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WELFARE, STATIC AND DYNAMIC SOLUTION CONCEPTS

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by

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Some problems concerning preferences and solutions in static and dynamic economic models of oligopoly and of welfare are discussed. The relationship between optimizing economic processes and behavioral processes is explored.

## 1. The Static Model of Economic Man

Most of the results of micro-economics and welfare economics have been obtained at the cost of using static models of rational economic man. These models have been analyzed with a variety of static solution concepts. In some instances, such as oligopoly theory, the static analysis no longer appears to offer the most fruitful avenue for further development. In this section some of the problems of and difficulties with the static analysis are examined.

Two major economic abstractions are used to model economic behavior. The first is that of the individual, or consumer, maximizing utility and the second is that of the firm maximizing profit. For either of these to be well defined, several severe modeling restrictions must be imposed. In particular we must assume that the individual has known fixed preferences; he knows all possibilities and can evaluate them.

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The situation must be regarded as static unless we wish to modify the models to handle several periods. In particular this can be done in a variety of ways. We may introduce a finite number of periods and consider a preference system which includes time-dated items. Nevertheless the preference systems of the individuals are assumed to be given and fixed from the start.

We could consider an infinite period and introduce a discount rate or a death probability or both. This enables us to consider bounded infinite series and still have an appropriately defined maximization problem.

If we introduce a death probability which is influenced by the actions of the decision-maker we may need to formulate a considerably different problem for the goals of the individual. The firm that faces bankruptcy and the individual who faces death may place a special value on these events which distinguishes them from all others. We may still be able to describe their behavior in terms of maximizing models, but they will be constrained maxima. In the case of the firm this leads to models such as:

Maximize expected discounted income during  
a specified number of periods  $T$  subject to  
maintaining the level of survival for each period  
above a specified level  $s_t$ .

It is also possible to consider maximizing the probability of survival subject to constraints on income.

Our difficulties with the individual do not stop at this point.

If we abandon complete information we must specify methods for search for and evaluation of new possibilities. This may include costs of search and feedbacks which modify goals in terms of results obtained to date. Consideration of these features leads us directly into behavioral theories and artificial intelligence.

There are several other sources of difficulty in describing individual preferences. What are the goals of managers and custodians or others who act in a fiduciary relationship with the property and funds of the public? Even the most utilitarian and rationally oriented theory of the firm or business enterprise must deal with the divergence of the goals of the individual from the apparent goals of the institution as he perceives them.

The models of individualistic man completely ignore the role of society whose influence on his preferences will come about in three ways. First, in his upbringing man's preferences will be molded and changed by his society. Setting aside these dynamic features two other influences remain. There are many public or joint goods and services which cannot be individually appropriated nor even privately paid for. The legal system, the armed forces, the national parks are a few. What are their dimensions, in what units do we measure them?

Man as a social animal may have preferences for states which involve not only what he obtains, but what is obtained by others. In this realm we encounter items which can more appropriately be said to form his value system rather than belong to his preferences. For example, he may have values concerning freedom, the democratic vote, the free enterprise system, prohibition, the taking of drugs. Formally, it is

easy to describe a utility function  $U_i(x_{1,1}, \dots, x_{1,n}, x_{2,1}, \dots, x_{m,n})$  where  $x_{i,j}$  is the amount of the  $j^{\text{th}}$  good or service received by the  $i^{\text{th}}$  individual. This has the welfare of an individual depend upon the distribution of goods and services to all. But to a great extent this is an empty formalism until we are willing to limit and specify properties of this utility function. If we are willing to specify only a complete ordering on preferences, then we encounter the Arrow voting paradox<sup>1/</sup> if group decisions are to take place by vote.

The most paradoxical feature to the Arrow paradox is not its conclusion concerning voting systems under the assumptions made, but that the assumptions have not been much more heavily challenged and that the concept of "solution" to a socio-economic problem has not been more intensively questioned. Is the voting paradox a paradox in a dynamic system? Beyond the observation that a one-shot static voting model of human affairs is a poor model, does it tell us very much more?

There is no monolithic theory of human behavior. Assumptions may be made, and models constructed which work well for a limited set of problems, may not for others. The elegance and apparent institution-free generality of the works of Hicks<sup>2/</sup>, Arrow<sup>3/</sup>, Debreu<sup>4/</sup>, and others can be dangerously deceptive as well as intellectually inspiring. Questions as to whether or not one can compare or add utilities among individuals, or whether a social preference ordering that is transitive can be found, are not questions which involve the eternal verities. They depend upon the problem at hand and the validity of the approximations implicit in

the assumptions.

## 2. Static Preferences and Solutions

In this section we confine ourselves to a highly restricted model of man; we leave out his role as a social being and assume that he is interested in individual ownership and the worth of goods and services to himself. His preferences are fixed and we assume that they can be represented by a complete ordering over all alternatives.

What is a solution to the economic problems in a society consisting of many well informed economic men of this variety? For even a static model we may specify many desiderata viewing the production and distribution of goods and services normatively. Otherwise we might postulate that the distribution of goods and services will be the result of behavior that we can predict. The same solution concept may be viewed normatively or behavioralistically.

Seven solution concepts are noted, all of which can be viewed statically. At least loosely, we may connect each one with an important societal value. They are presented in Table 1 below:

	<u>Solution</u>	<u>Social Relevance</u>
<u>Cooperative</u>		
	(1) Pareto optimality	economic efficiency
	(2) Core	the power of coalitions
	(3) Value	fairness or equity
	(4) Stable set	social stability
<u>Mechanistic</u>		
	(5) The price system	decentralization
<u>Noncooperative</u>		
	(6) The noncooperative equilibrium	the power of the individual
	(7) Beat-the-average	social status

Table 1

We briefly review the properties of these solutions:

- (1) Pareto Optimality: A division A of the proceeds from an economic system is Pareto optimal if there does not exist any other distribution B such that all members would obtain at least as much in B as in A and at least one would have his position improved. A Pareto optimal outcome is efficient inasmuch as no reallocation of resources can result in an increase in welfare to any individual without a decrease to another.
- (2) The Core: The core is the set of outcomes such that no group of individuals acting together can obtain by themselves more than they

are offered in any of the outcomes in the core<sup>5/</sup>.

Two very simple examples will serve to illustrate a game without a core and a game with a core. They are illustrated below in characteristic function form:

$$\begin{array}{cccccc}
 v(0)=0 & & & & v(0)=0 & \\
 v(\overline{1})=0 & v(\overline{2})=0 & v(\overline{3})=0 & v(\overline{1})=0 & v(\overline{2})=0 & v(\overline{3})=0 \\
 v(\overline{1,2})=2 \frac{1}{2} & v(\overline{1,3})=2 \frac{1}{2} & v(\overline{2,3})=2 \frac{1}{2} & v(\overline{1,2})=1 & v(\overline{1,3})=1 & v(\overline{2,3})=1 \\
 v(\overline{1,2,3})=3 & & & v(\overline{1,2,3})=3 & & 
 \end{array}$$

The characteristic function specifies the worth that a coalition can achieve if its members decide to act in unison by themselves. For a three person game there are eight coalitions (counting the one with no members). In the two examples above  $v(\overline{1,2}) = 2 \frac{1}{2}$  means that the coalition of players 1 and 2 in the first game are able to obtain  $2 \frac{1}{2}$  by independent action.  $v(\overline{1,3}) = 1$  means that the coalition of players 1 and 3 in the second game can obtain 1 by independent action.

An imputation of wealth is a vector  $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$  where  $\alpha_i$  is the share obtained by the  $i^{\text{th}}$  individual and  $\vec{\alpha}$  is a point on the Pareto optimal surface. (If there exists a transfer or side-payments mechanism, then  $\sum_{i=1}^n \alpha_i$  equals the joint maximum, a single number.)

The two games above can be described as follows: if any two individuals form an agreement, the referee gives them a sum  $k$  to share



( $k = 2 \frac{1}{2}$  or 1), the excluded individual obtains nothing. If all three form a coalition they obtain 3.

In both games, when viewed from society as a whole, it is evident that cooperation is the most profitable as they can obtain 3 to divide among them. In the first case, however, no matter how they divide the wealth there will always be a potential coalition that could have obtained more by failing to cooperate. This can be seen by trying to solve the following inequalities:

$$\begin{array}{l} \text{a.} \\ \text{b.} \end{array} \quad \left\{ \begin{array}{l} \alpha_1 + \alpha_2 \geq 2 \frac{1}{2} \\ \alpha_2 + \alpha_3 \geq 2 \frac{1}{2} \\ \alpha_1 + \alpha_3 \geq 2 \frac{1}{2} \end{array} \right. \\ \alpha_1 + \alpha_2 + \alpha_3 = 3 .$$

Adding the three inequalities together we obtain:

$$2(\alpha_1 + \alpha_2 + \alpha_3) \geq 7 \frac{1}{2}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 3 \frac{3}{4}$$

which is inconsistent with condition (b). This means that the core is empty. In the second game, however, we have:

$$\text{c.} \quad \left\{ \begin{array}{l} \alpha_1 + \alpha_2 \geq 1 \\ \alpha_2 + \alpha_3 \geq 1 \\ \alpha_1 + \alpha_2 \geq 1 \end{array} \right.$$

d.  $\alpha_1 + \alpha_2 + \alpha_3 = 3$  .

Adding the three inequalities together we obtain:

$$\alpha_1 + \alpha_2 + \alpha_3 \geq 1 \frac{1}{2}$$

which is consistent with (d). There will be many imputations in the core such as:  $(3/4, 3/4, 1 \frac{1}{2})$  or  $(1, 1, 1)$  or  $(1/2, 1 \frac{1}{2}, 1)$  .

The lack of existence of a core implies a great potential for social instability. Any division of the wealth of society will always be threatened by some group.

- (3) The Value: There are several different but highly related value solutions which have been suggested such as those of Shapley<sup>6/</sup>, Nash<sup>7/</sup>, Harsanyi<sup>8/</sup>, and Selten<sup>9/</sup>. They are all cooperative solutions (i.e., all participants will eventually cooperate but use the solution to determine their shares of the final proceeds). These solutions are based upon the axiomatization of concepts of symmetry, fairness, or equity.

The simplest solution to explain is the Shapley value. It can be calculated directly from the characteristic function of a game.

We consider every way in which an individual can enter every coalition and we credit him with his incremental contribution to the coalition. In terms of a coalition  $S$  containing player  $i$  this is  $v(S) - v(S - \{i\})$  where the symbol  $S - \{i\}$  stands

for the set of players  $S$  with player  $i$  removed. Adding all of his contributions together we average them and award him that amount as his "fair share." (This takes into account both his needs and his contribution to society.) If  $\psi_i$  is the fair share for the  $i^{\text{th}}$  player, then it can be shown that:

$$\psi_i = \sum_{\text{over all } S} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S - \{i\})] .$$

- (4) The Stable Set: The stable set was suggested as a solution concept by von Neumann and Morgenstern<sup>10/</sup>; several modifications and related solutions exist, such as those of Shapley<sup>11/</sup> and Vickrey<sup>12/</sup>.

A set of imputations  $S$  forms a solution if no imputation  $\vec{\alpha}$  in  $S$  dominates any other imputation  $\vec{\beta}$  in  $S$ ; and if there always exists an imputation in  $S$  which dominates an imputation not in  $S$ . In other words  $S$  consists of the set of imputations which do not dominate each other, but together dominate everything else.

The interpretation of this solution is in terms of social stability. Unlike an imputation in the core which satisfies conditions of group rationality in the sense that no group obtains less than it can obtain by independent behavior; it is quite possible that certain individuals fare badly in the solution. The complex interplay of a coalition structure may be stable and efficient and yet systematically discriminate against some members of the society.

The four solutions noted above may all be described as cooperative solutions inasmuch as no matter how the proceeds of society are divided in each instance it is assumed that the outcome will be Pareto optimal. Furthermore, they are all game-theoretic solutions in the sense that the view of the society as a whole is in terms of individuals exercising their strategic powers, forming coalitions, bargaining, threatening, and making settlements among themselves.

- (5) The Price System: The competitive equilibrium market model of the price system contrasts with both cooperative and noncooperative game solutions. It is basically a mechanistic solution which will (under the appropriate conditions) satisfy Pareto optimality and some important properties linked with decentralization.

Suppose there are a number of individuals trading in  $k$  commodities. Each individual  $i$  has preference system which can be represented by a function  $u_i(x_1^i, x_2^i, \dots, x_k^i)$  where  $x_k^i$  stands for the amount of the  $k^{\text{th}}$  commodity held by the  $i^{\text{th}}$  trader. The existence of a price system in competitive equilibrium amounts to there being a set of prices  $(p_1, p_2, \dots, p_k)$  such that if each individual merely accepts the prices as given and each tries to maximize his welfare subject to the budget constraint that income and expenditures must balance. (This can, of course, be modified for tax and credit conditions.) Then supply will exactly equal demand in all markets. Furthermore the result will be Pareto optimal.

The impact of the existence of a price system is that a complex multi-person optimization problem can be replaced by a host of individual decentralized optimization problems, all coordinated through the mechanism of prices.

- (6) The Noncooperative Equilibrium: This solution is noncooperative in the sense that no assumption is made that the outcome must be Pareto optimal. Often the noncooperative equilibrium solution is associated with situations with lack of communication, but there is no formal connection between communication conditions and the solution. This is discussed further when we turn to dynamics.

The spirit behind the noncooperative equilibrium is that all exercise their individual power in an introspective manner without any attempt at coordination or cooperation.

Consider a game with  $n$  players. Let the set of strategies for player  $i$  be  $S_i$  where  $s_i$  denotes a particular strategy. The payoff to the  $i^{\text{th}}$  player is  $P_i(s_1, s_2, s_3, \dots, s_n)$ . A set of strategies  $(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_n)$  is said to constitute an equilibrium point if for all  $i$ , the payoff

$$P_i(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_{i-1}, s_i, \bar{s}_{i+1}, \dots, \bar{s}_n)$$

is maximized by setting  $s_i = \bar{s}_i$ . In words, a set of strategies one for each player, forms an equilibrium point if each player, knowing the strategies of the others, is not motivated to change.

Noncooperative solutions form the basis for much of the work in oligopoly theory<sup>13/</sup>. They are in general not optimal. The individual exercise of large economic powers may easily cause an inefficient allocation of resources.

- (7) Beat-the-Average: Another type of noncooperative solution is one in which status or being better than the others is the major consideration. A natural formulation in the two-person game is where each player tries to maximize the difference between his payoff and that of his opponent. If  $P_1(s_1, s_2)$  is the payoff to the first and  $P_2(s_1, s_2)$  to the second, then the game can be described strategically by

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} (P_1(s_1, s_2) - P_2(s_1, s_2)) .$$

This can be generalized for  $n$  individuals<sup>14/</sup>.

## 2.1 Solutions and Economic Conditions

A desirable (but by no means necessary) property of a solution is that it should predict a unique outcome. Under most circumstances the solutions described above do not have this property. It is, however, remarkable that **given the** appropriate conditions\* the core, value, non-

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\* The conditions call for the choice sets of the individuals to be convex and that production conditions can be represented by a cone. Furthermore, that there be many individuals of every type in all markets for any good or service. There are some difficulties associated with the formalization of the economic model involving noncooperative behavior which we cannot deal with in this brief discussion<sup>16/</sup>.

cooperative equilibrium, and market price system solutions will all coincide<sup>15/</sup>. This means that at least if individuals had fixed independent preferences and that all goods were individually owned and could be traded, then a price mechanism could be viewed as a solution to not only decentralization but would satisfy the many other properties of the other solutions which all predict the same outcome.

When the appropriate conditions do not exist, then in general the solutions are all different. In any economy considered as a whole the necessary conditions do not exist. The presence of public goods and interrelated services and preferences cause economies and diseconomies external to the individual and the firm. In some cases a price system that balances all markets and results in an efficient allocation of goods and services may not exist. It is also possible that the core is empty.

What do we wish to consider as a solution when the conditions in an economy are not such that many different ways of viewing it would nevertheless lead to the same prediction? At least from the viewpoint of a static model of socio-economic behavior it appears that in some circumstances decentralization may be inconsistent with equity, equity with control over the power of groups, and so forth. Our list of desiderata cannot be jointly satisfied.

Do the different solutions suggested have an immediate interpretation in terms of dynamics, and if so do they help to resolve any of the paradoxes and inconsistencies encountered in the static theories?

3. A Preliminary to Dynamics

Our treatment of dynamics is limited to the consideration of fixed preferences with individuals making their decisions over a sequence of periods of indeterminate length. How does this influence the solution concepts we have described?

There are several modifications which must be made. First, as has been noted in Section 1, should a special value be attached to the possibility of ruin or death for the decision-maker? Either assumption poses no great difficulty in modeling. The greater difficulties lie elsewhere.

It is implicitly assumed in a static cooperative solution that the bargaining and negotiations have taken place. In particular the threats and counter-threats have established the power of every possible coalition. As it is implicitly assumed that any coalition can be formed costlessly and instantly the combinatorics of the powers of all coalitions are used to determine the value, the core, and the stable sets.

Given ruin possibilities, it is not possible to determine the power of coalitions for all time. Furthermore, it is unreasonable to assume that all coalitions can form at all times. It is far more reasonable to consider a sequential process by which it becomes feasible to update the power of a coalition as the process unfolds. Furthermore, it is more reasonable to consider the likelihood that coalitions can only change gradually.



The cooperative solutions, as static theories, exclude negotiation and bargaining as part of the game. A fruitful dynamic theory must include them as part of the step-by-step process.

The dynamic treatment of the price system poses many problems in the description of adjustment mechanisms. Arrow and Hurwicz<sup>17/</sup> and others have investigated these<sup>18/</sup>. As we are primarily concerned with game theoretic solutions, the price system as a solution concept is not considered further here.

Noncooperative solutions are more naturally linked with dynamics than are cooperative solutions inasmuch as the strategies used the players immediately determine their payoffs. In a cooperative solution the actual play of the game takes place only after the agreement concerning outcome.

Some variants of the iterated Prisoners' Dilemma serve to illustrate some of the problems in defining a dynamic noncooperative solution. We will see in Section 4 that threats, the role of language, and precommitment all play a critical role.

All the above comments have been made while limiting ourselves to models of man involving next to no psychological or socio-psychological features. A different approach than the one adopted here would be to seek a behavioral theory and immediately include search procedures<sup>19/</sup>, change in aspiration levels<sup>20/</sup>, dissonance reduction<sup>21/</sup>, uncertainty avoidance<sup>22/</sup>, and the host of other plausible and attractive observations and conjectures about organizational behavior. It is my belief that the game theoretic and behavioral models are complementary and not substitutes

for each other. As features, such as dynamics or lack of information, or survival, are added to the game theoretic model it draws closer to the behavioral model. When analysis is no longer feasible, simulation and artificial intelligence models of men<sup>23/</sup> and organizations<sup>24/</sup> become a natural next step.

Although undoubtedly a behavioral theory of the firm and other economic organizations shows great promise, it must be realized that it is in an extremely early stage. The work by March and Simon<sup>25/</sup> and by Cyert and March<sup>26/</sup>, for instance, provides more of a charter than an investigation. Many interesting conjectures and observations concerning human behavior are made but relatively little work has been done to date. These observations are not meant as a criticism of the efforts noted but merely as an effort to bring this development into focus with the other approaches.

Neither the behavioral nor the rationalistic theories of human group and intergroup behavior have been able to cope with the role of language and gesture. What is a threatening note or a threatening movement? Every motion in an Apache dance is designed to convey threat. How important is verbal interplay and personality in conflict resolution? Cooperative solutions in game theory avoid this problem by assuming it away. Any verbal exchanges are permitted "outside of the game." In a noncooperative theory it is both possible and reasonable to model the actual bargaining as part of the moves in the game. Unfortunately, the concept of move or strategy in a game theory model is far more easily

associated with a physical act such as moving a pawn, playing a card, dropping a bomb, or producing an automobile that it is with verbal statements, such as: "He is going to hit you," or gestures such as "raising your hand in anger." In the first set of instances there was no problem in coding the physical acts into moves. There was an easily discernible one-to-one relationship. In the second set of instances, the "truth value," the meaning or content of a sentence in a natural language, may depend heavily upon circumstances, tone, previous context, and so forth. It is not logically impossible to construct a game which has sentences as moves; it is however extremely difficult to do so in a fruitful manner.

The difficulty encountered in coding is not limited to game theory but applies to any theory of bargaining, negotiation, and communication in which words are used in the moves.

#### 4. Two Person Games of Economic and Social Survival

In previous writing<sup>27/</sup> a game of economic survival was defined and then interpreted for political or social application as well<sup>28/</sup>. This type of game is reviewed here using several examples and solution concepts which are then considered.

The classical Prisoners' Dilemma game serves as a natural introduction. Consider the game as shown in Table 2 below:

	1	2
1	10,10	-5,20
2	20,-5	-1,-1

Table 2

The joint maximum solution calls for each player to use his first strategy with an outcome of  $(10,10)$  resulting. The noncooperative equilibrium calls for the use of their second strategies with the disastrous result of  $(-1,-1)$ . Suppose that the game is to be played  $k$  times where  $k$  is any positive integer. It can be easily established by backward induction that the noncooperative equilibrium calls for the players to repeat their second strategies in each subgame with the net outcome of  $(-k,-k)$  in the overall game.

For a game of considerable length this repetition of a jointly damaging pair of strategies does not appear to be reasonable for several reasons. First, it leaves out the asset position of the players. Can they be expected to be able to survive lengthy periods with negative payoffs? Then the finite iteration of this game may not be the most fruitful approach to dynamics. Two alternatives present themselves, both of which avoid the paradox inherent in the Prisoners' dilemma where individual rationality leads to social disaster. Let the repetitions of the game be of indefinite duration with a probability  $p$  of ending in any particular period, and/or let there be a discount factor  $\rho < 1$  which modifies the worth of future income. Either or both of these devices serves to give a bounded payoffs to a game of indeterminate or infinite length.

A strategy in this supergame must contain instructions which tell a player how to behave in each subgame. For example:

"Play the first strategy in each subgame until your information indicates that your competitor has used his second strategy; from then on use your second strategy."

is a strategy in the supergame. If the first player employs this strategy, the second player can obtain:

$$10 \sum_{t=0}^{\infty} \rho^t = \frac{10}{1 - \rho}$$

by always using his first strategy. If he ever departs from his first strategy, say, at time  $\tau$ , the most he can hope to obtain is:

$$10 \sum_{t=0}^{\tau-1} \rho^t + 20\rho^{\tau} - \sum_{t=\tau+1}^{\infty} \rho^t$$

which is less than the first amount for  $\rho > \frac{10}{21}$ . This means that if both players used this strategy in the supergame when  $\rho$  is sufficiently high an equilibrium point which is jointly optimal will exist. There will be an infinite number of equilibrium points, each (except for the strategy calling for repeated plays of strategy two) with a statement containing one or more contingent clauses.

The paradox of joint disaster has been removed at the cost of the existence of an infinite number of noncooperative equilibrium points. This lack of determinacy is undesirable and leads to questioning the meaning of strategy in the game of indeterminate or infinite length. Do players present each other with their strategies prior to playing? Are they forced by the rules of the game to abide by their statements? We must reconsider the meaning of communication in many period games. Often in economic competition and in politics and diplomacy a strategy involves

a complex statement with many contingencies which needs to be understood by the opposition and believed for it to be effective. When there is no outside referee or enforcement agency the mere statement of one's strategy is not enough to give it credibility. A calculus of belief is needed so that a level of plausibility can be attached to any statement of intent involving actions in a multistage situation.

In the static theory of games, leaving aside problems of interpretation in communication, a commitment is a commitment; thus a threat is well defined in the sense that it is a statement of what will happen if certain wishes or demands are not complied with. It is like the perfect, flawless Doomsday machine. But perfect precommitment is a very limited aspect of threat as we use it in most human affairs.

There is the uncertainty of the individual about himself which may cause him to "feel threatened." There is the lack of knowledge concerning the goals of one's opponents which calls for a theory of optimal lying. Then there is the whole problem of the relationship between words and deeds. At this point the diplomat or political scientist might call for the study of protocol, national characteristics or institutional peculiarities. The behavioral scientist may stress learning or reinforcement mechanisms to explain the plausibility of threats. Both are undoubtedly right; these paths must be explored. Yet it still may be worth asking if it is possible to gain further insight by using a more straightforward extension of game theory. This we attempt to do.

Many of the cooperative solutions to an n-person static game

depend upon or are related with the characteristic function of the game, for example, stable sets, the core, and the value of the game. In order to calculate the characteristic function the threats of any coalition against the remaining participants and vice-versa must be considered. In a dynamic game, as we have already noted above, it is difficult to evaluate threats involving unenforceable commitments to strategies.

One approach to handling dynamics is to consider only behavioral strategies. Another is to try to measure plausibility in terms of the amounts of damage inflicted on both sides by the carrying out of a threat after non-compliance. Another is to introduce a "plausibility discount rate" such that the further away a move is in time, the less it influences current considerations.

In previous work<sup>29/</sup> it was suggested that threats could be divided into (1) suicidal, (2) killing, (3) weak, and (4) strong. The first arises where the implementation of the threat results in self annihilation. This could be used by madmen, geniuses, or blunderers. (1) and (2) are not mutually exclusive; (2) implies that there is a threat available that could kill the enemy. A weak threat is one which causes more damage to the threatener than the opponent. A strong threat would cause more damage to the opponent than the threatener.

Limiting ourselves to a narrow group of games which can be formulated as reasonable models of dynamic oligopolistic markets we find that there is sufficient structure to enable us to adopt the approaches noted above. In order to do so, we first offer a general description of

a game of economic survival and then limit ourselves to describing a dynamic version of the Cournot duopoly model.

A Game of Economic Survival is characterized by:

- (1) Asset conditions  $A_{1,t}, A_{2,t}, \dots, A_{n,t}$  which are divided into short and long term assets  $S_{i,t}$  and  $K_{i,t}$  where  $S_{i,t}$  are the short and  $K_{i,t}$  the long term assets.
- (2) Ruin and exit conditions, i.e., an end rule, are given as a function of  $A_{i,t}$  and a set of exogenous conditions  $R_{i,t}$ . A liquidation rule is given as a set of functions  $L_{i,t}(A_{i,t})$  paid to the bank accounts of the players. (This rule includes bankruptcy conditions to handle negative net worth.)
- (3) There are bank accounts  $B_{1,t}, B_{2,t}, \dots, B_{n,t}$  into which the players may make payments.
- (4) There are discount rates  $\rho_{i,t}$  which apply to all future monetary values. There are depreciation rates  $\alpha_{i,t}$  which apply to long term assets.
- (5) Each firm has a set of moves  $M_{1,t}, M_{2,t}, \dots, M_{n,t}$  where each  $M_{i,t}$  is a function of  $S_{i,t}$  and  $K_{i,t}$ . During a time period  $t$  each firm makes a market move,  $m_{i,t}$ , a dividend move  $b_{i,t}$  and an investment move  $Y_{i,t}$ .



- (6) There are a set of positional payoffs  $\Pi_{1,t}, \Pi_{2,t}, \dots, \Pi_{n,t}$  where  $\Pi_{i,t} = \Pi_{i,t}(m_{1,t}, m_{2,t}, \dots, m_{n,t})$ . (These functions contain the short term market structure).
- (7) Information conditions are specified.
- (8) An overall payoff function  $V_i$  is defined for each player  $i$ .

#### 4.1 A Cournot Duopoly as a Game of Economic Survival

- (1) Let  $A_{1,t}, A_{2,t}$  be the total asset ~~value~~ of the firms at time  $t$ .
- (2)  $R_1$  and  $R_2$  are the ruin conditions on the firms such that if  $A_{i,t} \leq R_i$  the firm is ruined and obtains an amount  $L_i(A_{i,t})$  paid to the bank account.
- (3)  $B_{1,t}$  and  $B_{2,t}$  are bank accounts into which the players may make payments. These payments are no longer available to the firm.
- (4)  $\rho$  is the discount rate.
- (5) Each firm has a capacity  $K_{1,t}, K_{2,t}$  which limits its moves in the market. A market move consists of selecting a production rate  $q_{i,t}$  such that  $0 \leq q_{i,t} \leq K_{i,t}$ .

Capacity is change: by investment and depreciation.

Let  $\alpha$  = the rate of depreciation,

$s$  = unit cost of capacity

$y_{i,t}$  = investment in period  $t$

$$K_{t+1} = (1-\alpha)K_t + y_{i,t}/s .$$

(6) Let there be a market given by the relation:

$$p = \alpha - \beta(q_1 + q_2) + \xi ,$$

where  $\xi$  is a random variable.

This is reconstituted each period. Each firm makes a move consisting of selecting a production rate simultaneously.

There is then a positional payoff  $\Pi_{i,t} = q_i[(\alpha-c) - \beta(q_1 + q_2) + \xi]$

where  $c$  is the (fixed) average cost of production. After this payoff each firm selects a dividend payment and a level of investment.

(7) After production the firms are informed about the acts of the other. After the dividend and investment decisions all firms are completely informed.

(8) Each firm attempts to maximize its expected discounted worth.

If the expected worth of the firm at time  $\tau$  is  $V_{i,\tau}$ , then

its overall expected discounted value is:

$$\sum_{t=0}^{\tau} \rho^t b_{i,t} + \rho^{\tau} V_{i,\tau} .$$

Where  $b_{i,t}$  is the amount paid into the bank account at time  $t$ .

This is the sum that the firm attempts to maximize.

The assets  $A_{i,t}$  of a firm are divided into two parts  $S_{i,t}$  and  $K_{i,t}$ .  $A_{i,t} = S_{i,t} + sK_{i,t}$ , where  $S_{i,t}$  is the liquid asset position of the firm. There is a limiting condition on investment that  $y_{i,t} \leq S_{i,t} + \Pi_{i,t} - b_{i,t}$ .

#### 4.2 Solutions to a Dynamic Cournot Duopoly

This game viewed in the market is a modified sequential repetition of a one-period simultaneous move game. Leaving out the random element  $\xi$ , we may illustrate the payoff possibilities as is shown in Figure 1. If capacity is large, one or both players

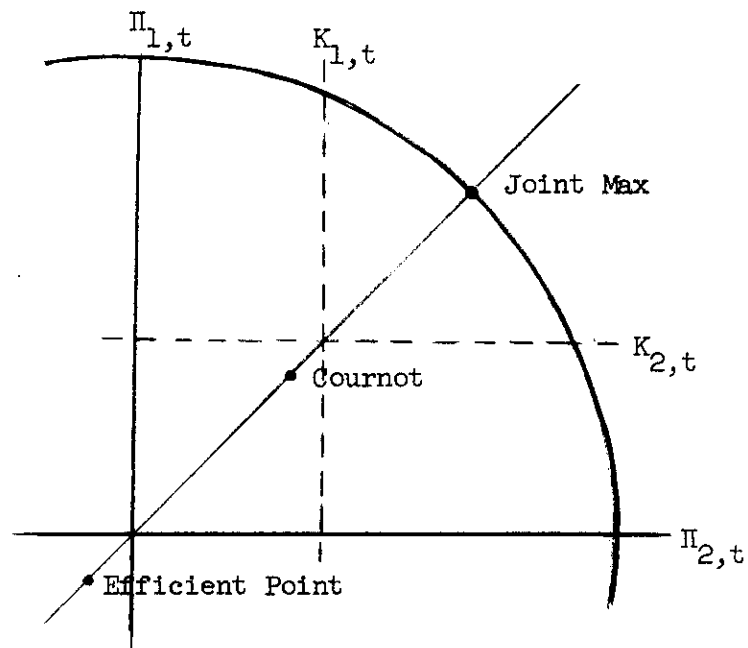


Figure 1

may suffer losses. If, however, capacities are highly restricted, for example  $K_{1,t} = K_{2,t} < \frac{\alpha-c}{3\beta}$  then no matter what they do, profits will be high. This is shown in Figure 1 where for limited capacity profits must lie within the area bounded by the Pareto optimal surface and the lines  $K_{1,t}$  and  $K_{2,t}$ . When capacity is low we may expect that there will be incentives for each player to increase the size of possible production rates, hence to invest in more plant.

Looking at the noncooperative equilibrium solution we see the following:

This game can be played by the use of behavioral strategies<sup>30/</sup> because after every period there is perfect information. If the players both lack capacity and have sufficient funds, the noncooperative equilibrium via behavioral strategies will call for an increase in capacities up to the static noncooperative equilibrium solution\*, i.e.,  $K_{1,t} = K_{2,t} = \frac{\alpha-c}{3\beta}$ . If the players both have considerable excess capacity, the equilibrium will depend upon their financial condition. If both have sufficient resources they will decrease capacity until it is reduced to that needed for the static noncooperative equilibrium\*.

If the firms have considerable excess capacity and there is

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\*/This is not quite correct. A small adjustment needs to be made depending on the carrying costs of the physical plant.

any asymmetry in financial conditions there is the possibility that one firm might find it profitable to drive out the other. This can best be illustrated by a 3x3 matrix example which nevertheless reflects the structure of the duopoly:

	1	2	3
1	5,5	1,6	-1,-2
2	6,1	2,2	-2,-3
3	-2,-1	-3,-2	-5,-5

Table 3

In Table 3 we note that if either firm uses its third strategy (the equivalent of "flooding the market"), then both will suffer losses. If one can bankrupt the other sufficiently quickly so that it can more than recoup the costs of the fight by its monopoly position, it will do so.

If a random element is introduced into the market then equilibria may have only a contingent stability as is shown by the example illustrated in Table 4. Given this payoff matrix with cer-

	1	2
1	1, 1	-1, 0
2	0,-1	-2,-2

Table 4

tainty and, say, a payoff of 2 per period for a monopolist, then the periodic use of (1,1) gives a long-run equilibrium if

$$\frac{1}{1-\rho} > \min \left[ \left\{ \frac{-2(1-\rho^{\tau+1})}{1-\rho} + \frac{2\rho^{\tau+1}}{1-\rho} \right\}, \left\{ \frac{2\rho^{\tau+1}}{1-\rho} \right\} \right]$$

or

$$1 > \min \left[ \left\{ 2(2\rho^{\tau+1} - 1) \right\}, \left\{ 2\rho^{2\tau+1} \right\} \right].$$

Suppose the second player has assets of 2, ruin is at zero and  $\rho = 7/8$ . The criterion gives:

$$\min \left[ \frac{68}{64}, \frac{85.75}{64} \right] > 1.$$

This means that it pays to drive the second firm out of business if it can be done in one period. If the firm has more assets available it will take longer and war is not profitable.

Suppose that the values in Table 4 were not certain but were subject to fluctuation. As long as the available assets of either firm are larger than 2 neither can be profitably ruined. If the fluctuation in payoff are sufficiently large so that although the expected outcome of (1,1) is 1 to each the actual outcome can be negative, the assets of one may be sufficiently reduced to upset the equilibrium. Thus we may say that (1,1) describes a contingent equilibrium dependent upon chance and the dividend policy.

When both firms are more or less equal and sufficiently rich we observe that the behavioral strategy noncooperative equilibrium is linked reasonably to the static noncooperative equilibrium. If one is financially weak, there is a possibility of war. We must now turn to the extensive strategies and examine these two different cases where threats are made.

We can even consider a  $\tau$ -period strategy which does not describe what the firm will do for all time, but for a specified number of periods in the future. The behavior strategy does not need to differentiate between "the word" and "the deed." No communication is needed; the moves themselves state the strategy. A  $\tau$ -period strategy ( $\tau > 1$ ) requires communication and belief if it is to be of use as a threat in enforcing peace in a market. If any statement is completely believed then almost any outcome can be enforced as an equilibrium point. This is unsatisfactory both from the viewpoint of a theory of solution and because the plausibility that all threats (including suicidal and weak ones) will be believed we are reduced to behavioral strategies. In the case of relatively equally powerful noncooperative oligopolists this might be reasonable; however, when one is in a position to destroy another it is quite likely that it may settle for a peaceful semi-cooperative outcome rather than fight. This says that strong and killing threats are likely to be believed.

We still need a calculus of plausibility. Furthermore,

it is quite possible that madmen, "hard bargainers," cheats, and those who do not honor their word may upset an equilibrium. Nevertheless by introducing a limit to the length and type of a strategy that will be believed, we may achieve an intermixture of a dynamic game theory and behavioral theory.

Communication between players may take many forms. If intent concerning behavior to be manifested after the next move must be conveyed, then written, oral, or other means of signalling must be used.

#### 5. N-Person Games of Economic Survival

It appears that among firms in oligopolistic competition a noncooperative equilibrium based upon behavioral strategies may be worth considering. In most other cases, such as welfare problems, labor-management bargaining, or international affairs, this is not so; we need to ~~consider~~ cooperative solutions.

Two somewhat related solution concepts for a dynamic n-person game have been suggested by Luce<sup>31/</sup> and Shubik<sup>32/</sup>. They are  $\psi$ -stability and k-r stability. The Luce solution is closely related to the core and makes use of the characteristic function of the game inasmuch as it evaluates the maximum amount that a coalition can obtain by itself. A function  $\psi$  is defined which limits coalitional change; then an outcome  $[x, \tau]$  is sought where  $x$  is an imputation and  $\tau$  a coalition structure such that for any



S in  $\psi(\tau)$  :

$$v(S) \leq \sum_{i \text{ in } S} x_i .$$

A further condition is suggested that if  $i$  is in a non-trivial coalition  $x_i > v(\{i\})$  .

The selection of the function  $\psi$  is ad hoc and must be presumed to reflect the inertias, costs, and customs of a society at a particular time. Thus, the Luce solution can be regarded as only a short term dynamic solution inasmuch as once coalition structure has changed, this may lead to other possibilities for further change or a new  $\psi_t$  .

Shubik's k-r stability is based upon dividing  $n$  players into three groups: the "violators," the "enforcers," and the passive. The enforcers state a joint strategy which contains a threat or specifies their behavior if the status quo is overthrown. The violators may use a joint strategy against this. The passive players are committed to adhering to the status quo. If the optimal behavior of the violators is to maintain the status quo, then the situation is said to be k-r stable. More generally we may describe the set of  $N$  players as being divided into sets  $K$  ,  $R$  and  $P$  where membership is mutually exclusive and  $N = K \cup R \cup P$  ; hence we may talk of  $K,R$  stability and a solution consists of a distribution  $x$  (not necessarily an imputation) and the coalition

structure described by  $K$ ,  $R$  and  $P$ .

This solution, like that of Luce, contains an ad hoc element. The coalition structure is given and assumed to reflect the institutions of society. We may extend the definition to stability against sets of different coalition structures. This appears to be worthwhile for symmetric games with economic structure, but otherwise the extension does not appear to be fruitful.

The major weakness, however, still lies with the definition of strategy and concept of threat. Why should the violators believe lengthy stated strategies? The status quo does not last for ever. One possible approach to the dynamic solution is to combine a behavioral mechanism with a modified game theory model by limiting the length of the strategies as follows: At any period a player may announce a strategy for  $\tau$  periods ahead. This  $\tau$ -period strategy amounts to a precommitment and is believed. In the subsequent period a player may announce a new strategy which consists of the still relevant branches of the old plus extra branches for the new period. For the calculation of equilibrium the violating coalition calculates the discounted sum of its payoffs for  $\tau$  periods.

As long as we have a satisfactory definition of strategy in a dynamic game and specify the possibilities for coalition formation, then a characteristic function can be defined and solution concepts such as the core or value may be applied.

It is unreasonable to expect that a single outcome is meant to last forever. Thus it is not sufficient to embed these solutions in a dynamic setting. They must be made dynamic themselves. Leaving aside changes in preferences and learning phenomena we reiterate, in conclusion, some of the basic problems in any attempt to create a dynamic theory.

- (1) What is meant by a strategy in a game of indefinite length?
- (2) What are conditions on credibility of strategies and what is meant by a threat? In particular, what calculations are used to determine the equivalent of the characteristic function (or no side-payment coalition values) for a dynamic game?
- (3) What are the goals of the players?
- (4) What is the positional payoff structure of the game?
- (5) How do positional payoffs, sidepayments, and coalition structures vary during the course of the game?

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