COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 239

ITERATIVE MULTI-LEVEL PLANNING WITH PRODUCTION TARGETS

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November 10, 1967

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1. INTRODUCTION

Because modern technological processes are so intricate, it is usually expedient, in a large centrally planned economic organization, to delegate responsibility. Formal mathematical models have been constructed which verify the intuitive notion that under certain assumptions the need for completely centralized knowledge can be obviated. Convergence to overall optimality can be achieved, these models show, if the appropriate information is iteratively calculated by each economic agent and transmitted to the others in the form of a suitable index. In many theoretical procedures the prospective indices sent out by the center are prices, while those received by it are in the form of quantities. Here the reverse order will be incorporated into an algorithm of the simplex family.

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* Most of the basic ideas reported here were worked out during the summer of 1967 while I was attending the Ford Foundation sponsored Berkeley Summer Workshop on Analytic Techniques for the Comparison of Economic Systems. Roy Radner and Edmond Malinvaud were especially helpful in criticizing an early draft. At Yale this work was supported by a grant from the National Science Foundation.

While such an algorithm may or may not be of interest as a computational device for mathematical programming, in this paper greater emphasis will be accorded its role as an abstract description of multi-level economic planning with incomplete information. In this context an algorithm which revolves around centrally prepared production quotas might be considered advantageous because quantity directives may be more appealing from a practical standpoint.

2. MOTIVATION

For many centrally planned economies, economic plans are prepared in accordance with the following rough format. As a result of past experience and a backlog of statistical information, the central planners possess an approximate but workable notion, usually in aggregate terms, of the technological possibilities confronting the various production units of the economy. Combining this knowledge with their own planners' preferences, highly tentative sets of control figures are prepared for key economic sectors. The control figures gradually seep down to the lowest economic echelons in the form of specific production quotas. Individual economic units will then engage in "counter planning," whereby they propose quota changes to their immediate superiors. Typically the basis for a proposed change in a quota is its alleged infeasibility. Economic units will attempt to convince their superiors that technological considerations preclude fulfillment of their assigned quota. In the process, they try to impart to the higher ups a notion of the technological constraints binding them and will often indicate the direction in which a new quota must move if it is to be producible ("we need at least _this_ much coal to produce _that_ much steel ..."). In turn, higher authorities attempt to cut slack from the padded production figures of their constituents. Soon after they are distributed, therefore, production figures start working their way back up the planning hierarchy so that inconsistencies can be resolved and slack removed. New targets are then reassigned on the basis of the increasingly accurate picture of overall production possibilities being continually revealed to the authorities by the planning process itself.

Especially at the highest level, target reassignment can be a complicated process, involving as it does the interaction of planners' preferences with intricate and continually changing reallocation possibilities. Eventually, when most of the quotas are neither overtight nor too slack, the plan will have converged to an operational stage and is ready to be implemented.

The principal aim of this study is to present a formal mathematical version of some aspects of the planning procedure just outlined and to examine its properties. Needless to say, a theoretical study of this sort cannot purport to reflect planning practice in any real economy. The aspect of reality most critically examined here, to the neglect of several others, is the learning game whereby the center iteratively comes closer to knowing the production sets as a result of the planning process itself.

3. A MODEL OF AN ECONOMY[1/]

The hypothetical economy studied here deals with n distinct and homogeneous commodities, identified by the subscript i taking the values 1 to n. Production is carried out by m distinct productive units or firms, indexed by the subscript k running from 1 to m. The n commodities under consideration refer only to items centrally traded and do not include commodities specific to any firm.

The net output of commodity i produced by firm k is denoted y_{ik}. It is negative if in fact firm k consumes this good. Firms transform inputs into outputs by laws of production indexed by the activities available to them. An activity is taken here in its most general sense to be merely the designation of one of the decision variables of the firm's production plan. The activity level of the j^{th} activity utilized by firm k is denoted v_{jk} $(j=1,\ldots,J_k)$.

Production possibilities for firm k are limited by a scarcity of fixed factors and other restraints which are reflected by the set of inequalities

$$f_{\ell k}(v_k, y_k) \leqq 0 \qquad \ell=1,\ldots, L_k \qquad (1)$$

[1/] This section describes some general concepts used in the modern theory of resource allocation and is necessarily brief. Fortunately some excellent references can be consulted. The entire framework including, whenever possible, the notation, has been adopted from Malinvaud [1967], which provides a general methodology for analyzing decentralized planning procedures and from which much of the inspiration for the present study has been derived. A comprehensive survey of the theory of resource allocation proper is contained in Koopmans [1957].

It is assumed that all of the functions $\{f_{\ell k}\}$ are differentiable[2]

The production set of all net outputs producible by firm k is denoted by Y_k and is formally defined as follows

$$Y_k \equiv \{y_k \mid v_k \text{ with } f_{\ell k}(v_k, y_k) \leqq 0 \text{ for } \ell = 1, \ldots, L_k\}$$

It is assumed that[3]

(i) Y_k is convex;

(ii) Y_k is closed and bounded from above;

(iii) if $y_k \in Y_k$ and $\hat{y}_k \leqq y_k$, then $\hat{y}_k \in Y_k$.

Final net output of commodity i is denoted by x_i. The final net output vector $\mathbf{x}$, which includes both consumption and investment goods, is feasible from the viewpoint of the planners if it belongs to a set X given *à priori* and assumed to be closed. The

[2] In addition, we assume that certain regularity conditions, the so-called "constraint qualifications" are met; Kuhn and Tucker [1951]. More general formulations, obviating the need for differentiability and phrasing the constraint qualifications in weaker terms are presented in Uzawa [1958].

[3] Assumption(i) is familiar from resource allocation theory and Koopmans [1957] should be consulted for an adequate discussion of its significance. We note here only that we are not requiring every operation performed within the firm to conform to the laws of decreasing or constant returns. We are merely presupposing that, together with possible decreasing returns in some operations, the "convexifying" effects of scarce fixed resources are strong enough to counteract the "deconvexifying" effects, if they are present, of increasing returns in other operations. The result is a convex set of input and output possibilities. Thus, the set of all vectors satisfying (1) need not be convex (if it were, we would not have to additionally postulate Y_k convex). Assumption(ii) can be thought of as being due essentially to the finiteness of fixed factors specific to firm k (like bolted-down capital). Assumption (iii) merely permits free disposal of commodities. The last two assumptions could be weakened but it would complicate the exposition without adding, in my opinion, much of economic significance.

relation of social preference is arithmetized by a welfare or utility function[4/], assumed to be continuous and defined for all $x \in X$.[5/] In addition, it is assumed that if $x \in X$, $\hat{x} \in X$ and $x \geqq \hat{x}$, then $U(x) \geqq U(\hat{x})$.

The resource stock of commodity i initially available to the economy is denoted ω_i.

The problem confronting the central planning agency is to [6/]

$$\text{maximize} \qquad U(x) \tag{2}$$

$$\text{subject to} \qquad x \in X \tag{3}$$

$$y_k \in Y_k \quad \text{for} \quad k = 1, \ldots, m \tag{4}$$

$$x \leqq \sum_{k=1}^{m} y_k + \omega \tag{5}$$

4/ It is obviously beyond the scope of this study to examine the conditions under which collective choices can be properly quantified. In this paper it will simply be postulated that social choices are representable by a welfare function which the planners know. Other important difficulties, including the problems of intertemporal choice, aggregation, veracity, and implementation are likewise being ignored here. A brief but excellent discussion of these matters is contained in Malinvaud [1967], sections I and II.

5/ Interestingly enough, this algorithm does not require that the welfare function $U(\ )$ be concave or that the set X be convex. I do not understand the practical implications for economic planning of this unorthodox feature; perhaps there are none.

6/ For the problem under consideration to be interesting we can neither assume a time period so short that the possibilities for substitution are negligible nor one so long as to warrant an explicit treatment of capital formation. An intermediate term plan, say of about four years' duration, is what we have in mind. This issue is discussed by Porwit [1963] pp. 8,9. For many East European socialist countries the outline of section 2 would actually be more appropriate as a description of short term planning; the intermediate term plan is often just a rough guideline and does not have the force of an operational document.

Under the assumptions made so far, this problem will possess a meaningful solution with maximum attainable utility U^*. The program $[x, y_1, \cdots, y_m]$ is called <u>feasible</u> if it satisfies (3), (4), (5). The program $[x^*, y_1^*, \ldots, y_m^*]$ is called <u>optimal</u> if it is feasible and if $U(x^*) = U^*$.

While the problem (2), (3), (4), (5) has been cast in a national planning setting, it should be clear that other interpretations are possible and that, in fact, many other important problems can be so structured. Even within the national planning framework, the concept of a firm is meant to be quite general. International trade, for example, could be accommodated by postulating two extra firms or departments. One would be in charge of exports, "producing" foreign exchange by "consuming" commodities sold abroad. The other, in charge of imports, "consumes" foreign exchange to "produce" commodities purchased from abroad. "Laws of production", for such firms would reflect supply and demand conditions on world markets.

4. THE IMPORTANT CONCEPT OF INCOMPLETE INFORMATION

Managers specialize in handling their own firm's problems and are therefore ignorant of the exact situation prevailing in other firms, of society's total available resources, or of the planners' preferences among net output possibilities. Nor are the managers of firm k likely to be explicitly aware of the set Y_k. Rather, they are accustomed to working directly with the activity constraints (1), and even these are familiar only for "customary" activity levels. Nevertheless, in the sense that they could map out the relevant sections of Y_k if they were asked to do so in an operationally meaningful way, the managers of firm k might be said to know it implicitly.

An analogous situation prevails at the level of the central planning agency. While the central planners can be considered to know explicitly the vector of available resources ω and the set of acceptable consumption vectors X, they are not likely to be acquainted with social welfare in the same way. However, it is assumed that, perhaps after some introspection, they can operationally choose unambiguously among various alternatives of social net output. In this sense, the planners can be thought of as implicitly possessing a utility function, even though such a function probably could not be explicitly displayed à priori.

When it comes to the activities $\{v_k\}$ specific to the firms, the center is considered to be completely ignorant. Nor does it know exactly the individual production sets $\{Y_k\}$ (otherwise the problem (2), (3), (4), (5) could be solved directly). But it would be unfair to characterize the planners as being completely ignorant of the production sets. From common sense awareness of what constitutes an unrealistic production possibility to copious statistics on present and past production performance, the central planning agency has ample ingredients for forming at least a moderately accurate picture of the technological options open to the firms. We can express this idea by saying that the planners are à priori aware of a closed, bounded-from-above[7] set of production possibilities for firm k, denoted Y_k°, such that

$$Y_k \subseteq Y_k^\circ \quad .$$

[7] Once again, the assumption of boundedness is excessively strong, but is retained for convenience.

If, for some reason, literally nothing were known about Y_k, the planners could always choose Y_k^o by fixing arbitrarily large positive bounds on the components of y_k.

5. A DECENTRALIZED PLANNING PROCEDURE

From what has just been said, it should be obvious that a workable planning algorithm cannot impose excessive informational requirements on any single economic agent. The approach taken here views the planning procedure as a learning process whereby the center iteratively comes to understand more and more exactly the relevant parts of the production possibilities sets without ever requiring any firm to transmit the entire set.

Suppose at stage s $(s=0,1,\ldots,S)$, the planners know of a closed bounded-from-above production set Y_k^s such that

$$Y_k \subseteq Y_k^s \quad .$$

At $s=0$, Y_k^s is given <u>à priori</u>; later it will become clear how a set with the required properties is recursively generated for other values of s. So far as the planners are aware, the set Y_k^s genuinely represents the technological options available to firm k. The central planning agency is therefore in a position to determine what it thinks is an optimal program $[x^s, q_1^s, \ldots, q_m^s]$ by solving the following master problem.

$$\text{Maximize} \quad U(x) \tag{6}$$

subject to

$$x \in X \tag{7}$$

$$q_k \in Y_k^s \tag{8}$$

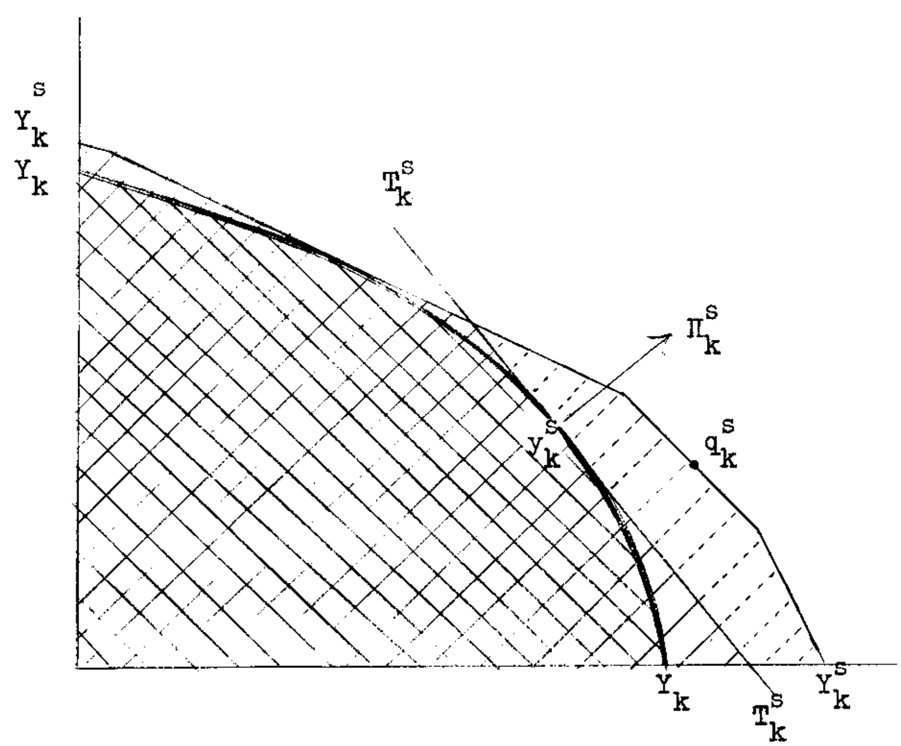
$$x \leqq \sum_{k=1}^{m} q_k + \omega \tag{9}$$

Under the assumptions, this is a well defined problem with maximum utility $U^s = U(x^s)$. The center now tries to impose the pseudo-optimal program $[x^s, q_1^s, \ldots, q_m^s]$ by assigning the vector q_k^s as a quota or target to firm k for $k=1,\ldots,m$. If, for each firm k, the quota q_k^s is producible $(q_k^s \in Y_k)$, the program $[x^s, q_1^s, \ldots, q_m^s]$ is also optimal for the center's original planning problem (2), (3), (4), (5). This follows immediately from the fact that $U^s \geqq U^*$ for all s. The planning agency thus has an easy way of identifying an optimal program when it has been attained.

If firm k cannot meet its target $(q_k^s \notin Y_k)$, it attempts to scale down the planners' overly optimistic impression of feasible input-output combinations so as to force them to reissue a new, hopefully feasible, target and to prevent them from repeating the error of thinking that the previous target is producible.

The situation can be geometrically portrayed in Figure 1. We formalize the notion that firm k, in order to show that q_k^s cannot be produced, imparts to the planners a knowledge of the more modest production alternatives <u>really</u> available to it by saying that the firm selects a hyperplane T_k^s tangent to Y_k which separate q_k^s from Y_k. Such a hyperplane is completely determined by specifying the point of tangency, y_k^s, and the normal to it at that point, Π_k^s, by the following definition,

$$T_k^s \equiv \{ y \mid \Pi_k^s y = \Pi_k^s y_k^s \} . \tag{10}$$



key: Y_k^s = area encompassing all positive sloping lines

Y_k^{s+1} = area encompassing all solid positive sloping lines

Y_k = cross-hatched area

FIGURE 1.
A Geometric Representation of the Production Target Procedure.

If $q_k^s \in Y_k$, define $\Pi_k^s \equiv \Pi_k^{s-1}$, and $y_k^s \equiv y_k^{s-1}$. In this case, the hyperplane T_k^s, while tangent to it, does not separate the production set Y_k from the quota q_k^s.

Let H_k^s stand for the half space defined by the tangent hyperplane T_k^s, i.e.,

$$H_k^s \equiv \{ q \mid \Pi_k^s q \leqq \Pi_k^s y_k^s \} .$$

By convexity, the planners know that Y_k must be contained within H_k^s, as well as within Y_k^s. We define Y_k^{s+1} as follows,

$$Y_k^{s+1} \equiv Y_k^s \cap H_k^s .$$

It will be true that

$$Y_k \subseteqq Y_k^{s+1} \subseteqq Y_k^s \cdots \subseteqq Y_k^1 \subseteqq Y_k^\circ ,$$

and that

$$U^* \leqq U^{s+1} \leqq U^s \ldots \leqq U^1 \leqq U^\circ .$$

In the case where $q_k^s \notin Y_k$, more than one hyperplane can be both tangent and separating. In addition to causing the algorithm to converge, such a hyperplane must, for the purposes of this paper, be selected as a result of an economically meaningful operation performed by the firms. In this context two methods for obtaining Π_k^s and y_k^s

seem to be especially interesting.[8/]

6. A "PRICE GUIDANCE" PROCEDURE

If the target q_k^s assigned by the master program (6), (7), (8), (9) is not feasible, firm k is given positive prices p_k^s and told to determine y_k^s and non-negative z_k^s which

$$\text{minimize} \qquad p_k^s z_k \tag{11}$$

$$\text{subject to} \qquad y_k \in Y_k \tag{12}$$

$$y_k + z_k \geqq q_k^s \tag{13}$$

Because q_k^s is not producible, the optimal value of the objective function must be positive. An interpretation of (11), (12), (13) is that the firms can purchase commodities to help meet their quotas at fixed transfer prices; their problem is to schedule production and arrange purchases so as to minimize the total "penalty cost" of meeting their quota. For the time being, the prices p_k^s are given exogenously. Later we discuss their significance in greater detail.

8/ However, others are certainly possible. It will not be difficult to see, for example, that various combinations of the two proposed variants would also work. Having experimented with other approaches, I can report that it would suffice to form a separating hyperplane from the optimal dual prices associated with minimizing any one of a variety of bona-fide infeasibility forms or distance measures (distance, that is, from the infeasible point to points in the production set), of which the two selected for detailed study are special cases. Unfortunately, some otherwise plausible distance measures would probably not fare well as devices of administrative control. For example, it might be needlessly difficult for the manager of a firm to choose a production point which minimizes the Euclidean distance from the assigned quota simply because he has little understanding of what it means. This is why we break away from mathematical generality at this time and confine ourselves to the further study of two ideas which seem somewhat more plausible from an organization viewpoint.

Firm k sees its problems as that of determining activity levels v_k^s, net outputs y_k^s, and <u>non-negative</u> purchases z_k^s to

$$\text{minimize} \qquad p_k^s \; z_k \tag{11}$$

$$\text{subject to} \qquad f_k(v_k, y_k) \leqq 0 \;\} \; \phi_k^s \tag{14}$$

$$y_k + z_k \geqq q_k^s \;\} \; \Pi_k^s \tag{13}$$

We annotate optimal values of the relevant variables at stage s with the superscript s . Define $f_k^s = f_k\,(v_k^s\,,\, y_k^s)$. Associated with an optimal solution of (11), (14), (13) are <u>non-negative</u> price vectors ϕ_k^s and Π_k^s such that, in vector notation, the following equations are fulfilled, [9]

$$\phi_k^s \, f_k^s = 0 \tag{15}$$

$$\Pi_k^s \,(y_k^s + z_k^s - q_k^s) = 0 \tag{16}$$

$$(p_k^s - \Pi_k^s)\, z_k^s = 0 \tag{17}$$

$$\Pi_k^s \leqq p_k^s \tag{18}$$

$$\phi_k^s \, \frac{\partial f_k^s}{\partial v_k^s} = 0 \tag{19}$$

$$\Pi_k^s = \phi_k^s \, \frac{\partial f_k^s}{\partial y_k^s} \tag{20}$$

9/ These conditions are derived in Kuhn and Tucker [1951]. See also Uzawa [1958].

The notation means, e.g., that the $(\ell, j)^{\text{th}}$ entry of the matrix

$$\frac{\partial f_k^s}{\partial y_k^s} \quad \text{is} \quad \left.\frac{\partial f_{\ell k}}{\partial y_{jk}}\right|_{\substack{v_k = v_k^s \\ y_k = y_k^s}}$$

Proposition 1. A vector Π_k^s which satisfies (15) - (20) is the normal to a hyperplane passing through y_k^s, tangent to Y_k, and separating q_k^s from Y_k.

That the vector Π_k^s satisfying (15) - (20) is the normal to the desired separating hyperplane can be a great convenience because optimal dual prices will often be automatically available as a by-product of the calculations which it was necessary to perform in order to obtain a solution to (11), (12), (13) in the first place. This is certainly the case with the simplex algorithm. Institutionally, of course, dual prices can be interpreted as marginal products.

proof:

$$\Pi_k^s y_k^s = \Pi_k^s q_k^s - \Pi_k^s z_k^s = \Pi_k^s q_k^s - p_k^s z_k^s < \Pi_k^s q_k^s$$

It remains to be demonstrated that if $y_k \in Y_k$, then $\Pi_k^s y_k \leqq \Pi_k^s y_k^s$.

Suppose not. Suppose there exists a vector $\tilde{y}_k$ such that $\tilde{y}_k \in Y_k$ and $\Pi_k^s \tilde{y}_k > \Pi_k^s y_k^s$. Define a vector function of λ, $y_k(\lambda)$, as follows

$$y_k(\lambda) \equiv \lambda \tilde{y}_k + (1-\lambda) y_k^s$$

Define $v_k^*(\lambda)$ and $R_k^*(\lambda)$ to be any solution of the problem

minimize $\| R_k(\lambda) \|^2$

$$\text{subject to} \quad f_k(v_k(\lambda),y_k(\lambda)) = f_k^s + \frac{\partial f_k^s}{\partial v_k^s}(v_k(\lambda) - v_k^s) + \frac{\partial f_k^s}{\partial y_k^s}(y_k(\lambda) - y_k^s) + R_k(\lambda) \qquad (21)$$

$$f_k(v_k(\lambda), y_k(\lambda)) \leqq \phi \qquad (22)$$

This is a meaningful problem for values of λ between 0 and 1 since, from convexity, $y_k(\lambda) \in Y_k$ for $0 \leqq \lambda \leqq 1$. It is tedious, but not difficult, to show that each component of $R_k^*(\lambda)$ must be of order λ^2 .

Using the fact that $\phi_k^s \geqq 0$ coupled with (15), (19), (20), (21), (22), we have that

$$0 \geqq \phi_k^s f_k(v_k^*(\lambda), y_k(\lambda))$$

$$= \phi_k^s f_k^s + \phi_k^s \frac{\partial f_k^s}{\partial v_k^s}(v_k^*(\lambda) - v_k^s) + \phi_k^s \frac{\partial f_k^s}{\partial y_k^s}(y_k(\lambda) - y_k^s) + \phi_k^s R_k^*(\lambda)$$

$$= \Pi_k^s (y_k(\lambda) - y_k^s) + \phi_k^s R_k^*(\lambda)$$

$$= \Pi_k^s (\lambda \tilde{y}_k + (1-\lambda) y_k^s - y_k^s) + \phi_k^s R_k^*(\lambda)$$

$$= \lambda \Pi_k^s (\tilde{y}_k - y_k^s) + \phi_k^s R_k^*(\lambda)$$

Since $\Pi_k^s(\tilde{y}_k - y_k^s) > 0$, passing to the limit as $\lambda \to 0$ provides the needed contradiction and concludes the proof.

The procedure under study would not be of much interest if it did not, in some sense, move closer and closer to an optimum. While pointwise convergence of this algorithm is not to be expected under the circumstances, it is sufficient to require that it converge to the production sets. Convergence to the production sets means that if $\bar{q}_k$ is a limit point of the sequence $\{q_k^s\}_{s=0,1,\ldots}$, then $\bar{q}_k \in Y_k$.

Proposition 2. If the (as yet arbitrarily specified) price sequence $\{p_k^s\}$ is bounded away from infinity ($p_{ik}^s \leqq M < \infty$ for some M, all i,k, s), and away from zero ($p_{ik}^s \geqq \epsilon > 0$ for some ϵ, all i, k, s), then the price guidance procedure is convergent to the production sets and $\lim_{s\to\infty} U^s = U^*$.

proof: Because each member of the sequences $\{q_k^s\}$, $\{y_k^s\}$, $\{p_k^s\}$, $\{\Pi_k^s\}$ belongs to a closed and bounded set, the existence of convergent subsequences is assured. Thus, if a subsequence of $\{q_k^s\}$ converges to $\bar{q}_k$, there must be at least one subsequence $\{q_k^t\}$, $\{y_k^t\}$, $\{p_k^t\}$, $\{\Pi_k^t\}$ of the original sequence, members of which are superscripted with the index t, such that $\lim_{t\to\infty} q_k^t = \bar{q}_k$, $\lim_{t\to\infty} y_k^t = \bar{y}_k$, $\lim_{t\to\infty} p_k^t = \bar{p}_k > 0$, and $\lim_{t\to\infty} \Pi_k^t = \bar{\Pi}_k$.

Suppose that $\bar{q}_k \notin Y_k$. From the separating hyperplane property,

$$\bar{\Pi}_k \bar{q}_k > \bar{\Pi}_k \bar{y}_k$$

Because of the way in which the sets $\{Y_k^s\}$ are defined, $\Pi_k^t q_k^{t+1} \leqq \Pi_k^t y_k^t$.

Passing to the limit as $t \to \infty$,

$$\bar{\Pi}_k \bar{q}_k \leqq \bar{\Pi}_k \bar{y}_k$$

This contradiction establishes that $\bar{q}_k \in Y_k$.

Since every limit point of $\{q_k^s\}$ must belong to Y_k, there exists a subsequence each member of which converges to a feasible plan $(\bar{x}, \bar{q}_1, \ldots \bar{q}_m)$. Because U^s is monotonically decreasing and $U^s \geqq U^*$ for each s, $\lim_{s \to \infty} U^s = U(\bar{x}) \geqq U^*$. But $\bar{x}$ is producible, implying $U(\bar{x}) \leqq U^*$. Thus, $\lim_{s \to \infty} U^s = U^*$.

<u>Proposition 3.</u> If, in addition to the other assumptions previously made, the production set Y_k is assumed polyhedral, the price guidance procedure converges in a finite number of stages.

The premises of this proposition would be fulfilled if the functions $f_{\ell k}(\ )$ of equation (1) were all linear. In this case, the firms possess what is often called a "linear programming technology", and the dual prices Π_k^s are readily obtainable from the simplex tableau.

<u>proof</u>: From Proposition 1, T_k^s as defined by equation (10) is a hyperplane tangent to Y_k. At every stage s for which the algorithm has not yet converged, there is at least one firm k such that $q_k^s \notin Y_k$. We now show that the facet $T_k^s \cap Y_k$ is different from the facet $T_k^r \cap Y_k$ for all $r < s$. Since $\Pi_k^r z_k^s \geqq 0$, two

cases can be distinguished:

$$(i) \quad \Pi_k^r z_k^s > 0$$

Then $\Pi_k^r (q_k^s - z_k^s) < \Pi_k^r q_k^s \leqq \Pi_k^r y_k^r$. Since $(q_k^s - z_k^s)$ belongs to the set $T_k^s \cap Y_k$ but not to T_k^r, it must be that $T_k^s \cap Y_k \neq T_k^r \cap Y_k$.

$$(ii) \quad \Pi_k^r z_k^s = 0$$

Because $\Pi_k^s z_k^s = p_k^s z_k^s > 0$, the following must hold simultaneously for at least one component i : $\Pi_{ik}^r = 0$, $z_{ik}^s > 0$, $\Pi_{ik}^s > 0$. Let u_i be an n-vector with the i^{th} component positive and every other component equal to zero. By the assumption of free disposal, the vector $(y_k^r - u_i)$ belongs to Y_k, and hence to $T_k^r \cap Y_k$, but not to T_k^s, and <u>à fortiori</u> not to $T_k^s \cap Y_k$.

Since there are only a finite number of facets for each production set and every stage calls forth at least one new facet, the procedure must terminate after a finite number of stages.[10]

While everything so far has been proved for arbitrary sequences of administered prices $\{p_k^s\}$ (provided only that they are bounded away from zero and infinity), it is more natural to think in terms of some candidates than others. A superior choice would appear to be the dual

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[10] If at stage s, $U^s < U^{s-1}$, non active constraints can be dropped from the master program without affecting the property of finite convergence -- they will be regenerated later if they are needed. This remark will also apply to the "quantity guidance" procedure, yet to be discussed.

prices associated with equation (9) of the master program. The price received by all the firms would be identical for a given commodity, reflecting marginal conditions throughout the economy in the limit as s approaches infinity, and approximating them before the limit is reached. Such prices would presumably help guide infeasible quotas toward feasibility in a way which would do minimal damage to overall utility, and for this reason the algorithm might be expected to be efficient.

Other possibilities readily suggest themselves. In a one product firm, the center might fix inputs at the quota level (by implicitly setting the prices of purchased inputs at very high values) and ask for the maximum attainable output. The opposite case is also conceivable -- fix output at the quota level (by setting its price at an arbitrarily high value) and ask for that combination of inputs which minimizes the total cost of inputs over and above the alloted quota. Or, one could envision a procedure that assigned fixed quotas for some commodities (perhaps allocatable primary resources like labor) by implicitly setting high administered prices and allowed the firms themselves to choose all other purchases by minimizing costs of fictitiously imported commodities. The common denominator of all these variants is the use, whether explicit or implicit, of a price p_k^s which is applied to excess demands over a target q_k^s in order to elicit a marginal productivity assessment Π_k^s from the firms.[11]

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[11] If $z_{ik}^s > 0$, then $p_{ik}^s = \Pi_{ik}^s$. Thus, it is not necessary to have the firms report back the marginal productivity of commodities which are purchased in positive amounts.

7. A "QUANTITY GUIDANCE" PROCEDURE

Given a producible production point $d_k^s \in Y_k$, firm k is told to determine y_k^s and λ_k^s to

$$\text{maximize} \qquad \lambda \qquad (23)$$

$$\text{subject to} \qquad y_k \in Y_k \qquad (12)$$

$$y_k \geqq \lambda\, q_k^s + (1-\lambda)\, d_k^s \qquad (24)$$

Firm k sees its problem with (14) replacing (12) .

Since d_k^s is producible, $\lambda_k^s \geqq 0$. To insure that the constraint qualification is fulfilled, d_k^s is required, for some $\delta > 0$, all k and s, to be a distance of at least δ from any plane passing through q_k^s and tangent to Y_k. This condition is superfluous for a linear programming technology.

The objective function (23) , along with (24) is of a form first popularized by Kantorovich.[12] The "minimum attainable output level" d_k^s is, for the time being, regarded as a datum.

<u>Proposition 4.</u> The dual price vector Π_k^s associated with equation (14) in the system (23), (14), (24) serves as the normal to a tangent hyperplane passing through y_k^s and, if $q_k^s \notin Y_k$, separating Y_k from q_k^s .

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[12] Kantorovich [1959].

proof: The argument establishing (Π_k^s , y_k^s) as a tangent hyperplane to the set Y_k is so similar to the proof of Proposition 1 that it is omitted here.

From duality theory, $\Pi_k^s(q_k^s - d_k^s) = 1$, and $\Pi_k^s y_k^s = \lambda_k^s \Pi_k^s q_k^s + (1 - \lambda_k^s) \Pi_k^s d_k^s$, with $\lambda_k^s < 1$ if $q_k^s \notin Y_k$. It follows that $\Pi_k^s y_k^s < \Pi_k^s q_k^s$, showing if $q_k^s \notin Y_k$, that q_k^s is separated from Y_k by a hyperplane tangent to Y_k and concluding the proof.

Proposition 5. Under the assumptions made, the quantity guidance procedure converges to the production sets and $\lim_{s\to\infty} U^s = U^*$.

proof: We need only prove convergence to the production sets, since the proof that $\lim_{s\to\infty} U^s = U^*$ is identical to the one presented in Proposition 2. As in that proposition, for any limit point $\bar{q}_k$ of $\{q_k^s\}$, there exist subsequences $\{q_k^t\}$, $\{y_k^t\}$, $\{d_k^t\}$, $\{\Pi_k^t\}$, and $\{\lambda_k^t\}$ superscripted with the index t such that $\lim_{t\to\infty} q_k^t = \bar{q}_k$, $\lim_{t\to\infty} y_k^t = \bar{y}_k$, $\lim_{t\to\infty} d_k^t = \bar{d}_k$, $\lim_{t\to\infty} \Pi_k^t = \bar{\Pi}_k$, and $\lim_{t\to\infty} \lambda_k^t = \bar{\lambda}_k$.

If $\bar{q}_k \notin Y_k$, then $\bar{\lambda}_k < 1$. From the separating hyperplane property,

$$\bar{\Pi}_k \bar{q}_k > \bar{\Pi}_k \bar{y}_k$$

From the way in which the sets Y_k^s are defined in the master problem, $\Pi_k^t q_k^{t+1} \leqq \Pi_k^t y_k^t$ for all t . Passing to the limit as $t \to \infty$,

$$\bar{\Pi}_k \bar{q}_k \leqq \bar{\Pi}_k \bar{y}_k$$

This contradiction establishes that $\bar{q}_k \in Y_k$.

<u>Proposition 6</u>. If in addition to the other assumptions previously made, the production set Y_k is assumed polyhedral, the "quantity guidance" procedure converges in a finite number of stages.

<u>proof:</u> As before, T_k^s is defined by equation (10). At every stage s for which the algorithm has not yet converged, there is at least one firm k such that $q_k^s \notin Y_k$, implying $0 < \lambda_k^s < 1$. Let r be any stage previous to s. We now show that the facet $T_k^s \cap Y_k$ is different from the facet $T_k^r \cap Y_k$.

This follows immediately if

$$\Pi_k^r y_k^r \neq \Pi_k^r (\lambda_k^s q_k^s + (1-\lambda_k^s) d_k^s)$$

because a point belonging to $T_k^s \cap Y_k$ does not belong to $T_k^r \cap Y_k$. Suppose, therefore, that

$$\Pi_k^r y_k^r = \Pi_k^r(\lambda_k^s q_k^s + (1-\lambda_k^s) d_k^s) .$$

Because $\Pi_k^r y_k^r \geqq \Pi_k^r q_k^s$ (from the way q_k^s must be chosen) and $\Pi_k^r y_k^r \geqq \Pi_k^r d_k^s$ (from the fact that (Π_k^r, y_k^r) is a hyperplane tangent to Y_k), it must be true that $\Pi_k^r y_k^r = \Pi_k^r q_k^s = \Pi_k^r d_k^s$.

For any number μ,

$$(\lambda_k^s - \mu) \Pi_k^r q_k^s + (1-\lambda_k^s + \mu) \Pi_k^r d_k^s$$

is equal to $\Pi_k^s y_k^r$ independent of μ. However, because $\Pi_k^s q_k^s > \Pi_k^s d_k^s$ (as was shown in the proof of the separating hyperplane property),

$$(\lambda_k^s - \mu)\, \Pi_k^s\, q_k^s + (1-\lambda_k^s + \mu)\, \Pi_k^s\, d_k^s$$

is a decreasing function of μ, equal to $\Pi_k^s y_k^s$ for $\mu = 0$.

Thus, for any μ satisfying $0 < \mu \leqq \lambda_k^s$ the point

$$(\lambda_k^s - \mu)\, q_k^s + (1-\lambda_k^s + \mu)\, d_k^s$$

belongs to $T_k^r \cap Y_k$ but not to T_k^s. This concludes the proof.

While there are many possible candidates for $\{d_k^s\}$, some are susceptible of a more interesting economic interpretation than others. Assuming a single product firm and choosing d_k^s to be the origin is equivalent to working with a command system in which the firms are required to maximize output but must consume inputs only in certain fixed proportions as prescribed by the vector q_k^s. This is reminiscent of those planning procedures involving "input norms" which are familiar to many students of centrally planned economies.

Another (but more complicated) procedure that might be expected to work efficiently is to choose d_k^s as the solution to the problem

maximize	$U(x)$	(25)
subject to	$x \in X$	(26)
	$x \geqq \sum_{k=1}^{m} d_k + \omega$	(27)
	$d_k \in D_k^s$	(28)

$$D_k^s = \sum_{r=0}^{s} \lambda_k^r y_k^r \tag{29}$$

$$\sum_{r=0}^{s} \lambda_k^r = 1 \tag{30}$$

$$\lambda_k^r \geqq 0 \tag{31}$$

for $k=1,\ldots,m$.

The problem (25) - (31) means that the center is choosing d_k^s to be the best production point from a set of convex combinations, each member of which set the planners know from convexity must be producible. A procedure forming d_k^s in this way might involve fewer stages because the center is working with a more accurate notion of the true production set Y_k, sandwiched between Y_k^s containing it and D_k^s contained by it. On the other hand, a disadvantage of this technique is that at each stage <u>two</u> central master programs must now be solved, instead of one.

8. SOME GENERAL REMARKS ON THE PRODUCTION TARGET PROCEDURE

The following comments will pertain to the general method of iteratively approximating production sets by tangent hyperplanes, called the production target procedure, which was outlined on pages 10, 11,12. The price guidance and quantity guidance algorithms are but two ways of implementing this approach, by employing specific infeasibility or distance functions.

It may be of interest to contrast the production target procedure with another model of decentralized planning which has been discussed in the literature. An algorithm first proposed by Dantzig and Wolfe [13] which was applied to an economic planning setting by Malinvaud [14] is an example of a type of procedure whereby the center approximates a production set by building it up from the inside, taking convex combinations of those feasible points which are recursively generated as part of the algorithm. The master program in the Dantzig-Wolfe-Malinvaud (D-W-M) approach is of the form (25) - (31). The procedure presented here is dual to the D-W-M approach in several respects. Here the production set is reconstructed via tangent hyperplanes (rather than boundary points as with D-W-M) and the center becomes progressively less (rather than more) optimistic about attainable utility because the production possibilities sets revealed to it are continually being narrowed down (rather than expanded out). In the D-W-M procedure, the center announces prices and the firms respond with quantities; the reverse sequence is more nearly the case with the procedure presented here.

In the linear programming case, a cutting plane algorithm which is somewhat similar to the one presented here can be shown to be formally dual to the D-W procedure in the sense that either one could

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[13] Dantzig and Wolfe [1961]

[14] Malinvaud [1967]

be derived from the other. [15] From our point of view, the chief deficiency of both these linear programming algorithms is that, because they have been created as computational aids and are dependent upon the simplex method for proving convergence, they do not lend themselves easily enough to a meaningful economic interpretation. It is important, as Malinvaud has done for D-W, to free the algorithmic approach of what to the economist is an unfortunate dependence upon primarily mathematical or computational concepts.[16]

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[15] We sketch an outline of how to obtain such an algorithm. First, find the dual to the original D-W problem. Introducing new artificial variables, rearrange the dual problem so as to be able to apply D-W decomposition to it. Dualize both the master and subproblems of this decomposition procedure back to primal form. After interpreting it, one has arrived at a cutting plane technique which looks somewhat similar to the one developed here. It can also be viewed as a generalization of the partitioning algorithm introduced by Benders [1962] and would be identical to it in the case of a one firm economy. This cutting plane algorithm differs from the one presented here because it requires the firms to maximize their part of the objective function for a given quota, a command which is vacuous unless the objective is a separable function of each firm's activities.

[16] For example, if a firm does not possess a unique profit maximizing combination the D-W approach calls for it to report back nevertheless a profit maximizing basic feasible solution (vertex of the simplex). Such an order would unfortunately not be operational in an organizational context. Finite convergence could be proved for the case of a polyhedral production set without ever explicitly requiring firms to report back vertices or extreme rays by utilizing the notion of a non-repeating facet, as was done here. The economist's need for re-writing is especially acute in the cutting plane approach because the main interest vis à vis national economic planning focuses precisely on the special case where the utility function is not linear and separable among the firms. In the language of activity analysis, the objective is a function of net aggregated output and not of individual activity levels per se. In this case the only real issue concerns the feasibility of a proposed quota; there is no natural objective function for the firms to maximize given their quotas. If a quota is infeasible, the dual sub-problem is unbounded. The dual cutting plane approach would then require the firms to report back a homogeneous basic feasible dual solution which could be made indefinitely negative (a member of an extreme ray of the set of dual feasible solutions). Such an order would, of course, be difficult for a manager to interpret, much less carry out, and we have endeavored in this paper to show how a more palatable sequence of economic interactions would also lead to convergence.

While a polyhedral production set can be described either as the intersection of half spaces formed by tangent hyperplanes or as the convex combination of extreme points, in more than two dimensions typically far fewer tangent hyperplanes than extreme points would be required. For this reason, at least in the case of polyhedral production sets, it might be hoped that the procedure presented here would converge in fewer stages than the D-W-M approach. However, from a programming point of view the subproblem and perhaps also the master may be more difficult to solve in the production target procedure. 17/

Although the production target algorithm has been shown to converge in the limit as the number of stages goes to infinity, any real-life planning procedure must cease after a finite number of stages. In practice, the central planning agency could probably call a halt to the proceedings whenever quotas were no longer overtight. For all practical purposes, this would undoubtedly be sufficient because in the real world the boundary of a production set is hardly an exact entity anyway. As far as the mechanics of the algorithm are concerned, the center could terminate at any stage by taking the best convex combination of previously proposed

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17/ While both share in common a rough similarity in the message sequencing-quantities from the center and marginal products from the firms -- this algorithm differs significantly from that proposed by Kornai and Liptak [1963]. Their algorithm is based on the method of fictitious play, a successive approximations approach, whereas the production target procedure is based on programming considerations not unlike those underlying the simplex method. Also, the K-L approach works only for an objective function which is separable among the firms.

production points as in (25) - (31). This would be the only time such a master problem would have to be solved. So long as at least one set of previously proposed production combinations satisfied (26) - (31) (which would, incidentally, also have to be the case for the proper operation of D-W-M), the utility attained as a result of solving the "termination problem" (25) - (31) would have to increase monotonically with the number of stages. In the sense that realizable utility monotonically increases, the production target algorithm, with the termination modification just described, could be thought of as having one of the advantages usually attributed to a primal algorithm.

In an institutional setting, we could dispense with such an exact formalization as has been postulated here. The basic idea is that the firms must correct the center's exaggerated notion of their technology sets in a way that leads to convergence. Whether this is done by relaying one separating hyperplane or several, formal curvilinear surfaces or mere verbal descriptions, is not important so long as it achieves the desired effect. The relevant feedback mechanism for the general case is flow-charted in Figure 2.

Finally we note that although everything in this paper has been presented in terms of but <u>two</u> levels of organization, represented symbolically by the center and the firms, generalization to three or more levels is certainly possible. While it is not examined in the present paper, such an extension contains an interesting interpretation in terms of a quota system with telescoped command levels.

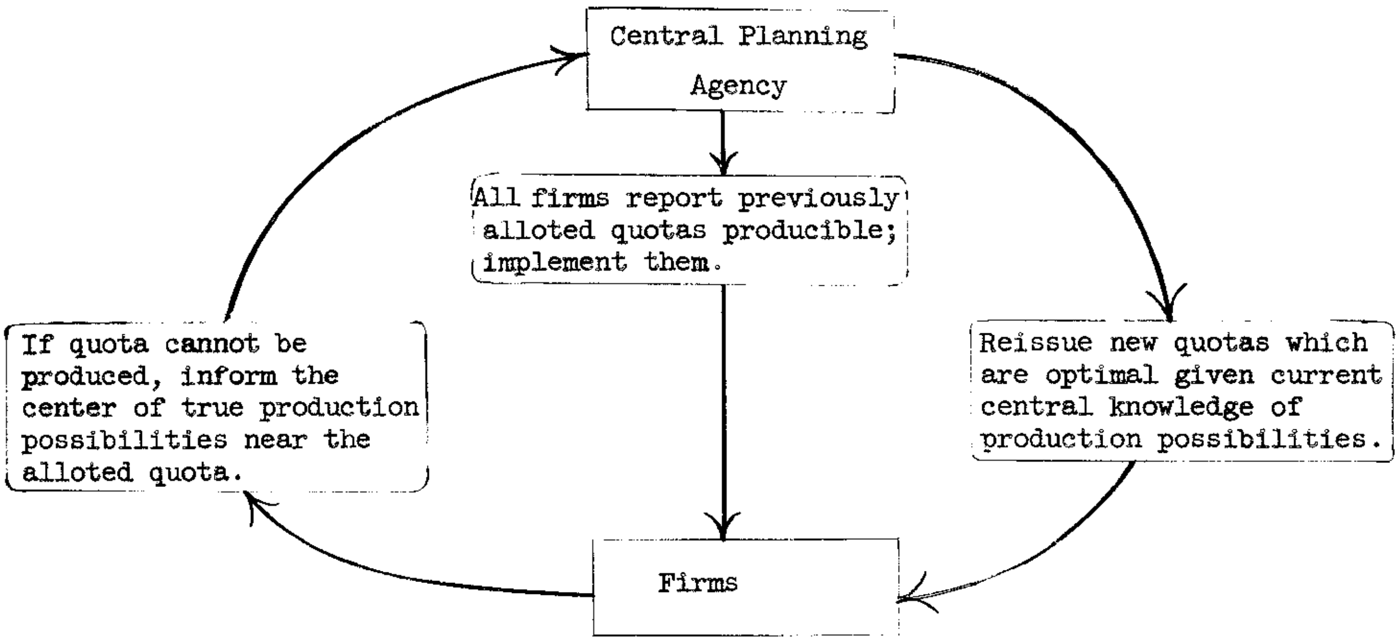

Figure 2.

The Flow of Information in the Production Target Procedure.

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