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A FURTHER COMPARISON OF SOME MODELS OF DUOPOLY

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1. Introduction

Ever since the work of Cournot [1/] various models of duopolistic competition have been formulated using different strategic variables. The best known are models with quantity as the strategic variable or price as the strategic variable. The former was the original model of Cournot, the latter has been formulated and studied by Bertrand, Edgeworth and others.[2/]

There are two major sets of considerations which must be taken into account. They are the formulation of the market model and the selection of a solution concept to apply to the model.

Solutions can be subdivided into four broad categories: static or dynamic; cooperative or noncooperative. Static solutions are usually concerned with equilibrium positions or other types of stability. Most of the economic writings on duopoly have been devoted to static or what can at best be described as "conversationally dynamic" models in which adjustments are described but time periods are not specified, nor are payoffs described in terms of discounted income streams.

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Dynamic solutions must deal explicitly with adjustment processes over time. A linkage between dynamics and statics can sometimes be made when steady state solutions exist, i.e., when the dynamic solution reaches a state in which the same actions are repeated every period. Behavioral models often associated with computer simulations have provided recent examples of dynamic duopoly solutions.[3/]

As a first approximation solutions can be broadly classified as cooperative or noncooperative. In the former, it is assumed that the firms will attempt to jointly optimize. Hence the outcome will be on the Pareto optimal surface of the profit possibility set for the firms. Other solutions, notably the noncooperative equilibrium of Cournot[4/] generalized by Nash[5/] do not involve an assumption that the outcome will be necessarily jointly optimal. It is evident that in the lengthy period of coexistence between two large firms any state between and including outright cooperation and warfare may exist. Quasi or partial cooperation is hard to define, but especially in dynamic behavioral models this type of distinction can be made.

In this paper we are primarily concerned with extending some static models and applying a noncooperative solution to them. Before doing so the problems of model construction and the definition of solution are considered in more detail.

2. Models of Duopoly

When we construct models of duopolistic markets we may divide difficulties into two categories, those concerned with dynamics or statics and those involving the selection of strategic variables.

2.1. Statics or Dynamics.

The simplest, most compact general description of a duopoly is given by the normalized form[6/] of a two person non-zero sum game. Let each player have a set of strategies S_1 and S_2 respectively. These strategies may involve pricing, production, product variation, promotion, advertising, or other actions or any combination of them. A payoff function is associated with each player. If the first selects a strategy s_1 and the second s_2 then their resultant payoffs are $P_1(s_1,s_2)$ and $P_2(s_1,s_2)$.

A simple and direct extension of this general static model to a dynamic case can be made if we consider time to be divided into a set of discrete intervals, say, years or quarters. Let there be a discount rate ρ_t applicable to the t^{th} time period. Let the period payoff to player i be given by $P_{i,t}(s_{1,t}, s_{2,t}, \eta_t)$ where $s_{1,t}$ and $s_{2,t}$ are the strategies for each player in the t^{th} period and η_t is any exogenous influence in this period. The overall payoff to each player is given by:

$$P_i = \sum_{\tau=0}^{\infty} \prod_{t=0}^{\tau} \rho_t \, P_{i,\tau} \, (s_{1,\tau}, \, s_{2,\tau}, \, \eta_\tau) \qquad i = 1, 2.$$

If the discount rate were constant this simplifies to:

$$P_i = \sum_{t=0}^{\infty} \rho^t P_{i,t}(s_{1,t}, \, s_{2,t}, \, \eta_t) \; .$$

For this to be adequately defined we may require conditions which bound the values of these sums. Implicit in a model such as this are the values attached to bankruptcy and to survival.

Even at this level of generality and abstraction it may be argued that a dynamic strategic or game theoretic model of competition obscures many of its major features. Lack of knowledge, uncertainty, learning, and the shifting of one's values and aspirations as functions of previous behavior are not reflected in this type of model.

2.2. The Economic Variables . Limiting ourselves to static models and accepting this degree of unreality we are still faced with the problem of describing the strategic variables which are manipulated by the firm and specifying the market structure. The criticism of Bertrand and Edgeworth of Cournot's model was on the point of the realism and relevance in using quantity as the strategic variable.

In different industries, different variables appear to be dominant. When products are highly similar, when they present few production, inventory, or transportation problems and have a demand that is relatively insensitive to advertising or other promotion, then price may be the dominant variable. Salt appears to be such a product. For various agricultural crops the problems of timing and technology make production the basic strategic consideration. With items such as fashion goods, production, promotion, and price may all be of importance simultaneously. In competition between brands location and inventory position may be the dominant competitive weapons.

A partial list of variables which are of strategic importance individually or in combinations in different markets is given by: price, production policy, product variation, advertising, promotion, distribution, location, service, quality control, and financing. Some of these categories overlap and the list is by no means exhaustive.

Economic theory and measurement in marketing are still at a sufficiently early stage of development that not a great deal is known about strategic aspects of advertising, promotion, and distribution. Although in many markets they are undoubtedly the most important factors, they are not discussed further here.

Most of the economic analysis of duopoly has been carried out in terms of price or quantity as the basic strategic variable where the firms are regarded as selling either identical products or products differentiated by location, technical properties or some other factors. It would seem that by now analysis based upon these variables would have been completed. There is, however, one important class of models which has scarcely been investigated. They are those in which both price and production are regarded as independent strategic variables simultaneously. This can happen when firms have to commit themselves to a production policy in advance, before selling to a price sensitive market. They have to run the risk of being caught with inventory or being caught out of stock.

Section 4 of this paper deals with price-quantity duopoly and contrasts it with models involving only price or quantity.

3. Solutions

In the discussion of duopoly it is often desirable to make clear the distinction between the structure of the market and the behavior of the firms. Referring back to 2.1, the sets of strategies S_1 and S_2 and the payoff functions $P_1(s_1, s_2)$ and $P_2(s_1, s_2)$ describe the market structure but they do not specify the behavior of the individuals. A solution concept applied to the structure of the market will serve to delineate the behavior of the participants.

A desirable property of a solution should be that it picks out a single outcome. If this is the case then the solution provides a unique prediction. Some solutions have this desirable property, others fail to possess it. An example of an important solution which fails to yield a unique outcome is the contract curve of Edgeworth.

In a previous paper 7/ the profits associated with a number of different solutions were indicated. Here we list some of the solutions which have been proposed.

Cooperative Solutions

(1) The Pareto Optimal Surface: This may be regarded as a weak form of solution which prescribes that the duopolists be efficient in their exploitation of their market. It should be noted that if the welfare of the customers is included explicitly then points which are on the Pareto optimal surface of the whole group are not necessarily on the optimal surface for the subset consisting only of the firms. This is illustrated in Figure 1 which is drawn for two firms and one customer. A_1A_2 is the optimal surface for the duopolists. It lies on the three dimensional surface EA_1A_2 where E is the efficient point which is efficient for society as a whole, but certainly not for the duopolists as it maps into the point O in the P_1P_2 plane.

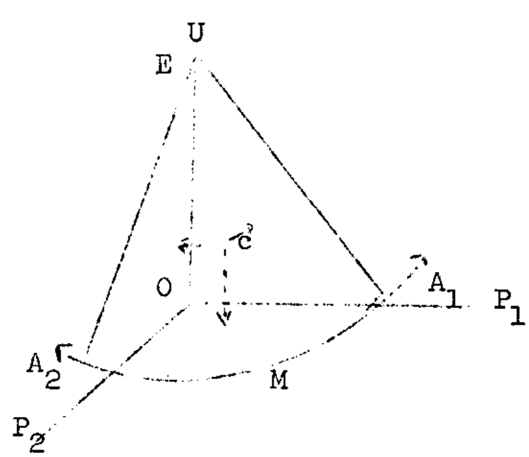


Figure 1

The Pareto optimal surface for the firms may contain extensions beyond A_1 or A_2 in the P_1P_2 plane. Group optimality does not preclude one individual accepting less than he could obtain by himself.

(2) The Contract Curve: This consists of that part of the Pareto optimal surface which satisfies conditions of individual rationality as well as joint optimality. This solution is more generally known as the core [8/].

We may note that the Pareto optimal surface may change depending upon whether the strategic model of the market involves price, quantity or price and quantity as strategic variables. The contract curve may also have somewhat different end points if the different market models have different threat possibilities.

(3) Joint Maximization: If the two firms could compare profits they would be in a position to select a specific outcome on their Pareto optimal surface as is indicated by the point M in Figure 1. This is usually not influenced by the strategic structure of the game.

(4) The Value: There are several different definitions for the value of an n-person game [9/]; however, they coincide for the two-person case. The calculation for the value has been illustrated in a previous paper [10/]. In essence the value embodies principles of "fair division," or equity as a method for "solving" the conflict in the market. The calculation of the value depends upon the

determination of the threats of the players and these depend explicitly upon their strategic possibilities. This is discussed further in Section 4.

Noncooperative and other Solutions

(1) The Noncooperative Equilibrium: This solution is very sensitive to the strategic formulation of the game. In Figure 1 the point c (which lies under the Pareto optimal surface) is the noncooperative equilibrium point for the Cournot or quantity strategy model of duopoly. The price and price-quantity models have considerably different solutions as is shown elsewhere 11/ and in Section 4.

(2) (a) Beat-the-Average, (b) Maxmin-the-Difference, (c) Maximize Profit Share: These three solutions are closely related inasmuch as (b) is a special case of (a) which occurs when there are only two players, and (a) is a special case of (c) 12/. All depend explicitly upon the strategic formulation of the game.

(3) The Threat Curve or Pareto Minimal Surface: One way in which the Pareto optimal surface is defined is by:

$$\text{maximize} \quad P_2(s_1, s_2)$$

subject to: $P_1 = c$.

This calls for the satisfaction of the first order condition:

$$\begin{vmatrix} \frac{\partial P_1}{\partial s_1} & \frac{\partial P_1}{\partial s_2} \\ \frac{\partial P_2}{\partial s_1} & \frac{\partial P_2}{\partial s_2} \end{vmatrix} = 0$$

The threat curve might be regarded as its inverse inasmuch as it may be defined by:

$$\text{minimize} \quad P_2(s_1, s_2)$$

subject to: $P_1 = c$.

This satisfies the same first order conditions but different second order conditions.

(4) The Competitive Equilibrium: In many ways the competitive equilibrium is a misnomer as it is obtained when each individual ceases to believe that he has any competitive influence whatsoever and acts as an individual maximizer taking prices as given... In general, duopoly models abstract from the entry of new firms whereas the competitive solution includes the possibility of entry. Without entry it is more appropriate to call the solution in which the individual equates marginal costs to price, the efficient production solution. It serves as a benchmark to indicate the production and prices which would prevail if the firms were run solely for the benefit of the consumers. This is indicated by point E in Figure 1 which lies on the Pareto optimal surface of society as a whole but gives all gain to the consumers.

(5) Partial Cooperation: A general solution which includes several of the solutions discussed above and may be used to reflect any level of cooperation can be obtained in the following manner.

Assume that the individual payoffs of the players are P_1 and P_2. However, their social payoffs are given by Π_1 and Π_2 where:

$$\Pi_1 = \theta_{11}P_1 + \theta_{12}P_2$$

$$\Pi_2 = \theta_{21}P_1 + \theta_{22}P_2$$

where the θ_{ij} is a coefficient of "concern" or interest that the i^{th} individual has in the size of the payoff to the j^{th} individual. If the players try to maximize Π_1 amd Π_2 it follows immediately that when all $\theta_{ij} = 1$ this is the joint maximum; when $\theta_{ii} = 1$ and $\theta_{ij} = 0$ for $i \neq j$ this is the non-cooperative equilibrium. The "maxmin the difference" solution is given by $\theta_{ij} = 1$ and $\theta_{ij} = -1$ when $i \neq j$. Other values of θ_{ij} reflect various levels of competition or cooperation.

Behavioral Solutions

Behavioral solutions are cast in a dynamic context. They involve the specification of reaction functions which may be based upon aspiration levels, learning, search procedures, survival, uncertainty minimization, the reduction of cognitive dissonance and so forth. As we limit ourselves to a static analysis at this time these types of solution are not discussed.

4. Price and Quantity Duopoly Models

For ease in discussion and computation in this section we use symmetric models with quadratic payoff functions. These involve constant average costs and a demand that may be represented by a linear function. Somewhat more general nonsymmetric models have been analyzed previously for the price and the quantity games.13/

4.1. Quantity or Cournot Duopoly

Let demand be given by:

$$p = \alpha - \beta(q_1 + q_2) \quad \text{or} \quad (q_1 + q_2) = \frac{\alpha - p}{\beta} \ .$$

If the average cost to each is k then their payoffs are:

$$P_i = (\alpha - \beta q - k)q_i \qquad \text{where} \quad q = q_1 + q_2 \ .$$

If we wish to consider a market where quantity is the strategic variable we need to specify the range of production over which the firms can operate. We select two possibilities, they each have a capacity limit of:

$$(1) \qquad \frac{\alpha - k}{\beta} \quad \text{or} \quad (2) \ \frac{\alpha - k}{2\beta} \ .$$

In the first case each is capable of supplying the whole market at the efficient or competitive price, in the second case each is only capable of supplying half of the market at that price.

If, in the first instance, both firms produced up to their capacity the demand function would give a negative price. As this is unreasonable we define demand conditions somewhat more precisely to be

$$p = \alpha - \beta q \quad \text{for} \quad q \leq \frac{\alpha}{\beta}, \qquad p = 0 \quad \text{for} \quad q > \frac{\alpha}{\beta}.$$

The Pareto optimal surface and the threat surface are both given by:

$$\begin{vmatrix} \alpha - k - 2\beta q_1 - \beta q_2 & \beta q_1 \\ -\beta q_2 & \alpha - k - 2\beta q_2 - \beta q_1 \end{vmatrix} = 0$$

or $$(\alpha - k - 2\beta(q_1 + q_2))\,(\alpha - k - \beta(q_1 + q_2)) = 0 .$$

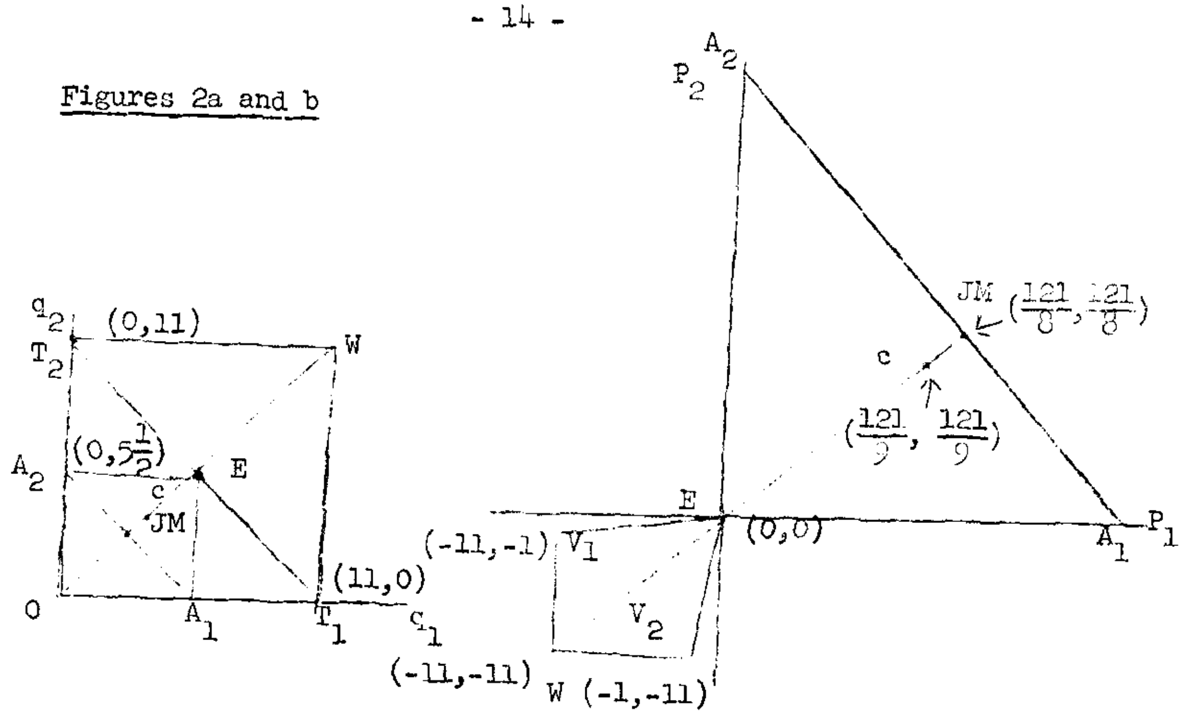
The first factor set equal to zero gives the Pareto optimal surface, and the second factor gives the threat curve. The joint maximum, noncooperative equilibrium and the efficient point are given respectively by:

$$q_1 = q_2 = \frac{\alpha - k}{4\beta}, \quad q_1 = q_2 = \frac{\alpha - k}{3\beta} \quad \text{and} \quad q_1 = q_2 = \frac{\alpha - k}{2\beta}$$

with profits of $\frac{(\alpha-k)^2}{8\beta}$ $\frac{(\alpha-k)^2}{9\beta}$ and 0 respectively.

These are shown in Figures 2 and 3 where we illustrate the solutions $\alpha = 12$, $\beta = 1$, $k = 1$ where the capacity of each firm is $M = 11$ and $M = 5\frac{1}{2}$ in the two cases we examine

Figures 2a and b



	Joint Max	Non Coop Eq	Efficiency
price	$6\frac{1}{2}$	$4\frac{2}{3}$	1
production	$5\frac{1}{2}$	$7\frac{1}{3}$	11
profit	$\frac{121}{8}$	$\frac{121}{9}$	0

In Figure 2a the strategy space for $M = 11$ is given by the large rectangle OT_1WT_2 and that for $M = 5\frac{1}{2}$ is OA_1EA_2. These give rise in the first instance, as is shown in Figure 2b, to a payoff space consisting of A_1A_2E and the area EV_1WV_2. In the second instance the payoff space consists only of the triangle EA_1A_2. The area for "corational" threats has been cut out owing to the lack of capacity for the firms. The various labeled points in Figure 2a map into the points labeled with the same letters in 2b. The line T_1ET_2 and the point O in Figure 2a all map onto the point E in Figure 2b.

4.2. Price or Bertrand-Edgeworth Models. For the price model we take precisely the same market structure as above, except we must specify demand when the same good is offered at different prices. The problem of contingent, rationed or oligopolistic market demand has been investigated previously by Shubik 14/ and Levitan 15/ . A general treatment has been given by Levitan. Here, as in a previous paper we make the simple assumption that unsatisfied demand is scaled down in proportion to the price named by the high-priced firm. This is indicated in Figure 3 where the point M is the point at which the firm charging the lower price, say p , reaches its capacity limit leaving an unsatisfied demand (at that price) of MA . At the price p_2 the unsatisfied demand is scaled down to EF .*

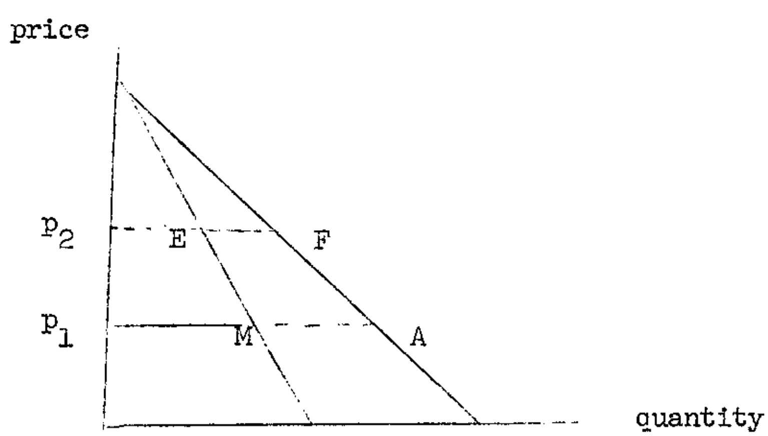


Figure 3

* Implicit in this analysis is that the customers do not set up secondary resale markets to any significant extent.

It is obvious that if each firm has the capacity to satisfy the market at a price equal to its cost then there will be no residual unsatisfied demand for the firm with the higher price. This implies that capacity considerations are critical to the threat possibilities when price is the strategic variable. In particular using the same parameters as for the quantity game, when $\alpha = 12$, $\beta = 1$ and $M = 11$ it follows that demand for the higher priced firm, at any price equal to or above cost for the lower priced firm, will be zero.

The three solutions presented for the quantity game, joint maximum, noncooperative equilibrium and the efficient point were the same for $M = 11$ and $M = 5\frac{1}{2}$ (if we had restricted capacity to even less than $M = 5\frac{1}{2}$ then the efficient point and eventually the noncooperative equilibrium would have been influenced). This is not the case for the price game. When $M = 11$ it is easy to see that the solutions are as follows:

	Joint Max.	Non Coop. Eq.	Efficiency
price	$6\frac{1}{2}$	1	1
production	$5\frac{1}{2}$	11	11
profit	$\frac{121}{8}$	0	0

The joint maximum and efficiency solutions are the same as for the quantity strategy game however the noncooperative equilibrium has been changed and now coincides with the efficient solution. This is the <u>Bertrand</u>

Solution to duopolistic competition.

When $M = 5\frac{1}{2}$ the joint maximum and efficiency solutions are not affected however the noncooperative equilibrium is changed. It no longer exists as a pure strategy but is manifested as a probability mix over a range of price. This solution is related to the existence of the Edgeworth Cycle or range of price fluctuation suggested by Edgeworth.

Before we calculate the Edgeworth Cycle and the noncooperative equilibrium it is necessary to consider the precise meaning of a strategy in a price strategy game. Having done so we will be in a position to draw diagrams similar to Figures 2a and b describing the strategy spaces and payoffs.

Given that each firm names a price as its strategy how are the payoffs to be calculated? There is nothing stopping a firm from naming a negative price, however as even the desire to damage one's competitor mitigates against this we will limit prices to being nonnegative.

Suppose that one firm has named the lower price for the identical commodity that both are selling; what are its commitments to its customers? We may adopt several conventions. The first is that found in the selling of items such as turbines or other heavy machinery. The firms each bid by naming a price; after one has obtained the order it then produces the items. The other has no inventory problem as production is not started before a bid has been accepted.

What happens if a firm wins a bid but does not have sufficient capacity to do the job? This is an insitutional question and depends in great detail upon the specific market mechanism. In some instances the bid may include a penalty for failure to deliver. In other situations, such as when a department store announces as a "special" a very attractive buy, it may also specify that it has only a limited number of units for sale at that price. If it does not do so then it may be liable to legal action if it fails to supply the demand. We examine three conventions:

(1) A firm makes a bid in terms of price, but it does not have to commit itself to production until it knows its demand. It is not penalized for failure to supply. This is the pure price or Bertrand-Edgeworth game [16/].

(2) A firm makes a bid in terms of price, simultaneously it commits itself to a level of production and a maximum amount that it is prepared to supply. In this case it may not supply all of its potential demand or it may be caught with excess inventories. This we call the price-quantity game.

(3) A firm makes a bid in terms of price and commits itself to production after it knows its demand. It is penalized for failure to supply.

Although we are presenting a static analysis it is evident that the actual problem calls for dynamic models. Thus the interpretation of the inventory and out-of-stock penalties must be in terms of future consequences.

4.2.1. The Price Game. We set up the price model first. Given this it becomes considerably easier to discuss the other two. Using the same structure as given in 4.1. for the quantity game, we must add conditions to describe the market for different prices. We select the capacity limit as $M = \frac{\alpha - k}{2\beta}$. This gives each individual precisely enough capacity to produce half of the total demand at the efficient price.

The payoff to Player 1, as a function of the two prices is given by:

$$(1) \quad p_1 < p_2 \qquad P_1 = M(p_1 - k)$$

$$(2) \quad p_1 = p_2 \qquad P_1 = \left(\frac{\alpha - p_1}{2\beta}\right)(p_1 - k)$$

$$(3) \quad p_1 > p_2 \qquad P_1 = \left(\frac{\alpha - p_2 - \beta M}{\alpha - \beta_2}\right)\left(\frac{\alpha - p_1}{\beta}\right)(p_1 - k) \; .$$

Taking the same example as before with $\alpha = 12$, $\beta = 1$, $k = 1$ and $M = 5.5$ we have:

$$(1) \quad p_1 < p_2 \qquad P_1 = 5.5(p_1 - 1) \qquad \text{for } p_1 > 1$$

$$(2) \quad p_1 = p_2 \qquad P_1 = \frac{1}{2}(12 - p_1)(p_1 - 1)$$

$$(3) \quad p_1 > p_2 \qquad P_1 = \left(\frac{6.5 - p_2}{12 - p_2}\right)(12 - p_1)(p_1 - 1) \qquad \text{for } p_1 > 1$$

Using this information we may draw two diagrams, Figures 3a and 3b showing the price strategy space and the payoff space. We may limit prices to the range $0 \leq p_i \leq 12$. Figure 3b which shows the payoff space illustrates that the Pareto optimal surface is not concave in this case (i.e.,the payoff set is not convex).

As the payoff functions are not continuous we cannot use the Jacobian condition to derive an equation for the surface. We know however that if the players charge different prices, the optimal price for the higher priced player is invariant at 6.5; however his profits decrease with the increase of the price of the other player. Suppose Player 2 charges the higher price, then for

$$M = 5.5 \qquad P_2 = \frac{(6.5 - p_1)}{(12 - p_1)} \left(\frac{121}{4}\right) \qquad \text{and} \quad P_1 = 5.5\,(p_1 - 1)$$

which gives the relationship $P_2 = \left(\frac{139 - 4P_1}{200 - 4P_1}\right)\left(\frac{121}{4}\right)$

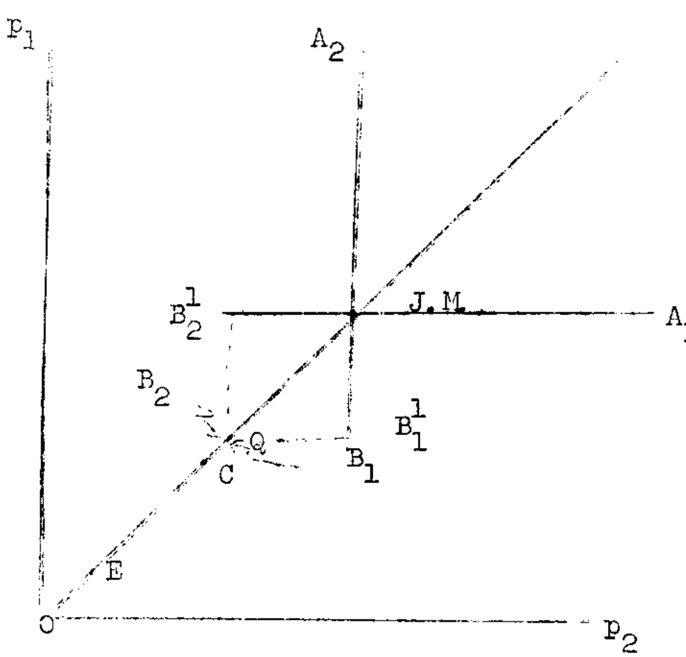


Figure 3a

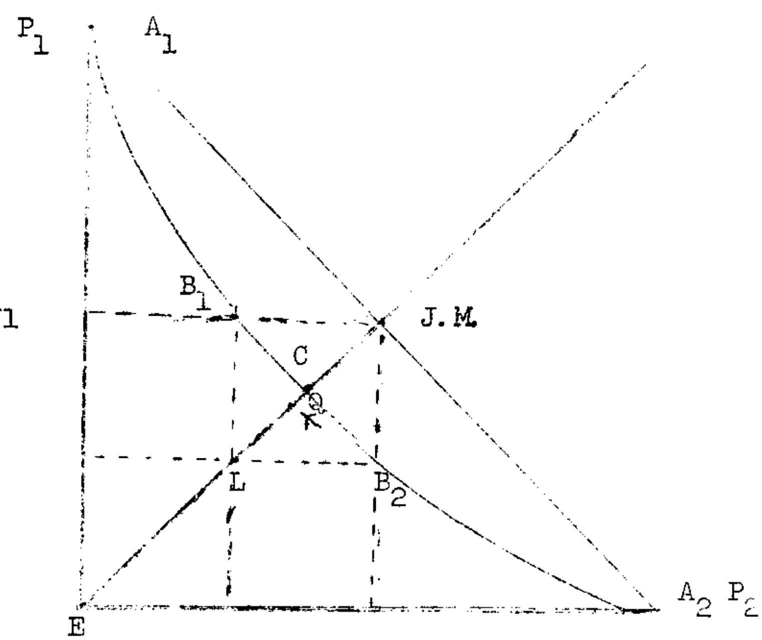


Figure 3b

The attainable set of payoffs are given by the area P_1EP_2L and the line L to J.M. This strange shape is caused by the discontinuity around $p_1 = p_2$. The Pareto optimal surface is not continuous, it consists of three parts, the lines A_1B_1 and A_2B_2 and the point J.M.

The points illustrated in Figure 3b are as follows:

		P_1	P_2	p_1	p_2
A_1(and A_2)	Monopolistic Maximization	121/4	0	6.5	>6.5
J.M.	Joint Maximization	121/8	121/8	6.5	6.5
B_1(and B_2)	Maxmin	121/16	121/8	$3.75-\epsilon$	3.75
Q		363/32	363/32	3.75	3.75
c	Lower bound of Edgeworth cycle	23.1/2	23.1/2	3.1	6.5

Figure 3a shows the strategy space for the two players. The points marked in 3a map into the points with the same marking in Figure 3b. Owing to the discontinuity in the payoffs some care must be taken in interpreting the diagrams. The half open interval A_2 to J.M. (excluding the point J.M.) in Figure 3a maps into A_2 in Figure 3b. The line J.M. to B_2^1 maps into A_1c. The point c (which is the lower bound of the Edgeworth Cycle) is not on the Pareto optimal surface as it is dominated by the equal price joint maximum, it is however the "connection" between the two parts A_1c and cA_2 of the surface which first have Player 1 and then Player 2 as the lower price competitor.

The Edgeworth cycle can be calculated by noting that either firm will be indifferent between just undercutting its competitor or raising its price considerably. Let the lower bound of the cycle be α . We have

$$\left(\frac{6.5 - \alpha}{12 - \alpha}\right) \quad (5.5)^2 = 5.5(\alpha - 1)$$

giving: $\alpha^2 - 18.5\alpha + 191/4 = 0$

or: $\alpha \simeq 3.1$.

The diagrams for the case when $M = 11$, i.e., when each firm is capable of satisfying the demand from the whole market by itself are somewhat different as is shown in Figures 4a and 4b. The only

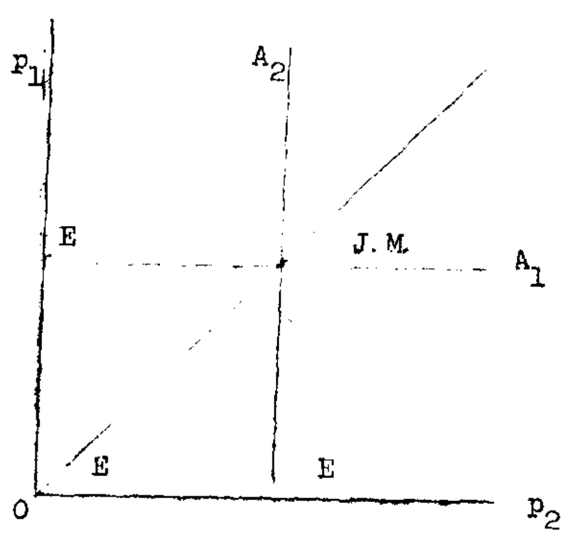


Figure 4a

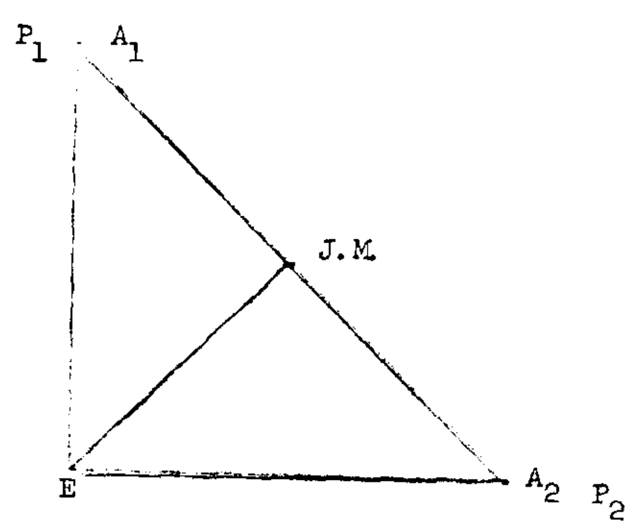


Figure 4b

points left on the Pareto optimal surface are A_1, A_2 and J.M. The payoff set consists only of the three lines EA_1, E J.M. and EA_2 (both in Figures 3 and 4 we have left out the small negative part of the payoff space which exists when a price of less than 1 is named). In Figure 4b E is not only the efficient point but the noncooperative equilibrium as well. In Figure 4a although the strategy set consists of all (p_1, p_2) such that $0 \leq p_i \leq 12$, $i = 1$ or 2 ; we only note some points which correspond with those in 4b. As before, the line A_2 to J.M. , excluding the point J.M. maps into the point A_2 .

It can be seen from Figures 3 and 4 that differences in capacity have considerable influence on the shape of the payoffs set in price competition.

Reverting to the case with $M = 5\frac{1}{2}$ we noted that there was no pure strategy noncooperative equilibrium solution. In order to obtain a noncooperative solution in this case it is necessary to solve for the mixed strategy equilibrium which amounts to solving the integral equation

$$P_1(p_1, p_2) = (p_1-1)\int_{p_2=a}^{p_1}\left(\frac{6.5-p_2}{12-p_2}\right)(12-p_1)d\varphi(p_2) + 5.5\int_{p_2=p_1}^{b}(p_1-1)d\varphi(p_2)$$

$$= 5.5(a-1)$$

for $a \leq p_i \leq b$. We investigate this equation further in the Appendix.

4.2.2. The Price-Quantity Game. In this game each player has a two-dimensional set of strategies. This gives them considerable flexibility in the obtaining of payoff combinations. It is evident that excess capacity has the same effect in the price-quantity game as in the price game. Reverting to our example, if $M = 11$ then the noncooperative solution is a pure strategy at the efficient point with $p_1 = p_2 = 1$ and $q_1 = q_2 = 5.5$.

Given that each individual names not only a price, but commits himself to a level of production we must specify conventions concerning out-of-stock and excess inventory situations. We pick the simple rule that if an individual is unable to supply a demand he suffers no penalty beyond that implicit in losing a part of his potential market. For ease in the treatment of inventories we may imagine the item to be perishable or a fashion good so that if it is not sold it is worthless at the end of the period. Given these conventions we may write the payoff functions for the case with $M = 5.5$ as:

$$(1) \quad p_1 < p_2 \qquad P_1 = q_1(p_1 - 1)$$

$$(2) \quad p_1 > p_2 \qquad P_1 = \min \left\{ \left(\frac{12 - p_2 - q_2}{12 - p_2} \right) (12 - p_1),\ q_1 \right\} p_1 - q_1 \ .$$

The same type of convention as before is needed for the case of $p_1 = p_2$. We omit making this explicit as it is not used in our subsequent discussion.

The condition which must be satisfied for a mixed strategy noncooperative equilibrium is as follows:

Let the ranges for the mixed strategy be $a_1 \leq p_1 \leq b_1$

$a_2 \leq q_2 \leq b_2$

then:

$$P_1(p_1, p_2, q_1, q_2) = \int_{p_2=p_1}^{b_1} q_1(p_1-1)d\varphi(p_2)$$

$$+ \int_{p_2=a_1}^{p_1} \left\{ \int_{q_2=a_2}^{(12-p_2)\frac{12-p_1-q_1}{12-p_1}} \left[\left(\frac{12-p_2-q_2}{12-p_2}\right)(12-p_1)p_1-q_1\right] d\Psi(q_2 \mid p_2) + \int_{q_2=(12-p_2)\left(\frac{12-p_1-q_1}{12-p_1}\right)}^{b_2} q_1(p_1-1)d\Psi(q_2 \mid p_2)\right\} d\varphi(p_2)$$

$$= 5.5\ (a_1-1)\ .$$

Where $\Psi(q_2 | p_2)$ is the conditional probability function that q_2 will be produced given p_2. This equation is discussed further in the appendix.

4.2.3. <u>The Price Game with an Out of Stock Penalty.</u> This alternative was suggested to the author by K. J. Arrow in a discussion. It is evident that if the firms each individually have enough capacity to satisfy the market this game is equivalent to the price game with large capacity. In terms of our example the case $M = 11$ is of little interest, however $M = 5.5$ poses a problem.

If we assume that there is a penalty of t per unit out of stock then the payoff function to this game is given by:

$$(1) \qquad P_1 = 5.5(p_1-1) - t(6.5-p_1) \qquad \text{for } p_1 < p_2$$

$$(2) \qquad P_1 = \tfrac{1}{2}(12-p_1)(p_1-1) \qquad p_1 = p_2$$

$$(3) \qquad P_1 = \left(\frac{6.5-p_2}{12-p_2}\right)(12-p_1)(p_1-1) \qquad p_1 > p_2 \;.$$

The tendency to cut price will be modified by the penalty.

Suppose that both players charge the same price, when will neither be inclined to undercut his competitor. This is given by:

$$\tfrac{1}{2}(12-p)(p-1) \geq 5.5\,(p-1) - t(6.5-p)$$

or

$$-\,p^2 + (2-2k)p - 1 + 13k \geq 0 \;.$$

Neither will be motivated to raise price if

$$\tfrac{1}{2}(12-p)(p-1) \geq \left(\frac{6.5-p}{12-p}\right)\left(\frac{121}{4}\right)$$

or $p \geq 3.51$ which yields a profit $P = 10.65$.

The previous inequality gives us

$$-\,12.33 + 3.51(2-2t) - 1 + 13t \geq 0$$

or $k \geq 1.05$.

This indicates that if the penalty is less than $t = 1.05$ per unit then there will be motivation for one firm to undercut the other at prices $p \leq 3.51$, however the optimal reaction for the other will be to raise his price to $p = 6.5$. This means that a modified

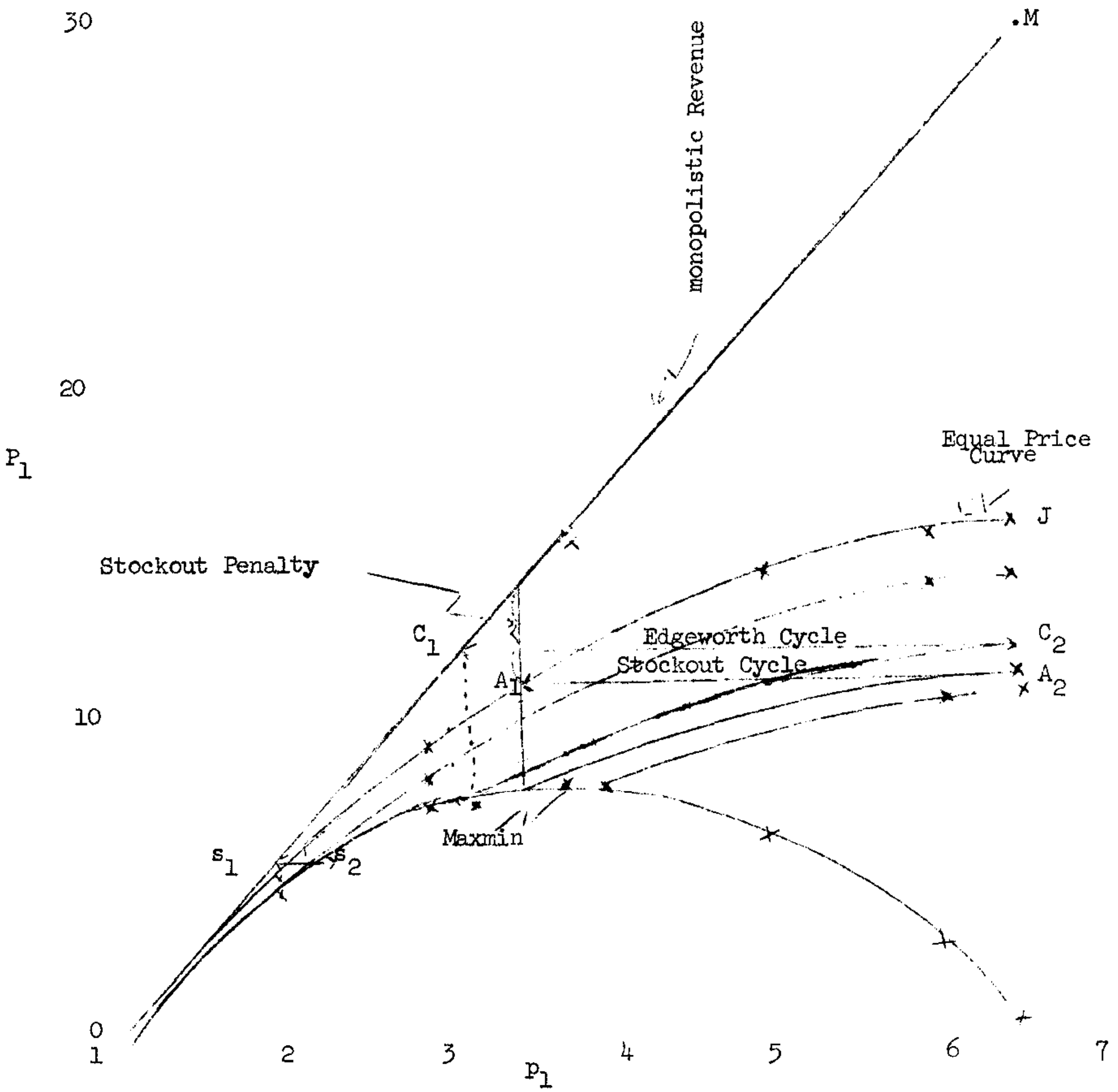

30
20
10
0
P_1
.M
monopolistic Revenue
Equal Price
Curve
J
Stockout Penalty
C_1
C_2
Edgeworth Cycle
Stockout Cycle
A_1
A_2
s_1
s_2
Maxmin
1
2
3
4
5
6
7
p_1

FIGURE 5

Edgeworth Cycle or stockout cycle exists for $0 \leq t \leq 1.05$. This is shown in Figure 5 where A_1 is the lower end of the cycle when $t = 1.05$. As is to be expected when $k \to 0$ the stockout cycle approaches the Edgeworth Cycle.

When $t > 1.05$ a new phenomenon is encountered, a continuum of equilibrium points appears whose range is given by $p = 3.51$ at the lower bound and $p = (1-t) + \sqrt{t^2 + 11t}$ at the upper bound. For example, for $k = 11$, all pairs of equal prices in the range:

$$3.51 \leq p \leq 5.56$$

are equilibrium points. As $t \to \infty$ we observe that $p \to 6.5$.

5. Conclusions

We have attempted to present an exhaustive examination of the models that can be constructed by considering price and/or quantity at the strategic variables. These included the Cournot, Edgeworth and Bertrand cases as well as two further models reflecting the effect of inventory costs and penalties for failure to supply. Each model had a different noncooperative solution. The difference between Bertrand and Edgeworth hinged upon capacity conditions. Given limited capacity the price, price-quantity and stock penalty models all gave rise to an instability manifested by the existence of the Edgeworth cycle (which is of the same length for both the price and price-quantity models) and the somewhat different "stockout cycles" when the stock penalty is sufficiently low in the stock penalty model.

When the stock penalty is high not only is the price instability removed but a continuum of equilibria is obtained. It is of interest to note that although in the duopolistic markets we find strikingly different solutions as the number of firms in the market is increased in the appropriate manner the price and price-quantity solutions converge to the competitive equilibrium (this is well known for the Cournot model; the other cases are covered by a theorem on the price model 17/) . Although each model is highly abstracted and "unrealistic" they all have their counterparts in different markets. The strategy space is dependent upon the specifics of technological and institutional structure. The stockout game does not converge. This appears to imply a weakness in modeling a fixed penalty for many firms.

A static analysis always strikes one as unsatisfactory in the description of duopoly. It has been offered here primarily in association with presenting an exhaustive description of the strategy sets and payoffs to the firms.

One may have a distinct discomfort with the Edgeworth Cycle as being too "unrealistic" even if we consider it as well defined dynamic model. It seems unreasonable to consider a competitor doubling his price. Customers might be influenced by too much price variability. Competitors might be induced to enter by a considerable rise in price. Ignoring entry, we limit our remarks to price change. It is possible to consider a limit to the amount that a price can be changed in a period, or alternatively we might wish to consider demand not only as a function of price but of previous price changes as well.

The limit on price change is considered. When we introduce a limit to the largest size for Δp we can see immediately from Figure 5 that the price instability is <u>not</u> removed in the price or price-quantity models. The range is merely limited by the maximum size of Δp.

(1) $\Delta p \geq C_1C_2$: This leaves the Edgeworth Cycle intact. Prices will fluctuate between C_1 and C_2.

(2) $\Delta p < C_1C_2$: The cycle exists but its base is moved down along OF_1C_1 and its length will be the largest Δp possible, for example suppose $\Delta p = s_1s_2$ then the cycle will be s_1s_2 .

All of the results above depend neither upon linear costs with capacity restrictions nor upon the firms selling identical products. Increasing costs with differentiated products would still lead to the same qualitative results.

It is of interest to note that when the stockout penalty is large the resultant equilibria yield higher payoffs to the firms than they would obtain noncooperatively without the penalty. We may regard the penalty as providing an extra threat available to help enforce the equilibrium. The power of competitive price-cutting becomes so great that neither wishes to risk using the weapon too much.

APPENDIX

The mixed strategy equilibrium for the price game can be obtained by solving the integral equation:

$$(p_1-1)(12-p_1)\int_{p_2=a}^{p_1}\left(\frac{6.5-p_2}{12-p_2}\right)d\Phi(p_2)+5.5(p_1-1)\int_{p_2=p_1}^{b}d\Phi(p_2)=5.5(a-1)\ .$$

It is necessary to solve for a, b and $\Phi(p)$. As this equation must have the same value for $a \leq p_1 \leq b$ we may differentiate it repeatedly with respect to p_1 obtaining:

$$(12-p_1)\left(\frac{6.5-p_1}{12-p_1}\right)\varphi(p_1)-\int_{p_2=a}^{p_1}\left(\frac{6.5-p_2}{12-p_2}\right)\varphi(p_2)dp_2-5.5\varphi(p_1)=-5.5\frac{(a-1)}{(p_1-1)^2}$$

or

$$(1-p_1)\varphi^1(p_1)-\varphi(p_1)-\frac{(6.5-p_1)}{(12-p_1)}\varphi(p_1)=11\frac{(a-1)}{(p_1-1)^3}$$

or

$$(1-p_1)\varphi^1(p_1)+\left(\frac{2p_1-18.5}{12-p_1}\right)\varphi(p_1)=11\frac{(a-1)}{(p_1-1)^3}$$

this can be evaluated.

The equation for the price-quantity equilibrium given in 4.2.2 can be somewhat simplified by observing that after differentiating twice with respect to q_1 it reduces to:

$$\int_{p_2=a}^{p_1}\psi(k|p_2)\varphi(p_2)dp_2=0$$

this implies that $\psi(k|p_2)$ is not a density, but that $q_2=f(p_2)$. In other words there is a specific level of production associated with any price selected.

FOOTNOTES

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   and

   Shapley, L. S., "A Value for N-Person Games," in H. Kuhn and A. W. Tucker, (eds.) Contributions to the Theory of Games, II, 28, Princeton, Princeton University Press, 1953, pp. 307-317.

10. See 7/.

11. Shubik, M., "A Comparison of Treatments of a Duopoly Situation, Part II," Econometrica, XXIII, October, 1955, pp. 417-431.

    and

    - - - , Strategy and Market Structure, pp. 109-117.

12. Levitan, Richard and M. Shubik, "The Nonsymmetric Game: The Generalized Beat-the-Average Solution," Cowles Commission for Research in Economics at Yale University, (forthcoming).

13. See 7/ and 11/.

14. See 2/.

15. Levitan, Richard E., "Oligopoly Demand," RC-1239, IBM Watson Research Center, Yorktown Heights, July 23, 1964.

16. See 11/.

17. Shubik, M., Strategy and Market Structure, Chapter VI.