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ON THE COMPUTATION OF EQUILIBRIUM PRICES

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by

Herbert Scarf[*]

1. Introduction

In Mathematical Investigations in the Theory of Value and
Prices, published in 1892, Irving Fisher described a mechanical and
hydraulic analogue device intended to calculate equilibrium prices
for a general competitive model. The present paper takes up the
same problem, and discusses an algorithm for a digital computer which
approximates equilibrium prices to an arbitrary degree of accuracy.

At least two versions of Fisher's device were actually
constructed and apparently performed successfully. The devices them-
selves have unfortunately been lost, but there are several photo-
graphs which may be seen in the edition of Fisher's volume reprinted
in 1961 by Yale University Press.

The equipment seems remarkably quaint and old-fashioned
in this era of high speed digital computers. Immersed in a large
tub filled with water are a number of canisters whose irregular
profiles are related to the consumers' marginal utilities for the

various commodities. Each canister is constructed partly of flex-
ible leather, looking somewhat like a bellows, which expands and
contracts in response to changes in prices. The canisters are con-
nected by an elaborate system of rods, hinges, and tubes filled
with water.

In order to specify the consumers' initial dollar incomes,
a row of plungers is adjusted to specific heights, and in the pure
exchange model, a similar series of adjustments is made to provide
information about the initial stocks of commodities before trade
takes place. The competitive price levels and allocations are then
determined when the system reaches a physical state of equilibrium.

In order to avoid elaborate engineering problems, Fisher
found it necessary to assume that the utility functions could be
written in a separable form so that the marginal utility of any
commodity was independent of the level of consumption of the remain-
ing commodities. In the model to which the algorithm of this paper
is applied, there is no need for an assumption of separable utility.
Each consumer will have a set of demand functions, which are con-
tinuous and homogeneous of degree zero in all prices, and in addi-
tion, a given vector of commodities which are owned prior to produc-
tion and trade. Fisher assumes that a specific dollar income appears
on the right hand side of each individual's budget constraint; recent
authors have preferred to work with a more general model in which
income is derived from the sale of factors whose prices are to be

determined at equilibrium.

Assuming in addition that no income is generated by profits arising from production, the market demand functions satisfy Walras's law, to the effect that the market value of demand at any set of prices is equal to the value of the stock of initially owned commodities.

Let the market demand functions be denoted by

$$\xi_1(\pi_1, \ldots, \pi_n)$$
$$\vdots$$
$$\xi_n(\pi_1, \ldots, \pi_n) \, ,$$

with $(\pi_1, \ldots, \pi_n)$ the vector of prices. Since the demand functions are naturally homogeneous of degree zero, it is sufficient to assume that they are defined only for prices which are nonnegative and sum to 1, and continuous on this set of prices. If the total stock of commodities prior to production and trade is given by the nonnegative vector $(w_1, \ldots w_n)$, then the Walras law states that $\pi_1 \xi_1(\pi) + \ldots + \pi_n \xi_n(\pi) \equiv \pi_1 w_1 + \ldots + \pi_n w_n$ identically for all prices.

While the consumption side of the economy is treated by Fisher in an essentially modern fashion, the model of production seems quite inadequate. There is no production function or transformation set, but instead Fisher uses a notion of the "marginal disutility of production" which is to be equated to the negative of

the marginal utility of consumption for each consumer and commodity. Moreover, factors do not seem to be used up in production nor is income generated by the sale of productive factors.

The algorithm of the present paper permits production to be described by an arbitrary activity analysis model with a finite list of activities. Each column of the matrix

$$
A = \begin{bmatrix}
-1 & 0 & \cdots & 0 & a_{1,n+1} & \cdots & a_{1,m} \\
0 & -1 & \cdots & 0 & & \vdots & \vdots \\
0 & 0 & \cdots & -1 & a_{n,n+1} & & a_{n,m}
\end{bmatrix}
$$

will refer to a specific productive process with inputs indicated by negative numbers and outputs by positive numbers.

The economy is assumed to be completely described by the market demand functions $\{\xi_i(\pi)\}$, the technology matrix A , and the stock of factors $(w_1, \ldots, w_n)$ prior to production. With this notation a competitive equilibrium is defined by a vector of prices $\pi_1, \ldots \pi_n$, and a collection of nonnegative activity levels, $x_1, \ldots x_m$, such that the following two sets of conditions are met:

1. Supply equals demand in each market, or mathematically,

$$
\xi_i(\pi) - \sum_{j=1}^{m} a_{ij} x_j = w_i \quad \text{for} \quad i = 1, \ldots n .
$$

2. Profit is maximized at the prices π , or

$$\sum_{i=1}^{n} \pi_i a_{ij} \leq 0 \quad \text{for all} \quad j = 1, \ldots m ,$$

with equality if $x_j > 0$.

The existence of equilibrium prices for models of this type has been investigated with great thoroughness by a number of writers, (for example, [1] or [3]) and we now know that if certain relatively mild conditions are placed upon the specification of the model, there will indeed be an equilibrium. In our model the conditions are quite simple. In addition to the assumptions already made, we require that the set of activity levels which gives rise to a nonnegative net supply of all commodities, forms a bounded set. In symbols, the set of nonnegative $(x_1, \ldots x_k)$ for which

$$\sum_{j=1}^{n} a_{ij} x_j + w_i \geq 0 \quad \text{for all} \quad i ,$$

shall be a bounded set. The assumption $w_i > 0$ for all i will also be used occasionally even though it can be replaced by more realistic requirements.

In order to demonstrate the existence of equilibrium prices in a model of this generality, it has been customary to make use of what are known as "fixed point" theorems, which describe conditions under which a continuous mapping leaves at least one point unchanged.

The arguments leading from a fixed point theorem to the existence
of equilibrium prices are often quite direct and economically sug-
gestive; they have, however, a major drawback of offering no reason-
able suggestion for the computation of such prices.

It may seem somewhat surprising, in view of the substan-
tial body of work in mathematical programming, that no techniques
have been proposed for what is one of the central problems in economic
theory -- the computation of equilibrium prices. This is undoubtedly
due to the preoccupation with models which are exclusively on the
production side of the economy and make no reference to the role
played by consumers in the determination of equilibrium prices.[*]

There is a sense, well-known to economists, in which the
model of competitive behavior does give rise to a nonlinear maximiza-
tion problem similar to those encountered in the theory of produc-
tion. If each consumer has a concave utility function, then the
maximization of a weighted sum of utilities subject to the constraints
of the technology and the existing stock of commodities, does pro-
duce a set of prices which have many of the properties of equili-
brium prices. Producers are maximizing profit at these prices, and
no consumer can receive a higher utility at lower cost. There is

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[*] In his thesis, Rolf Mantel [4], gives a proof of the existence of
equilibrium prices which is similar in many respects to the argu-
ments of the present paper. I have also received an unpublished
manuscript by H. Houthakker describing an algorithm which should
be very effective under certain restrictive assumptions about the
demand functions and the technology.

however one serious drawback, namely, that unless the utility weights
are selected in precisely the right way the consumption of each con-
sumer is in no way related to the income generated by the sale of
his productive factors. Unless we are willing to neglect this vital
link in the economic system completely, the problem has merely been
shifted from the determination of equilibrium prices to the deter-
mination of the appropriate utility weights, and the latter problem
is no simpler than the former.

Nor is any general computational approach to be found in
the literature about the stability of equilibrium, in which the pro-
cess of adjusting prices to excess demands may be viewed as a gra-
dient method for the computation of equilibrium prices. Even though
gradient methods are successful on the production side of the economy,
they need not be stable in a model involving consumers unless some
relatively stringent assumptions such as "gross substitutability,"
are placed on the market demand functions.

The basis for an effective algorithm for the computation
of equilibrium prices has come from a rather unexpected source.
Until recently, the existence of Nash or Cournot equilibrium points
in a finite, two person, nonzero sum game, has been treated by the
same nonconstructive topological methods as those used in equili-
brium analysis. But Lemke, working with a student Howson, has de-
vised a most ingenious algorithm, based on pivot steps as used in
linear programming, for calculating a Nash equilibrium point for a

two person game, [2]. Even though these problems have only a mathe-
matical connection, Lemke's basic idea may be combined with a differ-
ent notion of pivoting to give a constructive algorithm for approx-
imating fixed points of a continuous mapping, for finding a point
in the core of an economy, and for the algorithm of the present
paper, [5, 6].

The next three sections describe the details of our algor-
ithm. The reader whose interest is less in technical matters than
in applications may prefer to jump to Section 5 in which some ex-
amples are given.

2. Setting the Stage for the Algorithm

As we have seen, it is sufficient to consider only those
price vectors $\pi = (\pi_1, \ldots \pi_n)$ which lie on the simplex with

$\pi_i \geq 0$, and $\sum_{i=1}^{n} \pi_i = 1$. Rather than examine all vectors on this
simplex we shall assume that a large but finite list of vectors
$\pi^{n+1}, \ldots \pi^k$ has been selected, and restrict our attention to these
vectors as potential equilibrium prices. (The reason for beginning
the list with π^{n+1} rather than π^1 should become clear later in
the paper.) Since the actual equilibrium vector need not be found
in this list, the algorithm will provide only an approximation but
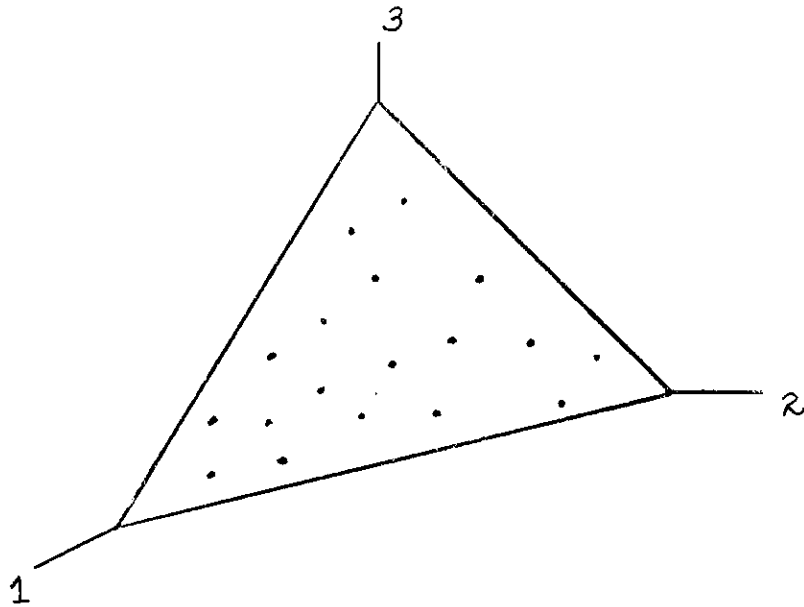one whose accuracy can be increased by enlarging the list of vectors.

Figure 1

The algorithm has been applied to a number of specific examples, some of which involved a set of vectors containing as many as 10^{16} members. Of course, if all of the vectors in such a list were to be examined in order to determine an approximate equilibrium price vector, we would substantially exceed the capabilities of even the fastest electronic computers; the algorithm however has rarely required an examination of more than 1500 such vectors, and has generally terminated with a far smaller number.

After the vectors $\pi^{n+1}, \ldots \pi^{k}$ have been selected, the next step is to construct a particular matrix B with n rows and

k columns. The first n columns of B will consist of a unit matrix. Column j , with $n + 1 \leq j \leq k$, will be related to the vector π^j , in the list of vectors, according to the following specific rules.

1. Let

$$\begin{matrix} a_{1\ell} \\ \vdots \\ a_{n\ell} \end{matrix}$$

be an

activity in the technology matrix A which yields a maximum profit at the prices π^j . If there are several activities which give the same maximum profit then an arbitrary selection of one of these is made.

2. If the largest profit obtainable at the prices π^j is positive, then the jth column of B is defined to be

$$\begin{matrix} -a_{1\ell} \\ \vdots \\ -a_{n\ell} \end{matrix} \, .$$

3. If the largest profit at the prices π^j is less than or equal to zero, then the jth column of B is defined to be

$$\begin{matrix} \xi_1(\pi^j) \\ \vdots \\ \xi_n(\pi^j) \end{matrix} \, .$$

The general appearance of the matrix B is as follows.
(The vector π^j is written above column j to indicate the connection between the two.):

$$
B = \begin{bmatrix} 1 & 0 \cdots 0 & & -a_{1\ell} & & \xi_1(\pi^{j_2}) \\ 0 & 1 \cdots 0 & & \vdots & & \vdots \\ 0 & 0 \cdots 1 & & -a_{n\ell} & & \xi_n(\pi^{j_2}) \end{bmatrix}
$$

with column headings $\pi^{n+1} \ldots \pi^{j_1} \ldots \qquad \pi^{j_2} \ldots \pi^k$

Aside from the first n columns, which will play the role of slack
variables, the columns of B will be composed of either the market
demands at a given set of prices, or the negative of that activity
vector which maximizes profit for the price vector appearing above
the column. Of course some of the activity vectors will be repeated
a substantial number of times.

The number of columns of B is apt to be very large, and
we are fortunate that the algorithm never requires an explicit re-
presentation of this matrix in the memory units of the computer.

We shall be concerned with nonnegative solutions of the
equations $Bx = w$, where

$$
w = (w_1, \ldots w_n)' ,
$$

the vector of factors available prior to production. When these
equations are written out explicitly they become

$$-\sum_{j} a_{i\ell} x_j + \sum_{j} \xi_i(\pi^j) y_j = w_i \ ,$$

where in the first set of terms, the subscript ℓ depends on j and refers to that activity which maximizes profit at the prices π^j , should that profit be positive. In the second set of terms the x_j's have been replaced by y_j's to emphasize the distinction between the two types of columns.

From the way in which the matrix B is constructed the following conditions are satisfied.

1. If $y_j > 0$, then $\displaystyle\sum_{i=1}^{n} \pi_i^j a_{i\ell} \leq 0$ **for every** ℓ .

2. If any activity ℓ , other than a disposal activity, has a positive weight x_j , then $\displaystyle\sum_{i=1}^{n} \pi_i^j a_{i\ell} > 0$.

The basic idea of our algorithm is to approximate an equilibrium price vector by determining a nonnegative solution to the equations $Bx = w$ with the property that **all** of the prices π^j corresponding to positive x_j or y_j are close to each other, and that the ith coordinate of all of these prices is close to zero, if the ith slack variable is positive.

In order to see that such a solution would indeed repre-

sent an approximation to a competitive equilibrium, let us imagine that the prices π^j corresponding to positive x_j and y_j are so close to each other that they can be replaced by a common price π in the above equations, and in conditions 1 and 2.[*] In addition π_i will be zero, if the ith slack is positive. Since all of the $\xi(\pi^j)$ with positive weights y_j are replaced by $\xi(\pi)$, the equations become

$$-\Sigma\, a_{i\ell}x_j + (\Sigma_j y_j)\xi_i(\pi) = w_i \ ,$$

which would describe the equality of supply and demand in all markets if it could be demonstrated that $\Sigma y_j = 1$.

Moreover, condition 1 becomes

1'. If $y_j > 0$, then $\Sigma\pi_i a_{i\ell} \leq 0$, for all ℓ , so that the fact that no activity makes a positive profit at prices π follows from the positivity of at least one y_j .

Condition 2 in conjunction with the fact that $\pi_i = 0$ if the ith slack variable is positive, may be restated as

_____

[*] A more precise mathematical treatment would involve taking successively more refined grids on the simplex and letting π be a limit point of those π^j corresponding to positive x^j or y^j .

2'. If any activity ℓ , including disposal activities, has a positive weight x_j , then $\Sigma\pi_i a_{i\ell} \geq 0$.

In order to show that we do indeed have an equilibrium price we first show that at least one y_j is strictly positive, for then conditions 1'. and 2'. imply that no activity makes a positive profit, and that those activities which are operated at a positive level have a zero profit. Moreover if the above equations are multiplied through by π_i and added we would then see that

$$(\Sigma_j y_j)(\Sigma_i \pi_i \xi_i(\pi)) = \Sigma_i \pi_i w_i ,$$

and it follows from the Walras law and the positivity of $\Sigma\pi_i w_i$ that $\Sigma_j y_j = 1$.

The only missing link in our argument is therefore the demonstration that at least one y_j is strictly positive. But if all y_j are zero it follows that

$$-\Sigma a_{i\ell} x_j = w_i , \quad \text{and}$$

$$\Sigma\pi_i a_{i\ell} \geq 0 \quad \text{for all} \quad \ell$$

with a positive x_j . But then

$$0 \leq \Sigma \Sigma\pi_i a_{i\ell} x_j = -\Sigma\pi_i w_i ,$$

which contradicts the positivity of $\Sigma \pi_i w_i$. (The assumption $w_i > 0$ for all i , is used here and in the previous paragraph.) This concludes our argument that π represents an equilibrium price.

Of course the fact that the vectors π^j corresponding to positive x_j and y_j are close to each other does not permit a literal replacement by a common vector π , but it should be clear from the continuity of the demand functions that an average of π^j will serve as an _approximate_ equilibrium price vector, since supply will be approximately equal to demand in all markets, profits if positive will be small, and the profits of those activities used at a positive level will be close to zero.

3. The Main Theorem

In the previous section, we constructed a matrix

$$
B = \begin{bmatrix}
1 & 0 & \dots & 0 & \overset{\pi^{n+1}}{b_{1,n+1}} & \dots & \overset{\pi^{k}}{b_{1,k}} \\
 & & & & & \vdots & \\
0 & 0 & \dots & 1 & b_{n,n+1} & \dots & b_{n,k}
\end{bmatrix} ,
$$

whose j^{th} column was either the market demand at prices π^j or the negative of the profit maximizing activity at these prices, with π^j a specific price on the simplex $\{\pi \mid \pi_i \geq 0 , \ \sum_i \pi_i = 1\}$. We then showed how an approximate equilibrium price vector could be

determined by finding a nonnegative solution to the equations $Bx = w$ such that all of the vectors π^j with positive x_j are close to each other, and with the ith coordinate of each of these vectors close to zero, if the ith slack is positive.

In order to be specific about this statement we must formulate a precise definition of the concept of closeness.

It will be useful to begin by making the following assumption which can easily be brought about by a perturbation of the vectors $\pi^{n+1}, \ldots \pi^k$.

Non Degeneracy Assumption: No two vectors in the set $\pi^{n+1}, \ldots \pi^k$ have the same i^{th} coordinate for any i, and no vector has a zero coordinate.
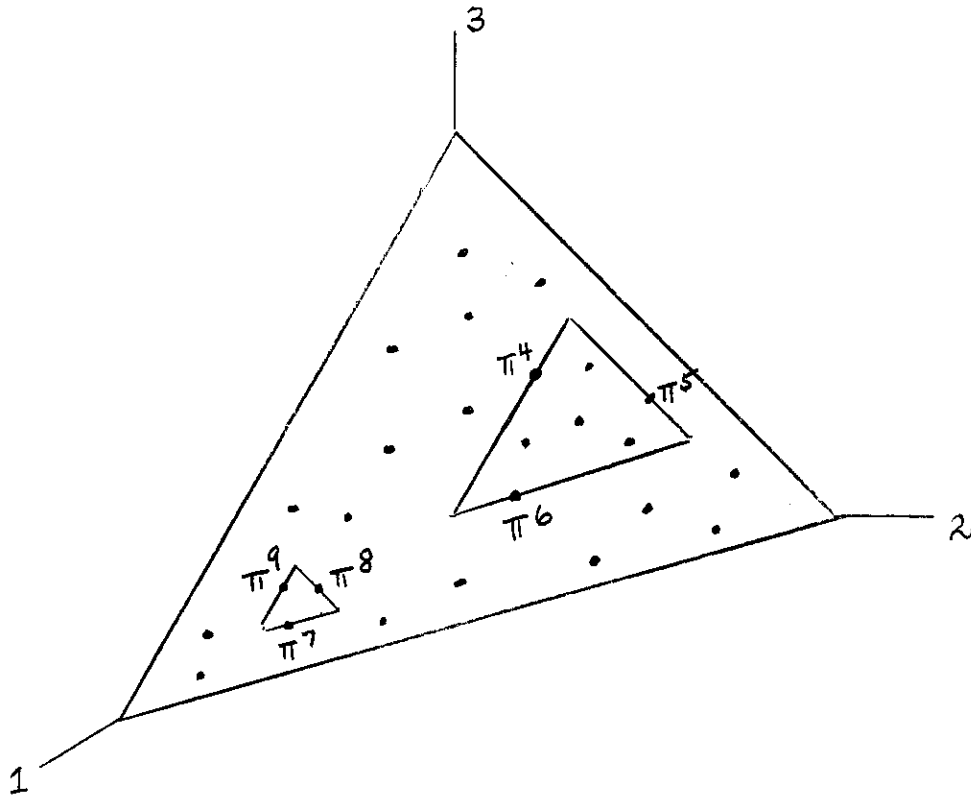
Consider n vectors $\pi^{j_1}, \ldots \pi^{j_n}$ selected from the list of vectors. These vectors may be used to generate a subsimplex in the following way: Begin by finding that one of the n vectors which has the smallest first coordinate. This will yield a unique vector because of the nondegeneracy assumption. Pass a hyperplane with a constant first coordinate through that vector. Then find that one of the n vectors with smallest second coordinate, and pass a hyperplane with constant second coordinate through that vector. If we continue in this fashion with each coordinate, the subsimplex which is then generated consists of all points $\pi = (\pi_1, \ldots \pi_n)$

with

$$\pi_1 \geq \min(\pi_1^{j_1}, \ldots \pi_1^{j_n})$$
$$\vdots$$
$$\pi_n \geq \min(\pi_n^{j_1}, \ldots \pi_n^{j_n}) \ ,$$

and

$$\pi_1 + \ldots + \pi_n = 1 \ .$$

In the above figure the list of vectors is given by $\pi^4, \ldots \pi^k$, and two subsimplices have been drawn, one generated by π^4, π^5 and π^6, and the other by π^7, π^8, and π^9. The first triple of vectors are fairly far apart, whereas the three vectors in the second triple are all quite close and this is indicated by the fact that there are several vectors in the list $\pi^4, \ldots \pi^k$ interior to the first subsimplex, but none interior to the second. In general if there are no vectors in the list $\pi^{n+1}, \ldots \pi^k$ interior to the subsimplex generated by $\pi^{j_1}, \ldots \pi^{j_n}$, then these n vectors must be close to each other. This is the concept of closeness which we shall adopt.

Before giving a formal definition, let us make one extension to accomodate the possibility of forming subsimplices some of whose edges are given by the coordinate hyperplanes $\pi_i = 0$. This can be accomplished by the device of defining n new vectors:

$$\pi^1 = (0, M_1, \ldots M_1)$$
$$\pi^2 = (M_2, 0, \ldots M_2)$$
$$\vdots$$
$$\pi^n = (M_n, M_n, \ldots 0) \, ,$$

with M_i different from each other and greater than one. These vectors, which are not on the simplex, are associated with the n slack variables in the matrix B. If we now consider n vectors

$\pi^{j_1}, \dots \pi^{j_n}$ from the extended list $\pi^1, \dots \pi^k$, and define the associated subsimplex as before to be the set of $\pi = (\pi_1, \dots \pi_n)$ with

$$\pi_1 \geq \min(\pi_1^{j_1}, \dots \pi_1^{j_n})$$
$$\vdots$$
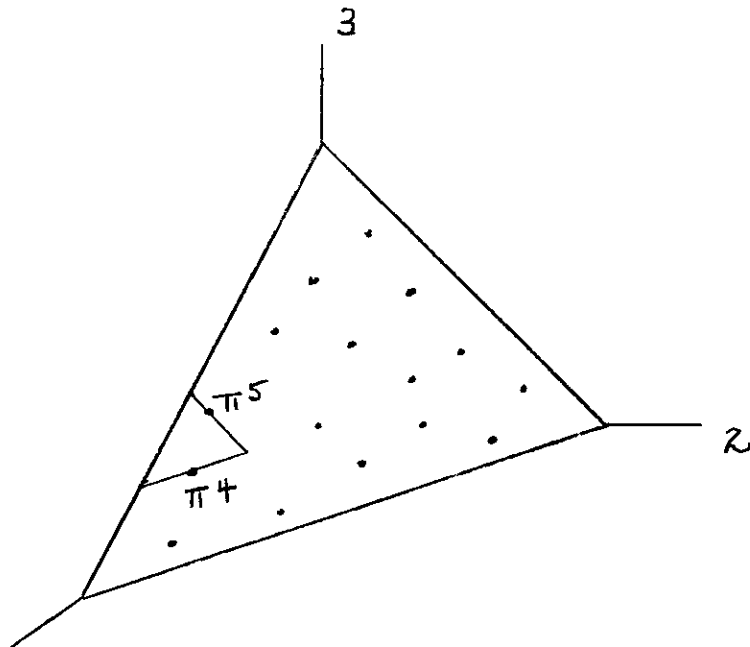$$\pi_n \geq \min(\pi_n^{j_1}, \dots \pi_n^{j_n}) ,$$

and

$$\pi_1 + \dots + \pi_n = 1 ,$$

then this subsimplex will be bounded by an edge $\pi_i = 0$ if π^i is one of the n vectors, but otherwise the definition of the subsimplex will be as before.

In the following figure the subsimplex is generated by π^2, π^4 and π^5,

and since there are no vectors in the list interior to this subsimplex we are justified in saying that the vectors π^4 and π^5 are close to each other, and have a second coordinate close to zero.

The following formal definition makes use of a term "primitive set," which I have used elsewhere, to describe the concept of closeness under discussion.

DEFINITION: A set of n vectors $\pi^{j_1}, \ldots \pi^{j_n}$ from the list $\pi^1, \ldots \pi^k$ will be said to form a <u>primitive set</u>, if no vectors in the list are interior to the subsimplex

$$\pi_1 \geq \min(\pi_1^{j_1}, \ldots \pi_1^{j_n})$$
$$\vdots$$
$$\pi_n \geq \min(\pi_n^{j_1}, \ldots \pi_n^{j_n}) \,,$$

$$\pi_1 + \ldots + \pi_n = 1 \,.$$

The main theorem of this paper, which will be demonstrated by means of a constructive algorithm, is

THEOREM: There exists a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$, such that the equations

$$Bx = w \,,$$

have a nonnegative solution with $x_j = 0$ if j is different from

$(j_1, \ldots j_n)$.

The algorithm behind this theorem will provide us with precisely the right type of solution to the problem discussed in the previous section. It will yield a nonnegative solution to $Bx = w$, with all of the π^j corresponding to positive x_j or y_j close to each other, and with the ith coordinate of each of these prices close to zero if the ith slack is positive.

4. The Algorithm

The reason for introducing the notion of a primitive set of vectors, is not only to define specifically when n vectors are to be considered close -- many other constructions would serve just as well for this -- but also because a type of operation, similar to a pivot step in linear programming, can be performed on a primitive set of vectors, and this operation is crucial for the development of our algorithm.

The operation consists in removing a specific vector from a primitive set of vectors and attempting to replace it by some other vector so that the new set of vectors is also a primitive set. As the following lemma indicates, this operation can, with one exception, always be performed and the replacement is uniquely determined.

LEMMA: Let $\pi^{j_1}, \ldots \pi^{j_n}$ be a primitive set and π^{j_α} a specific one of these vectors. Then aside from one exceptional case, there is a <u>unique</u> vector π^j in the list $\pi^1, \ldots \pi^k$, so that if π^j replaces π^{j_α}, the resulting collection of vectors forms a primitive set. The exceptional case arises when all of the
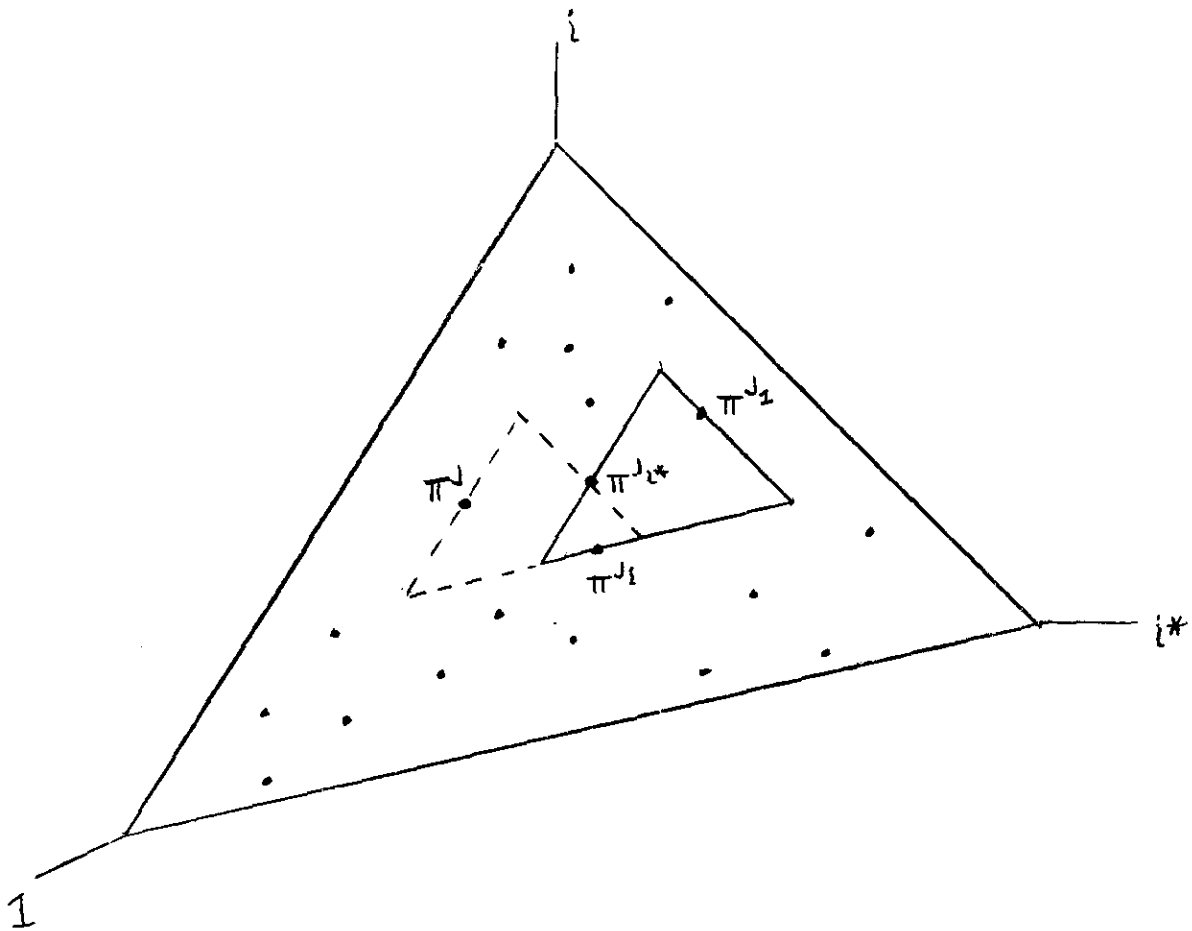
vectors π^{j_i} with $i \neq \alpha$, are from the first n vectors in the list, and in this case no replacement is possible.

The vector π^{j} which replaces π^{j_α} may be found by a simple geometrical construction. To illustrate this construction let us assume that

$$\pi_i^{j_i} = \min(\pi_i^{j_1}, \ldots \pi_i^{j_n}) \quad \text{so that} \quad \pi^{j_i} \text{ is on that}$$

face of the subsimplex on which the ith coordinate is constant. Assume moreover that π^{j_1} is being removed.

Let π^{j_i*} be that vector in the primitive set with the second smallest value of its first coordinate. To find the vector to replace π^{j_1} we move the face containing $\pi^{j_{i*}}$ parallel to it- self, lowering the i^{*th} coordinate until we first intersect a vector π^{j} in the list with

$$\pi_i^{j} > \pi_i^{j_i} \quad \text{for} \quad i \neq 1 , \quad i*$$

and

$$\pi_1^{j} > \pi_1^{j_{i*}} ,$$

or the face of the simplex S in which $\pi_{i*} = 0$.

The rule is applicable except when the vectors π^{j_i} with

$i \neq 1$ are all selected from the first n vectors of P_k, and it clearly produces a new primitive set. The details of the proof that π^j is the only possible replacement, and that no **replacement** is possible in the exceptional case may be found in [6].

To see the analogy between this type of replacement and a pivot step in linear programming, consider a system of linear equations in nonnegative variables, $Bx = w$, where

$$B = \begin{bmatrix} 1 \cdots 0 & b_{1,n+1} \cdots b_{1,k} \\ \vdots & \vdots \\ 0 \cdots 1 & b_{n,n+1} \cdots b_{n,k} \end{bmatrix} \; .$$

A feasible basis for this system of equations, is a collection of n linearly independent columns $j_1, \cdots j_n$ from the matrix B, such that the equations $Bx = w$ have a nonnegative solution with $x_j = 0$ if j is different from $j_1, \cdots j_n$.

In a pivot step, one takes a specific column outside of the basis and attempts to introduce it into the basis, while removing some column, so that the resulting collection of n columns is again a feasible basis. In linear programming one attempts to bring into the basis a specific column from outside, whereas with a primitive set one attempts to remove a column in the primitive set. If the set of nonnegative solutions to the equations $Bx = w$ forms a bounded set, then a pivot step can always be carried out, and if the problem is nondegenerate, in the sense used in linear program-

ming, there is a <u>unique</u> column to be eliminated from the basis for
the resulting collection of columns to form a feasible basis.

Our assumptions on the technology guarantee that the non-
negative solutions to $Bx = w$ form a bounded set when the matrix
B is constructed as in Section 2. And nondegeneracy can be brought
about by a small perturbation of w, or by using a lexicographic
ordering, so that in our case a pivot step on the matrix B is al-
ways possible and unique.

The main theorem of the previous section can now be re-
stated in a more specific and useful form.

THEOREM (RESTATEMENT) There exists a primitive set
$\pi^{j_1}, \ldots \pi^{j_n}$, so that the columns $j_1, \ldots j_n$ form a feasible
basis for $Bx = w$.

Let us now turn to a proof of this theorem. Consider the
set of vectors $(\pi^2, \ldots \pi^n, \pi^{j^*})$ with π^{j^*} selected from those
vectors beyond the first n so as to maximize its <u>first</u> coordinate.
This collection forms a primitive set, as the following figure clear-
ly indicates.
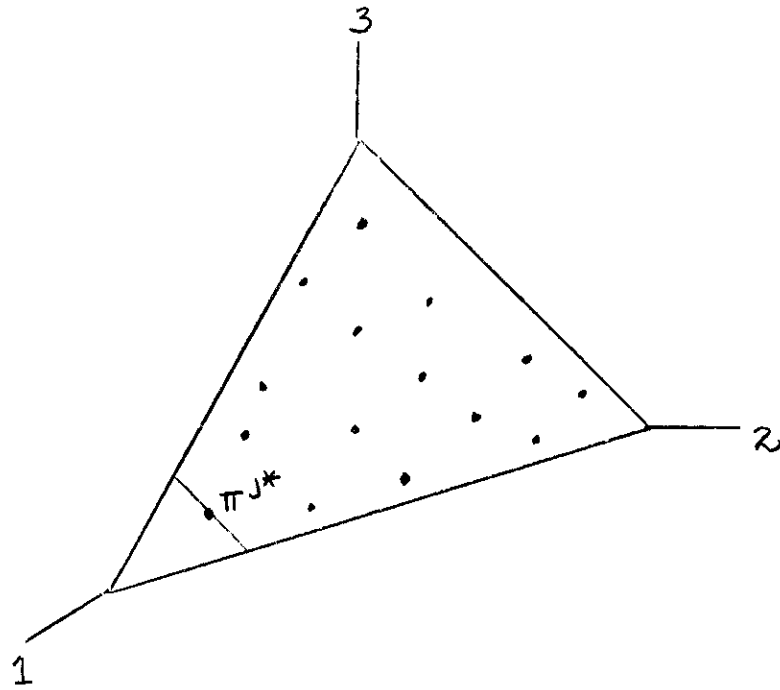
Moreover the columns 1, 2, ... n form a feasible basis for the matrix B since w > 0 . Let us perform a pivot step on B by introducing column j* . If column 1 is removed from the basis the problem is over since (2, ... n, j*) would be both a primitive set and a feasible basis for Bx = w . Generally this will not be the case and some column other than the first will be removed when j* is introduced. The next step in the algorithm is to remove from the primitive set that price which corresponds to the column just removed from the feasible basis for Bx = w .

The algorithm alternates between pivot steps on the B matrix and the analogous operation on the primitive sets. We take

into the feasible basis for B the column corresponding to that price just taken into the primitive set, followed by removing that price from the primitive set which corresponds to the column just removed from the basis for B.

It is easy to see that in any intermediary step in the computation, we shall be in a situation in which the feasible basis for B consists of column 1 and $n - 1$ other columns $j_2, \ldots j_n$, whereas the primitive set consists of vectors $\pi^{j_1}, \pi^{j_2}, \ldots \pi^{j_n}$ with $j_1 \neq 1$. The algorithm retains this relationship in which $n - 1$ of the indices are identical by performing either of <u>two</u> possible operations. If we are pivoting on the matrix B then column j_1 must be introduced, and if we are replacing an element in the primitive set then π^{j_1} must be removed. Except for the initial position where the primitive set is composed of the vectors $(\pi^2, \ldots \pi^n, \pi^{j^*})$ and the feasible basis consists of the first n columns, both of these operations can be performed; in the initial position only one operation can be taken since removing π^{j^*} is the exceptional case referred to in the Lemma.

In general some step has been taken to arrive at the present state. The algorithm then takes that other continuation open to it. The algorithm cannot cycle, since if the first state that it returns to is not the initial position there would have to be three ways to exit from this position rather than two, and if the

first position which is repeated is the initial position, there would be two ways to exit from this position rather than one.

Since the algorithm does not cycle, and there are a finite number of possible positions, the algorithm must terminate, and this can only happen when the prices corresponding to a feasible basis for $Bx = w$ also form a primitive set. This concludes the proof of the main theorem.

5. Some Examples

The algorithm has been programmed for an IBM 7094, and a number of examples have been tried. Before describing the results of a sample computation, it might be useful to indicate two of the special techniques that have been incorporated into the program.

In order to use the algorithm some specific set of vectors $\pi^{n+1}, \ldots \pi^{k}$ must be selected. I have found it convenient to form this list by taking all vectors on the simplex $\{\pi \mid \pi_i \geq 0 , \sum_{1}^{n} \pi_i = 1\}$ whose coordinates are positive fractions with a given denominator. In other words a denominator D is selected and we consider all vectors

$$\pi = (k_1/D, \ldots k_n/D) ,$$

with k_i positive integers such that $k_1 + \ldots + k_n = D$.

If the list of vectors has this special structure the operation of replacing a vector in a primitive set can be carried out by a simple algebraic computation and does not involve a lengthy search through all of the vectors in the list. Of course, the set $\pi^{n+1}, \ldots \pi^{k}$ will not satisfy the nondegeneracy assumption made in Section 3, since many vectors in the list have a common coordinate. There are a variety of techniques, however, for resolving degeneracy and insuring that the algorithm does not cycle.

The algorithm terminates with a primitive set $\pi^{j_1}, \ldots \pi^{j_n}$ such that the columns $j_1, \ldots j_n$ are a feasible basis for the equations $Bx = w$, and as we have seen, an average of the vectors in the primitive set will serve as an approximation to an equilibrium price vector. In order to determine the specific weights to be used in forming an average, I have assumed that the demand functions are locally linear in the neighborhood of the primitive set, and selected that vector which minimizes the maximum deviation between demand and supply. The efficiency of the algorithm is substantially increased by a device of this sort.

Let us consider, as a numerical example, with no pretense towards realism, an economy involving the following six commodities:

1. Capital available at the end of the current period.
2. Capital available at the beginning of the current period.
3. Skilled labor.

4. Unskilled labor.

5. Nondurable consumer goods.

6. Durable consumer goods.

During the particular time period, production may be carried out in each of three sectors; the construction of durable consumer goods, the production of nondurable consumer goods, and a sector for the construction of new capital available at the end of the period.

The durable consumer goods sector is assumed to be described by the two activities

$$
\begin{array}{cc}
4 & 4 \\
-5.3 & -5 \\
-2 & -1 \\
-1 & -6 \\
0 & 0 \\
4 & 3.5 \ ,
\end{array}
$$

with the commodities in the order given above. The first of these two activities represents a process which produces four units of durable consumer goods, and uses 5.3 units of capital, two units of skilled labor and one unit of unskilled labor. In addition, the 5.3 units of capital are partially depreciated during use and become four units of capital available at the end of the period. The second activity in this sector permits the substitution of unskilled labor for skilled labor.

There are three possible activities in the nondurable sector, i.e.,

$$
\begin{array}{ccc}
1.6 & 1.6 & 1.6 \\
-2 & -2 & -2 \\
-2 & -4 & -1 \\
-3 & -1 & -8 \\
6 & 8 & 7 \\
0 & 0 & 0 \;,
\end{array}
$$

with varying degrees of substitution between skilled and unskilled labor.

Finally the capital good sector involves the following three activities:

$$
\begin{array}{ccc}
.9 & 7 & 8 \\
-1 & -4 & -5 \\
0 & -3 & -2 \\
0 & -1 & -8 \\
0 & 0 & 0 \\
0 & 0 & 0 \;,
\end{array}
$$

the first of which represents the rate of capital depreciation if no investment is undertaken.

In addition to this activity analysis model of production, our hypothetical economy will involve five consumers each of whom has a distinct set of demand functions and vector of initial assets.

The following matrix describes the initial assets of each consumer:

	C. End	C. Beg.	Skil. L.	Unskil. L.	Non Dur.	Dur.
Consumer 1	0	3	5	.1	0	1
Consumer 2	0	.1	.1	7	0	2
Consumer 3	0	2	6	.1	0	1.5
Consumer 4	0	1	.1	8	0	1
Consumer 5	0	6	.1	.5	0	2

As we see, no consumer owns, prior to production, either nondurable goods or capital available at the end of the period. Consumer 5 is the largest owner of capital at the beginning of the period, and there are varying degrees of ownership of the two varieties of labor and of consumer durables.

I have assumed that each consumer has a set of demand functions derivable from a utility function with constant elasticity of substitution. This implies that at the prices $\pi_1, \ldots \pi_6$, the ith consumer will make the demands

$$\xi_j(\pi) = \frac{a_{ij} \cdot f_i(\pi_1, \ldots \pi_6)}{\pi_j^{b_i}} \quad,$$

where b_i is the elasticity of substitution for consumer i, a_{ij} measures the intensity of the ith consumer's demand for commodity j, and $f_i(\pi_1, \ldots \pi_6)$ is a complex function of the price vector

π selected so that the budget constraint is satisfied for each individual. The specific values of a_{ij} are given by

	C. End	C. Beg.	Skil. L.	Unskil. L.	Non Dur.	Dur.
Consumer 1	4	0	.2	0	2	3.2
Consumer 2	.4	0	0	.6	4	1
Consumer 3	2	0	.5	0	2	1.5
Consumer 4	5	0	0	.2	5	4.5
Consumer 5	3	0	0	.2	4	2

As we see, no consumer has a demand for capital at the beginning of the period, but there may be a substantial demand, depending of course on the prices, for capital at the end of the period. Since there is no explicit description of production after the end of the time period, this demand is to be interpreted as a demand for savings. The entries under the skilled and unskilled labor columns refer to a demand for leisure. Finally, the elasticities of substitution b_i are given by

Con.	b_i
1	1.2
2	1.6
3	.8
4	.5
5	.6

In the numerical solution of this example, the set $\pi^{n+1}, \ldots \pi^{k}$ was assumed to consist of all vectors of the form

$$(k_1/100, \; k_2/100, \; \ldots \; k_6/100) \; ,$$

with k_i positive integers summing to 100. There are an exceptionally large number of these vectors. The algorithm terminated after examining only 913 of them in a little over a minute of 7094 computing time, with the following primitive set

π^{j_1}	π^{j_2}	π^{j_3}	π^{j_4}	π^{j_5}	π^{j_6}
22	22	22	22	22	23
22	21	22	22	24	22
20	19	19	19	19	20
7	7	7	6	6	7
12	12	12	12	11	12
19	19	18	19	18	16 .

These six vectors are related, one by one, to six columns in the matrix B , which form a feasible basis for the equations $Bx = w$. The first four of these vectors are associated with the 9th, 7th, 10th, and 11th activities, in the order in which they are described above. The vector π^{j_5} gives rise to a negative profit for all possible activities, and therefore corresponds, in the matrix B to a column of demands. π^{6} is associated with the thirteenth

activity.

These six vectors were averaged by a set of weights ob-
tained as the solution of a specific linear program, and the follow-
ing price vector and activity levels were obtained:

$$\pi = (21.8,\ 21.8,\ 19.4,\ 7.4,\ 12.2,\ 17.4)\ ,$$

Activity	Level	Profit
7	.86	-.05
8	.0	-.25
9	.10	.03
10	1.41	.04
11	1.31	-.02
12	.0	-.02
13	.47	.00
14	.0	-.33

The profits in the final column are based on the price vector π
normalized so that its sum is one.

As a final summary, let us compare the market demand at
this set of prices, with the net supply obtained by using these
activity levels in conjunction with the initial stocks of commodities.

	C. End	C. Beg.	Skil. Lab.	Unsk. Lab.	Non. Dur.	Dur.
Demand	11.27	.00	1.02	2.17	21.08	10.98
Supply	11.27	-.01	1.01	2.14	21.06	10.96 .

The price vector π and the activity levels given above
seem to be a fine approximation to an equilibrium insofar as the
equality of supply and demand in each market. The profits, which
should be zero for those activities in use and less than or equal
to zero for the remaining activities, seem a bit less satisfactory.
This is undoubtedly due to the preoccupation of the final linear
programming problem with minimizing the maximum deviation between
supply and demand, a goal which is not directly responsive to con-
siderations of profit. Many other averaging processes can be used
and they deserve to be explored before substantially larger problems
are tried. It should be pointed out that the final linear program-
ming problem, which takes no more than one or two seconds of com-
puting time, is a very minor part of the algorithm. The important
work in the algorithm is done in determining the primitive set whose
associated columns form a feasible basis for $Bx = w$. It is this
calculation which indicates the neighborhood in which approximate
equilibrium prices are to be found.

In examining the above example one sees that the price of
capital available today is identical with the price to be paid today
for capital delivered tomorrow, so that the real rate of interest
should naturally be taken as zero. This is reflected in the fact
that the initial stock of capital falls during the period from 12.1
units to 11.3 units, even though activity 13, a capital producing
activity is used at the level of .47.

Let us compare this model with one which differs from it by the introduction of one new productive activity in the capital goods sector. The activity

$$6.4$$
$$-3.5$$
$$-1$$
$$-5$$
$$0$$
$$0$$

has a profit of .13 at the previous equilibrium prices, normalized so that their sum is one, so that if this activity is available it will surely be used. It would also seem reasonable to suspect that the use of this activity would have a tendency to increase the interest rate from its previous level of zero.

If we adopt the same grid of prices on the simplex, the calculations for this model terminate after the examination of 1185 price vectors in about one minute and 20 seconds of 7094 computing time. After averaging, the following price vectors and activity levels are obtained:

$$\pi = (18.8, \ 22.0, \ 19.6, \ 7.1, \ 13.4, \ 19.1) \ ,$$

Activity	Level	Profit
7	.69	-.11
8	.0	-.30
9	.0	.06
10	1.79	.08
11	.76	.04
12	.0	-.05
13	.0	-.22
14	.0	-.56
15	.95	-.12

The relation between demand and net supply is given by the table

	C. End	C. Beg.	Skil. L.	Unsk. L.	Non. Dur.	Dur.
Demand	12.98	.00	1.03	2.41	19.73	10.29
Supply	12.93	-.01	1.02	2.40	19.68	10.28 .

The new activity is used at the expense of activity 11 in which a substantial quantity of unskilled labor was required to produce nondurable goods. As might be imagined, the price of nondurable goods rises and its consumption falls. The expected rise in the interest rate takes place, along with an increase in savings.

These examples give an indication of the speed and accuracy with which the algorithm can solve a moderately difficult problem. I feel quite sure that the performance of the algorithm can be substantially

improved in both of these dimensions by more subtle programming techniques, and that eventually problems involving as many as twenty commodities will be feasible without an excessive use of computing time.

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