIANI-MILLER THEOREM

Joseph E. Stiglitz

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# ERRATA

p. 5, footnote 8, line 7. $(EU'r_j - \lambda)B_{ij} = 0$ instead of $(EU'r_j - \lambda)B_{ij} < 0$ .

p. 14, line 20. "It is required that ..." instead of "Keep that ..."

p. 16, line 2. $= \int_{\mathcal{S}} U'\hat{r}_j(\theta)f(\theta)d\theta + \int_{\mathcal{S}} U' \frac{X(\theta)}{B_j} f(\theta)d\theta$ instead of

$$= \int_{\mathcal{S}} U'\hat{r}_j(\theta)d\theta + \int_{\mathcal{S}} U' \frac{X(\theta)}{B_j} f(\theta)d\theta$$

line 4. $\hat{r}_jB_j = \psi(B_j)$ . $\psi' > 0$ instead of $\hat{r}_jB_j = X(B_j)$ . $X' > 0$ .

line 5. $\psi(B_j)$ instead of $X(B_j)$

line 6. delete

line 7. $[U'(\theta^*)\{(\hat{r}_jB_j - X(\theta^*))f(\theta^*)\} + \ldots$ instead of

$[U'(\theta^*)\{\hat{r}_jB_jf(\theta^*) - X(\theta^*)\}$ +

p. 17, line 18. $\theta \in \mathcal{S}$

p. 18, line 3. $V = \frac{EU'X_j}{rEU'}$ instead of $V = \frac{rEU'X_j}{EU'}$

line 14. $V = \frac{EU'X_j}{rEU'}$ instead of $V = \frac{EU'}{rEU'}$

p. 28, footnote 21, line 15 $\sum_{\mathcal{S}} X_j(\theta_k)p_k^* - B_j \ldots$ instead of $\sum_{\mathcal{S}} X_j(\theta_k)p_k^* - eB_j$

p. 32, line 13. $\frac{\Sigma U'(\hat{Y}(\beta))\hat{Y}(\beta)}{\Sigma U'(\hat{Y}(\beta))} = r$

# A RE-EXAMINATION OF THE MODIGLIANI-MILLER THEOREM*

by

Joseph E. Stiglitz

(Paper to be presented at December 1967 Meetings of the Econometric Society)

## 1. Introduction

In their now classic paper of 1958 [13], Modigliani and Miller demonstrated that the cost of capital for a firm was independent of the debt-equity ratio. Although much of the subsequent discussion has focused on the realism of particular assumptions,[1] there have been few attempts to delineate under exactly how wide a class of assumptions the MM theorem obtains.[2] The purpose of this paper is to show, in the context of a general equilibrium state pre-

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1/ See, e.g., [8, 3].

2/ An exception is the work of Hirschleifer [ 10], who used the Arrow-Debreu model (which assumes at least as many securities as states of the world); after an earlier version of this paper had been circulated, my attention was drawn to the unpublished Ph.D. dissertation of G. Pye, where results quite similar to several contained here are shown in the context of a somewhat different model. For other general equilibrium portfolio (stock-market) models, see Sharp [ 20 ], Lintner [ 11 ], and Mossin [ 17 ], all of whom use a mean-variance approach; and Diamond [ 7 ] who assumes multiplicative uncertainty and focuses on the problem of Pareto optimality.

ference model[3] with consumers who maximize expected utility that the Modigliani-Miller theorem holds under much more general conditions than those assumed in their original studies.[4]

Section 2 presents the basic model; Section 3 considers the case of no bankruptcy; Sections 4 and 5 are concerned with the problems of bankruptcy. In Section 6 we discuss the relationship of our model to the Arrow-Debreu model and the implications for Pareto optimality; in Section 7, we investigate investment under uncertainty, and in Section 8 we make some concluding comments.

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3/ For a discussion of state-preferences, see [1, 2, 6, 7, 9,10].

4/ It should be emphasized that we are considering the "ideal" world of Modigliani and Miller, where there is, for instance, no preferential tax treatment of interest charges. We shall see, however, that many of the features usually associated with "imperfect" capital markets, such as increasing cost of borrowing as the number of bonds issued increases, appear even in our purely competitive capital markets.

## 2. The Basic Model

### a. The Firm

We consider firm j which wishes to raise a given amount of capital, $A_j$ . It can do this in two ways: issuing new shares $S_j$, which it sells at a price, $p_j$, or new bonds, $B_j$.[5] Thus $A_j = p_j S_j + B_j$ . For convenience, and without loss of generality, we assume that the firm has no outstanding bonds and one outstanding share. The profits (before paying bondholders) are uncertain; we can consider the profits, $X_j$, as a function of the state of the world $\theta$, $X_j(\theta)$ . It should be emphasized that this is the only source of uncertainty in the model; depending on the particular model being examined, we shall assume either that there are a discrete finite number, n, of states of the world or that $X_j(\theta)$ is a random variable defined over the interval $0 \leq \theta \leq 1$ , and $f(\theta)$ will be the density function of $\theta$ . There is a government bond which is perfectly certain and yields a return, r . (The rate of return on any safe asset will be denoted by an unsubscripted r .) In general, the rate $r_j$ which the $j^{th}$ firm must pay will depend on the number of bonds issued.

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5/ This is essentially the simple problem posed in the standard elementary textbooks in the development of the corporate firm: the entrepreneur who founded the company finds that he must raise new capital, $A_j$ . We ask, how should he raise this new capital.

In Figure 1, we have drawn one possible configuration of $X_j(\theta)$ . If the firm issues $B_j$ bonds, then the probability of bankruptcy is given by OA .[6] Note that as $B_j$ increases, the probability of bankruptcy increases. Accordingly, we would expect $r_j$ to increase.

$$r_j = r(B_j) \qquad r'(B) \geq 0 .$$

When the firm does not go bankrupt, i.e., when $X_j(\theta) \geq r_j B_j$ , the return on the bond is $r_j$ ; when it does, the return is $\frac{X_j(\theta)}{B_j}$ . (Note that the return $r_j$ is the principal plus interest.)

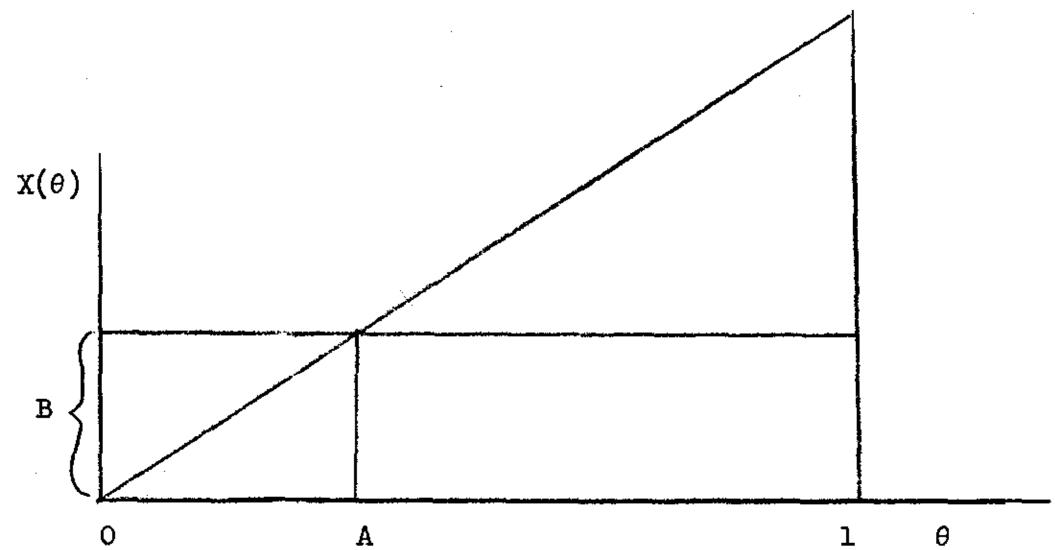


Figure 1

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[6] Throughout the discussion, we limit ourselves to essentially a two-period problem. In a two-period model, a firm either makes its interest payments or goes bankrupt: there is nothing in between. In the real world there are however alternatives. It can, in particular, defer the interest payments or the principle payments. If there is a positive probability of such deferal, the market will of course force the firm to pay a higher rate of interest. But while the "transactions" costs in deferment are relatively small, there are likely to be very large costs involved in the dissolution of the firm. This will provide a strong incentive against issuing "too many" bonds. In fact, if there are large transactions costs involved in bankruptcy or deferral, the MM theorem would not hold: value maximizing firms would never issue enough bonds to go bankrupt.

Multi-period models also raise the problem of price uncertainty (capital gains) which we do not treat explicitly.

Finally, earnings per share, $e_j$ , in state $\theta$ is given by

$$e_j(\theta) = \frac{X_j(\theta) - r_j(B_j)B_j}{S_j + 1} \qquad r_j(B_j)B_j \leq X_j(\theta)$$
$$= 0 \qquad r_j(B_j)B_j \geq X_j(\theta)$$

The firm is assumed to choose its debt-equity ratio to maximize the price of a share or the value of the firm. To see that these are equivalent, observe that $V = p(S + 1) + B = p + A$ . Maximizing $p$ maximizes $V$ . If there are perfect capital markets, this is equivalent to maximizing the owners' utility.

b. <u>The Consumer</u>

It will be assumed that the $i^{th}$ individual has a given wealth, $W_i$, which he allocates to alternative investment opportunities to maximize expected utility; that is, if there are a large number of alternative bonds and stocks, $B_j$ and $S_j$ , with earnings $e_j$ and $r_j$ , he purchases $B_{ij}$ of bond $j$ and $S_{ij}$ of stock $j$ , to maximize his expected utility of wealth,

$$EU(\sum_j e_j S_{ij} + \sum_j r_j B_{ij})$$

subject to the constraint that

$$W_i = \sum_j p_j S_{ij} + \sum_j B_{ij}$$

where $p_j$ is the price of the $j^{th}$ stock, and E is the expectation operator.[7/]

The conditions for expected utility maximization can be written

$$EU'e_j = \lambda p_j$$

$$EU'r_j = \lambda$$

where $\lambda$ is the marginal utility of wealth.[8/]

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[7/] As usual, we assume that $U' > 0$ and $U'' < 0$. We are assuming that the portfolio decision can be studied separately from the consumption decision.

[8/] The restriction that $U'' < 0$, guarantees that the above conditions are not only necessary but also sufficient, provided $S_{ij}$ and $B_{ij}$ may be negative (short sales are allowed) or provided the individual is at an interior solution with $S_{ij} > 0$, $B_{ij} > 0$. If $S_{ij}$ and $B_{ij}$ are constrained to be non-negative, then the above conditions are replaced by the Kuhn-Tucker conditions, $EU'e_j \leq \lambda p_j$, $EU'r_j \leq \lambda$, $S_{ij}(EU'e_j - \lambda p_j) = 0$, $(EU'r_j - \lambda)B_{ij} < 0$.

## 3. The Case of No Bankruptcy

Two of the aspects of the MM proof that theoretical economists found most disturbing were (a) the dependence of the proof on the concept of risk classes and (b) the partial equilibrium nature of the argument. We shall now show in a perfect capital market with no bankruptcy, the MM theorem is perfectly general. More precisely, we shall prove

Proposition 1. If individuals can borrow and lend at the market rate of interest and there is no bankruptcy, if there exists a general equilibrium with each firm having a particular debt-equity ratio, a particular value, and a particular price of its shares, then there exists another general equilibrium solution with any firm having any other debt-equity ratio, but with the value of all firms, the price of all shares and the interest rate on bonds unchanged.

Proof. For convenience, let us consider a firm which in the initial general equilibrium issues B bonds. We shall show that if it issues no bonds, all markets will clear if the price of the share and the return on bonds is unchanged.

The income pattern from a dollar invested in equity in the initial situation when B bonds are issued, is simply[9/]

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[9/] For definition of symbols, see Section 2.

$$\frac{X(\theta) - rB}{pS} = \frac{X(\theta) - rB}{A - B + p}$$

Consider now the second situation, where the firm issues no bonds. If the price of a share is the same in the two situations and the individual borrows

$$\hat{B} = B/(A - B + p)$$

and with the proceeds and his original dollar buys securities, his income in state of the world $\theta$ will be

$$\frac{X(\theta)(1 + \hat{B})}{A + p} - r\hat{B} = X(\theta)\left(\frac{1}{A - B + p}\right) - \frac{rB}{A - B + p}$$

so that the individual is clearly indifferent between the two situations. If the demand for shares previously had been $D^*$, the demand is now $(1 + \hat{B})D^* = \frac{A + p}{A + p - B} D^*$. But if the demand equalled the supply $D^* = A + p - B$, so that the demand when the firm issues no bonds is $A + p$. But the supply of shares (value of equity) if the firm issues no bonds is $A + p$, so demand equals supply.[10]

It is important to observe that the only assumption made in this proof was that in equilibrium, there is a single interest

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[10] Not only is the demand for securities equal to its supply, but the total demand for bonds is unchanged at the same interest rate. In the first situation the demand for loans by the firm is $B$. In the second situation this is reduced to zero, but demands by individuals increase. For each dollar of his own that the individual invests, he borrows $\hat{B}$, so the increment is $\hat{B}D^* = \frac{B}{A + p - B}(A + p - B) = B$. This completes the proof.

rate at which perfectly safe bonds are bought and sold both by individuals and firms. In particular, no assumptions about the behavior of individuals under uncertainty (e.g., expected utility maximization), no assumptions about the size of firms, and no assumptions about the nature of risk (e.g., risk classes) have been made.

The MM theorem has, however, also come under attack because of certain imperfections in the capital market. The particular manifestations of this which have caused the most concern are: (1) individuals cannot borrow at the same rate as firms; (2) there are transactions costs; (3) the cost of borrowing for the firm increases as the number of bonds issued increases.[11/] This last problem is essentially the problem of bankruptcy, and is treated in subsequent sections. Here, we treat the first two.

First, it should be noted that individuals do not actually need to borrow from the market, but only to change their holdings of corporate bonds in their portfolios.[12/] A problem can arise then only if an individual has no bonds in his portfolio, which cannot be ruled out on *a priori* grounds but surely is not very important

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[11/] The most important imperfections probably arise from taxation. From an analytic point of view, it is important to understand clearly the workings of a capital market in the absence of taxes. In a separate paper, we shall treat the particular complications introduced by taxes.

[12/] This point was made by MM in their original article.

empirically.

Second, even if there were individuals who had no bonds in their portfolio, and individuals could not borrow, if there were two companies in the same risk class with different debt-equity ratios, the individual could "arbitrage" simply by changing the proportions in which he held the two stocks. More precisely, we shall now show

Proposition 2. If there are three or more companies with the given income pattern $X(\theta)$ , and the most levered and the least levered companies have the same value, then the value of all other firms must be the same.

Proof. There are two firms with identical income patterns across the states of nature, denoted by $X(\theta)$ . We shall assume for simplicity that the first firm has no debt and the second a debt of $B_2$ . The returns per dollar of equity holdings in the first company in state $\theta$ is given by

$$\frac{X(\theta)}{E_1}$$

Similarly, the return per dollar for the second firm

$$\frac{X - rB_2}{E_2}$$

where $F_i$ is the value of the equity of the $i^{th}$ firm. Assume there were a third firm with debt between the other two

$$0 < B_3 < B_2$$

and which gave a return in state $\theta$ of

$$\frac{X(\theta) - rB_3}{E_3}$$

We shall show that the third firm must have a value less than or equal to that of the first two (which are assumed to be identical $V_1=V_2=V$). Assume that the value of the third firm were greater than $V$. If an individual took a dollar, invested $(B_3/E_3)/(B_2/E_2)$ in the second firm and the remainder in the first, his income in state $\theta$ would be

$$X(\theta)\left[\left(\frac{1}{E_2} - \frac{1}{E_1}\right)\frac{B_3/E_3}{B_2/E_2} + \frac{1}{E_1}\right] - r\,\frac{B_3/E_3}{B_2/E_2}\,\frac{B_2}{E_2}$$

$$= \frac{X(\theta)}{E_1}\{(E_1 - E_2)\frac{B_3/E_3}{B_2} + 1\} - r\,\frac{B_3}{E_3}$$

We must compare this with the return per dollar invested in firm 3.

$$\frac{X(\theta) - rB_3}{E_3}$$

But note that if $V_3 > V$, the return from investing in firm 3 is always smaller than from the combination of firm 1 and 2, i.e.,

$$\frac{(E_1 - E_2)}{E_1} \frac{B_3/E_3}{B_2} + \frac{1}{E_1} > \frac{1}{E_3}$$

This can be seen by observing that

$$\frac{E_1 - E_2}{E_1} \frac{B_3/E_3}{B_2} + \frac{1}{E_1} = \frac{E_1 - E_2}{E_3 E_1} \frac{V_3 - E_2}{V - E_2} + \frac{1}{E_1}$$

$$> \frac{E_1 - E_2}{E_3 E_1} \frac{V - E_3}{V - E_2} + \frac{1}{E_1} > \frac{1}{E_3} ,$$

since $$(E_1 - E_2) \frac{(V - E_3)}{(V - E_2)} > E_1 - E_3$$

i.e. $$E_1 V - E_2 V - E_1 E_3 + E_2 E_3 > E_1 V - E_3 V - E_1 E_2 + E_2 E_3$$

or $$(E_3 - E_2)(V - E_1) > 0$$

Similarly, it can be shown that the value of the third firm cannot be less than the value of the first two.

The problem of transactions costs may be handled in a similar manner to that of limitations on an individual's borrowing.[13/] Baumol

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[13/] Throughout, we assume that transactions costs are proportional to the size of the transaction, and are the same for firms and individuals. Differences in costs act just like taxes, and will not be treated here.

and Malkiel [3] have suggested that, if in order to undertake the arbitrage operations the individual had to borrow to provide "homemade" leverage, the total value of transactions would be greater than if the company already provided the desired leverage.

If there are sizeable transactions costs then the net income from the "homemade" leverage, if the value of the firms were the same. Thus Baumol and Malkiel argue that the levered company will have a higher value than the unlevered company.

But if we return to our proof of proposition 1, it will be recalled that the total value of transactions was unaffected by the debt-equity ratio of the firm: changes in individuals' demands for bonds exactly equaled the change in the demands by firms. To see this another way, if the individual has bonds in his portfolio or if there are two companies one with high leverage and one with low leverage, as in proposition 2, the individual can simply change his portfolio composition without undertaking any additional transactions.

## 4. Bankruptcy in Competitive Markets

Bankruptcy presents a problem for the usual proofs of the MM theorem on two accounts: first, it means that the rate of interest which the firm must pay on its bonds will increase as the number of bonds increases. MM have treated the case where it increases at exactly the same rate for all firms and individuals, but this is clearly a very special case. Second, if a firm goes bankrupt, it is no longer possible for an individual to replicate the exact pattern of returns, except if he can buy on margin, using the security as collateral; and if he defaults, he only forfeits the security and not any of his other assets.[14] To see this, consider the two alternative policies considered in Section 3; in the one case, the firm issues no bonds (hence no chance of default) and in the other he issues $B$ bonds. If, in the first case, the individual sells $\hat{B}$ bonds, where $\hat{B} = \frac{B}{A - B + p}$, and if the price is the same in the two situations, he can exactly replicate the returns in those states where the firm does not go bankrupt, as we have already shown. But if the firm goes bankrupt in some state, $\theta'$, in the latter case his return is zero, while in the former his return is $\frac{X(\theta')}{A + p}(1 + \hat{B}) - r\hat{B} < 0$. If however, he can forfeit the security then his return will again be zero.

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[14] Although it is possible for individuals to make such limited liability arrangements by, for instance, putting the particular assets in the wife's name, by setting up holding corporations, etc., it is not clear empirically how important such arrangements are.

Of course, if the firm has a positive probability of going bankrupt, it will have to pay a higher nominal rate of interest. But if the individual is to use the security as collateral, he, too, will have to pay a higher nominal rate of interest. And indeed, it is clear that the two will be exactly the same, since the pattern of returns on the bonds in bankruptcy will be the same. Thus, we have

Proposition 3. If a firm has a positive probability of going bankrupt, and an individual can borrow using those securities as collateral (so that if his return from the securities is less than his borrowings, he can forfeit the securities) there is no optimal debt-equity policy.

It should be noted that this does not require 100 percent margins. The required margin is only $\hat{B}/1 + \hat{B} = \frac{B}{A + p}$ .

Individuals may, of course, not be able to obtain even the level of margin required by the above analysis. What happens then? We shall now show

Proposition 4. In a competitive capital market, the price of a share is invariant to the debt-equity ratio even if the firm goes bankrupt, provided at least one group of individuals does not specialize in bonds or securities of that firm. Keep that the total issue of shares and bonds of the company be sufficiently small that the marginal utility of income in each state of the world is unaffected by the consumer's purchase of the firm's shares or bonds, regardless of the debt-equity policy pursued by the firm; this is just what we mean by perfect capital markets.

In Section 2 we showed that, provided individuals do not specialize in bonds or securities,[15]

$$p_j = \frac{EU'e_j}{\lambda} \qquad \text{and} \qquad 1 = \frac{EU'r_j}{\lambda}$$

But

$$e_j(\theta) = \begin{cases} \dfrac{X(\theta) - \hat{r}_j B_j}{S_j} = \dfrac{X(\theta) - \hat{r}_j B_j}{(A - B + p_j)/p_j} & X(\theta) > \hat{r}_j B_j \\ 0 & X(\theta) \leq \hat{r}_j B_j \end{cases}$$

where

$$r_j(\theta) = \hat{r}_j \qquad \text{for} \quad \theta \in X(\theta) > \hat{r}_j B_j$$

$$= \frac{X(\theta)}{B_j} \qquad \text{for} \quad \theta \in X(\theta) \leq \hat{r}_j B_j$$

Thus, if $f(\theta)$ is the density function for the states of nature, and $\mathcal{L}$ is defined as the set of states under which bankruptcy does not occur, i.e., $\mathcal{L} = \{\theta | X(\theta) \geq \hat{r}_j B_j\}$

$$p_j = \frac{\int_{\mathcal{L}} \frac{U'(\theta)[X(\theta) - \hat{r}_j B_j]}{(A - B + p_j)/p_j} f(\theta) d\theta}{\lambda} = \int_{\mathcal{L}} \frac{U'(X(\theta) - r_j B_j) f(\theta) d}{\lambda} + B - A$$

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[15] All we require for the proof is that the individual's demand price for both the firm's bonds and stock equal the market price. Whether he in fact purchases the securities is irrelevant.

and if $\mathcal{A}' \equiv \{\theta | X(\theta) < r_j B_j\}$

$$\lambda = \int_{\mathcal{A}} U' \hat{r}_j(\theta) d\theta + \int_{\mathcal{A}} U' \frac{X(\theta)}{B_j} f(\theta) d\theta$$

Multiplying this last equation by $B_j$, we obtain a functional relationship between $B_j$ and $\hat{r}_j B_j$; $\hat{r}_j B_j = X(B_j)$. $X' > 0$, as can be seen by taking the total differential of $X(B_j)$:

$$\{(\hat{r}_j B_j - X(\theta^*) f(\theta^*)\}$$

$$[U'(\theta^*)\{\hat{r}_j B_j f(\theta^*) - X(\theta^*)\} + \int_{\mathcal{A}} U' f(\theta) d\theta] d(\hat{r}_j B_j) = \lambda dB_j$$

where $\theta^*$ is defined by $X(\theta^*) = \hat{r}_j B_j$, and therefore

$$\frac{d\hat{r}_j B_j}{dB_j} = \frac{\lambda}{\int_{\mathcal{A}} U' f(\theta) d\theta} > 0$$

It should be clear that if $\int_{\mathcal{A}} f(\theta) d\theta < 1$, $\frac{d\hat{r}_j}{dB_j} > 0$.

We can now calculate the optimal debt-equity ratio. Since $B_j$ and $\hat{r}_j B_j$ are monotonically related, we can consider $p_j$ as a function of one variable, say $\hat{r}_j B_j$:

$$\frac{dp_j}{d\hat{r}B_j} = \frac{U'(\theta^*)\{X(\theta^*) - \hat{r}_j B_j\} f(\theta^*)}{\lambda} - \frac{\int_{\mathcal{A}} U'(\theta) f(\theta) d\theta}{\lambda} + \frac{dB}{d\hat{r}_j B_j}$$

$$= [-\int_{\mathcal{A}} U'(\theta) f(\theta) d\theta + \int_{\mathcal{A}} U'(\theta) f(\theta) d\theta]/\lambda$$

$$= 0$$

Thus the price of the firm, $p_j$, and the value of the firm are independent of the debt-equity ratio, even if the firm issues a sufficiently large number of bonds that there is a positive probability of going bankrupt. Observe that as it issues more bonds, it pays a higher nominal rate, i.e., it pays a higher rate of interest when it pays back its bonds; but the probability of paying back the bonds is decreased as B increases.

Average Returns. What is the relationship between the average return on the value of a firm and the market rate of interest on a safe asset? i.e., if (using the Modigliani-Miller notation)

$$\rho_j = \bar{X}_j/V_j = EX_j/V_j$$

then is $\rho_j \gtrless r$ ? The presumption thus far has been that $\rho > r$ because while the return on the bond is perfectly safe, that on the security is not. We wish to show that this is not necessarily the case.

$$V_j = p_j S_j + B_j$$

the value of a firm is equal to its equity plus debt. But we have already shown that $p_j = \frac{E\{U'(X_j - r_j B_j)}{\lambda S_j}$ for $\theta \in$ , so

$$p_j S_j = \frac{1}{\lambda}[E\{U'(X_j - r_j B_j))\}]$$

But $\lambda = EU'r_j$ , so

$$V_j = \frac{1}{\lambda} EU_j'X_j$$

But $\lambda = rEU'$ , so

$$V = \frac{rEU'X_j}{EU'}$$

and thus we have

Proposition 5

$$\rho_j = \frac{EX_j EU'}{EU'X_j} r \gtrless r \quad \text{as} \quad E(U' - \overline{U'})(X_j - \overline{X}_j) \lessgtr 0$$

that is, $\rho_j$ is equal to $r$ if $U'$ and $X$ have zero covariance,[16] greater than $r$ if they are negatively correlated and less than $r$ if they are positively correlated.

The point of this is to emphasize the importance of the correlation of the returns from any given security with the returns from other sources. The desirability of an asset cannot be summarized (in general) by just its mean and variance.[17]

It should be noted that the result that $V = \frac{EU'}{rEU'}$ provides

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[16] Or if $U$ is linear, so $U'$ is constant.

[17] Cf. [10, 11]. The implications of this result for the use of discount factors in public projects should be clear; if a flood control project has, for instance, a high return in those states where $U'$ is high, a discount rate less than $r$ should be used.

an alternative proof that the value of a firm and the price of a share is independent of the debt-equity ratio even if bankruptcy occurs, provided that the marginal utility $U'$ in every state is unaffected by the firm.[18]

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[18] The Representative Individual. There is one more condition under which the Modigliani-Miller theorem still holds: where the market acts as if there were $n$ identical consumers. If there were $n$ identical consumers, each of them would purchase $1/n$th of the bonds and $1/n$th of the securities of the firm, and so the income from bonds and securities of the firm in state $\theta$ is just $X(\theta)/n$. Thus the marginal utility of the representative individual is independent of the debt-equity ratio (although it may not be the same after the stocks and bonds have been issued as before) and thus the value of the firm, which we have shown to be $V = \frac{EU'X}{rEU'}$ is independent of the debt-equity ratio.

## 5. Bankruptcy: An Example

In order to understand the role played by the assumption that the firm be small relative to the size of the market, let us consider an example where the firm is large.

We shall assume that there are two groups of individuals; the endowments and utility functions are identical within groups but differ between the groups. We shall compare the equilibrium price of the firm under the assumption that it issues no bonds and that it issues a sufficiently large number of bonds to go into bankruptcy in two of the three equally probable states of the world. The government follows a monetary policy which maintains the rate of interest on government bonds at unity. In Table 5.1, we have set out the details on the firm.

Table 5.1

| Total Income of Firm | State 1 | State 2 | State 3 |
|---|---|---|---|
| A = 100 | 500 | 50 | 0 |

Our first order conditions for utility maximization impose the restrictions that in the no bankruptcy case (letting $U_i$ denote utility of an individual in the $i^{th}$ group.) (e(3) is zero.)

$$p = \frac{e(1)U_1'(1) + e(2)U_1'(2)}{r\sum_\theta U_1'(\theta)} = \frac{e(1)U_2'(1) + e(2)U_2'(2)}{r\sum_\theta U_2'(\theta)}$$

Assume the firm declares that it will sell 5 shares at a price of 25 per share. Then earnings per share in state 1 are 100 and in state 2 are 10. The above equation then reduces to the condition that

$$-3U_i'(1) + .6U_i'(2) + U_i'(3) = 0 \qquad i = 1, 2$$

If we denote by a caret the values of the various variables in the bankruptcy case, we obtain as our necessary and sufficient conditions for utility maximization that

$$\hat{p} = \hat{e}(1)\hat{U}_1'(1)/r\sum_\theta U_1'(\theta) = \hat{e}(1)\hat{U}_2'(1)/r\sum_\theta \hat{U}_2'(\theta)$$

$$1 = [\hat{r}(1)\hat{U}_1'(1) + \hat{r}(2)\hat{U}_1'(2)]/r\sum_\theta \hat{U}_1'(\theta)$$

$$= [\hat{r}(1)U_2'(1) + \hat{r}(2)U_2'(2)]/r\sum_\theta \hat{U}_2'(\theta)$$

where $\hat{r}(\theta)$ is the return on the corporate bond in state $\theta$. Assume that the firm declares that it will issue 3.8 shares at a price 15.99 and 58.3 bonds at a nominal interest rate of 3.4. Thus the earnings per share and per bond in different states are as given in Table 5.2.

Table 5.2

| | | State 1 | State 2 | State 3 |
|---|---|---|---|---|
| Earnings per share | | 79.5 | 0 | 0 |
| Earnings per bond | | 3.4 | .86 | 0 |
| Price of share: | 15.99 | | | |
| No. of shares: | 3.8 | | | |
| Value of firm: | 115.9 | | | |

Then, in our example, the above equations reduce to

$$\hat{U}_i'(1) = .2\sum_\theta \hat{U}_i'(\theta) \qquad \theta = 1, 2$$

$$\hat{U}_i'(2) = .35\sum_\theta \hat{U}'(\theta) \qquad \theta = 1, 2$$

$$\hat{U}_i'(3) = .45\sum_\theta \hat{U}'(\theta) \qquad \theta = 1, 2$$

The first group of individuals has 91.8 to invest, the second 80 to invest. Consider the following allocations in the bankruptcy and no bankruptcy situations (Table 5.3).

Table 5.3

| | No Bonds | Bankruptcy |
|---|---|---|
| **First Individual** | | |
| Shares | 3.65 @ 25 = 87.5 | 3 @ 15.9 = 47.7 |
| Corp. Bonds | 0    0 | 30 @ 1   30.0 |
| Gov. Bonds | 4.31 @ 1 = 4.3 | 14.1 @ 1   14.1 |
| Total | 91.8 | 91.8 |
| **Second Individual** | | |
| Shares | 1.5 @ 25 = 37.5 | .8 @ 15.9 = 12.7 |
| Corp. Bonds | 0    0 | 28.3 @ 1   28.3 |
| Gov. Bonds | 42.5 @ 1 = 42.5 | 39.0 @ 1   39.0 |
| Total | 80.0 | 80.0 |

Assume that they have zero income from other sources, (if their income from other sources is not zero, the analysis could easily be adapted). Then their income in each state of the world is given in the following table.

Table 5.4

| | State 1 | State 2 | State 3 |
|---|---|---|---|
| No Bonds | | | |
| First individual | 354.3 | 39.3 | 4.3 |
| Second individual | 192.5 | 57.5 | 42.5 |
| Bankruptcy | | | |
| First individual | 354.6 | 39.9 | 14.1 |
| Second individual | 198.9 | 63.3 | 39.0 |

It is clear that if the utility function of each individual is such that

| First Individual | Second Individual |
|---|---|
| $U'(354.6) = 1$ | $U'(198.9) = 1$ |
| $U'(354.3) = 1.1$ | $U'(192.5) = 1.05$ |
| $U'(39.9) = 1.75$ | $U'(63.3) = 1.75$ |
| $U'(39.3) = 1.95$ | $U'(57.5) = 1.9$ |
| $U'(14.1) = 2.25$ | $U'(42.5) = 2.0$ |
| $U'(4.3) = 2.28$ | $U'(39.0) = 2.25$ |

not only will all the first order conditions of utility maximization be satisfied, but, since their utility functions exhibit diminishing marginal utility, these allocations will in fact be utility maximizing (at least locally).

So far, we have not considered the possibility of individuals buying securities on margin in the no bankruptcy situation. Without loss of generality, we consider the case where an individual borrows a dollar and buys one dollar's worth of securities. The return per dollar of a security in state 1 is 4, in state 2 is .4 and in state 3 is 0. The borrower will be willing to pay a nominal interest rate up to 4. It is easy to show in our example that the minimum nominal interest rate at which anyone would lend is also 4,[18] so 4 is the only possible interest rate for such loans. Moreover, at an interest rate of 4, the first order conditions of utility maximization are satisfied when all individuals (in both groups) make zero loans.

What we have shown is that there is more than one general equilibrium solution to this economy (which is not surprising). And although there may be multiple equilibria in competitive markets as well as in non-competitive markets, the important point to observe is that in the big-firm case, the firm can decide which of the possible equilibria the economy will be in. In this case, if the firm were to maximize value, it would pick the no bankruptcy solution, where the value is 125, over the bankruptcy solution where the value is 115, even though the latter situation is Pareto optimal (the ratio

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[18] Assume that the individual took \$1 away from safe bonds to make such a loan. If $\tilde{r}$ is the nominal interest rate, then the change in his utility is given by $(\tilde{r} - 1)U'(1) - .6U'(2) - U'(3)$, which at our equilibrium point is positive for $\tilde{r}$ greater than 4, zero for $\tilde{r} = 4$.

of the marginal utilities in the different states for the two individuals are the same), while the former is not. Although in competitive capital markets it is necessary for firms to maximize their value if they are to maximize the utility of their owners, this is not necessarily true in non-competitive markets.[19/]

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[19/] In our example, the government has followed the policy of maintaining the rate of interest on government bonds at unity. This required a greater number of bonds in the bankruptcy situation than in the no bankruptcy case. But similar results can be obtained if the government issued a fixed number of bonds, and let the interest rate vary to clear the market.

## 6. Specialization, Arrow-Debreu Securities and Pareto Optimality

Arrow and Debreu have formulated a model of general equilibrium under uncertainty in which individuals can buy and sell promises to pay if a given state of the world occurs [1, 6]. Such securities are referred to as Arrow-Debreu securities. It is shown that (under the usual assumptions, which include convexity of preferences, a sufficient condition for which is $U'' < 0$ ) the economy will be Pareto optimal if there are these securities.

In this section, we investigate the relationship between our model and the Arrow-Debreu model and the implications of our analysis for Pareto optimality (where there are no production decisions to be made; in the next section, we consider the case where $A$ , the level of investment, is variable).[20] Throughout this section we shall assume that there are a finite number of equally probable states of the world. We shall also assume that the individual consumes a positive amount in every state of the world (or the limit of the marginal rate of substitution between a unit of consumption in state $i$ and a unit in state $j$ goes to zero as consumption in state $i$ goes to zero).

Under these conditions, then, Pareto optimality requires that the ratio of the marginal utility in state $k$ to the sum of

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[20] See also [7, 10].

his marginal utilities be the same, i.e.,

$$U_i'(\theta_k) = \alpha_k \sum_k U_i'(\theta_k) \qquad \text{all } k, \; i.$$

Thus all individuals will have the same demand prices,

$$p_j = \frac{\Sigma \alpha_k e_j(\theta_k)}{r} = \frac{\sum_k \alpha_k X_j(\theta_k)}{r} - A$$

It immediately follows that,

<u>Proposition 6</u>: provided the $\alpha_k$'s are independent of the firms debt equity ratio (i.e. the firm is "small"), Pareto optimality implies that the price of a firm is independent of its debt-equity ratio.

The converse of this proposition is however not true. The Modigliani-Miller theorem may hold, and yet the economy may not be Pareto Optimal. For in the absence of the Arrow-Debreu securities individuals cannot in general equate their marginal rates of substitution between different states of the world.

The question naturally arises, to what extent can the stock and bond markets serve as a substitute for these Arrow-Debreu securities. A stock or a bond can be thought of as a bundle, a market basket of Arrow-Debreu securities. If we have enough of these securities, they can serve as a complete substitute for the Arrow-Debreu securities. If there are $n$ states of the world, and $n$ independent kinds of securities, then the states of the world can be spanned by the securities. If $d$ is the vector of purchases of the various securities by an individual, and $E$ is the matrix of returns from the different securities ( $e_{jk}$ is the return of the $j^{th}$ security in state $k$ ) then the total returns in each state are given by $d^* = Ed$ . If an individual wants a pattern of returns $d^*$ , then his vector of purchases is given by $d = E^{-1}d^*$ .

If we have a sufficiently large number of securities and a finite number of states of the world, we can use the Arrow-Debreu model to provide us with a particularly simple proof of the Modigliani-Miller theorem.[21/] (Cf. [10].)

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[21/] We first introduce the fictitious Arrow-Debreu securities, and find the general equilibrium solution. A promise to pay 1 dollar in state $k$ has a price $p_k^*$ and the $i^{th}$ individual purchases a vector $d_i^*$ of these securities. We have already shown however that if we have $n$ independent securities, the individual can exactly replicate his income stream across the states of nature by buying the bundle $d = E^{-1}d^*$ . The price of a share of the $j^{th}$ firm is just

$$p_j = \Sigma e_j(\theta_k)p_k^*$$

But $$e_j = \frac{X_j - r_jB_j}{S_j} \quad \text{if} \quad X_j(\theta_k) \geq r_jB_j$$

$$= 0 \qquad\qquad \text{if} \quad X_j(\theta_k) \leq r_jB_j$$

and $1 = r_j \sum_{\mathcal{S}} p_k^* + \sum_{\mathcal{S}} X_j(\theta_k)p_k^*$ where $\mathcal{S} = \{\theta_k | X_j(\theta_k) \geq r_jB_j\}$ and $S'$ is the complement of $S$ .

or $r_jB_j \sum_{\mathcal{S}} p_k^* = B_j - \sum_{\mathcal{S}'} X_j(\theta_k)p_k^*$ .

Thus

$$p_jS_j = \sum_{\mathcal{S}} X_j(\theta_k)p_k^* - eB_j + \sum_{\mathcal{S}'} X_j(\theta_k)p_k^*$$

or

$$V = p_jS_j + B_j = \Sigma X_j(\theta_k)p_k^*$$

It should be emphasized that if there are fewer securities than states of nature, not only will it be true that the economy is not Pareto optimal, but also the number of independent securities will depend on the debt-equity policy pursued by the firm. If the firm does not go bankrupt, its bonds are exactly like the bonds of any firm, so it issues (at most) one independent security. But if it does go bankrupt, its bonds may provide a second independent security. As we noted in the example of the previous section, however, there is no reason to assume that the firm will issue the amount of bonds that lead to Pareto optimality.

The markets for each kind of security must, of course, be competitive. Thus, if there were $n$ states of nature and only $n$ securities, there must be a large number of firms issuing each kind of security. But if we take literally the Arrow-Debreu definition of a state of nature, it becomes immediately clear that there undoubtedly will be more states of nature than firms. For every time we create a new firm, and (say) the corresponding two new securities, we create far more than two new states of the world: the state of the world where the president of the firm sleeps well on November 23 is different from that in which he does not, the state of the world when worker X in the firm is absent on a given day is different from the state when he is not. In fact, there are many relevant characteristics of a state of the world which can be described by a continuous variable and hence there must be an infinity of states.

Yet, in some sense, most of these states are not very different from one another; and this would suggest that the equilibrium solution might look very much like it would if there were as many securities as states. For example, much of the variation in the return on stocks can be explained by the business cycle. If in any given business cycle state, the variance of the return were very small, and there were a small number of identifiable business cycle states, then the economy might look very much as if it were described by an Arrow-Debreu securities market.

We shall now attempt to make these statements somewhat more precise. Without loss of generality, we can partition the states of the world and write

$$X(\theta) = X(\beta) + \epsilon(\gamma|\beta)$$

with $E\,\epsilon(\gamma|\beta) = 0$

We now make the following further assumptions about $\epsilon$ for each firm

$$E\,\epsilon_i^2 < \sigma$$

$$\text{Cov}\,(\epsilon_i,\ \epsilon_j) = 0$$ [22]

[22] All we need to make is the assumption that the covariance be less than unity. But this simplifies the proofs considerably. It is important to observe that the upper-bound on the variance of any given security does not increase as the number of firms increase. $\gamma$ can be either continuous over an interval or discrete. We also assume that $U''$ is bounded.

Assume the individual looks only at the return $X(\beta)$ and ignores the $\epsilon$ terms, i.e. he only looks at the mean return in each state of the business cycle. Then if there are more securities than states in the business cycle, the demand prices for securities will be determined in the manner described above (pp.26-8), and the equilibrium will be essentially that which would have occurred had there been Arrow-Debreu securities (in terms of the states of the business cycle). Because he has ignored all the other characteristics of the economy, the individual's expected utility will be lower than if he makes the complete calculation. But if the loss is sufficiently small, and there is a cost of calculation, it is reasonable to assume that the individual will in fact ignore these other characteristics. We shall now show that as the number of securities over which the individual allocates his portfolio gets large, so that the maximum allocation to any particular risky asset gets small, the loss of utility gets arbitrarily small.

Let $a_i$ be the allocation to the $i^{th}$ firm if the individual makes the complete calculation; $\hat{a}_i$ be the allocation if he pays attention only to the mean return in each business cycle state; Thus, for purchased securities

$$EU'(\sum_i a_i[X_i(\beta) + \epsilon_i(\gamma|\beta)])(X_j(\beta) + \epsilon_j(\gamma|\beta)) = \lambda$$

$$EU'(\sum_i \hat{a}_i X_i(\beta))X_j(\beta) = \hat{\lambda}$$

We are interested in the resulting differences in expected utility

$$EU(\Sigma a_i[X_i(\beta) + \epsilon_i(\gamma|\beta)]) - EU(\Sigma \hat{a}_i[X_i(\beta) + \epsilon_i(\gamma|\beta)])$$

$$= \sum_{\beta}\{\int U(\Sigma(a_i[X_i(\beta) + \epsilon_i(\gamma|\beta)]) - U(\Sigma\hat{a}_i[X_i(\beta) + \epsilon_i(\gamma|\beta])d\gamma\}$$

$$= \sum_{\beta}\{U(\Sigma(a_iX_i(\beta)))-U(\Sigma(\hat{a}_iX_i(\beta))) + U''(\Sigma(a_i(X_i(\beta) + \mu\epsilon_i(\gamma|\beta)))\Sigma a_i^2 E\epsilon_i^2$$

$$+ U''(\Sigma\hat{a}_i(X_i(\beta) + \hat{\mu}\epsilon_i(\gamma|\beta)))\Sigma a_i^2 E\epsilon_i^2$$

for some $\mu$ , $\hat{\mu}$ $0 \leq \mu < 1$ , $0 \leq \hat{\mu} \leq 1$ .

But from the first order conditions we obtain, if we define $\Sigma a_i X_i(\beta) = Y(\beta)$ ,

$$E\{[U'(Y(\beta) + \Sigma a_i\epsilon_i)](X_j + \epsilon_j)\} = \sum_{\beta} U'(Y(\beta))X_j + U''(Y(\beta) + \tilde{\mu}\Sigma a_i\epsilon_i)a_j E\epsilon_j^2 = \lambda$$

or

$$\sum_{\beta}\{U'(Y(\beta))Y(\beta) + \Sigma U''(Y(\beta) + \tilde{\mu}\Sigma a_i\epsilon_i)a_j^2 E\epsilon_j^2\} = \lambda$$

Assume the individual has bonds in his portfolio. Then

$$\sum_{\beta}\frac{U'(Y(\beta))Y(\beta)}{\Sigma U'(Y(\beta))} + \sum_{\beta}\sum_{i}\frac{\Sigma U''(Y(\beta) + \tilde{\mu}\Sigma a_i\epsilon_i)a_i^2 E[\epsilon_i(\gamma|\beta)]^2}{\Sigma U'(Y(\beta))} = r$$

and, if we define $\hat{Y}(\beta) = \Sigma\hat{a}_i X_i(\beta)$ ,

$$\frac{\Sigma U'(\hat{Y}(\beta))Y(\beta)}{\Sigma U'(\hat{Y}(\beta))}$$

so as $a_j$ gets small,

$$\left| \frac{\Sigma U'(\hat{Y}(\beta))\hat{Y}(\beta)}{\Sigma U'(\hat{Y}(\beta))} - \frac{\Sigma U'(Y(\beta))Y(\beta)}{\Sigma U'(Y(\beta))} \right| < \delta$$

$\delta$ arbitrarly small, which is equivalent to

$$|\Sigma U'(Y)(Y - \hat{Y}) + \Sigma U''(Y)(Y - \hat{Y})^2 \ldots | < \delta(\Sigma U'(\hat{Y}(\beta)) = \frac{\delta\hat{\lambda}}{r} < \eta$$

where $\eta$ can be made arbitrarly small since $\hat{\lambda}$ is bounded. But

$$EU(Y + \Sigma a_i \epsilon_i) - EU(\hat{Y} + \Sigma \hat{a}_i \epsilon_i) = \sum_{\beta} U'(Y)(Y - \hat{Y}) + U''(Y)(Y - \hat{Y})^2 + \ldots . \; o(a_i)$$

and hence the loss of utility is arbitrarily small.[23/]

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[23/] One should be careful to observe that in order for the equilibrium to be even "approximately" described by Arrow-Debreu markets in the sense given above, capital markets must be perfect, i.e., the change in the marginal utility of income from the introduction of a new security into the portfolio of any individual must be arbitrarily small. The converse, however, is not true; capital markets may be perfect in the above sense, but the Arrow-Debreu model may not apply even "approximately."

## 7. Optimal Investment Policy and Pareto Optimality

Let us assume that we have a competitive capital market in which the Modigliani-Miller theorem holds. So far we have assumed that the level of investment of the firm, $A$ , is given. Now we ask, how ought $A$ be chosen?

Since $V = p + A$ , to maximize $p$ , the price of a share, it must maximize $V - A$ , i.e.,

$$V' = 1$$

But

$$\rho V = EX$$

so that maximization of $p$ leads to the condition that

$$EX' = \frac{dEX}{dA} = \rho + \rho' V$$

the expected increase in profits (equals the increase in expected profits) must equal the marginal cost of capital, which is equal to the average cost of capital $\rho$ , plus the change in the cost $\rho'$ , times the value of the firm. (Cf. [19].)[24]

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[24] There is a problem concerning the observability of $\rho'$ . Let $\hat{X}_j(\theta\hat{A}_j)$ be the return at some contemplated investment level $\hat{A}_j$ if there exists a firm $i$ , whose return $X_i(\theta)$ is perfectly correlated with $\hat{X}_j(\theta)$ , then the $j^{th}$ firm immediately knows $\rho(\hat{A}_j)$ ; if this is true for all levels of investment, $\rho'$ can be calculated in a straightforward manner. If there exists no such firm, we run into exactly the same problems that we encounter in the usual static models of competitive general equilibrium: prices for non-produced as well as produced commodities must be specified. Alternatively we could interpret $\rho'$ as the expected change in $\rho_j$ the second-best Pareto optimality argument then requires expected and actual $\rho'$ to be the same, i.e., perfect foresight.

The expression we have obtained is identical to Modigliani and Miller's Proposition III when $\rho' = 0$, i.e., the marginal cost of capital is equal to the average cost. But from the fact that $\rho = \frac{EXEU'}{EU'X}r$, this implies that

$$\frac{EU'X'}{EX'} = \frac{EU'X}{EX}$$

which requires in general that $X$ be proportional to $X'$, i.e.

$$X(\theta, A) = I(A)X(\theta)$$

$$\frac{dX(\theta, A)}{dA} = I'(A)X(\theta) = X(\theta, A)\frac{I'(A)}{I(A)}$$

This condition is, of course, exactly equivalent to the Modigliani-Miller definition of a risk class, and is clearly very restrictive. There is no reason to believe that the pattern of returns across the states of nature from a large investment should be exactly the same as that from a small investment.[25] Indeed there is no presumption that $\rho'$ be either positive or negative. Note that $\frac{EU'X'}{EX'}$ is the weighted sum of marginal utilities, where the weights are $X'$, and $\frac{EU'X}{EX}$ is the weighted sum of marginal utilities where the weights are $X$. If the former is greater than the latter, $\rho'$ is negative. In this case, problems analogous to those facing the usual decreasing

[25] Indeed, this is not just a question of scale; there is no reason to believe the new capital goods will be exactly like the old.

costs (increasing returns) industry may arise.

The question naturally arises, what implications does this have for the efficiency of the economy? If the increasing cost of capital were due to "market" imperfections, then one might think that this might result in a misallocation of resources. But in fact the changing cost of capital is due to the different evaluations of the streams of output produced at different levels of investment.

We follow Diamond [7] in defining the following constrained Pareto optimum problem: the planner is allowed to choose the level of investment and the allocation of the output of the different firms, subject to the constraint that each individual's consumption must be a linear function of the outputs of the different firms. This constraint arises naturally in a market economy. Assume we have a general equilibrium where the $i^{th}$ individual owns $\alpha_{ij}$ of the $j^{th}$ firm, and $B_i$ bonds, and where the $j^{th}$ form has issued $B_j$ bonds. We have shown that this general equilibrium is equivalent to the on where firms issue no bonds, the $i^{th}$ individual owns the same share of each firm but buys $\hat{B}_i$ bonds. His income in each state then is described by the equation $Y_i(\theta) = \hat{B}_i + \sum_{j=1}^{n} \alpha_{ij} X_j(\theta)$ , $i = 1, \ldots, m$

where $n$ is the number of firms and $m$ the number of individuals. We shall show that if firms follow the rule specified above, the economy will attain such a constrained Pareto optimum. Diamond has shown this to be true if there is multiplicative uncertainty, i.e.

the type of production function we have defined above as $X(\theta, A) = I(A)X(\theta)$ and which Modigliani and Miller also assumed. If there is not this kind of uncertainty, then there is the difficult problem of calculating exactly how $\rho$ changes with the level of investment; the individuals who own shares in the company after the change may be an entirely different group from those who owned it before the change. We shall assume that all individuals buy all securities.[26] It is easy to show that the constrained optimum requires

$$\frac{EU_1'X_j'}{EU_1'} + \sum_{2}^{m}\alpha_{ij}\left\{\frac{EU_i'X_j'}{EU_i'} - \frac{EU_1'X_j'}{EU_1'}\right\} = \frac{\text{constant}}{EU_1'}$$

Now, if the competitive industry follows our investment rule, it will set

$$EX' = \rho + \rho' V$$

But from the definition of $\rho$, under the assumption that no individual specializes, this is equivalent to

$$EU_i'X_j' = rEU_i' \qquad i = 1, \ldots, m$$

Thus constrained Pareto optimality requires

$$EU_1'X_j' = \text{constant}$$

---

[26] i.e., that all the marginal equalities rather than the relevant inequalities hold. This is the same assumption make by Diamond. If individuals can sell short, then, of course, the marginal equalities will always hold.

But under the competitive investment rule

$$EU_1'X_j' = rEU_1'$$

and, hence it is clear that the condition for Pareto optimality holds.

This result says, given the constraints of the market economy where an individual must get the same proportion of the returns from a company independent of the state of nature, the competitive economy following the above investment rule is Pareto Optimal. But from the point of view of social welfare, this constraint is not a very natural one. A socialist economy is not so constrained, and hence can in general improve the welfare of each individual in the economy.[27] This will not be true, of course, if the ratio of marginal utilities in state i to that in state j for all individuals is the same, e.g. if we have a sufficiently large number of securities to insure a "first best" Pareto optimal.[28]

We can summarize the results of this section in the following propositions:

Proposition 7: To maximize the price of a share, the firm must set

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[27] In fact, however, the socialist economy may face much more binding constraints, such as that the output in every state of nature be divided equally, or according to needs. This egalitarian rule, however, does not allow for the different attitudes of different individuals towards risk, and will, of course, result in a non Pareto ptimal situation.

[28] I am not convinced that the second-best Pareto optimality of the market is a particularly important property: (a) the linear consumption constraint can be removed by the introduction of adequate insurance markets; (b) its welfare implications are minimal, since a socialist economy is not so constrained; (c) if there are as many securities as states a first-best Pareto optimum can be attained.

$$EX' = \frac{dEX}{dA} = \rho + \rho'V$$

The increment in the expected return equal to the marginal cost of capital.

Proposition 8: If there are "constant returns to scale" in investment across the states of natures (multiplicative uncertainty) so

$$X(\theta, A) = X(\theta)I(A)$$

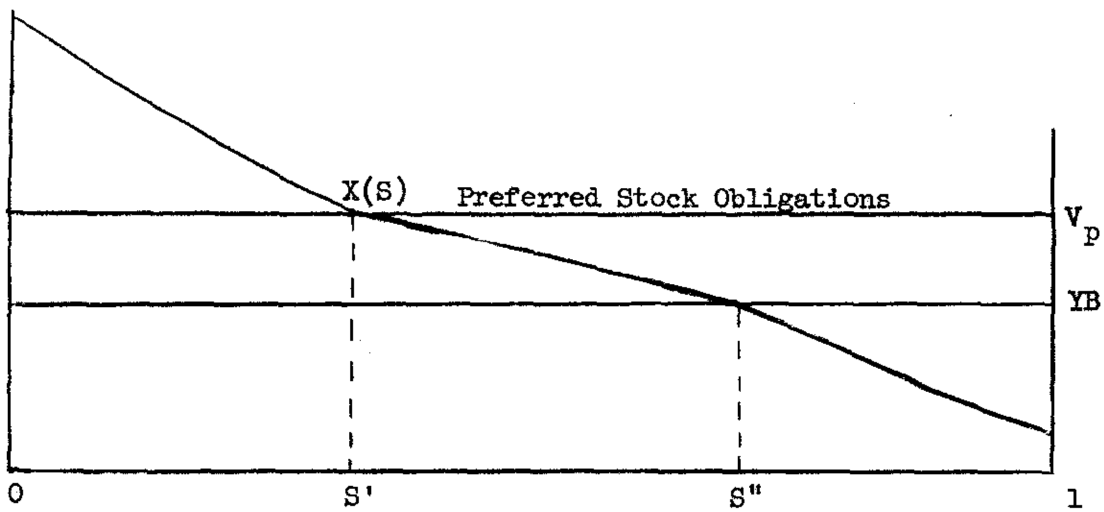
the average and marginal cost of capital are the same.

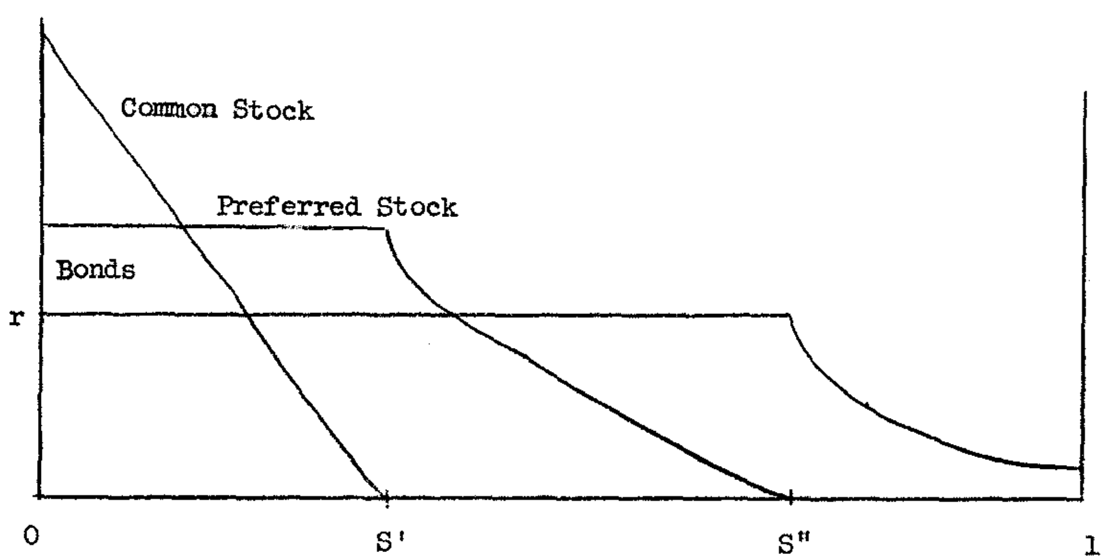
Proposition 9: Firms following the investment rule of Proposition 7, will lead the economy to a second best Pareto optimum, where the economy is constrained to give the same proportion of returns of the company to an individual independent of the state of nature. If there are fewer securities than states of nature, a socialist economy can in general make everyone better off without making anyone worse off (than in a competitive economy).

## 8. Concluding Comments: Perfect and Imperfect Capital Markets

We have all been brought up to believe that the sure test of a competitive market is that the price of the commodity an individual or firm sells or of the factor it buys be independent of the amount is sells or buys and be the same for all firms or individuals in the economy. On this basis, we have been led to believe that the capital market is imperfectly competitive: (a) as the firms issue more bonds the rate of interest it pays may go up; (b) individuals may have to pay a higher rate than firms, and some firms higher than others; (c) lending rates may differ from borrowing rates.

One of the purposes of this paper has been to show that this so-called evidence for an imperfect capital market is nothing of the sort (see also [21]). The crucial fallacy lies in the implicit assumption that one firm's bond is exactly the same commodity as another firm's bond, and that bonds a firm issues when it has a low debt-equity ratio and those which it issues when it has a high debt-equity ratio are the same commodities. But they are not. They give different patterns of returns. If there is any chance of default, a bond gives a variable return (i.e. is a risky asset) just as a security is, although it gives a very different pattern of return. In the figure below we have shown the patterns for a common stock, preferred stock, and bond. And just as there is no reason to expect butter, and cheese even though they are "related commodities" to have the same price, so there is no reason to expect the rate of interest when there is a low debt equity ratio to be the same where there is a high debt equity ratio. Indeed, we have shown that in a perfect capital market, as the number of bonds increases a firm will have to pay a higher nominal interest rate. Even the discrepancy between borrowing and lending rates does not imply imperfect capital markets. For when I lend to the bank, and my account is ensured by FDIC, I



assume there is a zero probability of bankruptcy. But when the bank lends back to me, it cannot make the same assumption.

The question then arises, how can be meaningfully define a "perfectly competitive" capital market. The usual characterization of a competitive market is a large number of firms selling the same commodity. But what if the bonds and securities of each corporation are really different commodities, i.e. give different patterns of income across the states of nature. The introduction of the concept of risk classes is one way of getting around this difficulty, since then there are a large number of securities giving the same income pattern. But the risk class notion is unnecessarily restrictive. The MM theorem holds as long as an individualcan borrow and lend at the market rate of interest in the no bankruptcy cases, or if he can borrow on collateral in the bankruptcy case. But even if individuals cannot borrow on collateral, if capital markets are competitive in the sense that the marginal utility of income in each state of the world is unaffected by the issue of a new security, and at least one group of individuals is willing to buy both bonds and stocks of a given corporation, the Modigliani-Miller theorem holds. The crucial question is to what extent these assumptions are reasonably approximated.

## References

[1] Arrow, K. J., "The Role of Securities in the Optimal Allocation of Risk Bearing," Review of Economic Studies, April 1964, pp. 91-96.

[2] __________, Some Aspects of the Theory of Risk Bearing.

[3] Baumol W. and B. Malkiel, "The Firm's Optimal Debt-Equity Combination and the Cost of Capital," Quarterly Journal of Economics, November 1967.

[4] Borch, K., "Equilibrium in a Reinsurance Market," Econometrica, July 1962.

[5] Brewer, D. E. and J. B. Michaelson, "The Cost of Capital, Corporation Finance, and the Theory of Investment: Comment," American Economic Review, June 1965, pp. 516-523.

[6] Debreu, G., Theory of Value, New York, 1959.

[7] Diamond, P., "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty," American Economic Review, September 1967, pp. 759-776.

[8] Durand, D., "Cost of Capital, Corporation Finance, and the Theory of Investment: Comment," American Economic Review, September 1967, pp. 639-655.

[9] Hirshleifer, J., "Investment Decision under Uncertainty Choice Theoretic Approaches," QJE, November 1965, pp. 509-536.

[10] __________, "Investment Decision under Uncertainty: Applications of the State-Preference Approach," QJE, May 1966, pp. 237-277.

[11] Lintner, J., "Dividends, Earnings, Leverage, Stock Prices and the Supply of Capital to Corporations," Review of Economics and Statistics, August 1962, pp. 243-269.

[12] Markowitz, H., Portfolio Selection, New York, John Wiley and Sons, 1959.

[13] Modigliani F. and M. H. Miller, "The Cost of Capital, Corporation Finance, and the Theory of Investment," American Economic Review, June 1958, pp. 261-297.

[14] __________, "Reply to Rose and Durand," American Economic Review, September 1959, pp. 665-669.

[15] ______________, "Corporate Income Taxes and the Cost of Capital: A Correction," American Economic Review, June 1963, pp. 433-443.

[16] ______________, "Reply to D. E. Brewer and J. B. Michaelson," American Economic Review, June 1965, pp.

[17] Mossin, J., "Equilibrium in a Capital Asset Market," Econometrica, Octooer 1966, pp. 768-783.

[18] Pratt, J. W., "Risk Aversion in the Small and in the Large," Econometrica, January-April 1964, pp. 122-136.

[19] Pye, G., unpublished Ph.D. dissertation, Department of Economics, Massachusetts Institute of Technology.

[20] Sharp, W. F., "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," The Journal of Finance, September 1964.

[21] Stigler, G., "Imperfections in the Capital Market," Journal of Political Economy, June 1967, pp. 287-293.