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OPTIMAL EMPLOYMENT AND INFLATION OVER TIME

Edmund S. Phelps

University of Pennsylvania

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This paper presents dynamic models from which are derived the "socially optimal" time-path of aggregate employment or capacity utilization. Given this employment path and the initially expected rate of inflation, the time-path of the actual rate of inflation (positive or negative) can also be derived, so that one can say equally well that the present paper studies the optimal path of inflation over time. These models are oversimplified in several respects; for example, a closed, nonstochastic economy is postulated in which monetary policy is exogenous and of a special kind. In view of such simplifications, I should not want this paper taken as a declaration on macro-economic policy. Nevertheless I feel that many economists -- including, I hope, many policy-makers -- will want to know what these models teach.

The principal ingredients of the basic model are the following. First, a sort of Phillips Curve in terms of price change, rather than wage change, that shifts with variations in the expected rate of inflation. Second, a mechanism by which the expected inflation rate adjusts to the actual inflation rate. Third, a social utility function that is the sum or integral of the instantaneous "rate of utility" (possibly discounted) at each point in time now and in the future. Last, a derived dependence of the rate of utility at any time upon current "utilization" -- the decision variable under fiscal control -- and upon the money rate of interest, hence, given the real rate of interest, upon the expected rate of inflation -- the state variable in the problem at hand. (These novelties will later be elaborated and defended.) An optimal utilization or employment path is one which maximizes the social utility integral subject to the Quasi-Phillips Curve and the adaptive expectations mechanism.

The choice problem just sketched is dynamical: An optimal utilization policy by the government must weigh the consequences for future utility possibilities of today's utilization decision. By contrast, the conventional approach to the employment-inflation problem -- if there is a conventional approach -- is wholly statical.[2] I can show where I believe the conventional approach goes wrong in this respect, and thus explain the idea that motivated this paper, by the following simple argument.

Visualize a diagram on which the unemployment-labor force ratio is measured along the right-hand horizontal axis and the proportionate rate of price increase or rate of inflation is measured on the vertical axis. Now represent a locus of unemployment-inflation combinations available to the government <u>when the expected rate of inflation equals zero</u> by the characteristically shaped Phillips Curve.[3] This curve (defined only for positive unemployment) is negatively sloped, strictly convex (bowed in toward the origin) and it intersects the horizontal axis at some unemployment ratio, say u^*, $0 < u^* < 1$. The quantity u^* measures the "equilibrium" unemployment ratio when the expected rate of inflation is zero since it is that unemployment at which the actual rate of inflation equals the expected rate of inflation so that the expected inflation rate remains unchanged. Now superimpose a family of social indifference curves onto the diagram. These curves are negatively sloped (at least in the positive quadrant) and strictly concave (bowed out from the origin). Suppose that one of these indifference curves is tangent to the Phillips Curve at some unemployment ratio, say $\tilde{u}$, smaller than u^*. The quantity $\tilde{u}$ measures the (statical) optimum in the conventional approach: At this unemployment ratio, the "benefits" of a small reduction of unemployment are offset by the "costs" of the increase of the inflation rate that, according to the Phillips Curve, would be necessitated by the reduction of unemployment. The inequality $\tilde{u} < u^*$ stems from the customary (though not unanimous) judgment that there is some reduction of unemployment below u^* that is worth the little inflation it entails.

Since the statical "optimum" will produce some positive rate of inflation, the question arises: Will this inflation not eventually create the expectation of inflation by the participants in product and labor markets? It is reasonable to believe so. If it does, it is further reasonable to argue, on the ground that labor and management think in terms of real wages and real costs, that the Phillips Curve will gradually shift upward (in a uniform vertical displacement) by the full amount of the newly expected and previously actual rate of inflation.

Alert policy-makers will now recalculate the "optimum" in the light of the new, higher Phillips Curve. If the indifference curves are shaped in the unlikely way that the "optimal" unemployment ratio does not change, then the actual rate of inflation associated with this decision will be higher by the amount of the upward shift of the Phillips Curve. The pattern will now repeat: The Phillips Curve will shift upward again as expectations are again revised. If the policy-makers persist in their decision to have an unchanged unemployment ratio, there will occur what is popularly called a "wage-price spiral" that is "explosive" or "hyperinflationary" in character. But it is not an endless spiral for once the rate of inflation has increased to a certain upper bound, the monetary system will break down.

It is more likely that the indifference curves are shaped in such a way that any upward, uniformly vertical displacement of the Phillips Curve will cause the policy-makers to "take out" the loss in the form of an increase of the unemployment ratio as well as an increase of the rate of inflation. Thus the actually chosen rate of inflation will increase by less than the increase in the expected rate of inflation. As a consequence, though the Phillips Curve will shift up in the "second round," because a higher rate of inflation was chosen, it will shift upward by less than the amount of the first shift. As the pattern repeats, both the actual and the expected rates of inflation will be increasing all the time, but at a decreasing rate. The rate of inflation will be increasing because the unemployment ratio is smaller than u^*, so that the actual rate of inflation always exceeds the expected rate with the consequence that the Phillips Curve is always rising. However, as the statically "optimal" $\tilde{u}$ approaches u^*, a stationary equilibrium will be asymptotically approached in which $\tilde{u} = u^*$ and there is equality between the expected and actual rates of inflation -- except in the unlikely event that the indifference curves make $\tilde{u}$ approach u^* so slowly that the rate of inflation causes the monetary system to break down before equilibrium can be achieved. Hence, even though a state of steady inflation is eventually achieved, it is likely to be a very high rate of inflation -- much higher, probably, than the policy-makers would have chosen had they not acted myopically in always choosing the static "optimum." More precisely, the conventional approach goes wrong in implicitly discounting future utilities infinitely heavily. (This is not the only amendment to the conventional approach that I shall make.)

Of course, my criticism is founded upon the postulated "instability" of the Phillips Curve. In fact, a situation of sustained "overemployment" -- more precisely, unemployment less than u^* by a nonvanishing amount -- has been supposed to produce an explosive spiral through its effects upon the Phillips Curve. On my assumptions, the only steady-state Phillips Curve is a vertical line intersecting the horizontal axis at u^*.[4] Now some econometric work over the past ten years might suggest that, especially on a fairly aggregative level, the Phillips Curve is a tolerably stable empirical relationship.[5] But these studies probably estimate some average of different Phillips Curves, corresponding to different expected rates of inflation which have varied only over a small range. Further, some writers have found the actual rate of inflation to have a weak influence on wage (or price) change and this may be explained by the view that the actual rate of inflation is a proxy, but a very poor one, for the expected rate of inflation.[6] The notion that the steady-state Phillips Curve differs from the nonsteady-state curves is intuitively reasonable and it is all that is required for justification of a dynamical analysis rather than a statical one such as previously described. This is shown by the fact that, in the dynamic models to be presented, the optimal employment or utilization policy depends critically upon the rate of time preference, a concept not really contained in the conventional approach.

I. THE STANDARD MODEL

What I call the "standard model" is the most convenient one with which to begin. The subsequent three models, which will require only brief attention once the standard model has been understood, can be viewed as variations on this model. The "standard model" is one with an infinite-time decision-making horizon, a smooth utility function and no money illusion. Part II will investigate the model with a finite time horizon, a pathological case in which the utility function is not continuous and a rather peculiar case of "inflation illusion."

A. Possibilities and Preferences

In this section I develop the model and state the optimization problem. The solution will be discussed in Sections B and C.

1. The 'virtual' golden age, utilization and interest. I wanted these preliminary models to have three convenient properties. First, to simplify the preference side of the model, I wanted the money rate of interest to be a stationary function of employment or utilization, given the expected rate of inflation. Second, I wanted consumption alone to vary with utilization, not investment, again in order to simplify preferences. Third, I wanted the marginal productivity of labor rising at the same constant proportionate rate for every employment or utilization ratio, in order that the notion of a

stationary family of Phillips Curves in terms of prices have greater plausibility. I can obtain each of these properties from the postulate that the economy, thanks to a suitably chosen monetary policy and to the nature of population growth and technological progress, is undergoing "virtual" golden-age growth. By this I mean that actual golden-age growth would be observed in the economy if the employment-labor force ratio or utilization ratio were constant (at any level). Golden-age growth is said to occur when all variables change exponentially, so that investment, consumption and output grow at the same rate (which may exceed the rate of increase of labor).

To generate virtual golden-age growth I make the following assumption. Let us suppose that the homogeneous labor force (or competitive supply of labor) depends only upon population, the real wage rate, disposable real income per capita and real wealth per capita, and is hence independent of the real and money rates of interest.[7] (Taxes will be lump-sum so that we need not work with after-tax wage rates and after-tax interest rates.) Population will be supposed to grow exponentially at some rate $\lambda \geq 0$. It will be supposed that the labor supply function is homogeneous of degree one in population and homogeneous of degree zero in the real wage, disposable real income per head and real wealth per head. Hence, whenever the latter three variables are in constant ratios to one another -- whenever they are changing equiproportionately -- the labor supply will grow at rate γ . More general assumptions are apt to impair the feasibility of golden-age growth.

As for production, let us think in terms of an aggregate production function. It would be possible, but inconvenient, to express the necessary assumptions in terms of a "vintage" aggregative model; more serious problems would be encountered if we were to allow heterogeneous kinds of investments. Suppose that the aggregate production function exhibits constant returns to scale in capital and employment and that technical progress, if any, enters in a purely labor-augmenting way, so that output is a linear homogeneous function of capital and augmented employment (or employment measured in "efficiency units"). Suppose further that the proportionate rate of labor augmentation is a nonnegative constant $\lambda \geq 0$. Then augmented labor supply will grow exponentially at the "natural" rate, $\gamma + \lambda \geq 0$, whenever the real wage rate, disposable real income per capita and real per capita wealth grow in the same proportion.

As for capital, we require that the capital stock grow exponentially at the rate $\gamma + \lambda$. Then (and only then) output will grow exponentially, as will investment and hence consumption, at the rate $\gamma + \lambda$ for any constant augmented employment-capital ratio. This implies that the government, by monetary actions I shall assume, always brings about the right level of (exponentially growing) investment necessary for exponential growth of capital at the natural rate and thus for virtual golden-age growth.

On these assumptions, then, there is virtual golden-age growth. At any constant ratio of augmented employment to capital -- which I shall call the <u>utilization ratio</u> -- output, investment, consumption, capital, augmented employment and, under marginal productivity pricing, real profits and real wages will all grow exponentially at the natural rate. Marginal and average product of labor and, under marginal productivity pricing, the real wage rate, real income per capita and real wealth per capita will all grow at the rate λ . Disposable real income per head will also grow at rate λ on plausible assumptions (e.g., a constant average propensity to consume) such that the taxes per head necessary for the exponential growth of consumption per head also grow at rate λ . Thus the labor supply will grow at rate γ , like population and employment. The marginal product of capital and the equilibrium competitive real interest rate will be constant over time. But of course this is only virtual golden-age growth: If the augmented employment-capital ratio is changing over time, most of these variables will not be growing exponentially. It is only population, labor augmentation and capital (hence investment) that grow exponentially come what may.

As already indicated, the monetary authority is postulated to guide investment along its programmed, exponential path. It is as if the Bank were bent on (virtual) balanced growth, independently of the fiscal authority, which will be assumed to exert no direct influence on investment demand by any devices like business income taxes or investment subsidies. The fiscal authority does, however, have control over consumption demand and hence, given the programmed investment demand, aggregate demand and employment. Since employment is the decision variable in the present problem, fiscal devices are the policy instruments by which consumption demand and thus employment are controlled. I postulate the use of lump-sum taxes on households for this purpose. While the Fisc is therefore the decision-maker of interest here, I suppose that the Fisc maximizes (as it sees it) a social utility function that reflects the preferences of the individuals in the economy.

The monetary instruments by which the Bank keeps investment on its programmed path are assumed to be devices like open-market operations which operate through the rate of interest or directly upon the demand for capital. The Bank must be alert therefore to adjust interest rates in the face of changes in aggregate demand or utilization engineered by the Fisc. What kind of interest rate policy is therefore required by the Bank? On the usual assumption of diminishing marginal productivities, the higher the augmented employment-capital ratio, the higher will be the marginal product of capital and hence, if capital receives its marginal product, the higher will be the real profit rate on capital and real profits; according to the usual neoclassical theory, therefore, the demand for capital by firms will also be higher at any given real rate of interest. On one kind of theory, it will then be necessary (and sufficient) for the Bank to raise the real rate of interest by a certain amount in order to keep the quantity of capital demanded, and hence the volume of investment undertaken, from rising above its programmed path. On another monetary theory holding that the competitive real rate of interest must always equal the marginal product of capital (in equilibrium), the real interest rate will automatically rise (with marginal productivity) but open-market sales must still be undertaken to decrease the "demand price" for capital, lest investment exceed its programmed path. The gist of the argument, then, is that the real rate of interest will be higher the greater is the ratio of augmented employment to capital -- since investment is being guided along the exponential path appropriate to virtual golden-age growth.[8] Now let us pass to the details.

The real rate of interest is the money rate of interest minus the expected rate of inflation. I assume here that expectations of the current price trend are "single valued" and held unanimously by the public (but not necessarily by the policy-makers who, from this point of view, lead an unreal existence). By the money rate of interest is meant the rate of interest charged on the short-term lending of money, hence the nominal yield on short-term bonds. If equity shares are homogeneous with bonds (i.e., no risk considerations), it is also the expected yield (including capital gain) on shares. If shares are homogeneous with physical capital, it is also the nominal money yield on physical capital in financial or portfolio equilibrium; then, in equilibrium, the real rate of interest is equal to the marginal product of capital. If we let i denote the money rate of interest and let r denote the real rate of interest, we obtain upon transposing terms in the previously stated relation,

$$(1) \qquad i = r - x , \qquad 0 \leq i < i_b ,$$

where x is the expected rate of algebraic <u>deflation</u>. Thus $-x$, which is to be added to r to obtain i, is the expected rate of inflation.[9] Equation (1) says, therefore, that as x becomes algebraically small, say negative -- i.e., as inflation becomes expected -- the money rate of interest becomes high, given the real rate of interest; for given the physical or real yield on capital, the prospects of high nominal capital gains on physical assets (and hence on equities) produced by the expectation of inflation will induce people to ask a high interest rate on the lending of money, while borrowers will be prepared to pay a high rate since the loan will be expected to be repaid in money of a lower purchasing power.

Since no one will lend money at a negative money rate of interest when he can hold money without physical cost, the money rate of interest must be nonnegative. Further, it is assumed that there is a constant, i_b , to be called the "barter point," such that at any money interest rate equal to or in excess of it money ceases to be held so that the monetary system breaks down; this is because such a high money rate of interest imposes excessive opportunity costs on the holding of noninterest-bearing money instead of earning assets like bonds and capital.

As indicated previously, the real rate of interest will be taken to be an increasing function of the utilization ratio, the ratio of augmented employment to capital, denoted by y :

$$(2) \qquad r = r(y) \; , \quad r(y) > 0 \; , \quad r'(y) > 0 \; , \quad r''(y) \geq 0 \; ,$$
$$0 < \mu \leq y \leq \bar{y} < \infty \; .$$

Consider the bounds on the utilization ratio. If positive employment is required for positive output then, by virtue of diminishing marginal productivity of labor, there is some small utilization ratio, denoted by μ , such that output will be only large enough to permit production of the programmed investment, leaving no employed resources for the production of consumption goods. Since negative consumption is infeasible,. no value of y less than μ is feasible. The value μ is a constant by implication of the previous postulates. In the other direction, there is clearly, at any time, an upper bound on (augmented) employment arising from the supply of labor function and the size of population. This explains the upper bound $\bar{y}$ which, quite plausibly in view of the previous assumptions, is taken to be a constant.

Consider now the $r(y)$ function itself in the feasible range of the utilization ratio. The postulate that $r(y) > 0$ for all feasible y is perhaps not unreasonable; it could be relaxed. The curvature of $r(y)$ is of greater importance. (Figure 2 gives a picture of this function.) On the view that r is equal to the marginal product of capital, one is in some difficulty for there are innumerable production functions that make the marginal product of capital a strictly concave (increasing) function of the labor-capital ratio, e.g., the Cobb-Douglas. Fortunately, I do not really require convexity of $r(y)$; $r''(y) \geq 0$ is overly strong for my purpose which, it will later be clear, is the concavity of U in y in (8). (Even the latter concavity could probably be dispensed with by one more expert than the present author in dynamic control theory, though probably the solutions would be somewhat affected.) I shall later indicate the minimum requirement on $r''(y)$. At this moment let it be said that there are countless production functions which make $r''(y) \geq 0$; for example, any production function which makes the marginal-product-of-labor curve linear or strictly convex in labor (which is uncustomary in textbooks) will suffice and even some concavity is consistent with (2).

Finally, a word about the use of the ratio of augmented labor to capital as a strategic variable in the model. Since capital is growing like $e^{(\gamma+\lambda)t}$ while employment is multiplied by $e^{\lambda t}$ to obtain augmented employment, it can be seen that, if N denotes employment and K denotes capital, then, with suitable choice of units,

$$y = \frac{e^{\lambda t} N}{K} = \frac{e^{\lambda t} N}{e^{(\gamma+\lambda)t}} = \frac{N}{e^{\gamma t}} .$$

Hence the definition of the utilization ratio used here does not imply a neoclassical model with aggregate "capital" in the background. Only neoclassical <u>properties</u> like diminishing marginal productivities need to be postulated and these are much more general than the neoclassical model. The previous relation shows that we could as well define the utilization ratio as the employment-population ratio (since population is growing like $e^{\gamma t}$) which, in the present model, is a linear transformation of the augmented employment-capital ratio. Thus the utilization ratio here measures not only the intensity with which the capital stock is utilized (the number of augmented men working with a unit of capital) but also the utilization of the population in productive employment.

2. <u>Inflation, utilization and expectations</u>.

I am going to postulate that the rate of inflation depends upon the utilization ratio and upon the expected rate of inflation. In particular, the rate of inflation is an increasing, strictly convex function of the utilization ratio. When the expected rate of inflation is zero, the actual rate of inflation will be zero when the utilization ratio equals some constant y^* between μ and $\bar{y}$, will be positive for any greater utilization ratio and negative for any smaller utilization ratio. As $\bar{y}$ is approached, the rate of inflation approaches infinity. Finally, every increase of the expected rate of inflation by one point will increase by one point the actual rate of inflation associated with any given utilization ratio. Remembering that $-x$ is the expected rate of inflation, one therefore may write

$$\frac{\dot{p}}{p} = f(y) - x\ , \qquad \mu \leq y \leq \bar{y}\ ,$$

$$(3) \qquad f'(y) > 0\ ,\ \ f''(y) > 0\ ,\ \ f(\bar{y}) = \infty\ ,\ \ f(y^*) = 0\ ,$$

$$\mu < y^* < \bar{y}\ ,$$

where p is the price level and $\dot{p}$ its absolute time-rate of change so that $\dot{p}/p$ is the rate of inflation. Thus we must add the expected rate of inflation to the function $f(y)$ to obtain the actual rate of inflation. For every x we have a Quasi-Phillips Curve relation between $\dot{p}/p$ and y. The relationship is pictured in Figure 1.

I believe there can be no real question that, if the somewhat Phillipsian notion of the $f(y)$ function is accepted, the expected rate of inflation must be added to it as in (3). For if, as postulated, there is no money illusion and the supply of labor is independent of the real and money rates of interest and hence independent of the expected rate of inflation, then, however fast wage rates and prices are advancing at a given utilization ratio when there is no expected inflation, they must each be advancing one percentage point faster when the expected rate of inflation is one percent. (If labor supply depends upon x we would have to write $f(y,x)$ so that y^* is no longer a constant, independent of x. The last part of this paper contains a model based on such a case.) Note that no assumption of any kind concerning the formation of expectations has yet been made here; no assumption of perfect foresight or the like is implied in the formulation of this inflation function.

The concept of the function $f(y)$ is more vulnerable to criticism.

At bottom, I am simply postulating it, hoping that some approximation to it (at least) can sometime be derived from basic theoretical considerations and can be empirically supported. It is surely true that prices tend to rise most rapidly in periods of high capacity utilization, given inflation expectations. But it may be that other variables belong in this inflation function.

One line of defense would be to derive equation (3) from the ordinary Phillips Curve in terms of the wage-rate change and the employment-labor force ratio, adjusted for the rate of expected inflation in the proper manner. As I shall now show, such an adjusted, ordinary Phillips Curve together with marginal cost pricing yields <u>almost</u> the formulation in (3).

First of all, it will be recalled that constancy of the utilization ratio gives 'virtual' golden-age growth and, in particular, produces exponential growth of employment and the supply of labor at rate γ, the population growth rate, if the real wage rate equals the marginal product of labor (for then the real wage rate, real disposable per capita income and real per capita wealth will be growing equiproportionately, namely at rate λ). Hence the employment-labor supply ratio is constant whenever the utilization ratio is constant. On almost any reasonable assumption about the shape of the labor supply function it will follow that these two ratios are monotonically increasing functions of one another. Therefore we can probably regard the utilization ratio as a proxy for the employment-labor supply ratio, which is essentially the independent variable used in the ordinary Phillips Curve.

On this basis we can write our family of Phillips Curves in the form

$$\frac{\dot{w}}{w} = h(y) - x$$

$$h'(y) > 0 \ , \quad h''(y) > 0 \ , \quad h(\bar{y}) = \infty \ , \quad h(y^{\dagger}) = 0$$

$$\mu < y^{\dagger} < \bar{y}$$

where w is the money wage rate and hence $\dot{w}/w$ is the proportionate rate of wage increase.[10] If $\dot{w}/w$ is convex in the employment-labor force ratio, as usually inferred from data, then $h(y)$ will be convex (as assumed) if the employment-labor force ratio is convex in the utilization ratio or at least not "too concave."[11]

Now let us suppose, somewhat unrealistically, that price is instantaneously equated to marginal cost so that the money wage rate is continuously equal to the money marginal value product of labor; that is, $w = pm$, where m is labor's marginal product. Differentiation of this with respect to time together with the marginal product-of-labor relation of the form $m = g(y)e^{\lambda t}$ yields

$$\frac{\dot{p}}{p} = \frac{\dot{w}}{w} - \lambda - \frac{\dot{y}}{y}\left[\frac{\partial m}{\partial y} \cdot \frac{y}{m}\right]$$

and therefore, upon substituting the Phillips Curve equation for $\dot{w}/w$,

$$\frac{\dot{p}}{p} = h(y) - \lambda - x - \frac{\dot{y}}{y}\left[\frac{\partial m}{\partial y} \cdot \frac{y}{m}\right] .$$

The bracketed expression is the elasticity of labor's marginal product with respect to the utilization ratio and must be negative if there is diminishing marginal productivity of labor, that is, rising marginal costs. We can obtain our Quasi-Phillips Curve in terms of price change from the above relation if and only if we are willing to disregard this term, taking it to be negligibly different from zero; then, letting $f(y)$ denote $h(y) - \lambda$, we have exactly equation (3).

If the last equation is the true relationship, rather than (3), then my decision to ignore the last term must be reckoned a flaw in my model. But I believe that inclusion of this last term might divert attention from propositions and insights which are more important to develop at this stage of inquiry.

Looking at Figure 1 or equation (3) we see that y^* can be regarded as the equilibrium utilization ratio for at $y = y^*$ (and only there) the actual rate of inflation will equal the expected rate of inflation. Mathematically, $\dot{p}/p = -x$ at $y = y^*$ since $f(y^*) = 0$. The diagram likewise shows that all the points on the vertical dashed line intersecting y^* are equilibrium points. Without intending normative significance, we may refer to $y > y^*$ as "over-utilization" and refer to $y < y^*$ as "under-utilization," merely from the point of view of equilibrium.

When there is over-utilization, the actual rate of inflation exceeds the expected rate and vice versa when there is under-utilization. In either of these situations there will presumably be an adjustment of the expected rate of inflation. I shall adopt the mechanism of "adaptive expectations" first used in this context by Phillip Cagan.[12] The (algebraic) absolute time-rate of increase of the expected rate of inflation will be supposed to be an increasing function of the (algebraic) excess of the actual rate of inflation over the expected rate, being equal to zero when the latter excess equals zero. Symbolically, if $(\dot{p}/p)^e$ denotes the expected rate of inflation, the postulate is

$$\frac{d}{dt}\left(\frac{\dot{p}}{p}\right)^e = a\left[\frac{\dot{p}}{p} - \left(\frac{\dot{p}}{p}\right)^e\right]$$

or, in terms of the expected rate of deflation,

$$(4) \qquad -\overset{\circ}{x} = a\left(\frac{\overset{\circ}{p}}{p} + x\right) \quad ,$$

$$a(0) = 0 \ , \quad a'(\ ) > 0 \ , \quad a''(\ ) > \frac{-a'(\ )\, f''(\ )}{f'(\ )\, f'(\ )} \quad .$$

Concerning the curvature of the function $a(\overset{\circ}{p}/p + x)$, it might be thought to be linear or it might be conjectured to be strictly convex for positive $\frac{\overset{\circ}{p}}{p} + x$ and strictly concave for negative $\frac{\overset{\circ}{p}}{p} + x$. All I am requiring is that the function not be "too concave" in the feasible range of y; in particular, it must not be more concave then the f function is convex, loosely speaking.

Substitution of (3) into (4) yields

$$-\overset{\circ}{x} = a(f(y) - x + x) = a(f(y)) \ .$$

If we let $G(y)$ denote $-a(f(y))$, then, by virtue of (3) and (4) we may write

$$(5) \qquad \overset{\circ}{x} = G(y) \ , \quad \mu \leq y \leq \bar{y} \ ,$$

$$G(y^{*}) = 0 \ , \quad G'(y) < 0 \ , \quad G''(y) < 0 \ .$$

Thus, when $y = y^{*}$, the actual and expected inflation rates are equal so that there is no change in the expected rate of inflation. When $y > y^{*}$, so that the actual inflation rate exceeds the expected rate, the expected rate of inflation will be rising or, equivalently, the expected rate of deflation will be falling. The opposite results hold when $y < y^{*}$. Note that as y is increased, the rate at which the expected rate of inflation is increasing over time will increase with y at an increasing rate.

In order to determine the path of x over time as a function of the chosen y path, we need to know the (initial) x at time zero, $x(0)$, which we take to be a datum:

(6) $$x(0) = x_o$$

We now have to consider restrictions on x_o arising from the upper and lower bounds on the money interest rate given in (1). First, for our analytical problem to be interesting, we require that x_o not be so algebraically small -- that the initially expected inflation rate not be so great -- that no feasible y decision by the Fisc can save the monetary system from breaking down in the first instant; that is, x_o must be sufficiently large algebraically that $i = r(y) - x_o < i_b$ for sufficiently small $y \geq \mu$. Hence we require that $r(\mu) - x_o < i_b$ (or, in later notation, $x_o > x_b(\mu)$).

As for the nonnegativity of the money interest rate, by analogous reasoning I should require only that x_o not be so large -- that the initially expected deflation rate not be so great -- that there is no y that will permit the Bank to make the real rate of interest low enough to induce the programmed volume of investment; that is, x_o must be sufficiently small that $i = r(y) - x_o \geq 0$ for sufficiently large $y < \bar{y}$, hence that $r(\bar{y}) - x_o \geq 0$. But I have to confess that I do not take seriously the nonnegativity constraint in my analysis. To justify this neglect I want somewhat stronger assumptions that will prevent the constraint from becoming binding when an optimal policy is followed. The constraint will not be binding initially if $r(\mu) - x_o \geq 0$, since the chosen y must be at least as great as μ . If, further, we postulate that $r(\mu) - \hat{x}(y^*) \geq 0$, where $\hat{x}(y^*)$ is a "satiation" concept later defined, then the constraint will not be binding in the future either, for our solution will be seen to imply that the optimal $x(t) \leq \max [x_o, \hat{x}(y^*)]$ for all t . I believe these conditions are fairly innocuous (as well as over-strong) and that it is wise not to complicate the problem at this stage by serious consideration of the nonnegativity constraint. The principal point of these last two paragraphs is that the initially expected deflation rate must be "admissible" in view of the two constraints on the money rate of interest.

3. <u>Utilization, liquidity and utility.</u> The problem of the Fisc is to choose a path $y(t)$, $t \geq 0$, or, equivalently, a policy function, $y(x, \ldots)$ subject to (5), (6) and the information in (1), (2) and (3). For this the Fisc requires preferences. I shall follow Frank Ramsey in adopting a "social utility function" that is the integral over time of the possibly discounted instantaneous "rate of utility."[13]

On what variables should the (undiscounted) rate of utility, U , at any time t be taken to depend? I am going to suppose that the only two basic desiderata are consumption and leisure. On this ground I am going to write the twice-differentiable function

$$(7) \qquad U = \phi(i, y) = \phi[r(y) - x, y]$$

where

(a)

$$\phi_2 > 0 \text{ for } y < y^o , \quad y^* < y^o < \bar{y} ,$$
$$\phi_2 < 0 \text{ for } y > y^o ,$$
where $\phi_2(i,y^o) = 0$ for all i, y^o a constant.
$$\phi_{22} < 0 \text{ for all } y .$$
$$\lim_{y \to \mu} \phi = -\infty , \quad \lim_{y \to \bar{y}} \phi = -\infty$$

(b)

$$\phi_1 = \phi_{11} = \phi_{12} = 0 \text{ for } i \le \hat{i} , \quad o < \hat{i} < i_b ,$$
$$\phi_1 < 0 , \quad \phi_{11} < 0 , \quad \phi_{21} = \phi_{12} = 0 \text{ for } i > \hat{i} ,$$
where $\phi_1(\hat{i},y) = 0$ for all y, $\hat{i}$ a constant.
$$\lim_{i \to i_b} \phi = -\infty .$$

It should be noted that the function ϕ is taken to be determined up to a linear transformation so that the assumptions on the signs of the second partial derivatives are meaningful. Figure 2 shows the contours of constant U .[14] Now the explanation.

Consider first the dependence of the rate of utility upon utilization for a fixed money rate of interest. I.e., consider (7a). Clearly, as y is increased, there will be more output, assuming always positive marginal productivity of labor, so that, given exogenous investment, there will be more consumption. In addition, there will be a reduction of involuntary unemployment, at least in a certain range. But, on the other hand, there will also be a reduction of leisure. Further, a discrepancy between y and y^* implies the failure of expectations to be realized, which suggests that

that people will have wished they had made different decisions.[15]

To make order out of this tangle of conflicting influences on the utility rate, I suggest the following view. Suppose for the moment that there were a homogeneous national labor market. Then y^* would be the market-clearing utilization ratio at which the gain from a little more income (or consumption) was just outweighed by the loss of leisure necessary to produce it; thus the utility peak would be at y^*. Since consumption is strictly concave in y while effort increases linearly with y, we would expect the curve to be strictly concave everywhere, i.e., dome-shaped. Moreover, as y approaches μ, so zero consumption is approached, the rate of utility can reasonably be supposed to go to minus infinity; similarly, as y approaches $\bar{y}$, it is perhaps natural to suppose that the rate of utility again goes to minus infinity (although nothing in the solution hinges on this strong assumption). In such a world, what permits the Fisc to coax employment in excess of y^* is the failure of people to predict the magnitude of the rate of inflation; in this world, some real normative significance attaches to "over-utilization" or "over-employment."

But in the real world, where there are countless imperfections and immobilities among heterogeneous sub-markets for different skills of labor in different industries, an additional consideration is operative. In such a world, there is substantial involuntary unemployment in some (presumably not all) sectors of the economy and among certain skill categories of labor even in utilization equilibrium; the point y^* is characterized by a balance between excess demand in some sectors and excess supply in others. (See, for example, Bent Hansen's analysis, op. cit.). In view of this and the social undesirability (ceteris paribus) of involuntary unemployment, I have supposed in (7) that the dome-shaped utility curve reaches a peak at some constant y^o greater than y^* but less than $\bar{\bar{y}}$ [16]; but the rate of utility does decline with y beyond this point as the involuntary over-employment in some labor markets and other misallocations by individuals due to their failure to expect the resulting inflation become increasingly weighty. I shall indicate later the effect of making $y^o = y^*$ contrary to my postulate. Note that y^o is a constant, independent of the money interest rate; this simplifying assumption seems advisable for consistency with the earlier postulate that the supply of labor is independent of the money interest rate.

I have discussed (7a) which is to say the profile of ϕ against utilization for a given money rate of interest. (A diagram of the relation between U and y for a given x will be shown later.) Consider now the dependence of the rate of utility on the money interest rate for a given utilization ratio. The money rate of interest measures the opportunity cost of holding money in preference to earning assets since, in the absence of own-interest on money, the money interest rate measures the spread between the yield on earning assets and the yield on money. After a point, an increase of the money interest rate increases incentives to economize on money for transactions purposes by means of frequent trips to banks and the like. I shall suppose for simplicity that these time-consuming efforts fall on leisure rather than on labor supply as indicated earlier. As the money rate of interest approaches the "barter point," i_b, these activities become so onerous that money ceases to be held and the monetary system breaks down. At a sufficiently small (but positive) money interest rate, $\hat{i}$, or at any smaller interest rate, incentives to economize are weak enough to permit a state of "full liquidity" in which all transactions balances are held in the form of money.[17]

Thus, concerning the relation between ϕ and i for given y, I suppose that the curve is flat in the full-liquidity range, $0 \leq i \leq \hat{i}$, negatively sloped and strictly concave for greater i and that the curve approaches minus infinity as i approaches i_b. I do not care how close $\hat{i}$ and i_b are to one another as long as they are separated. By making the curve go to minus infinity I insure that the optimal policy is not one producing the breakdown of the monetary system. I have now explained (7b) except for the condition that $\phi_{21} = \phi_{12} \leq 0$. This means that an increase of the money interest rate (outside the full-liquidity range) decreases or leaves unchanged the marginal utility of utilization; this seems reasonable since both an increase of i and of y imply a reduction of leisure, making leisure more or at least not less valuable at the margin.

It is clear from Figure 2 that, given the dependence of the interest rate on utilization, neither the value of y such that $i = \hat{i}$ (full liquidity) nor $y = y^o$ is generally a statical optimum, i.e., gives the maximum current rate of utility. The decision to make $i = \hat{i}$ may cost too much in terms of underutilization while the decision $y = y^o$ may entail too high an interest rate. As Figure 2 shows, the static optimum is at $\tilde{y}$ which is an increasing function of x up to y^o. If the Fisc sought to maximize the current rate of utility (which is not optimal to do), it would (except in the case of a no-tangency, full-liquidity solution) equate the marginal rate of substitution, $-\phi_2/\phi_1$, to the slope of the i-function, $r'(y)$, taking out any gain from a downward shift of the i-function -- of an increase of x -- in the form of greater y and smaller i; for all x greater than or equal to some large x, say $\hat{x}(\tilde{y})$, $\tilde{y}$ is identical of y^o and $i \leq \hat{i}$ (full liquidity) as the diagram shows.

We need now to describe the rate of utility as a function of x and y, i.e., taking both the direct effect and the indirect effect through i, given x, of a change of y. From (2) and (7) we obtain

(8)

$$U = U(x,y), \quad 0 \leq r(y) - x < i_b, \quad \mu \leq y \leq \bar{y}$$

(8)

$$U_y = \phi_1 r'(y) + \phi_2 > 0 \quad \text{for} \quad y < \tilde{y}(x)$$
$$U_y < 0 \quad \text{for} \quad y > \tilde{y}(x), \quad \mu < \tilde{y}(x) \leq y^o,$$

$$\text{where} \quad U_y(x,\tilde{y}) = \phi_1 [r(\tilde{y}) - x] r'(\tilde{y}) + \phi_2 [r(\tilde{y}) - x, \tilde{y}] = 0 .$$

$$U_{yy} = \phi_{11} r'(y) r'(y) + 2\phi_{21} r'(y) + \phi_{22} + \phi_1 r''(y) < 0 \quad (\text{for all } y).$$

$$U_{yx} = -\phi_{11} r'(y) - \phi_{21} \begin{Bmatrix} \geq \\ = \end{Bmatrix} 0 \quad \text{as} \quad x \begin{Bmatrix} < \\ \geq \end{Bmatrix} \hat{x}(y) \quad \text{or} \quad i \begin{Bmatrix} > \\ \leq \end{Bmatrix} \hat{k} .$$

$$\tilde{y}'(x) = - U_{yx}/U_{yy} \geq 0 .$$

$$\lim_{y \to \mu} U = -\infty, \quad \lim_{y \to \min[y_b(x), \bar{y}]} U = -\infty$$

$$\text{where} \quad r(y_b) - x = i_b, \quad y_b'(x) > 0 .$$

(b) $$U_x = U_{xx} = U_{xy} = 0 \quad \text{for} \quad x \geq \hat{x}(y)$$

$$U_x = - \phi_1 > 0 \ , \ U_{xx} = \phi_{11} < 0 \ , \ U_{xy} = - \phi_{11} r'(y) - \phi_{12} < 0 \quad \text{for} \quad x < \hat{x}(y) \ ,$$

where $r(y) - \hat{x} = \hat{i}$, $\hat{x}'(y) > 0$.

$$\lim_{x \to x_b(y)} U = - \infty \quad \text{where} \quad r(y) \quad x_b = i_b, \ x_b'(y) > 0 \ .$$

(c) $$U(\hat{x}, y^*) = \phi(\hat{i}, y^*) = \hat{U} \ .$$

Let us first interpret the new notation before looking at the diagrams. The function $\tilde{y}(x)$ has already been explained; it denotes the y at which the rate of utility is at a maximum with respect to y , taking into account the influence of y upon i , given x . The quantity y_b , also an increasing function of x , is that value of y which, given x , is just large enough to cause a breakdown of the monetary system by virtue of its causing $i = i_b$ through the $r(y)$ function; of course, x may be large enough to make $y_b > \bar{y}$ in which case y_b is irrelevant; it will be relevant if x is so negative that the economy is teetering on the edge of barter. The quantity $\hat{x}$, which is an increasing function of y , is that value of x just sufficiently great, given y , to permit full liquidity, i.e., to permit $i = \hat{i}$; since an increase of y entails a higher r , i.e., $r'(y) > 0$, we shall need **greater** x to maintain $i = \hat{i}$ the higher is y ; of course, any $x > \hat{x}(y)$ is also consistent with full liquidity, as $\hat{x}$ is the <u>minimum</u> x consistent with full liquidity. The quantity x_b , which is certainly negative even for large y , is that value of x so small algebraically that, given y , $i = i_b$ so that the monetary system breaks down; since $r'(y) > 0$, an increase of y causes an algebraic increase of x_b for we then need a smaller expected <u>inflation</u> rate to save the economy from barter. Finally, as a matter of notation, $\hat{U}$ denotes the rate of utility at equilibrium utilization and full liquidity, i.e., at $y = y^*$ and $x \geq \hat{x}(y^*)$; $\hat{U}$ is the maximum <u>sustainable</u> rate of utility.

Figure 3 illustrates the dependence of the utility rate on y , allowing for the interest effect of utilization, for two particular values of x : First, $x = \hat{x}(y^*)$ so that there will be full liquidity at $y = y^*$ (and at smaller y); second, $x = x_1 < \hat{x}(y^*)$, i.e., at a smaller x . I have supposed for the sake of definiteness that x_1 is so small -- very negative -- that when $x = x_1$ full liquidity is not realizable even at very small y so that the two curves never coincide; and that $x_b(y^*) < x_1$ so that the right-hand asymptotic lies to the right of y^* .

Both curves are strictly concave since $U_{yy} < 0$. (It can now be pointed out that $r''(y) > 0$ is unnecessarily strong for U_{yy} everywhere, let alone for $U_{yy} < 0$ in the neighborhood of $\tilde{y}$ as consideration of Figure 2 will show. One can simply postulate $U_{yy} < 0$ noting that this prohibits $r''(y)$ from being excessively negative.) Both curves reach a peak -- the static optimum -- left of y^o since $x < \hat{x}(y^o)$ in both cases. The top curve reaches a peak to the right of y^* because at $y = y^*$ there is full liquidity, so $\phi_1 = 0$ (right-hand as well as left-hand derivative), while $\phi_2 > 0$ because $y^o > y^*$, so that $U_y(\hat{x}(y^*), y^*) > 0$, i.e., the curve must still be rising at y^* . For purposes of illustration it was assumed that $y_b(\hat{x}(y^*)) \geq \bar{y}$ so that the right-hand asymptote is $\bar{y}$. The lower curve, corresponding to a much smaller x, has the same shape but reaches a peak, $\tilde{y}(x_1)$, to the left of y^*. This is because, in the case illustrated (if x is very small), the marginal gain from higher utilization at $y = y^* < y^o$ is not worth the concomitant increase of interest rate because the interest rate is already so high in this case. (It should be remarked that the portion of the solution (discussed later) which can be regarded as "deflationist" is not in any way dependent upon the fact that, for sufficiently small x, $\tilde{y}(x) < y^*$; deflation (or at least $y < y^*$) can be optimal even for x much higher than the aforementioned value, i.e., even when the static optimum is always above y^*.) Looking at the right-hand asymptote, this reflects the fact that for sufficiently small x, $y_b(x) < \bar{y}$. I have assumed for definiteness that $y_b(x_1) > y^*$ but the reverse inequality is certainly possible. Note finally, for completeness, that $y_b(x)$ approaches μ aymptotically as x falls and approaches $x_b(\mu)$.

Figure 4 illustrates the dependence of the utility rate on x for two given values of y: First, $y = y^*$ so that there will be full liquidity at $x \geq \hat{x}(y^*)$; second, $y = y_1 < y^*$. Both curves are, loosely speaking, reverse images of the curve (not drawn but fully discussed) of ϕ against i since, with y fixed, every one point increase of x is a one point decrease of i. Both curves are concave, strictly concave outside the full-liquidity range. Consider the former curve. It is assumed for illustration only that $\hat{x}(y^*) > 0$ meaning that, in equilibrium, deflation is necessary for full liquidity. As x is decreased -- the expected <u>inflation</u> rate increased -- the money rate in interest is increased (at a constant rate) so the rate of utility falls -- at an increasing rate by virtue of the strict concavity of ϕ in i. As x approaches $x_b(y^*)$, so that i approaches the barter point, the rate of utility goes to minus infinity. The other curve, corresponding to a smaller y, has the same shape. However, because y is smaller in this case and therefore i is smaller for every x, the critical rate x_b which drives the system into barter is algebraically smaller than in the previous case; i.e., a higher expected inflation rate is consistent with $i < i_b$ when y is smaller. Similarly, a smaller algebraic deflation rate, namely $\hat{x}(y_1)$, is needed for full liquidity. Note that since $y_1 < y^* < y^o$, full liquidity $(i \leq \hat{i})$ in this case gives a lower rate of utility than does full liquidity in the previous case where $y = y^*$. While it is of no significance, these considerations imply that the two curves cross: At algebraically very small x, $y^* > y_1 > \tilde{y}(x)$ so that $y^* > y_1$ actually reduces the rate of utility in that range of x.

Before (8) is utilized, some defense of it and consideration of alternatives is in order. Consider the poor German worker of the early 1920's. He was not in the market for equities so that for him the real interest rate was zero; or, rather, for him the real interest rate was only the convenience yield of holding a stock of consumer durables (cigarettes, bottled beer, etc.) which we might regard as becoming rapidly negligible as this stock is increased. It could be argued that for such people the appropriate utility-rate function is better described by

$$U = \psi(-x, y)$$

on the ground that the opportunity cost of holding money is simply the expected rate of inflation. If we make assumptions like $\psi_{21} < 0$ in the spirit of (7) we can still arrive at (8). There is little to be gained except simplicity from this approach at the cost of neglecting altogether the role of the real rate of interest for those people who participate in the capital market and who own a substantial amount of the wealth.

Another issue is my omission of the actual inflation rate from (7). Observe that, by virtue of (3) which makes the inflation rate a function of x and y, the utility rate must ultimately depend on x and y, as in (8). We could write

$$U = \psi(\dot{p}/p, i, y) = \psi[f(y) - x, r(y) - x, y]$$

and still obtain some version of (8). The issue therefore revolves only around the shape of the function in (8).

I have already given full weight to the loss of utility arising from a discrepancy between the actual and expected rates of inflation. It is in large part this discrepancy that motivates opposition to inflation. It is not really inflation *per se* that many economists oppose but rather an unexpectedly high rate of inflation. Nevertheless it might be argued that it is of no consolation to fixed-income groups to guess correctly the current rate of inflation if they did not anticipate when they contracted their fixed money incomes the bulk of the inflation that has occurred in the intervening time!

On one interpretation, this is a distributional argument: The real incomes or real wealth of widows and orphans on previously contracted fixed incomes will be eroded to socially undesirable levels by inflation. My grounds for omitting the actual inflation rate, from this point of view, is that the government has other means than the depressing of the utilization ratio to rectify tolerably the distribution of income.[18]

To the extent that appropriate redistribution efforts still leave such groups too poor, there is certainly a case for introducing the actual rate of inflation into the utility-rate function, ψ. But it is enormously difficult to introduce it appropriately. For if the actual and expected inflation rates should be equal for a long time then the actual rate of inflation deserves less and less weight over time; for eventually the inflation will have become a <u>fully anticipated</u> one. Thus an appropriate utility-rate function must be a nonstationary function. No simple possibilities satisfy me. But I wish to point out that since the optimal path in my model produces asymptotically a steady rate of algebraic inflation, hence an asymptotically anticipated inflation, and since the rate of an anticipated inflation makes no difference distributionally (apart from its liquidity effect already recognized), the asymptotic properties of the solution here are immune to criticism from this point of view.

The actual inflation rate has another influence which, it could be argued, is time-independent and hence persisting for all time. This is the nuisance cost of adjusting price lists up or down. If the rate of inflation is twenty percent or minus twenty percent per annum, every firm in every industry will have to revise its price lists very frequently, which again has its leisure or production costs. This suggests giving the actual rate of inflation a weak role in the utility-rate function. $\psi(\ )$ can be made a dome-shaped function of $\dot{p}/p$. The concavity of U in y would be threatened a little -- precautions would be needed to insure that $U_{yy} < 0$ everywhere -- but not much of (8) would be lost. The main difference is that instead of having a U maximum in the x plane for all $x \geq \hat{x}(y)$ we would have a unique, nonflat peak in Figure 4 since too high an expected rate of deflation would cause too high an actual deflation rate from the point of view of price lists. I shall mention in the next section an instance where it would be useful to introduce such a modification.[19]

My greatest reservations center on the stationarity of the utility-rate function in (7). Suppose first that $\lambda = 0$. Due to virtual golden-age growth, aggregate consumption and leisure will be growing at rate γ , like population, at any constant utilization ratio. Since the "pie" is getting bigger over time, should not U be made to depend upon t ? Fortunately, however, per capita consumption and per capita leisure -- which depend only on i and y -- will be constant so that the use of a stationary utility-rate function is not wholly unreasonable. The real issue here is "discounting."

More serious difficulties arise when $\lambda > 0$. Then a constant i and y imply exponentially growing consumption per head and constant leisure per head (by virtue of the labor supply function's properties). In this case it does seem a little strange that time should not appear as an argument of the utility-rate function. But I believe that examples of underlying utility functions could be found such that time would not appear in the derive utility-rate function ϕ in (7).[20]

I shall however allow the rate of utility to be "discounted" at a nonnegative rate in the usual multiplication way. No solution to our problem in its present formulation will exist if there is negative discounting.

In deciding which of two (x,y) paths to take -- actually $x(t)$ alone suffices to describe a path -- the Fisc is postulated to compare the integrals of the possibly discounted rates of utility produced by the two paths. Hence the "social utility," W, of a path (x,y) is given by

$$(9) \qquad W = \int_0^\infty e^{-\delta t} U(x,y)\, dt \,, \qquad \delta \geq 0 \,,$$

where t is time, $e^{-\delta t}$ is the discount factor applied to the rate of utility t years hence, and δ is the rate of utility discount. (It is understood in (9) that $x = x(t)$, $y = y(t)$.) The case $\delta = 0$ will receive special consideration in a moment.

The optimization problem of the Fisc can now be stated as: Maximize (9) subject to (5) and (6). The "optimal policy" is the function $y = y(x)$ which gives the greatest feasible W. Given $x(0) = x_0$, there is an optimal path $x = x(t)$ which describes the state of the system at each time. From this information one can also derive $y = y(t)$, since $\dot{x}(t)$ gives $y(t)$ by (5).

In the case $\delta = 0$, there may be many feasible paths which cause the integral in (9) to diverge to infinity, which give infinite W; intuitively, it is unreasonable to regard all of these paths as "optimal" so that a different criterion of preferences and of optimality is wanted in this case. Such a criterion will be described briefly in the next section, which also gives the solution to the zero-discount case. (Nevertheless the above formulation of the mathematics of optimization is essentially correct.) The subsequent section gives the solution to the case of a positive utility discount rate.

B. Optimal Policy when no Utility Discounting

The optimality criterion now widely used by economists to deal with no-discount, infinite-horizon problems of this sort has been called the "overtaking principle." A path $(x_1(t), y_1(t))$ is said to be preferred or indifferent to another path $(x_2(t), y_2(t))$ if and only if one can find a time T^o sufficiently large that, for all $T > T^o$,

$$\int_0^T U(x_1, y_1)\, dt \geq \int_0^T U(x_2, y_2)\, dt \ .$$

The former path is preferred because it eventually "overtakes" the latter path. A feasible path is said to be optimal if it is preferred or indifferent to all other feasible paths. If one then obtains a solution to the maximization problem now to be described, this solution is the optimum in this sense.[21]

The above optimality criterion justifies the use of a device first employed by Ramsey in his analysis of the somewhat analogous problem of optimal saving over time:[22] Choose the units in which the utility rate is measured in such a way that $\hat{U} = 0$, i.e., $U(\hat{x}(y^*), y^*) = 0$. This is merely a linear transformation of the function U that will not affect the preference orderings implied by the integral comparisons just described. Now go ahead with the problem

$$(10)\qquad \begin{aligned} &\underset{y}{\text{Max}}\ W = \int_0^\infty U(x,y)\,dt\ , \qquad \hat{U} = 0 \\ &\text{subject to}\ \dot{x} = G(y)\ , \qquad x(0) = x_o\ . \end{aligned}$$

The divergence problem cannot now arise. This is not to say however that an optimal policy will exist for all x_o .

Readers familiar with the Ramsey problem will recognize (10) as rather like the optimal saving problem. There x is "capital" and y is "consumption."[23] There is a zero-interest capital-saturation level in Ramsey that is analogous to our liquidity satiation level, $\hat{x}(y)$; his income -- the maximum consumption subject to constant capital -- is analogous to our y^* . His solution was the following: If initial capital is short of capital saturation, consume less than income, driving capital up to the saturation level; if initial capital exceeds the saturation level, consume more than income, driving capital down to the saturation level; if initial capital equals the capital-saturation level, stay there by consuming all capital-saturation income. Thus capital either equals for all time or approaches asymptotically and monotonically the capital-saturation level while consumption either equals or approaches asymptotically (and monotonically) the capital-saturation level.

The solution to the problem here is similar in part. If $x_o < \hat{x}(y^*)$ it is optimal to make $y < y^*$ for all <u>t</u> , causing x to rise and approach $\hat{x}(y^*)$ asymptotically, while y approaches y^* asymptotically and monotonically. In other words, if the economy "inherits" an initially expected algebraic deflation rate that is insufficient for full liquidity when the utilization ratio is at its equilibrium value, then, for an optimum, the Fisc must engineer under-utilization for all time so as to cause a gradual, asymptotic movement of the expected deflation rate up to the level consistent with full liquidity and equilibirum utilization; in the limit, as time increases, under-utilization vanishes and a full-liquidity equilibrium is realized.

If $x_o = \hat{x}(y^*)$ then $y = y^*$ is optimal for all t, and therefore $x = \hat{x}(y^*)$ for all t. Should the economy inherit the minimum expected deflation rate consistent with full liquidity at equilibrium utilization, then equilibrium utilization with full liquidity is optimal for all time. The case $x_o > \hat{x}(y^*)$ will be discussed later.

What will be remarkable to those steeped in the statical approach is that, when $x_o \leq \hat{x}(y^*)$, over-utilization is not optimal whether or not x is large enough to make $\tilde{y}(x) > y^*$. Further it can be shown that optimal y is always smaller than $\tilde{y}$ even when $\tilde{y} < y^*$.

Analogous to the Ramsey-Keynes equation that gives optimal consumption as a function of capital is the following equation that describes optimal utilization as a function of the current expected deflation rate:[24]

$$U(x,y) + G(y)\frac{U_y(x,y)}{-G'(y)} = 0 . \tag{11}$$

For purposes of diagrammatics it is helpful to write $U = V(x,\dot{x}) = V[x, G(y)]$, which we may do since $G(y)$ is monotone decreasing in y, and then to express (11) in the form

(12) $$V(x,G) - G\,V_G(x,G) = 0$$

where
$$V_G = \frac{U_y(x,y)}{G'(y)}\,, \qquad V_x = U_x\,,$$

$$V_{GG} = \frac{U_{yy}\,G' - G''\,U_y}{G'G'G'}\,, \qquad V_{Gx} = \frac{U_{yx}}{G'(y)}\,.$$

If we think of $\dot{x} = G(y)$ as "investment," then (12) says that the optimal policy equates the rate of utility to investment multiplied by the (negative) marginal utility of investment, V_G ; this is essentially the Ramsey-Keynes rule.

From the information above on derivatives we see that V increases as G is increased (i.e., as y is decreased from $\bar{y}$ of $y_b(x)$, whichever is smaller) up to $G(\tilde{y})$ whereupon V then decreases, going to minus infinity as G approaches $G(\mu)$. Only this latter decreasing region, where $V_G < 0$ or $U_y > 0$ is of relevance; in that region, $V_{GG} < 0$ unambiguously.

In Figure 5 the solid curve depicts the possibly realistic case of x_o great enough that $\tilde{y}(x_o) > y^*$, so that $G(\tilde{y}(x_o)) < 0$, but not great enough for full liquidity when $y = y^*$, i.e., $x_o < \hat{x}(y^*)$. Thus the solid utility curve, for $x = x_o$, has a peak left of the origin but it passes under the origin, since $U(x_o,y^*) < \hat{U} = 0$. The tangency point, at (V_o, G_o) , shows the optimal initial $G(y)$ and hence the optimal y . Since optimal $G(y) > 0$ (i.e., $y < y^*$) , x will be increasing and the V curve will therefore shift up and possibly to the left; as this process occurs, the tangency point approaches the origin, so that $y = y^*$ and $x = \hat{x}(y^*)$ in the limit.[25] The dashed curve represents the asymptotic location of the V curve. Just as equilibrium utilization is approached only asymptotically, it can be shown that full liquidity $(i \leq \hat{i})$ is approached only asymptotically. (This follows from $r'(y) > 0$ and the results that $U_y > 0$ along the optimal path.)

The case $x_o = \hat{x}(y^*)$ is now obvious. Here we are in long-run equilibrium to begin with, as shown by the dashed V curve in Figure 5. The tangency point occurs at the origin so $y = y^*$ is optimal initially; this means that the equality $x(t) = \hat{x}(y^*)$ continues so that $y = y^*$ continues to be optimal for all t.

Consider now the case $x_o > \hat{x}(y^*)$. Since there cannot be more than full liquidity when $y = y^*$, i.e., $U(x_o, y^*) = \hat{U}$ even when $x_o > \hat{x}(y^*)$, the tangency point continues to be at the origin. Yet the implied policy $y(t) = y^*$, $x(t) = x_o > \hat{x}(y^*)$ for all t cannot be optimal. For there is a "surplus" of expected deflation here; i.e., $i < \hat{i}$ when $y = y^*$. Since V reaches a peak to the left of y^*, there are clearly policies of at least temporary over-utilization $(y > y^*)$ which will permit $U > \hat{U}$ for at least a while and yet allow $U = \hat{U}$ forever after; this is because $x = \hat{x}(y^*) < x_o$ is sufficient for $U(x, y^*) = \hat{U}$. In other words, there is room for a "binge" of at least temporary over-utilization while all the time enjoying full liquidity and while never driving x below $\hat{x}(y^*)$.

But it cannot be concluded that over-utilization is optimal when $x_o > \hat{x}(y^*)$. For no such temporary or even asymptotically vanishing binge of over-utilization can satisfy (12), which is a necessary condition for an optimum; in terms of Figure 5, there is no way that such a policy can satisfy the necessary tangency condition.

Since neither $y > y^*$, $y = y^*$ nor $y < y^*$ is optimal, the inescapable conclusion is that there exists no optimum in this case. An intuitive explanation is the following: For every binge that you specify which makes $x(t)$ approach $\hat{x}(y^*)$ (as y approaches y^*), I can, by virtue of the strict concavity of the V curve, specify another binge that makes x approach $\hat{x}(y^*)$ more slowly which will be even better. There is no "best binge" (or even set of "best binges") just as there is no number closest to unity yet not equal to it. Hence there is no path preferred or indifferent to all other feasible paths.

There are at least four avenues of escape from this disconcerting situation. Let us first ask, how did Ramsey avoid it? He could avoid it (actually he never recognized it) by postulating that the net marginal product of capital became negative beyond the capital saturation point so that there was an immediate and positive loss from having too much capital. (This is fair enough if capital depreciates even in storage.) In our model there is no immediate loss from having "too high" an expected deflation rate; $i < \hat{i}$ is as good as $i = \hat{i}$. To introduce a loss we need to suppose that U in (8) is strictly concave in x, reaching a peak at some $\hat{\hat{x}}(y)$ and falling off thereafter. As mentioned earlier, this postulate could be justified by the price-list consideration that it is a nuisance to have to reduce prices with great frequency. (But a previous footnote indicates my uneasiness with this consideration.) Alternatively one could make assumptions such that $G_x(x,y) < 0$ as is done in the last section of this paper.

Another avenue of escape is the introduction of a positive utility discount, as I have done in the next section. Then there will be a "best binge" so there will be an optimum for all x_o (in the admissible range).

A third avenue is to employ a finite-time horizon. Then any binge must come to an end at the end of some given number of years. There will be a "best binge" and an optimum will always exist. See the second part of this paper for such a model.

The fourth avenue of escape is to postulate that $y^o = y^*$ so that $\tilde{y}(x) \leq y^*$ for all x and therefore the V peak cannot occur to the left of the origin. I find this unsatisfactory although some readers may not. The reader can now work out this case using a diagram like Figure 5. If $x_o < \hat{x}(y^*)$, under-utilization is optimal as before; if $x_o \geq \hat{x}(y^*)$, equilibrium utilization is optimal. Anyone who wants to go as far as postulating $y^o < y^*$ will encounter problems of the nonexistence of an optimum.

Some of the qualitative results of this section may be expressed by the following "policy function" derived from (12):

$$(13) \qquad y = y(x) \, , \qquad x \leq \hat{x}(y^*) \, ,$$

where
$$y'(x) \;\{\gtreqless\}\; 0 \quad \text{as} \quad x \;\{\lesseqgtr\}\; \hat{x}(y^*) \, ,$$

$$y(x) \begin{cases} = y^* & \text{if} \quad x = \hat{x}(y^*) \, , \\ < y^* & \text{if} \quad x < \hat{x}(y^*) \, , \end{cases}$$

$$\lim_{x \to x_b(\mu)} y(x) = \mu \, .$$

Let us turn now to the mathematically more congenial case of a positive discount.

C. Optimal Policy When Positive Utility Discounting

Our problem now is

(14)
$$\underset{y}{\text{Max}} \quad W = \int_0^{\infty} e^{-\delta t}\, U(x,y)\, dt\ , \qquad \delta > 0\ ,$$
$$\text{subject to} \quad \dot{x} = G(y)\ , \quad x(0) = x_o\ .$$

A mathematical analysis, in which (14) is a special case, is contained in the appendix of this paper. I shall describe the solution here.

The optimal path of the variable $x(t)$ either coincides with or asymptotically and monotonically approaches (from every x_o) a "long-run equilibrium" value, x^* , which is uniquely determined by

(15)
$$\delta = \frac{-U_x(x^*,y^*)\ G'(y^*)}{U_y(x^*,y^*)}$$

It is easy to see from (15), the inequality $G'(y) < 0$ and the observation that an optimal path would never make $U_y(x,y) < 0$, that $U_x(x^*,y^*) > 0$. This and (8) yield the result that $x^* < \hat{x}(y^*)$. Thus, in the long run, there will be less than full liquidity when there is positive discounting of future utility rates. This is because the current gain from high utilization always offsets the discounted future loss due to a short fall from full liquidity.

If $x_o < x^*$, so that the expected deflation rate is below its long-run optimal value, then, to drive $x(t)$ monotonically toward x^* we require $y < y^*$, i.e., under-utilization; $y(t)$ will approach y^* only asymptotically as $x(t)$ approaches x^* . If $x_o = x^*$, then $y = y^*$ is optimal for all t .

If $x_o > x^*$, then, to drive $x(t)$ monotonically toward x^* we require $y > y^*$, i.e., over-utilization; but, again, $y(t)$ will approach y^* asymptotically. (It does not appear that the path $y(t)$ is necessarily monotonic but this is of little importance.)

This last result -- the optimality of over-utilization in some circumstances -- is of considerable interest. The previous section laid a possible foundation for a "deflationist" policy when the initially expected deflation rate was insufficient for full liquidity with equilibrium utilization; more precisely, under-utilization was optimal in that circumstance so that the actual rate of inflation resulting would be less than the expected rate, though it need not be negative initially (or even asymptotically if $\hat{x}(y^*) \geq 0$). Moreover, an "inflationist" policy of over-utilization, though it might be better than any under-utilization policy, was never optimal for there could never exist an over-utilization optimum. We see here that, when there is a positive utility discount, over-utilization will be optimal when $x_o > x^*$; since $x^* < \hat{x}(y^*)$, this embraces the case $x_o = \hat{x}(y^*)$, i.e., the case in which there would be full liquidity at equilbrium utilization.

The greater is the utility discount rate, the smaller algebraically will be the equilibrium deflation rate. Differentiation of (15) yields

$$\text{(16)} \qquad \frac{dx^*}{d\delta} = \frac{[U_y(x^*,y^*)]^2}{[U_{yx}(x^*,y^*)U_x(x^*,y^*) - U_{xx}(x^*,y^*)U_y(x^*,y^*)]G'(y^*)} < 0 ,$$

since the denominator is unambigously negative for all $x^* < \hat{x}(y^*)$, hence for

$\delta > 0$. This indicates that, given some x_o , we are more likely to find over-utilization initially optimal $(x_o > x^*)$ the larger is the utility discount rate.

Nevertheless one cannot, by choosing sufficiently large δ , make x^* arbitrarily small (algebraically), not even as small as $x_b(y^*)$. It is the inequality $\tilde{y}(x^*) > y^*$ that lies behind the optimality of $y > y^*$ when $x_o > x^*$. It can be shown that x^* cannot be made larger than $\tilde{x}(y^*)$, where $\tilde{x}$ is defined by $\tilde{y}(x) = y^*$; for as δ goes to infinity, the derivative $U_y(x^*,y^*)$ in (15) goes to zero (while $U_x(x^*,y^*)$ stays finite), indicating that x^* approaches the value such that $U_y(x,y^*) = 0$, hence approaches the value $\tilde{x}(y^*)$.

The value $\tilde{x}(y^*)$ is precisely the level of x to which the myopic, statical approach would drive $x(t)$. That approach, which maximizes the current rate of utility at each time, leads to a policy $y = \tilde{y}(x)$; under that policy, equilibrium is realized only when (asymptotically) $x = \tilde{x}(y^*)$ so that $\tilde{y}(x) = y^*$. Thus the statical approach and the case of an infinitely high discount rate lead to the same equilibrium value of x . Indeed, it can be seen from the appendix (in particular, the equivalent of the Euler condition) that infinite utility discounting makes $U_y(y,x) = 0$ always, which means $y = \tilde{y}(x)$, so that the statical approach and infinitely heavy discounting lead to identical policies throughout time.

But optimal behavior in the limit as δ goes to infinity is of little interest. Given any (finite) value of δ , the dynamic approach yields different results from the statical policy $y = \tilde{y}(x)$. First, since $U_y(x,y) > 0$ along any dynamically optimal path, the optimal $y < \tilde{y}$ for all x . Second, and this needs emphasis, even if x_o is such that $\tilde{y}(x_o) > y^*$, so that myopic maximization of the initial rate of utility would call for $y > y^*$, the truly optimal $y < y^*$ if (and only if) $x_o < x^*$. Thus, if the currently expected rate of inflation is 2 percent while the long-run equilibrium (asymptotically optimal) expected inflation rate is less, say 1 percent, then under-utilization is optimal whether or not the current utility-rate curve peaks to the right of y^*. This theme is essentially a repetition of a theme of the previous section: A dynamical approach can lead to a qualitatively different optimal policy than a myopic, statical approach. In particular, a "deflationist" policy of under-utilization (and hence a rise of x over time) may be optimal even when myopic maximization of the current rate of utility calls for over-utilization (and hence a fall of x over time).

The above results may be summarized in a qualitative way as follows:

$$y = y(x)$$

where

$$(17) \qquad y(x) \begin{cases} > y^* & \text{if } x_o > x^* , \\ = y^* & \text{if } x_o = x^* , \\ < y^* & \text{if } x_o < x^* , \end{cases}$$

$$\lim_{x \to x_b(\mu)} y(x) = \mu , \quad y(x) < \tilde{y}(x) \quad \text{for all } x ,$$

$$\text{with } \tilde{x} < x^*(\delta) < \hat{x}(y^*) \quad \text{for all } \delta > 0 , \quad x^{*\prime}(\delta) < 0 .$$

Once again we may ask, what if $y^o = y^*$? Then $\tilde{y}(x) \leq y^*$ for all x. In this event, $y < y^*$ when $x_o < x^*$ as above. And if $x_o \geq x^*$, then $y = y^*$;

hence there is no over-utilization, because there is no gain to be had in the present (from over-utilization) that is worth a discounted future loss (from a reduction of future liquidity).

II. VARIATIONS ON THE STANDARD MODEL

Consider first the finite-horizon version of the model just analyzed.

A. A Finite Time Horizon

Suppose the economy is believed to come to an end after a length of time T . Or suppose that people or planners wish to optimize only over a finite interval of time beginning at time zero and ending at T . Suppose further that it is decided to leave a certain terminal expected deflation rate, $x(T)$, to posterity. Then our problem is

$$\max_{y} \int_0^T e^{-\delta t} U(x,y)\, dt$$

(18) $$\text{subject to } \dot{x} = G(y) \text{ and}$$

$$x(0) = x_o , \quad x(T) = x_T .$$

We have to assume, as earlier, that our end-points are admissible, i.e., x_o, $x_T > x_b(\mu)$; and once again we wish to dodge the interest-rate nonnegativity constraint by assuming that x_o and x_T are not too large. Note that there is no longer a nonnegativity restriction on δ .

I shall make three brief points about the problem in (18), not giving it the attention that it perhaps deserves. First, an optimal policy will exist for all admissible x_o and x_T . We need not worry about the size of

x_0 nor whether $\delta = 0$ or even $\delta < 0$.

Second, the initially optimal y will be approximately equal to the initially optimal y (should an optimum exist) in the infinite horizon problem if T is "large." More precisely, by choosing T sufficiently large one can make the two values of y (the optimal values in the two models) as close to one another as one desires. This is, after all, one justification for studying infinite-horizon models: The optimal policy in the early years is likely to be a good approximation to the truly optimal policy in the early years even if the world is going to come to an end -- as long as the end is far away.[26] Thus, if T is sufficiently large and $\delta = 0$, under-utilization will be optimal initially if $x_0 < \hat{x}(y^*)$. But, in addition, over-utilization will be optimal if $x_0 > \hat{x}(y^*)$, the case in which there was no optimum in the infinite-horizon model.

The third remark concerns the optimal path of $x(t)$ over the whole interval. The problem stated in (18) is reminiscent of the Cass-Samuelson problem of optimal saving over a finite time interval, out of which emerged the "catenary turnpike theorem," just as the infinite horizon model was reminiscent of the Ramsey problem.[27] Analogous to that theorem is the following conjecture: There exists a value $x^* = x^*(\delta)$, given by (15) for the case $\delta \geq 0$ and determined otherwise for the case $\delta < 0$, such that the optimal $x(t)$ path will be seen as "arching" toward that value in catenary fashion; by making T sufficiently large one can cause the optimal path $x(t)$ to spend an arbitrarily large proportion of the time interval arbitrarily near the x^* value. Thus, when $\delta = 0$, the value $x^* = \hat{x}(y^*)$ is a kind of turnpike, just as the Golden Rule path is a "turnpike" when there is no utility discounting in the analogous Cass-Samuelson problem. When $\delta > 0$, $x^*(\delta)$ from (15) is the turnpike, with $x^*(\delta) < \hat{x}(y^*)$. When $\delta < 0$, $x^*(\delta)$ will be greater than $\hat{x}(y^*)$, reflecting the optimality of accumulating a "surplus" of x prior to the negatively-discounted later-year feast of over-utilization and drawing down of x to its stipulated, terminal value.

B. A "Full Liquidity or Barter" Model

In the standard model (and in its finite-horizon variant) the rate of utility was a smooth function of the expected deflation rate, given the utilization ratio; i.e., even the second derivatives $U_{xx}(x,y)$ and $U_{yx}(x,y)$ were continuous. One could obtain somewhat different results by supposing that only the first derivatives were continuous, that the first derivative $U_x(x,y)$ has a kink at $x = \hat{x}(y)$. Here I shall go to the extreme of postulating that $U(x,y)$ itself is discontinuous in a particular way.

As explained earlier, x could have appeared in (8) by making ϕ in (7) depend upon $\mathring{p}/p = f(y) - x$. Let us continue to suppose that the actual rate of inflation is a matter of indifference, given utilization (hence the discrepancy between the actual and expected inflation rates) and the money interest rate. Let us suppose further that the money interest rate affects the rate of utility only as it permits or prohibits the functioning of the money economy. In particular, either there is barter (no liquidity) or there is full liquidity, the money interest rate being a matter of indifference:

$$(19) \qquad U = \begin{cases} \phi(y) & \text{if } i < i_b , \\ -\infty & \text{if } i \geq i_b . \end{cases}$$

In other words, the economy is not sensitive to the money interest so long as the latter variable is short of the barter point. As before, barter is postulated to be an incalculable disaster which it could not be optimal to bring about.

Using the relation $i = r(y) - x$ in (2) we then obtain in place of (8) the utility-rate function

$$U = U(x,y) = \begin{cases} \phi(y) & \text{if } x > x_b(y) , \\ -\infty & \text{otherwise.} \end{cases} \tag{20}$$

Thus the expected deflation rate has no effect in the range $x > x_b(y)$ where x_b is defined by $r(y) - x_b = i_b$ as in (8); but when $x \leq x_b(y)$, the rate of utility is really not even defined.

Consider now the infinite-horizon problem

$$\begin{aligned} &\underset{y}{\text{Max}} \quad W = \int_0^{\infty} e^{-\delta t} \, U(x,y) \, dt , \qquad \delta \geq 0 , \\ &\text{subject to } \dot{x} = G(y) , \quad x(0) = x_o . \end{aligned} \tag{21}$$

Will an optimum exist for this problem? Not if we suppose, in the spirit of (7), that $\phi'(y) = 0$ at $y = y^o > y^*$. Assume initially that $x_o > x_b(y^o)$. Certainly the statical, myopic policy $y = \tilde{y} = y^o > y^*$ will not be optimal for that policy will cause $x(t)$ to fall eventually to $x_b(y^o)$ with barter thus resulting. An optimal policy must cause $y(t)$ to fall gradually to y^* sufficiently fast that $x(t) > x_b(y(t))$ for all t . But for every such policy there will be another, better policy which causes the economy to come a little closer to the brink of barter. There is no "brinkmanship" optimum. Similarly, even if $x_o < x_b(y^o)$, it is best for $y(t)$ to be always as close as possible to $y_b(x(t))$, where y_b is defined by $r(y_b) - x = i_b$; but there is again no closest path of pursuit. Hence no optimum exists for any x_o.

The message of this section is therefore the following: Unless the expected deflation rate enters continuously into the utility-rate function ϕ, either through the door of the money interest rate as in (7) or through the door of the actual rate of inflation, there is the possibility that no optimum will exist, even if there is discounting of future utilities, when an infinite time-horizon is used; there will be no optimum if barter is regarded as infinitely disastrous. Even the myopic, statical approach produces absurd results in the discontinuous case just considered.

If barter is regarded as producing only a finite disutility, a great portion of the utility-rate function in (8) would have to be revised. It seems a plausible guess that if there is positive utility discounting, an optimal path will exist, one which will presumably produce a barter economy after years of happy over-utilization. For most readers, however, such a model will seem an instance of thinking about the unthinkable.

C. "Inflation Illusion"

Throughout this paper I have worked with an inflation function that yields a vertical steady-state or equilibrium relation between the inflation rate (on the vertical axis) and the utilization ratio (on the horizontal axis.)

Though I am skeptical myself, I wish now to satisfy those readers who believe that the steady-state Quasi-Phillips Curve is positively sloped, that a rapid, steady, expected inflation is compatible with a higher utilization ratio than a creeping, steady, expected inflation. Presumably the thought behind this belief is that when the inflation rate (both actual and expected) is increased, a larger volume of labor is supplied so that the same unemployment ratio corresponds to a larger utilization ratio. (Note the absence of any implied difference in the unemployment ratio in the high-inflation and low-inflation situations.) Provided that the labor supply is, if dependent on the money interest rate at all, a nonincreasing function of the money interest rate -- and this seems reasonable -- such postulated behavior should perhaps be regarded as irrational on the part of labor suppliers.[28] In any case, perhaps the following model will portray such behavior satisfactorily.

Let us postulate in place of (3) the relation

$$\frac{\dot{p}}{p} = f(y) - \theta x , \qquad 0 < \theta < 1 \tag{22}$$

where $f(x)$ has the same properties as in (3) and where θ is a constant. Since $\theta > 0$, the Quasi-Phillips Curve still shifts upward with a one percentage point increase of the expected inflation rate; but since $\theta < 1$, it does not shift up by the whole percentage point.

As before, the equilibrium utilization ratio, y^*, is defined as that ratio which makes the actual inflation rate equal to the (given) expected inflation rate. But now we have a locus of equilibrium utilization ratios, $y^* = y^*(x)$, one for each x . This locus is defined by (22) upon substituting the expected inflation rate, $-x$, for the actual inflation rate, $\mathring{p}/p$. Thus y^* is determined by the relation

$$0 = f(y^*) + (1 - \theta)\, x$$

(23) $$\text{where} \quad \frac{d(-x)}{dy^*} = \frac{f'(y)}{1-\theta} > 0 ,$$

$$\frac{d^2(-x)}{dy^{*2}} = \frac{f''(y)}{1-\theta} > 0 .$$

Thus an increase of the expected inflation rate is associated with an increase of the equilibrium utilization ratio; the equilibrium or steady-state locus is positively sloped; but since $(1 - \theta) < 1$, it is steeper than the individual Quasi-Phillips Curves, each of which corresponds to a particular x . The equilibrium locus shares the convexity property of the Quasi-Phillips Curves.

Before introducing this inflation function into the model, I shall indicate briefly that an equilibrium locus having the properties just mentioned can be derived by postulating that labor supply is an increasing function of the money interest rate so that, given any utilization ratio, the rate of inflation decreases with the money interest rate. This is, of course, the opposite of what we should expect on liquidity considerations for a high interest rate causes time-consuming efforts to economize on money. For simplicity, consider the simple relation

$$\frac{\dot{p}}{p} = f(y) - x - \beta[r(y) - x] , \qquad \beta > 0 .$$

Then the equilibrium locus is defined by

$$0 = f(y^*) - \beta r(y^*) + \beta x .$$

For sufficiently small β , the locus will be increasing, convex and (if desired) steeper than the Quasi-Phillips Curves.

I am going to linearize the adaptive expections function in (4) so that no hopelessly complex conditions for the existence of an optimum and its uniqueness will arise. Thus, the adjustment coefficient is now a constant, α , as used by Cagan[29]:

(24) $$-\dot{x} = \alpha(\frac{\dot{p}}{p} + x) , \qquad \alpha > 0 .$$

Combining (22) and (24) yields

$$-\dot{x} = \alpha[f(y) + (1 - \theta) x]$$

Thus, when $y = y^*(x)$, so that $f(y) + (1 - \theta)x = 0$ [by (23)], $\dot{x} = 0$; when $y > y^*(x)$, $-\dot{x} > 0$ (the expected inflation rate is increasing) and when $y < y^*(x)$, $-\dot{x} < 0$, both by virtue of the inequality $f'(y) > 0$.

We may write the previous equation in the form

(25) $$\dot{x} = G(x,y)$$
where $G(x,y^*) = 0$, $G_x(x,y) < 0$, $G_{xx}(x,y) = 0$, $G_y(x,y) < 0$, $G_{yy}(x,y) < 0$, $G_{xy}(x,y) = 0$.

Note that $G(x,y) = 0$ is an equivalent way of defining the $y^*(x)$ locus. By differentiation one finds that $d(-x)/dy^* = G_y(x,y)/G_x(x,y)$.

Our utility-rate function in (8) remains intact in the face of the above revision of (5). Hence we may state the infinite-horizon version of our problem as

(26)
$$\max_y W = \int_0^\infty e^{-\delta t} U(x,y)\, dt , \qquad \delta \geq 0$$
$$\text{subject to } \dot{x} = G(x,y) , \qquad x(0) = x_o .$$

I shall consider first the case $\delta > 0$, although all the statements made about this case are valid for $\delta = 0$ as well. In this case there is a long-run equilibrium value of x , say x^* , that is uniquely defined by

(27)
$$\delta - G_x(x^*,y^*(x^*)) = \frac{-U_x(x^*,y^*(x^*))G_y(x^*,y^*(x))}{U_y(x^*,y^*(x^*))}$$

Since $G_x < 0$, $G_y < 0$ and $U_y > 0$ along any optimal path, (27) determines a value of x^* such that $U_x > 0$; thus the long-run equilibrium is not one of full liquidity (even in the limit as δ approaches zero).

It is shown in the appendix that either optimal $x(t)$ coincides with x^* -- this is the case when $x_o = x^*$ -- or else $x(t)$ approaches x^* asymptotically and monotonically, the optimal path being unique. Thus, if $x_o < x^*$, under-utilization is optimal, meaning that $y < y^*(x_o)$ so that x will be increasing toward x^* ; under-utilization will vanish in the limit as time goes to infinity. If $x_o = x^*$, equilibrium utilization is optimal, meaning $y = y^*(x_o)$ so that x will be constant. If $x_o > x^*$, over-utilization is optimal so that x will be decreasing toward x^*.

As in the standard model, the equilibrium and asymptotically optimal expected deflection rate is a decreasing function of δ. As δ goes to infinity, x approaches the lower limit $\tilde{x}^*$ which is defined by $U_y(x, y^*(x)) = 0$. This is the value of x which the statical policy $y = \tilde{y}(x)$, where $U_y(\tilde{y}, x) = 0$, would produce asymptotically. Along a dynamically optimal path (with δ finite of course), $y < \tilde{y}$ for all x and hence for all t.

Since x^* is decreasing in δ, we are more likely to find a policy of over-utilization to be optimal the greater is δ. Further, the greater δ the greater will be the asymptotic level of optimal y since y^* is a decreasing function of x, i.e., the equilibrium relation between y^* and $-x$ is positively sloped.

Consider now the special but important case $\delta = 0$. We adopt the "overtaking principle" of optimality and choose the unit of utility such that $U(x^*, y^*(x^*)) = 0$ where x^* is defined by

$$(28) \qquad \frac{U_x(x^*, y^*(x^*))}{U_y(x^*, y^*(x^*))} = \frac{G_x(x^*, y^*(x^*))}{G_y(x^*, y^*(x^*))}$$

This value x^* is determined by the tangency of the $y^*(x)$ locus with a utility contour in the (x, y) plane. It is the value of x such that the marginal rate of substitution between x and y, U_x/U_y, equals the marginal rate of transformation between x and y^*, namely $G_x/G_y = y^{*\prime}(x)$. Our convexity assumptions make it unique. We see therefore that indifference-curve analysis has a place here, as in the static approach.[30] But the tangency is along the equilibrium locus, $y^*(x)$, not along a Quasi-Phillips Curve corresponding to a particular x . Moreover, this tangency point at x^* tells us only the direction in which $x(t)$ should be made to move, not the optimal initial y . Finally, concerning x^* , note that (28) is simply (27 with $\delta = 0$.

The value x^* is that steady rate of deflation that gives the highest rate of utility among all paths of constant x . Hence $U(x^*, y^*(x^*))$, which has been made equal to zero, is the highest sustainable rate of utility. It takes the place of $\hat{U}$ in the standard model. But the path $(x^*, y^*(x^*))$ does not yield full liquidity for $U_x(x^*, y^*(x^*)) > 0$ in (28) since $G_x(x,y) < 0$ for all x and y . There is, of course, a range of the $y^*(x)$ locus giving full liquidity; but since $U_y(x, y^*(x)) > 0$ in this range, it pays to "buy" an increase of y^* -- moving rightward along the equilibrium locus -- at the cost of a short-fall from full liquidity.

The value x^* in (28) is the asymptotically optimal or long-run equilibrium value of x . Thus if $x_o < x^*$, under-utilization is optimal; if $x_o = x^*$ then equilibrium utilization is optimal; if $x_o > x^*$ then over-utilization is optimal.

An optimum continues to exist even if $x_o > x^*$, unlike the result found in the standard model where no optimum exists if $x_o > \hat{x}(y^*)$. The reason an optimum exists here is that it is possible to have "too much" expected deflation in a steady state; any movement in either direction of x from x^* reduces the steady rate of utility produced by $y = y^*(x)$. This is essentially because x has a negative "marginal productivity" in the present model, $G_x(x,y) < 0$, unlike the standard model; this feature makes the present model rather closer to the Ramsey model in which, when capital exceeds the saturation level, the net marginal productivity of capital is negative, close enough, in fact, to insure the existence of an optimum.

Analogous to the Ramsey-Keynes formula for the optimal y as a function of x and to the previous equation (11) is the relation[31]

$$(29) \qquad U(x,y) - G(x,y)\,\frac{U_y(x,y)}{G_y(x,y)} = 0$$

This can be portrayed geometrically, as (11) was. When $x < x^*$, so that $U_xG_y - U_yG_x > 0$, then there is a tangency point as in Figure 5 with $G > 0$, $U < 0$; when $x = x^*$, so that $U_xG_y - U_yG_x = 0$, the tangency point occurs at $G = 0$, $U = U(x^*,y^*(x^*)) = 0$; when $x > x^*$, so that $U_xG_y - U_yG_x < 0$, then the left-hand tangency with $G < 0$, $U > 0$, describes the optimal policy. Noting that G and $U_xG_y - U_yG_x$ have always the same algebraic sign, it can be seen from differentiation of (29) that optimal y is a monotone increasing function of x.

In partial summary of the foregoing, we may say that the optimal utilization ratio has the following properties both when $\delta > 0$ and when $\delta = 0$:

$$y = y(x) \quad \text{for all admissible} \quad x ,$$

(30)
$$\text{where} \quad y(x) \begin{cases} < y^*(x) & \text{if} \quad x < x^* , \\ = y^*(x) & \text{if} \quad x = x^* , \\ > y^*(x) & \text{if} \quad x > x^* \end{cases}$$

$$\text{with} \quad \hat{x}^* < x^*(\delta) < \hat{x}(y^*(\hat{x})) \quad \text{for all} \quad \delta \geq 0 , \quad x^{*\prime}(\delta) < 0 .$$

III. CONCLUSIONS AND RESERVATIONS

Perhaps the principal theme of this paper is the following: A tight fiscal policy producing under-utilization, and hence producing an actual algebraic inflation rate that is smaller than the currently expected inflation rate, will be optimal if and only if the currently expected inflation rate exceeds the long-run equilibrium (asymptotically optimal) inflation rate. The latter is determined by liquidity considerations and by social time preference (the utility discount rate), not by the strength of preferences for high or low utilization at a given rate of interest. In the standard model possessing a vertical steady-state Quasi-Phillips Curve, where the equilibrium utilization is independent of the expected inflation rate, the long-run equilibrium inflation rate is simply the maximum expected inflation rate consistent with full liquidity at equilibrium utilization if the utility discount rate is zero. (This was a disconcerting case in one respect for if the initially expected inflation rate exceeds this full-liquidity rate, then no optimum exists.) If there is positive discounting of future utility rates, in which case an optimum always exists, the long-run equilibrium inflation rate is greater than the full-liquidity rate; the equilibrium rate is greater the larger is the utility discount rate. From this point of view, therefore, what characterizes the advocates of a "high-pressure" economic policy of over-utilization is their implicit adoption of a large discount rate. In favoring high utilization today at the cost of high inflation in the eventual future equilibrium, they reveal high "time preference" or "impatience."

Three variants of the standard model were presented. A finite time-horizon model and its "turnpike" property was discussed. The necessity of introducing the expected inflation rate into the utility-rate function in a continuous way for the existence of an optimum in an infinite time-horizon model was shown. Finally, a model was presented with a positively-sloped steady-state Quasi-Phillips Curve in terms of the utilization ratio. In this model it is optimal to give up full liquidity, even if there is no discounting of future utility rates, in order to enjoy asymptotically the high equilibrium utilization ratio that a high expected inflation rate permits. But there is no hard evidence or strong theoretical case for such a positive slope; indeed, liquidity considerations suggest that as the expected inflation ratio becomes large, the equilibrium utilization locus will exhibit some negative slope.

I believe these dynamical models, or at least dynamical models of this sort, to be a methodological step forward from the conventional, statical approach to aggregate demand policy discussed at the outset of this paper. But I wish to disclaim any immediate policy significance for the above models. A number of modifications and extensions of these models ought to be made before their implications for policy are seriously considered by governments.

I shall first list some simplifying properties of the above models which it may be desirable to remove, even though doing so may not have any critical effects. (1) Monetary policy was exogenous in a special way, viz., the real interest rate function, $r(y)$, made investment independent of utilization so that virtual golden age growth could take place. It would be more realistic to suppose that the monetary authority incompletely offsets the effect on investment of variations in utilization, so that there is some "induced investment."[32] (2) A related point is that the possible requirement by the monetary authority of a negative money interest rate was ignored, which is justifiable only for algebraically large expected inflation rates. (3) Also related to the efficacy of monetary policy is the postulated non-stochastic character of the economy. There were no exogenous shocks and inventory stocks, which may be destabilizing, were also ignored. (4) The utility-rate function ϕ was taken to be stationary (apart from discounting). Probably this is not generally legitimate, even under virtual golden-age growth, if productivity is rising. Relaxation of this stationarity postulate may produce a gain in permitting negative utility discounting to be entertained. (5) The approach was aggregative, while the very notion of the Phillips Curve may require a disaggregative model. (6) The utility rate function U was made strictly concave in utilization, which required what may be a counterfactual assumption on $r''(y)$. (7) The function ϕ was smooth in i whereas the first derivative may have a kink at $\hat{i}$. (8) Taxes were postulated to be lump-sum, the wealth-capital distinction was ignored and labor supply was taken to be homogeneous of degree zero in real wage rate, disposable per capita income and per capita wealth; this insured that labor supply would grow exponentially when the utilization ratio is constant (under marginal productivity pricing). (9) The employment-output relation was stationary without lagged adjustments due to hiring-and-firing costs or due to lagged adaptation of the capital structure to a change of the output level. (10) Somewhat related is the postulated stationarity of $r(y)$. (11) "Nonbarter" was a result of the model due to the postulate that barter was "infinitely disastrous." It could be argued that nonbarter should be made a constraint in the problem rather than an implication; or even argued that barter is only finitely disastrous and therefore an eventuality that the model should entertain. (12) Even though allowance was made for the effect of a discrepancy between the actual and expected inflation rates on the rate of social utility, the actual inflation rate itself was excluded from the utility-rate function ϕ . The nuisance of revising price lists provides some justification for the inclusion of this variable in ϕ , although its introduction there can affect only the shape of the U function, not the identity of its arguments.

There are other extensions or modifications that may have a more profound effect on the conclusions drawn from the model. (1) Monetary policy was taken as exogenous, the monetary authorities living a life apart from the fiscal authorities. It would be interesting to suppose that the Bank and the Fisc recognize that their policies combine to affect both utilization and growth, that both affect social welfare and that, for an over-all optimum, both growth and utilization need to be optimized simultaneously. I suspect that if this is done, the asymptotically optimal real interest rate (if such exists) will be higher the larger is the utility discount rate; but then the asymptotic money interest rate will be higher (and liquidity lower) for any given, asymptotic (actual and expected) inflation rate. Now the asymptotically optimal money interest rate will presumably be higher the larger is the utility discount rate, as in the models of this paper; but unless it rises at least as much as the asymptotically optimal real interest rate, the asymptotic inflation rate will decrease, rather than increase, with the increase of the utility discount rate, contrary to the case in this paper where the real interest rate is independent of the utility discount rate. Thus the conclusion of this paper that high time preference is associated with over-utilization and a high equilibrium inflation rate may have to be revised if investment is an endogenous variable under the control of a monetary authority that maximizes, jointly with the fiscal authority, a common social utility function. (2) The economy of this paper was closed to foreign trade and capital movements. The models here need substantial extensions if they are to accommodate an open economy. There are two approaches: One may take exchange-rate policy as exogenous, with the exchange rate being either flexible, discontinuously adjustable or absolutely inflexible (never an adjustment of the exchange rate); or one may make exchange-rate policy endogeneous, like fiscal policy and monetary policy so that one is optimizing utilization, growth and the payments balance simultaneously. (3) The rate of inflation was independent of the <u>rate of change</u> of the utilization ratio in this paper. If, as is possible, the Phillips Curves in terms of wage change, rather then the Quasi-Phillips Curves in terms of price change, are the correct relations, then our constraint is of the form $\dot{x} = G(x, y, \dot{y})$ and the nature of the optimal policy will be affected. At a minimum, there will be some inertia in the optimal path of the utilization ratio and it seems plausible that even the long-run equilibrium will depend somewhat on initial conditions. (4) It may be desirable to consider, simultaneously with optimal utilization policy, optimal policies aimed at shifting downward the Phillips Curves by structural reform of the labor and goods markets. For suppose that one's models generates a policy of substantial under-utilization. It should be pointed out again that $y < y^*$ implies deflation only if the expected inflation rate is zero or negative; under-utilization means only that the actual algebraic inflation rate will be smaller than the expected rate; the attention given by some economists to the size of the unemployment ratio necessary for stable prices when positive inflation is expected appears to be without reason. Nevertheless, substantial under-utilization would be unpalatable. Such an analytical finding might only serve to emphasize the desirability of policy instruments to shift down the Phillips Curves, so that it may be wise to incorporate such instruments into the model. (5) Fiscal instruments -- rates of taxation -- were continuously variable over time in this paper. But frequent "spinning of the dials" by the government would be politically infeasible and perhaps undesirable: too much uncertainty and speculative behavior leading to misallocation would result. The government might want therefore to announce its fiscal intentions over the future and this should be incorporated into the model. But this leads to the following related problem. (6) The currently expected inflation rate was taken to depend only upon past inflation rates. It is likely to depend also upon what the population believes to be the government's intentions toward current and future aggregate demand and inflation. Optimal fiscal policies may be greatly affected by this consideration. To take an extreme case, if the public cannot be deceived, if it can predict the inflation rate correctly because it knows the government's aims, then the government will have no choice but to establish equilibrium utilization.

The last point is related to another with which it may be appropriate to conclude. The rationale for a policy of under-utilization or over-utilization in the models considered here is that such a policy will cause the actual inflation rate to fall short of or to exceed the expected inflation rate and thus cause people to revise downward or upward the inflation rate that they expect; the government finds it optimal to do this when, in a sense, the inflation rate that people expect is "not good for them." Even if one has no qualms about paternalism -- after all, the government needs <u>some</u> policy toward aggregate demand -- one may be troubled by the deception that such a policy involves. It can be objected that the only moral policy open to the government is to produce that inflation rate which people expect, hence immediately to establish equilibrium utilization. This is a problem that one encounters elsewhere in the area of macroeconomic policy. Presumably the answer to this objection is that most people are willing to be deceived if the deception is in their own interest.

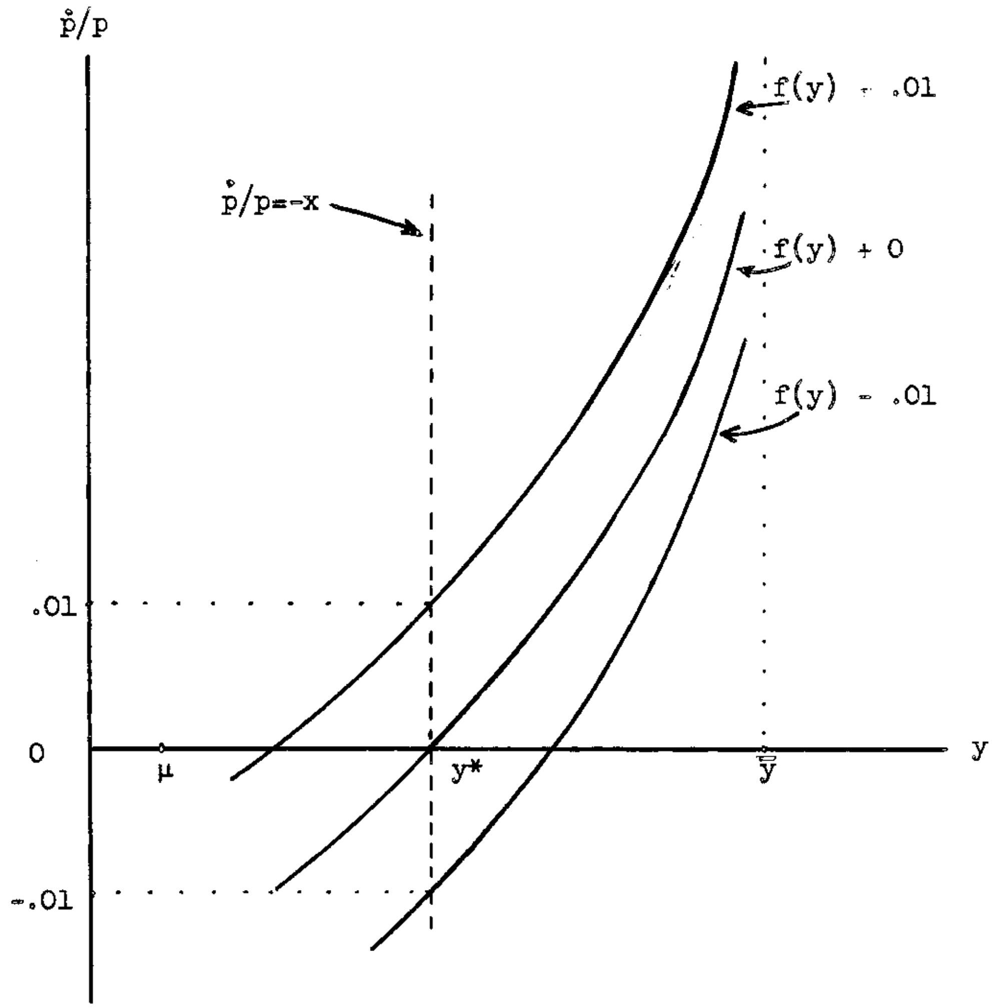

Figure 1: Quasi-Phillips Curves for -x=.01,0,-.01

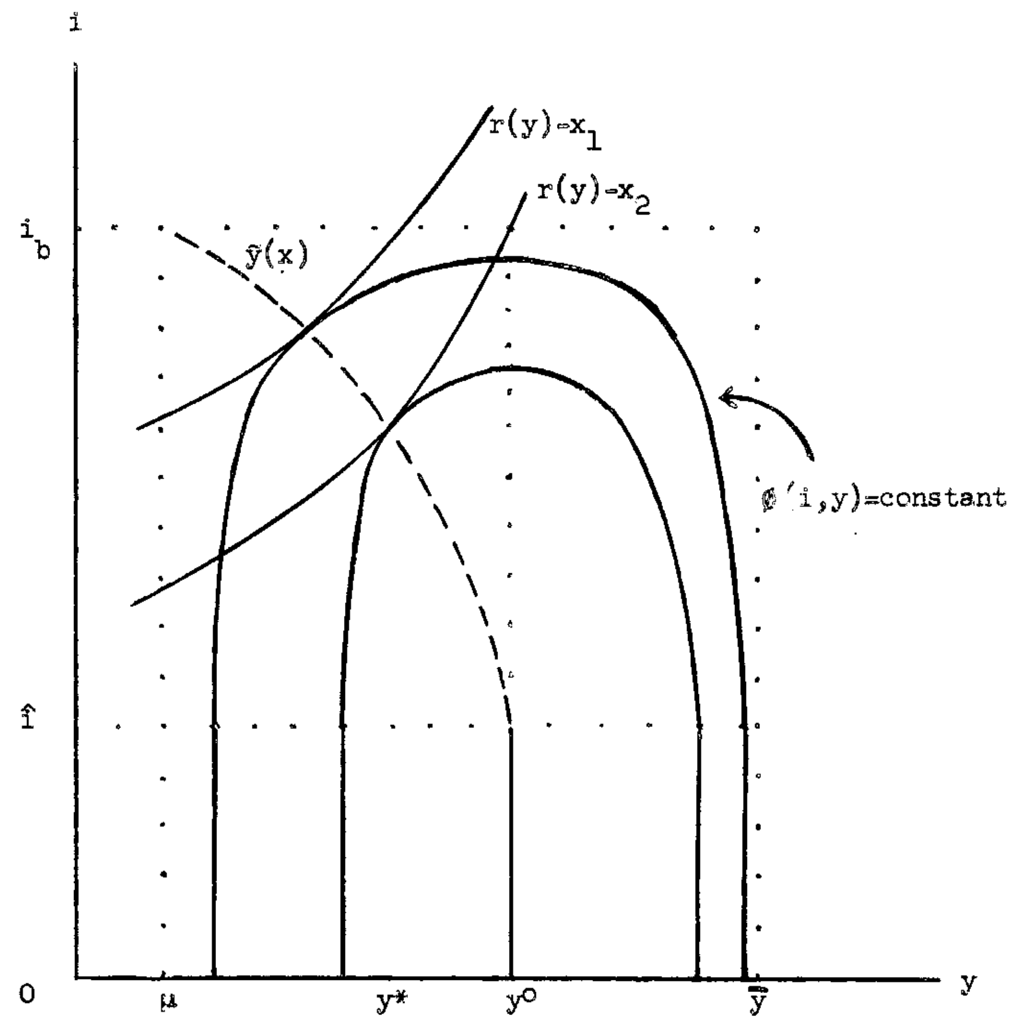

Figure 2: Contours of Constant $\phi(i,y)$, the Interest Rate Function and the Function $\tilde{y}(x)$.

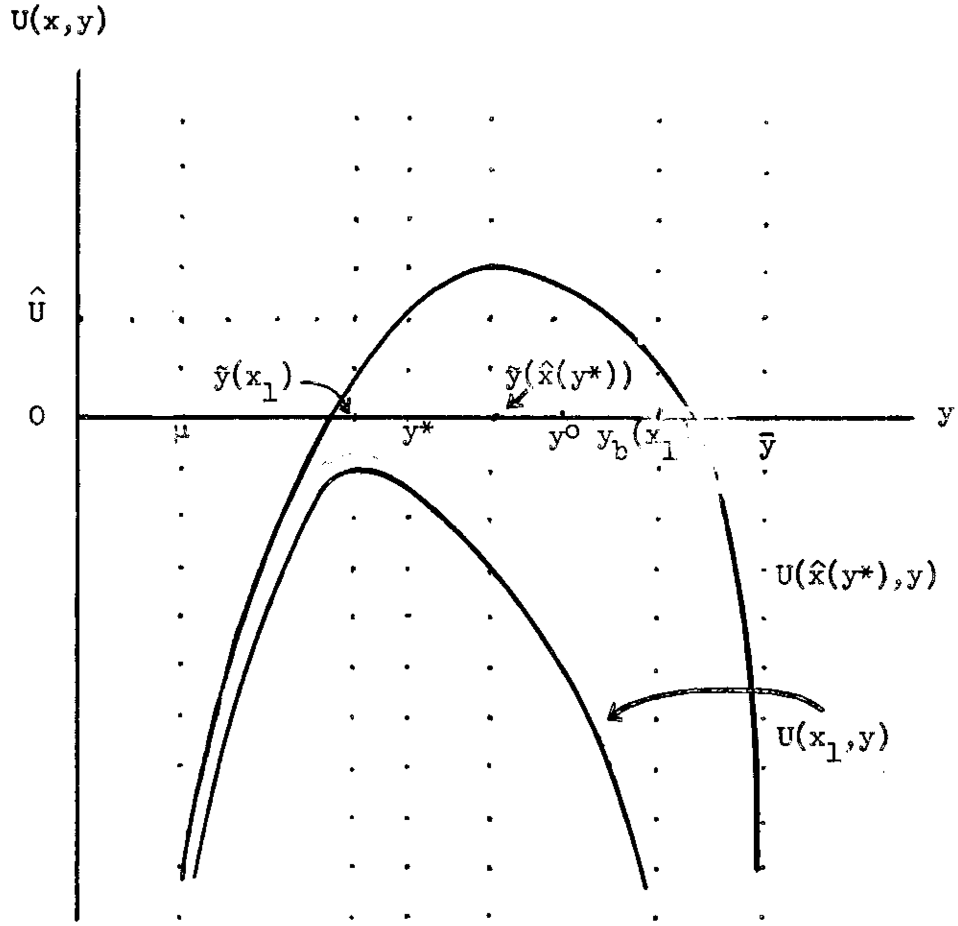

Figure 3: Dependence of the Utility Rate on the Utilization Ratio when $x=\hat{x}(y^*)$ and when $x=x_1$.

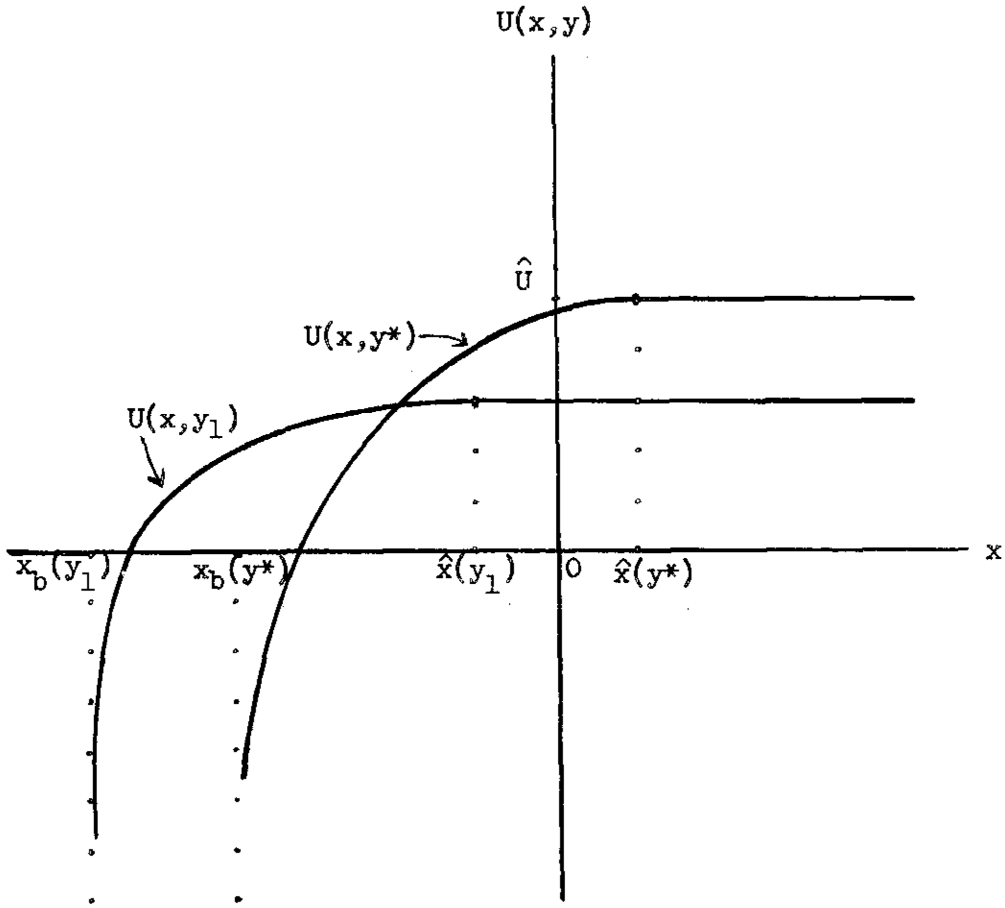

Figure 4: Dependence of the Utility Rate on Expected Deflation Rate when $y=y^*$ and when $y=y_1$

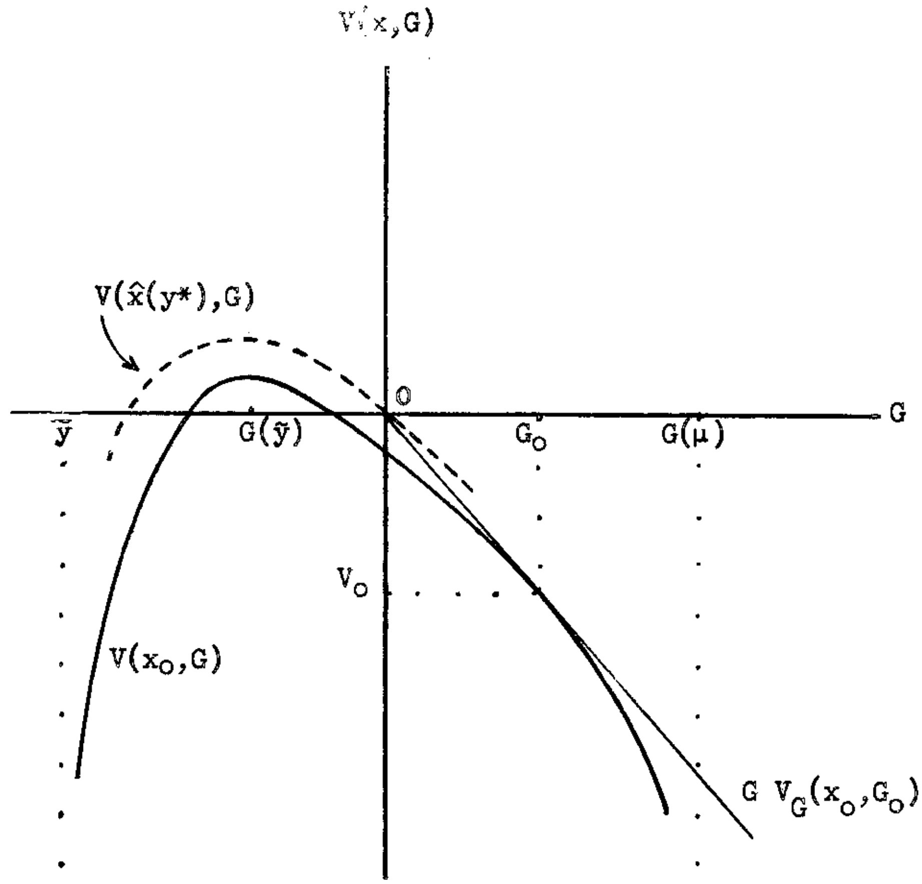

Figure 5: The No-Discount Utilization Optimum when $x_0 < \hat{x}(y^*)$.

APPENDIX

The problem analyzed here is

(A.1)
$$\text{Max} \int_0^\infty e^{-\delta t}\, U(x,y)\, dt , \qquad \delta > 0$$
$$\text{subject to } \dot{x} = G(x,y) , \qquad x(0) = x_o$$

where (8a,b) gives the properties of U and (25) gives the properties of G ; in addition, the consequences of using the simpler (5) in place of (25) will be indicated. The approach to this classical variational problem will be that of L.S. Pontryagin and associates, The Mathematical Theory of Optimal Processes (New York and London: Interscience Publishers, 1962).

From (A.1) we form the Hamiltonian expression

(A.2)
$$H(x,y,q) = U(x,y)e^{-\delta t} + qe^{-\delta t}\, G(x,y)$$

From Pontryagin's Theorem 6 (p.69) and additional discussion (esp. pp. 189-191) it follows that the conditions which an optimal path must satisfy are that there exists a continuous function $q = q(t)$ such that x and $qe^{-\delta t}$ obey the relations

(A.3)
$$\frac{dx}{dt} = \frac{\partial H(x,y,q)}{\partial (qe^{-\delta t})} \quad \text{or} \quad \dot{x} = G(x,y) ,$$

(A.4)
$$\frac{d(qe^{-\delta t})}{dt} = \frac{\partial H(x,y,q)}{\partial x}$$
$$\text{or} \qquad \dot{q} = \delta q - G_x(x,y)q - U_x(x,y) ,$$

(A.5) $$\lim_{t\to\infty} qe^{-\delta t} = 0 ,$$

and such that y always maximizes

(A. 6) $$H(x,y,q) \quad \text{or} \quad \psi(x,y,q) = U(x,y) + q\,G(x,y) .$$

At this maximum

$$\psi_y(x,y,q) = U_y(x,y) + q\,G_y(x,y) = 0$$

$$\psi_{yy}(x,y,q) = U_{yy}(x,y) + q\,G_{yy}(x,y) < 0$$

whence

(A.7) $$q = \frac{U_y(x,y)}{-G_y(x,y)} ,$$

which is always nonnegative since no account of the constraint $i \geq 0$ is taken (for $U_y < 0$ only if there were a constraint $y \geq y_c(x)$ where $y_c > \tilde{y}(x)$). It is implied that $\mu < y \leq \tilde{y}(x)$.

There is a stationary path, $x = x^*$, $y = y^*$, $q = q^*$ which satisfies (A.3) and (A.4). This is given by

(A.8) $$0 = G(x^*,y^*)$$

(A.9) $$0 = [\delta - G_x(x^*,y^*)]\,q - U_x(x^*,y^*)$$

Using (A.7) we therefore obtain

(A.10) $$\delta - G_x(x^*,y^*(x)) = \frac{-U_x(x^*,y^*(x))\,G_y(x^*,y^*(x))}{U_y(x^*,y^*(x))}$$

which determines x^*. It follows from (A.10) and (A.9) that $0 < q^* < \infty$ so that (A.5) is satisfied.

We turn now to the solution of the pair of differential equations (A.3) and (A.4) subject to (A.5) and (A.7). We may eliminate y from (A.3) and (A.4) since, from (A.7) we may write

(A.11) $$y = \varphi(x,q)$$

where

$$\varphi_x = \frac{U_{yx} G_y}{-U_{yy} G_y + U_y G_{yy}} = \frac{-U_{yx}}{U_{yy} + q G_{yy}} ,$$

$$\varphi_q = \frac{G_y G_y}{-U_{yy} G_y + U_y G_{yy}} = \frac{-G_y}{U_{yy} + q G_{yy}} .$$

It can be seen that $\varphi_q < 0$ while $\varphi_x > 0$ when $x < \hat{x}(y)$ and $\varphi_x = 0$ otherwise (full liquidity).

Now let

(A.12) $$g(x,q) = \dot{x} = G(x,y)$$

(A.13) $$h(x,q) = \dot{q} = [\delta - G_x(x,y)]\, q - U_x(x,y)$$

It is shown now that there is a saddle-point solution, $(x(t), q(t))$, which asymptotically approaches the stationary equilibrium (x^*,q^*).

Linear terms of the expansion of (A.12)-(A.13) around the point (x^*,q^*) yield the two characteristic roots

(A.14) $$\lambda = \frac{g_x(x^*,q^*) + h_q(x^*,q^*) \pm \sqrt{[g_x(x^*,q^*) + h_q(x^*,q^*)]^2 - 4|A|}}{2}$$

where

$$|A| = \begin{vmatrix} g_x(x^*,q^*) & g_q(x^*,q^*) \\ h_x(x^*, q^*) & h_q(x^*,q^*) \end{vmatrix}$$

Using (A.11), it can be shown that

(A.15) $$|A| = \varphi_q[G_y U_{xx} - G_x U_{xy}] + (\delta - G_x)[\varphi_x G_y + G_x]$$

which is negative for $\delta > 0$ (even if $G_x = 0$) so that the characteristic roots are real and opposite in sign, from which it follows that (x^*,q^*) is a saddle-point. (See L. S. Pontryagin, Ordinary Differential Equations (Reading, Mass.: Addison-Wesley, 1962), pp. 244-261.)

It can also be shown that x approaches x^* monotonically, so that either $y < y^*(x)$ or $y > y^*(x)$ for all t (unless $x_0 = x^*$ in which case $y = y^*(x^*)$). The equation

(A.16) $$g(x,q) = 0 \; ,$$

which gives all the singular points of (A.12), determines the q such that $\dot{x} = 0$ as a function of x. Implicit differentiation of (A.16) yields

(A.17) $$\left(\frac{dq}{dx}\right)_{\dot{x}=0} = \frac{-(G_x + G_y \varphi_x)}{G_y \varphi_q}$$

Thus, if we diagram this relation between q and x, with q on the vertical axis, the curve will be positively sloped if $G_x < 0$ as in (25); if $G_x = 0$, as in (5), then the curve becomes horizontal at $x = \hat{x}(y^*)$ and elsewhere to the right. (A.16) may be expressed equivalently in the form

(A.16a) $$q = \frac{U_y(x,y^*(x))}{-G_y(x,y^*(x))}$$

since $y = y^*(x)$ if $\dot{x} = 0$, from which it may be seen that the curve approaches minus infinity as x decreases and approaches the value x_b satisfying $U_x(x,y^*(x)) = \infty$, which is the largest x giving barter when $y = y^*(x)$. The curve crosses the x-axis at $x = \tilde{x}^*$ defined by $\tilde{y}(x) = y^*(x)$.

The equation

(A.18) $$h(x,q) = 0$$

describes the locus of singular points making $\dot{q} = 0$. Differentiation yields

(A.19) $$\left(\frac{dq}{dx}\right)_{\dot{q}=0} = \frac{U_{xx} + U_{xy}\,\varphi_x}{(\delta - G_x) - U_{xy}\,\varphi_q}$$

For x sufficiently large for full liquidity the numerator equals zero so this curve is flat beyond a certain large x. In this range, $q = 0$ by (A.13). Elsewhere, the numerator may be of either sign so the curve need not be monotonic. By (A.13) we may write (A.18) in the form

(A.18a) $$q = \frac{U(x,\varphi(x,q))}{\delta - G_x(x,\varphi(x,q))}$$

It may be seen that as $x \to x_b(\mu)$, $q \to \infty$ so the curve is asymptotic to $x_b(\mu)$

with negative slope. Admissible x_o must satisfy $x_o > x_b(\mu)$. While the slope may be of either sign for intermediate x, it can be seen that the $\dot{q} = 0$ curve intersects the $\dot{x} = 0$ curve from above, there being just one intersection. Using (A.17), we may write (A.19) in the form

$$\text{(A.20)} \qquad \left(\frac{dq}{dx}\right)_{\dot{q}=0} = \frac{U_{xx}}{\delta - G_x - U_{xy}\,\varphi_q} + \frac{\left(\frac{dq}{dx}\right)_{\dot{x}=0} + \frac{G_x}{G_y \varphi_q}}{\left[\frac{\delta - G_x}{U_{xy}\,\varphi_x}\right]\left(\frac{-\varphi_x}{\varphi_q}\right) + 1}$$

Since at (x^*, q^*) there is not full liquidity, the first term is negative at (x^*, q^*). The numerator of the second term exceeds (dq/dx), $\dot{x} = 0$, if $G_x < 0$, and is equal if $G_x = 0$. The denominator of the second term exceeds 1. Therefore

$$\left(\frac{dq}{dx}\right)_{\dot{q}=0} < \left(\frac{dq}{dx}\right)_{\dot{x}=0} \qquad \text{at } (x^*, q^*),$$

so there is just one intersection.

Since $h_q(x,q) = \delta - G_x - U_{xy}\,\varphi_q > 0$, we have $\dot{q} > 0$ above the $\dot{q} = 0$ curve and $\dot{q} < 0$ below. Since $g_q(x,q) = G_y\,\varphi_q > 0$ and $g_x(x,q) = G_x + G_y\,\varphi_x < 0$ we have $\dot{x} > 0$ above and left of $\dot{x} = 0$ and $\dot{x} < 0$ below and to the right. Thus it can be seen that x is monotonically increasing if $x < x^*$ and monotonically decreasing if $x > x^*$ along the hair-line, saddle-point solution path. The variable q need not be monotonic in x so that apparently the policy function $y = y(x)$ need not be monotonic.

As for the case $\delta = 0$, it may be noted that (A.4) and (A.7) give the Euler equation whose first integral yields (29). In this case at least y is monotonic in x as can be seen from differentiation of (29).

FOOTNOTES

1. This paper was written during my tenure of a Faculty Research Grant from the Social Science Research Council.

2. A recent example of the approach I have in mind is R. G. Lipsey, "Structural and Deficient-Demand Unemployment Reconsidered" in A. M. Ross, ed., Employment Policy and the Labor Market (Berkeley: University of California Press, 1965). See also A. M. Okun, "The Role of Aggregate Demand in Alleviating Unemployment," Princeton University Conference, The Unemployment Problem in the United States: Trends and Proposals, Princeton, New Jersey, May 13, 1965.

3. The locus classicus is, of course, W. A. Phillips, "The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861-1957," Economica, Vol. 25 (November, 1958), pp. 283-299.

4. If the Phillips Curve shifts upward with a one point increase of the expected inflation rate by less than one point then the steady-state Phillips Curve will be negatively sloped. But it will be steeper than the nonsteady-state Phillips Curves so that my criticism of the conventional approach remains qualitatively valid. It is true, however, that the criticism loses more of its force the less steep is the steady-state curve in relation to the nonsteady-state curves. But even the introduction of money illusion may not be capable of generating a nonvertical steady-state Phillips Curve in terms of the unemployment ratio if the labor

force (or supply of labor) is correctly measured for then the expansion of labor supply which unperceived inflation may bring about will be reflected in a higher unemployment ratio for every given level of employment. If, however, we construct a Quasi-Phillips Curve in terms of the ratio of unemployment to population, then the steady-state curve might well have such a negative slope under money illusion or inflation illusion. This case is analyzed in the last part of this paper.

5. Of a hundred references that could be cited, some half of which tend to support the Phillips Curve with some alterations, see, for example, with reference to British data, R. G. Lipsey, "The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1862-1957: A Further Analysis," Economica, Vol. 27 (February, 1960), pp. 1-31; and, with reference to American data, G. L. Perry, "The Determinants of Wage Rate Changes and the Inflation-Unemployment Trade-off in the United States," Review of Economic Studies, Vol. 31 (October, 1964), pp. 287-308.

6. See, for example, Lipsey, ibid., and Perry, op. cit.

7. Throughout this paper I neglect the difference between wealth and capital, i.e., I neglect the government debt. This is really acceptable only if the wealth-capital ratio is constant over time. While this will not occur in my model, that ratio will become constant asymptotically as any golden-age path is approached. I suggest therefore that my error is small enough to be neglected safely.

8. Two comments are in order here. First, we do usually observe, I believe, that interest rates are relatively high in "good times," but evidently they are not sufficiently high or high soon enough to prevent pro-cyclical variations of investment expenditures. Possibly the reason is that business fluctuations are too sharp and imperfectly foreseen to permit the monetary authorities to stabilize investment. But if fiscal weapons were used effectively to control consumption demand, as they are assumed to be in this paper, then the Bank's job of controlling investment would be much facilitated. It must be admitted however that the whole question of optimal fiscal and monetary policy in the presence of exogenous stochastic shocks and policy lags is beyond the scope of this paper.
Second, it should be mentioned that the exclusive assignment of investment control to the monetary authority is quite unessential to this paper. Indeed, it might be more realistic to suppose that investment was controllable in the desired manner through fiscal weapons. But then one could not identify the real rate of interest loosely with the pre-tax marginal product of capital so there would be no simple interpretation of the shape of the $r(y)$ function in equation (2).

9. I know that I owe the reader an apology for inflicting this notation on him. I have chosen to work in terms of expected deflation in order to emphasize its resemblance to capital in the well-known problem of optimal saving, a problem having some similarity to the present one.

10. In this formulation, I have excluded the "Lipsey term" which makes the

rate of wage increase depend also upon the rate of change of the employment ratio. It may be that the "loops" which Lipsey sought to eliminate by this term could be explained by cyclical variations in the expected rate of inflation so that no such Lipsey term really belongs. See R. G. Lipsey, ibid.

11. Some readers may at this point demand an answer to the question: Can the Phillips Curve be given a theoretical foundation or is it merely a spurious empirical relationship which cannot be explained by other tested relationships and which the theorist should not attempt to dignify with an explanation? I confess I am unable to give answers at the present time. I do believe that the Phillips Curve has validity as an approximation in competitive models of heterogeneous goods and labor in which there is imperfect mobility of resources. See, for example, B. Hansen, "Full Employment and Wage Stability" in J. T. Dunlop, ed., The Theory of Wage Determination: Proceedings of a Conference of the International Economic Association, (London: Macmillan, 1957), pp. 66-78. Such models can explain the coexistence (at least temporarily) of an upward wage trend in the face of aggregate involuntary unemployment. Nevertheless those models need further analysis. Even in a model with a homogeneous but frictional national labor market producing only one good, it may be possible to generat a relation between employment and wage-rate change by consideration of an "over-employment" situation in which last period's wage is found to have less purchasing power than expected because the price level has risen faster than was expected when last period's wage bargain was made.

12. P. Cagan, "The Monetary Dynamics of Hyperinflation," in M. Friedman, ed., Studies in the Quantity Theory of Money (Chicago: University of Chicago Press, 1956), pp. 25-117.

13. F. P. Ramsey, "A Mathematical Theory of Saving," Economic Journal, Vol. 38, (September, 1928), pp. 543-559. For a discussion in a different context of the axiomatic basis for such a utility function, see T. C. Koopmans, "Stationary Ordinal Utility and Impatience," Econometrica, Vol. 28, (April, 1960), pp. 287-309.

14. The assumptions in (7) guarantee strictly diminishing marginal rate of substitution above $\hat{i}$ and to the left of y^o. But for convexity to the right of y^o we require that ϕ_{21} not be "too negative." Fortunately the contours are of no interest to the right of y^o so we need not bother to place a lower bound on ϕ_{21}.

15. With aggregate investment being fixed, people cannot save too much or too little in the aggregate. But they can work too much or too little as a consequence of incorrect expectations.

16. In polling people to determine y^o the Fisc does not reveal to people that the level of the money rate of interest depends upon their social choice of y; y^o is, like y^* earlier, a utility peak for a fixed money interest rate.

17. A formal analysis of interest and "full liquidity" is contained in my paper, "Anticipated Inflation and Economic Welfare," Journal of Political

Economy, Vol. 73, (January, 1966), pp. 1-13. That paper deliberately neglects the steps necessary to establish the desired expected inflation rate in the particular case where, as here, no interest can be paid on money; it is entirely comparative statics, unlike the present paper. Incidentally, it is assumed there too that the lost time from economizing on money is "taken out" in the form of a leisure reduction rather than a labor-supply reduction (in order to facilitate diagrammatic analysis). Understanding of the present paper does not require knowledge of that paper.

18. On another view, the government has a moral obligation to validate the expectations held by groups who have contracted for fixed incomes (whether or not they are poor), even to the extent that if inflation has occurred recently the government now owes these groups a little deflation. The government of my model treats such obligations as "bygones," worrying only about the consequences of current deceptions, not past ones.

19. This price-list consideration perhaps ought also to enter in a complicated, nonstationary way since a high, steady rate of inflation might eventually call forth institutional changes in the nature of money or perhaps even some system of "compounded prices."

20. It would be somewhat unfair of the "statical economists" (whom I attacked in the introduction) to reproach me on this issue of stationarity since they have not explained how their indifference curves shift with population increase or technical progress.

21. See, for example, "The Ramsey Problem and the Golden Rule of Accumulation" in E. S. Phelps, Golden Rules of Economic Growth, (New York: W. W. Norton, 1966) and the references cited there.

22. Ramsey, op. cit.

23. Some differences are that his utility rate was independent of capital; his investment-consumption relation, G, depended upon capital; utility was everywhere increasing in consumption; and $G'(y) = -1$ in his case.

24. For a simple derivation, in which the differentiability necessary for the Euler condition is not assumed, see R. E. Bellman, Dynamic Programming (Princeton: Princeton University Press, 1956), pp. 249-250. See also the present paper's Mathematical Appendix which is largely concerned with the $\delta > 0$ case.

25. The reader may have noticed a second tangency point with $G < 0$. Pursuit of that policy would lead asymptotically to $y = y^*$ with $x = \tilde{x}$ where $\tilde{y}(\tilde{x}) = y^*$; since $\tilde{x} < \hat{x}(y^*)$, such a policy must cause W to diverge to minus infinity so that it cannot be optimal.

26. If it is taken for granted that someday the world will come to an end, then there has been a great deal of misplaced worry over the failure of some infinite-horizon models to possess an optimum. Yet finite-horizon models raise problems of their own, or, perhaps more accurately, they bring to light problems that are hard to solve but which require facing; for example, it may be that for T we have only a probability distribution.

27. See D. Cass, Studies in the Theory of Optimal Growth (Doctoral Dissertation, Stanford University, 1965) and P. A. Samuelson, "A Catenary Turnpike Theorem Involving Consumption and the Golden Rule," American Economic Review, Vol. 51 (June, 1965), pp. 486-496.

28. Possibly a "redistribution effect" can explain such a postulated (but not verified) behavioral relation. The shift to a higher expected inflation rate will, if it was not full anticipated, cause a loss to fixed-money-income groups and a gain to those individuals and institutions (e.g., the government) for whom the fixed incomes are a liability (e.g., bonds and other paper obligations). If the losers increase their supply of labor due to their loss of expected future real income by more than the gainers reduce their labor supply due to their enrichment, then aggregate labor supply will increase.

29. Cagan, op. cit.

30. The reader may find it useful to draw a diagram with $\mathring{p}/p$ and $-x$ measured on the vertical and y on the horizontal axis, drawing in the Quasi-Phillips Curves, the upward sloping $y^*(x)$ curve and the somewhat dome-shaped utility contours in the $(-x,y)$ plane. The latter will have maxima along a negatively sloped $\tilde{y}$ curve, and have vertical portions below a negatively sloped $\hat{x}$ curve, and be defined only in a certain domain.

31. Bellman, loc. cit.

32. In his paper on the optimal increase of aggregate demand, J. Blackk took account of induced investment but his approach is statical at bottom. See J. Black, "Inflation and Long-run Growth," Economica, Vol. 26 (May, 1959), pp. 145-153.