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PART V: THE NON-SYMMETRIC GAME:

JOINT MAXIMUM EFFICIENT SOLUTION AND

MEASURES OF COLLUSION AND WELFARE

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PART V: THE NONSYMMETRIC GAME:
JOINT MAXIMUM, EFFICIENT SOLUTION AND
MEASURES OF COLLUSION AND WELFARE*

by

Richard Levitan and Martin Shubik

1. Introduction

In the previous part^{1/} the mathematical structure for a nonsymmetric model of a market was given and the resulting n-person nonconstant sum game was examined and solved for the non-cooperative equilibrium point. Two further solutions are considered here and the problem of comparing them is discussed.

2. Solutions and their Comparison

2.1 The Symmetric Game: General Discussion

The symmetric game was solved for three different solutions, they were respectively the joint maximum; the non-cooperative equilibrium and the beat-the-average outcome. The Pareto optimal surface could be counted as a fourth solution. A fifth solution, the efficient point, is closely related to the beat-the-average in the symmetric case but requires some extra specification, which will be given below.

For ease of discussion, Figure 1 is drawn for the case with only two firms in competition and the remarks immediately following are based upon two firms in competition although they generalize immediately for any number of firms.

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Figure 1 displays

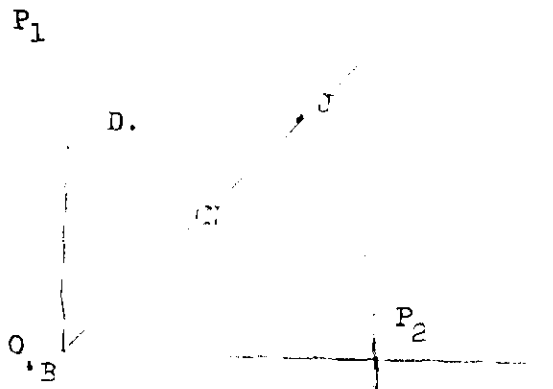


Figure 1

the steady state profits for the two firms under various conditions. The Pareto optimal surface is given by $P_1 J P_2$, the point J is the joint maximum. The point N is the outcome of the non-cooperative equilibrium, and B is the beat-the-average outcome. We note that B actually yields negative profits to both hence could not be stable in the long run as the firms would be wiped out. The negative profits are equal to the overheads of the firms and advertising costs. If these fixed costs and advertising were zero then the profits of the firms when following a beat-the-average policy would be zero.

We note that the points B , N and J lie on a straight line. This is not surprising given the symmetry in the game. Given the definitions of the three solutions it appears to be reasonable to try to use the relationship between the distance BJ and BN as a measure for the degree of innate cooperation or competition built into the structure of the game. Our index of the cooperative structure of this type of game is given by

$$\rho = BN / BJ \quad \text{where } 0 \leq \rho \leq 1$$

Where a totally cooperative structure has N coincide with J and hence $\rho = 1$. A totally noncooperative structure has N coincide with B hence $\rho = 0$. It is of interest to note that the symmetric market structure as described in the previous papers^{2/} has a different structural level of cooperation as the number of participants is increased. The changes in the payoffs associated with the three main solutions is indicated in Figure 2 for different numbers of participants. For one participant J and N_1

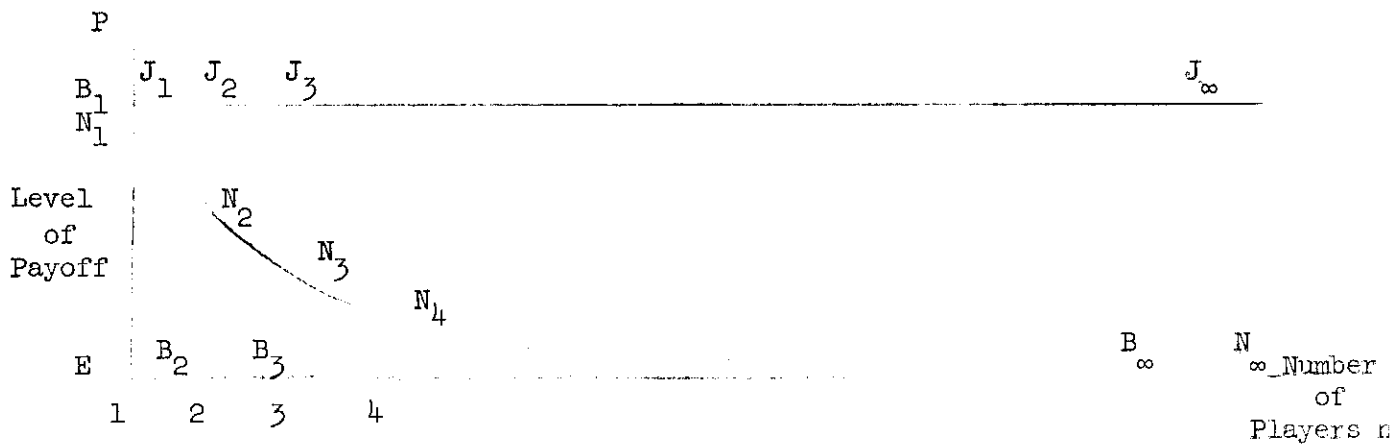


Figure 2

coincide. The B_1 solution is hardly meaningful, but would also coincide with J_1 and N_1 leaving ρ indeterminate. This is not unreasonable if we wish to regard ρ as a coefficient defined only for situations with more than one individual. If we wish to use the efficient point solution, then as is shown in Figure 2, E may be used instead of B_1 in the one-person case and will yield a $\rho = 1$. For $n \geq 2$ we observe that all solutions B_2, B_3, \dots give the same level of profit, and for $n \geq 1$ all solutions J_1, J_2, \dots give the same level of profit. The noncooperative level of profit changes as the number of players n increases. For $n=1$, N_1 coincides with J_1 but as n becomes large N_n

approaches B_n asymptotically hence the range of structural cooperation varies from 1 to 0 as the number of participants changes from 1 to a large number.

The speed of convergence of N_n to B_n is controlled by the selection of the parameters in the game, as is the distance between B_n and J_n . This means that although the introduction of more players changes the structural index of cooperation, the structure may also be changed by adjusting parameters, leaving the number of players fixed.

Nonsymmetric outcomes such as the point D in Figure 1 can be explained in a variety of ways. The success of the explanations will depend upon experimentation and socio-psychological and economic theories which have as yet been scarcely explored. Formally we may observe that if we regard the subjective payoff of each player to be given by $\Pi_i = \sum_{j=1}^n \Theta_{i,j} P_j$ where the P_j are the objective payoffs and the $\Theta_{i,j}$ are parameters then any outcome can be achieved by the appropriate selection of $\Theta_{i,j}$. However there are many other features of human behavior such as ability to learn; level of I.Q.; aspiration level; change of attitude in competitive situations and so forth which might offer better explanations. From the viewpoint of the economist if the structure and the payoffs are completely symmetric in this model we would expect the steady state outcome to be symmetric as well if the players are similar as the solutions suggested are unique and symmetric.

2.2 The Efficient Point as a Solution

Another solution which has been mentioned is the efficient point. Suppose that the industry represented by the model were nationalized and (in an ideal world) were run so as to optimize the benefits to society as a whole.

It is well known that such a policy would call for all of the firms to operate with marginal costs set equal to the price of their product.^{2/} In the symmetric market this can be shown to be the equivalent to the beat-the-average solution^{4/} except in the interpretation of the role of advertising.

Economic theory does not yet have a satisfactory accepted body of knowledge concerning the role of advertising. This is mainly because advertising is evidently a phenomenon associated with change in tastes, with information processes and with dynamics in general. It has been humorously observed that: "Come the revolution, the consumers will all have strawberries and cream and like them!". Do we wish to include in the study of the economics of efficient allocation the possibilities that to achieve this we might change the tastes of the consumers? The demand for the product of the individual firm in the symmetric market was given by:

$$(1) \quad d_i = \frac{1}{n} \left[\alpha - \beta (p_i + \gamma [p_i - \bar{p}]) \right] \left(\theta + (1-\theta) \frac{na_i}{\sum a_i} \right) \left(1 + \eta \sqrt{\sum a_i} \right)$$

This contains two advertising terms. The first term represents the competitive aspects of advertising, or the expenditures devoted to switching customers from one brand of a like commodity to another without any overall increase in market size. In this simple symmetric market it is easy to show that efficiency conditions would call for no advertising if this were the only effect.

The second advertising term $(1 + \eta \sqrt{\sum a_i})$ represents the cooperative effect of advertising which enlarges the demand for the complete class of products. It might be reasonable to expect even a nationalized industry to advertise if it could be argued that such an increase in demand represented an increase in consumer welfare that more than offset the expenditures on advertising. We close this Pandora's Box for the time being and limit ourselves to the more special case where we assume that $\eta = 0$. In this instance the efficient solution will differ from the beat-the-average solution inasmuch as the former will have no advertising expenditures while the latter, in general will. As θ approaches 1 the beat-the-average solution approaches the efficient point.

2.3. Efficiency and Welfare

The payoffs in Figure 1 are drawn from the viewpoint of the firms with no explicit indication of the welfare of the customers. The customers' welfare was implicitly accounted for by the demand functions. In Figure 1 we have placed the origin at E, the efficient point which corresponds not to zero profits to each firm but to profits of $-K$ where K is the fixed costs of each firm. We could at least attempt to illustrate the welfare of the consumers by introducing an aggregate consumer represented by a utility function which gives rise to the demand functions in the model. The three dimensional diagram in Figure 3 has as one axis the welfare of the consumers. We note that neither N nor B lie on this three dimensional Pareto optimal surface. They are all inefficient when viewed by the customers and the firms considered as a whole. The efficient point lies on the surface and is the point at which all gain accrues to the consumer.

The previous Pareto optimal surface P_1JP_2 lies on the new surface and represents the locus of outcomes at which the firms share all of the gains

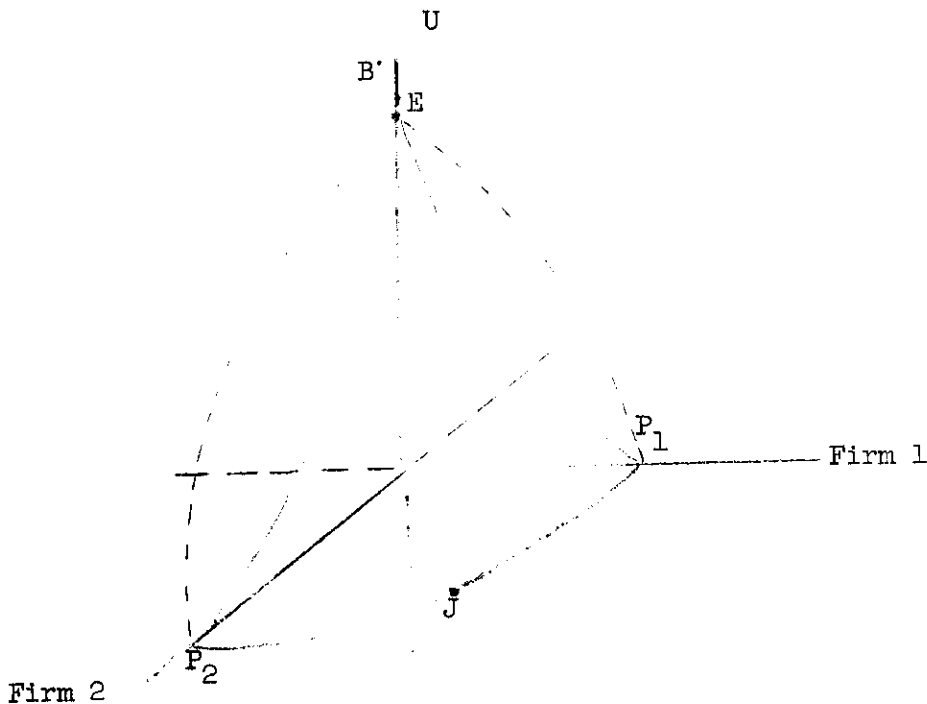


Figure 3

3. Non-symmetric Solutions

3.1. Preliminary Comments

In order to extend our analysis of the non-symmetric game we propose to obtain the joint maximum and efficient solutions and to compare them with the non-cooperative solution by evaluating both the profits of the firms and consumer welfare. In order to do this we shall need to derive a utility function for the consumers based upon the observed demand structure. This is

done in Section 4.

3.2. Joint Maximization

The profit of the i^{th} firm is given by:

$$(2) \quad \Pi_i = \beta w_i (p_i - c_i) (V - p_i - \gamma(p_i - \bar{p}))$$

where w_i is the market share for the i^{th} firm,

c_i is the average cost of production for the i^{th} firm

and \bar{p} is the weighted average of prices. ($\bar{p} = \sum_{j=1}^n w_j p_j$, $\sum w_j = 1$)

We omit advertising from the analysis at this point. As it enters in a multiplicative manner it may be determined after price has been determined.

Taking derivatives of Π_i with respect to i and j we obtain:

$$(3) \quad \frac{\partial \Pi_i}{\partial p_i} = \beta w_i \left[(p_i - c_i) (-1 - \gamma(1 - w_i)) + (V - p_i - \gamma(p_i - \bar{p})) \right]$$

$$= \beta w_i \left[V + (1 + \gamma - \gamma w_i) c_i - (2(1 + \gamma) p_i - \gamma w_i) p_i + \gamma \bar{p} \right]$$

$$(4) \quad \frac{\partial \Pi_i}{\partial p_j} = \beta w_i (p_i - c_i) w_j \gamma$$

Summing (4) we obtain $\sum_{j \neq i} \frac{\partial \Pi_j}{\partial p_i} = \gamma \beta w_i \sum w_j (p_j - c_j)$;

adding this to (3) gives:

$$\begin{aligned}
 \Sigma \frac{\partial \pi_j}{\partial p_i} &= \beta w_i \left[\gamma \Sigma_{j \neq i} w_j (p_j - c_j) + \gamma w_i (p_i - c_i) + V + (1+\gamma)c_i - 2(1+\gamma)p_i + \gamma \bar{p} \right] \\
 (5) \qquad &= \beta w_i \left[\gamma \Sigma w_j (p_j - c_j) + V + (1+\gamma)c_i - 2(1+\gamma)p_i + \gamma \bar{p} \right] \\
 &= \beta w_i \left[V + (1+\gamma)c_i - \Sigma w_j c_j - 2(1+\gamma)p_i + 2 \gamma \bar{p} \right] = 0
 \end{aligned}$$

Rearranging (5) and rewriting it in matrix notation gives

$$(6) \qquad 2 \left[(1+\gamma) I - \gamma S w \right] p = v \hat{1} + \left((1+\gamma) I - \gamma S w \right) c$$

where

$$S = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix}, \quad \hat{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

This gives:

$$\begin{aligned}
 (7) \qquad p &= \frac{V}{2} \left((1+\gamma) I - \gamma S w \right)^{-1} + \frac{1}{2} c \\
 &= \frac{V}{2} \frac{\gamma}{1+\gamma} \left(\frac{1}{\gamma} I + S w \right) \hat{1} + \frac{1}{2} c \\
 &= \frac{1}{2} c + \frac{V}{2} \frac{1}{1+\gamma} (\hat{1} + \gamma \hat{1}) = \frac{1}{2} (c + v \hat{1}).
 \end{aligned}$$

$$(8) \quad p_i = \frac{c_i + V}{2}$$

In conformity with our model in Part IV we limit our consideration of advertising to the single multiplicative term

$$\left(\Theta \frac{a_i}{\Sigma a_i} + (1+\Theta)w_i \right) .$$

This involves only the competitive aspects of advertising. Given prices determined, payoffs are of the form

$$(9) \quad \Pi_i = k_i \left(\Theta \frac{a_i}{\Sigma a_i} + (1+\Theta)w_i \right) - a_i .$$

The first order condition for a joint maximum is:

$$- \Theta \frac{\sum_{j=1} k_j a_j}{(\Sigma a_i)^2} + \Theta k_i \frac{(\Sigma a_j - a_i)}{(\Sigma a_j)^2} = 0$$

which reduces to:

$$(10) \quad \boxed{k_i \Sigma a_i - \Sigma k_j a_j = (\Sigma a_j)^2} .$$

In matrix form this may be written as:

$$KSa - SKa = (\Sigma a_j)^2 \hat{1}$$

or
$$a = (\Sigma a)^2 (KS - SK)^{-1} \hat{1};$$

however $(KS - SK)^{-1}$ is singular if it is of order greater than 2. This is not surprising when we consider the meaning of the basic model. If advertising only reallocates the shares of an overall market of fixed size then if all firms are

acting in a cooperative manner only the one with the lowest costs need advertise. The nature of the advertising term is such that we obtain an unrealistic solution for only one active firm. Any infinitesimal amount ϵ will be sufficient to direct any market share that can be moved by advertising to the only firm with a non-zero expenditure.

Possible a more reasonable approach would be to define a special functional form for the advertising term when only one firm is active. The shape of this function involves empirical and institutional considerations. At this point we leave the determination of the function as an open problem.

3.3. Efficient Point

Given constant average costs for all firms, suppose that they were all being run to maximize the welfare of the consumers. This would call for a price policy such that

(11)

$$p_i = c_i$$

If we make the assumption that society does not wish to change the tastes of its members in the market then we have

(12)

$$a_i = 0$$

We see immediately that both the joint maximum and efficient point solutions are extremely simple generalizations of the results from the symmetric game.

4. A Welfare Measure

In the symmetric game, the various solutions were colinear. It would be at least reasonable to contemplate the construction of measures both of welfare and collusion in terms of indices involving distances or ratios of distances along the line BONJ in Figure 1. When we abandon symmetry the problem becomes far more difficult as is indicated in Figure 4. This illustrates the payoff set and various solutions when there are two nonsymmetric competitors (without fixed costs).

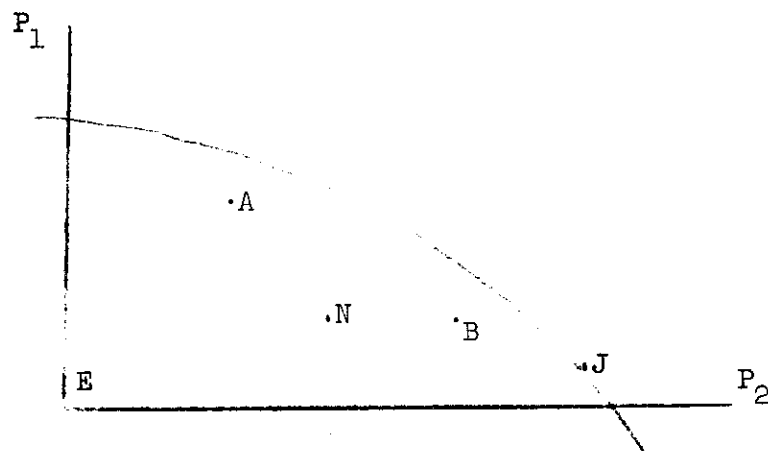


Figure 4

Suppose that we observe two outcomes A and B. How can we compare them with the joint maximum J or the noncooperative equilibrium N? A vector measure of the level of collusion can be obtained by considering:

$$\Pi_i = \sum_{j=1}^n \Theta_{ij} P_j$$

to be the subjective payoffs, evaluate the Θ_{ij} for the outcomes and use $(\Theta_{i1}, \Theta_{i2}, \dots, \Theta_{in})$ as the measure of the level of cooperation of the i^{th} firm.

Apart from the problem of the measurement of collusion, the evaluation of the welfare effects of the policies of the firms remains. As a crude first order approximation we can derive a community utility function from the demand conditions and evaluate the profits of the firms and the customer welfare for the various outcomes. We derive the aggregate utility in two ways, in terms of quantities of goods and in terms of prices. Both may be useful to work with.

4.1. Aggregate Utility: Quantity Function

$$(13) \quad \text{Let } U(\mathbf{x}) = \frac{1}{2} \mathbf{x}^t \mathbf{A} \mathbf{x} + \mathbf{b}^t \mathbf{x}$$

where \mathbf{x} is the vector of quantities of commodities
 \mathbf{x}^t is the transpose of \mathbf{x}

Given \mathbf{p} a vector of prices for all commodities, then

$$(14) \quad U'(\mathbf{x}) = \mathbf{A} \mathbf{x} + \mathbf{b} = \lambda \mathbf{p}$$

where λ is a Lagrangian multiplier.

From (14) and (2) we obtain:

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1} (\lambda \mathbf{p} - \mathbf{b}) = \beta \mathbf{w} \left[\mathbf{V} - \mathbf{p} - \gamma (\mathbf{p} - \mathbf{S} \mathbf{w} \mathbf{p}) \right] \\ &= \beta \mathbf{w} \left[(\gamma \mathbf{S} \mathbf{w} - (1+\gamma) \mathbf{I}) \mathbf{p} + \mathbf{V} \hat{\mathbf{1}} \right] \end{aligned}$$

giving:

$$(15) \quad \lambda \mathbf{A}^{-1} = \beta \mathbf{w} \left\{ \gamma \mathbf{S} \mathbf{w} - (1+\gamma) \mathbf{I} \right\}$$

$$(16) \quad \mathbf{A}^{-1} \mathbf{b} = - \beta \mathbf{V} \mathbf{w} \mathbf{I}$$

hence

$$(17) \quad A = \frac{\lambda}{\beta} \left\{ \gamma S w - (1+\gamma) I \right\}^{-1} w^{-1} = \frac{\lambda}{\gamma\beta} w^{-1} \left\{ S - \frac{1+\gamma}{\gamma} w^{-1} \right\}^{-1} w^{-1} .$$

$$(18) \quad b = - \lambda V \left\{ \gamma S w - (1+\gamma) I \right\}^{-1} \hat{1} .$$

call $\left(\frac{1+\gamma}{\gamma} \right) w^{-1} = Z$ then from (17)

$$(19) \quad A = \frac{\lambda}{\gamma\beta} w^{-1} (S-Z)^{-1} w^{-1} .$$

It can be shown that

$$(20) \quad (S-Z)^{-1} = - \left(\frac{\gamma}{1+\gamma} w + \frac{\gamma^2}{1+\gamma} w S w \right)$$

giving (21)
$$A = - \frac{\lambda}{\beta} \left(\frac{1}{1+\gamma} w^{-1} + \frac{\gamma}{1+\gamma} S \right) = - \frac{\lambda}{(1+\gamma)\beta} (w^{-1} + \gamma S) ,$$

and (22)
$$b = - \beta V A w \hat{1} = \frac{\lambda V}{1+\gamma} (w^{-1} + \gamma S) w \hat{1}$$

$$= \frac{\lambda V}{(1+\gamma)} \left[(1+\gamma) \hat{1} \right] = \lambda V \hat{1} .$$

giving

$$(23) \quad \frac{U(x)}{\lambda} = V \sum_{i=1}^n x_i - \frac{1}{2\beta(1+\gamma)} \left\{ \frac{\sum_{i=1}^n x_i^2}{w_i} + \gamma (\sum x_i)^2 \right\}$$

where λ may be interpreted as the marginal utility of money (assumed constant over the range under examination).

4.2. Aggregate Utility: Price Function

From (21) and (22) we have:

$$(24) \quad U(x) = -\frac{\lambda}{2\beta} x^t \left(\frac{1}{1+\gamma} w^{-1} + \frac{\gamma}{1+\gamma} S \right) x + \lambda V \hat{1}^t x .$$

We may write the demand function as:

$$(25) \quad x = r - Q p$$

where $r = \beta v w \hat{1}$ and $Q = \beta w \left((1+\gamma) I - \gamma S w \right)$ as can be seen from differentiating (13) and obtaining

$$b - Ax = \lambda p \quad \text{or} \quad x = A^{-1}(b - \lambda p) = r - Q p .$$

Let the utility function in terms of price be $\phi(p)$, then:

$$(26) \quad \phi(p) = b^t x - \frac{1}{2} x^t A x ;$$

Divide by λ and substitute for x to obtain:

$$(27) \quad \begin{aligned} \frac{\phi(p)}{\lambda} &= r^t Q^{-1}(r - Qp) - \frac{1}{2} (r - Qp)^t Q^{-1}(r - Qp) \\ &= r^t Q^{-1} r - r^t p - \frac{1}{2} \left[r^t Q^{-1} r - 2 r^t Q^{-1} Qp + p^t Q p \right] \\ &= \frac{1}{2} (r^t Q^{-1} r - p^t Q p) \end{aligned}$$

hence (28)
$$\frac{2}{\lambda} \phi = \beta V^2 \hat{l}^t w \left(\frac{1}{1+\gamma} w^{-1} + \frac{\gamma}{1+\gamma} S \right) w \hat{l} - \beta p^t w \left((1+\gamma) I - \gamma S w \right) p$$

or
$$\begin{aligned} \frac{2}{\beta \lambda} \phi &= V^2 \left(\frac{1}{1+\gamma} \hat{l}^t w \hat{l} + \frac{\gamma}{1+\gamma} \hat{l}^t w \hat{l} \hat{l}^t w \hat{l} \right) - p^t w \left((1+\gamma) I - \gamma S w \right) p \\ &= V^2 \left(\frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma} \right) - (1+\gamma) \sum_{j=1}^n w_j p_j^2 + \gamma (p^t w \hat{l}) (\hat{l}^t w p) \\ &= V^2 - (1+\gamma) \sum_{j=1}^n w_j p_j^2 + \gamma \left(\sum_{j=1}^n w_j p_j \right)^2 \\ &= V^2 - (1+\gamma) \left(\sigma_p^2 - \bar{p}^2 \right) + \gamma \bar{p}^2 \\ &= V^2 - \bar{p}^2 - (1+\gamma) \sigma_p^2 \end{aligned}$$

Thus we may write the utility function in terms of price as:

(29)

$$\phi(p) = \frac{\lambda \beta}{2} V^2 - \bar{p}^2 - (1+\gamma) \sigma_p^2$$

where σ_p^2 is the weighted variance $= \sum w_j p_j^2 - (\sum w_j p_j)^2$

and \bar{p} is the weighted average $= \sum w_j p_j$.

Consumer surplus is given by:

(30)
$$S = \phi - \lambda p^T x,$$

where $x = \beta w (V I - ((1+\gamma) I - \gamma S w) p)$.

Total surplus or gain to all parties is:

$$\begin{aligned}
 (31) \quad T &= S + \Pi = S + (p-c)^T x = \varphi - c^T x . \\
 &= \varphi - c^T \beta w (V1 - ((1+\gamma)I - \gamma S w)p) \\
 &= \varphi - \beta (V\bar{c} - (1+\gamma) \sum w_i c_i p_i + \gamma \bar{c} \bar{p}) \\
 &= \varphi - \beta (V\bar{c} - (1+\gamma)(\sigma_{cp}^2 + \bar{c} \bar{p}) + \gamma \bar{c} \bar{p}) \\
 &= \varphi - \beta (V\bar{c} - \bar{c} \bar{p} - (1+\gamma)\sigma_{cp}^2) \\
 &= \frac{\beta}{2} (V^2 - \bar{p}^2 - 2V\bar{c} + \bar{c} \bar{p} - (1+\gamma)(\sigma_p^2 - 2\sigma_{cp}^2)) ;
 \end{aligned}$$

where σ_{cp} is the covariance of price and cost.

In particular for any set of prices the departure from efficiency is measured by

$$\begin{aligned}
 (32) \quad T(c) - T(p) &= \frac{\beta}{2} (\bar{p}^2 + \bar{c}^2 + 2\bar{c}\bar{p} + (1+\gamma)(\sigma_c^2 + \sigma_p^2 - 2\sigma_{cp}^2)) \\
 &= \frac{\beta}{2} ((\bar{p}-\bar{c})^2 + (1+\gamma)(\sigma_p^2 - 2\sigma_{cp}^2 - \sigma_c^2)) .
 \end{aligned}$$

5. An Example: A Simulated Automobile Market

Using some very crude figures obtained in a previous paper^{5/} based on highly aggregated information on the automobile market we compare the joint maximum, noncooperative equilibrium and the efficient solution with the actual results in the market. The calculations were performed by a computer program not presented here, but available.

	Weights w_i	Average Costs
General Motors	.65	1870*
Ford	.236	1927
Chrysler	.114	1984

These figures give $\beta = 5764$ and $\gamma = 4.6$

	p_1	p_2	p_3	P_1	P_2	P_3	ϕ	Total Surplus
Joint Max.	3486	3515	3543	10156	3272	1394	39730	22233
Actual	2767	2511	2552	6528	2683	1180	54077	27492
Noncoop. Eq.	2703	2572	2565	6813	2560	1133	54625	27813
Efficient Pt.	1870	1927	1984	0	0	0	64637	29644

The p_i are prices, P_i are net variable revenues (no overheads are subtracted and the ϕ is the consumer utility.

We make no pretense at accuracy; the calculations are offered only to suggest the relative sizes of the different solutions. For example it appears that the actual market is very close to and slightly more inefficient than the price noncooperative equilibrium.

The relative efficiencies of the solutions are approximately in the ratio of 100:94:92.5:75 ranging from the efficient solution to total cooperation by the firms. The loss to the public is, of course greatest when the firms collude.

* Our crude calculations give the average costs of General Motors higher than the others. We believe this to be due to the crudeness of our aggregation (2,095). For this calculation we reduce General Motors' costs.

FOOTNOTES

- 1/ Levitan, Richard and Martin Shubik, "Part IV: Mathematical Structure and Analysis of the Nonsymmetric Game," CFDP No. 219, March 2, 1967.
- 2/ _____, "A Business Game for Teaching and Research Purposes, Part I., General Description of the Game," RC-730, IBM Watson Research Center, Yorktown Heights, July 17, 1962.

_____, "A Business Game for Teaching and Research Purposes, Part IIA (Revised), Theory and Mathematical Structure of the Game," RC-1221, IBM Watson Research Center, Yorktown Heights, June 22, 1964.

Shubik, Martin, "A Business Game for Teaching and Research Purposes, Part III, Discussion and Manual for Users," Yale University, New Haven, November 1964.

Levitan, Richard and Martin Shubik, "Part IV: Mathematical Structure and Analysis of the Nonsymmetric Game," CFDP No. 219, March 2, 1967.
- 3/ Shubik, Martin, Strategy and Market Structure, Chapter 3, John Wiley and Sons, 1959.
- 4/ Ibid.
- 5/ Levitan, Richard and Martin Shubik, "Part IV: Mathematical Structure and Analysis of the Nonsymmetric Game."