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Structure of Preference Over Time

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Structure of Preference Over Time^{*}

by

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1. Introductory remarks.

A standard model in the theory of consumer's choice assumes that the consumer maximizes a utility function under given budgetary constraints. In thinking about optimal economic growth, it seems natural to extend this idea to the maximization of a social welfare function under constraints of technology and initial capital stock.

Even in the case of the individual consumer planning for a single period's consumption, however, the time-honored concept of a utility function is not an entirely satisfactory primary concept. One may wish to look on it as a numerical representation of an underlying preference ordering, a more basic concept to be more fully defined below. Once this step is made, one will also want to know which class of preference orderings permits such a representation. Moreover, one will not want to exclude a priori the consideration of preference orderings that do not permit such a representation.

These considerations have still greater force with regard to problems of evaluative comparison of growth paths for an indefinite future. If one interprets this as an infinite future, neither the concept of a utility function

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depending on infinitely many variables, nor that of a preference ordering on a space of infinitely many dimensions, have an obvious intuitive understandability about them. To start from the more basic one -- the preference ordering -- is therefore even more desirable in that case, in that it helps avoid implicit assumptions one is not aware of.

This paper is a discussion of the choice of a criterion for the evaluation of growth paths that starts from postulates about a preference ordering. Technical parts of the reasoning in some sections are set off in starred subsections bearing the same number. These can be passed up by readers interested in results rather than proofs. Equality by definition will be denoted by \equiv

2. Preference orderings and representations thereof.

We shall now define and describe the mathematical concept of a preference ordering on a prospect space.

The prospect space \mathcal{X} is the set of all alternative prospects between which choice may conceivably arise. The term "space" is, of course, a geometric metaphor, and the prospects will sometimes also be called "points." In the static model of consumer's choice, the prospects are bundles (or vectors) of consumption goods imagined available for consumption in a stated period. (A "bundle" specifies an amount for each good on the list). In the simplest model of preference over time, the prospects are programs, that is, sequences of bundles imagined available in successive periods (finite or infinite in number as the analysis requires).

A complete preference ordering is a relation (to be denoted \succeq) between the prospects x, y, \dots in \mathcal{X} , considered pairwise, such that

(transitivity) if $x \succeq y$ and $y \succeq z$ then $x \succeq z$,

(completeness) for any pair of prospects x, y of \mathcal{X}
either $x \succeq y$ or $y \succeq x$ or both.

The relation $x \succeq y$ is interpreted as " x is at least as good as y ," or synonymously " x is preferred or equivalent to y ." Preference (\succ) and equivalence (\sim) are again transitive relations, derived from \succeq by

" $x \succ y$ " means " $x \succeq y$ but not $y \succeq x$ ", and is also denoted
" $y \prec x$ ",

" $x \sim y$ " means " $x \succeq y$ and also $y \succeq x$ ".

A partial preference ordering is obtained if we substitute for the completeness requirement above

(reflexivity) for all x of \mathcal{X} , $x \succeq x$.

Completeness implies reflexivity (take $x = y$), but the converse is, of course, not true. Hence, in a partially ordered space there may be pairs of prospects that are not comparable.¹

¹ What is called a "preference ordering" here is called a "preordering" by Debreu [1959, p. 7]. Arrow [1963, pp. 13, 35] uses "weak ordering" for our "complete preference ordering," and "quasi-ordering" for our "partial preference ordering." In mathematical literature, the term "weak order," or "weak ordering," is used whenever (as above) equivalence ($x \sim y$) does not necessarily imply equality ($x = y$).

By a numerical representation of a complete² preference ordering \succsim we mean a function f , defined in all points x of the prospect space \mathcal{X} , and whose values $f(x)$ are real numbers, such that

$$(2.1) \quad f(x) \geq f(y) \quad \text{if and only if} \quad x \succsim y.$$

Using the above definitions of preference and of equivalence, one sees readily that this is logically equivalent to

$$(2.2) \quad \begin{cases} f(x) > f(y) & \text{if and only if} \quad x \succ y, \text{ and} \\ f(x) = f(y) & \text{if and only if} \quad x \sim y. \end{cases}$$

The usefulness of a representation by a continuous function, if one exists, lies primarily in the availability of stronger mathematical techniques in that case. There is a temptation to look on the values, and the differences between values, assumed by a representing "utility function" as measures of satisfaction levels, and of differences thereof, associated with the prospects in question. Such interpretations may have heuristic usefulness because of the brevity of phrasing they make possible. However, their observational basis is not really clear. An observed choice between two prospects reveals at best the fact and the direction of preference, not its strength. A descriptive theory of choice thus stays somewhat closer to what is verifiable by observation

² If the preference ordering is not complete, a numerical representation is a function f such that $f(x) \geq f(y)$ if $x \succsim y$, together with a specification of the set of pairs (x, y) of prospects x, y in \mathcal{X} which are indeed comparable. Such representations have been considered by Aumann [1964].

if it is built on postulates about the underlying preference ordering. A similar remark applies to normative theory. One can better inspect and appraise a recommendation coached in terms of actual choices in various situations, than one derived from measures of "satisfaction" whose operational significance is unclear.

We shall now describe the results of two postulational studies in the literature, as illustrations of the points just made, and for use in what follows. In Sections 3, 4, we shall discuss only complete preference orderings, without always repeating the adjective.

3. Representation of a continuous preference ordering.

Intuitively, one would call a preference ordering continuous if a small change in any prospect can not drastically change the position of that prospect in the ranking of all other prospects. Starting from a sharp definition of this concept, Debreu [1959, Section 4,6] has shown conditions under which a continuous preference ordering can be represented by a continuous utility function.* In subsection 3* we show that the definition used by Debreu is logically equivalent to the following one.

The notion of a "small" change in a prospect can be made precise by assuming a given distance function in the prospect space. This is a function $d(x, y)$, defined for all pairs (x, y) of points in \mathcal{X} , with the following properties usually associated with a distance:

$$(3.1) \quad \begin{cases} d(x, y) = d(y, x) \geq 0 & \text{for all } x, y, \\ d(x, y) = 0 & \text{implies } x = y, \\ d(x, z) \leq d(x, y) + d(y, z) & \text{for all } x, y, z. \end{cases}$$

* See also Wold [1943].

We shall call the preference ordering \succsim continuous on \mathcal{X} if

$$(3.2) \left\{ \begin{array}{l} \text{for any } x, y \text{ of } \mathcal{X} \text{ such that } x \succ y, \text{ there exists a number } \delta > 0 \text{ such that} \\ \text{(a) } z \succ y \text{ for all } z \text{ in } \mathcal{X} \text{ such that } d(z, x) \leq \delta. \\ \text{(b) } x \succ w \text{ for all } w \text{ in } \mathcal{X} \text{ such that } d(y, w) \leq \delta. \end{array} \right.$$

(In particular, this is the case if all prospects in \mathcal{X} are equivalent). The same continuity concept may be obtained from many, but not from all, different choices of the distance function. We now have

Proposition 1 [Debreu, 1959]. A continuous³ complete preference ordering \succsim defined on a connected subset⁴ \mathcal{X} of n -dimensional Euclidean space E^n (n finite) can be represented by a utility function $u(x)$ defined and continuous³ in \mathcal{X} .

Not every conceivable preference ordering is continuous. If any increase in this year's food supply, however small, is deemed preferable to any increase in next year's food supply, however large, we have an example of the discontinuous lexicographic ordering.

If $u(x)$ is a continuous representation of \succsim , and if ϕ is any

³ Continuity of \succsim and of $u(x)$ is defined using the same distance function, for instance $d(x, y) = \max |x_i - y_i|$, if x_i , $i = 1, \dots, n$, are the coordinates of x . While this distance function depends on the units of measurement of the amounts x_i , $i = 1, \dots, n$, the continuity concept is independent of these units.

⁴ Depending on the interpretation, the prospect space \mathcal{X} may be the set of all points x with all coordinates $x_i \geq 0$, or any other representation of the range of alternative prospects suitable in a given problem. \mathcal{X} is called (arcwise) connected if any two points of \mathcal{X} can be connected by a continuous curve contained in \mathcal{X} .

continuous increasing function defined for all values assumed by $u(x)$ on \mathcal{X} , then

$$(3.3) \quad u^*(x) \equiv \varphi(u(x))$$

is likewise a continuous representation of \succsim . Conversely, if $u(x)$ and $u^*(x)$ are two continuous representations of \succsim , then such a function φ exists for which (3.3) holds. Therefore, only the notion of higher or lower among the levels of $u(x)$ has significance, not the numerical values $u(x)$ themselves or the differences thereof. In particular, even if \succsim should possess a differentiable representation $u(x)$, there is no intrinsic meaning in the "marginal utility" $\frac{\partial u}{\partial x_i}$ of any commodity. This is often expressed by the statement that $u(x)$ is an ordinal, not a cardinal utility. However, even if $u(x)$ is only ordinal, for given units of commodities i, j , the ratio

$$(3.4) \quad \frac{\partial u^*(x)}{\partial x_i} \bigg/ \frac{\partial u^*(x')}{\partial x_j}$$

of two "marginal utilities" in the same point ($x = x'$), or in two equivalent points ($x \sim x'$), is invariant. That is, the ratio (3.4) is independent of the choice of a differentiable φ in (3.3), hence is a quantity meaningful in terms of the given ordering \succsim .

By suitable choice of φ in (3.3) one can make the range $u^*(\mathcal{X})$ of $u^*(x)$ coincide with any finite nondegenerate interval, including the left and/or right endpoint depending on whether \mathcal{X} contains a worst and/or best element of \succsim . $u^*(\mathcal{X})$ can be unbounded from below and/or above only if no worst and/or best element exists.

3* Equivalence of two definitions of continuity of an ordering. The definitions to be compared are:

D If $\lim_{n \rightarrow \infty} y_n = y$ and $x \preceq y_n \preceq z$ for all n , then $x \preceq y \preceq z$.

D' If $y \succ x$ there exists $\delta > 0$ such that

(a) $d(y, w) \leq \delta$ implies $w \succ x$

(b) $d(w, x) \leq \delta$ implies $y \succ w$

Assume "not D." Then there exists y_n with $x \preceq y_n \preceq z$ for all n but either $\lim_{n \rightarrow \infty} y_n = y \succ x$ or $z \succ y$. Taking the case $y \succ x$, we choose δ in D' such that $d(y, y_n) \leq \delta$ implies $y_n \succ x$, and N in the definition of limit such that $d(y, y_N) \leq \delta$. Then $y_N \succ x \preceq y_N$, a contradiction. The case $z \succ y$ is similar.

Assume next "not D'," and take $\delta_n = \frac{1}{2^n}$. Then, though $y \succ x$, there exists either a sequence y_n such that $d(y, y_n) \leq \delta_n$ but $x \preceq y_n$, or a sequence x_n such that $d(x_n, x) \leq \delta_n$ but $x_n \succeq y$. By D, both cases imply $x \preceq y$, contradicting $y \succ x$.

Two statements such that the negation of either contradicts the other are equivalent.

4. "Cardinal" representation in the presence of independent sets of commodities

The problem of deriving special forms for a utility function from assumptions about independence in groups of commodities has been studied by several authors, including Leontief [1947 a, b] and Samuelson [1948, Ch. VII]. We shall follow Debreu [1960] because he avoids assumptions of differentiability of the utility function that seem unrelated to the essence of the problem.

To illustrate the independence concept to be defined more sharply below, one may wish to assume that preferences between food bundles are independent of the amounts of clothing and of other commodities consumed, and similarly for preferences between clothing bundles, etc.; furthermore that preferences between food-and-clothing bundles are independent of the amount of other commodities consumed, etc.

In this Section we shall consider only the case of a partition of the list S into three independent sublists, P, Q, R , indicating at the end of the subsection 2* why a similar result holds for a partition into more than three sublists. Let $x = (x_P, x_Q, x_R)$ denote a commodity bundle extending over the entire list S , with $x_P = (x_{P1}, \dots, x_{Pn_P})$ extending over P , etc. Let \succsim denote a preference ordering on the space \mathcal{X} of all bundles x such that x_P is in a given space \mathcal{X}_P , x_Q in \mathcal{X}_Q , x_R in \mathcal{X}_R . We shall call \succsim sensitive in P if there exist x_P, y_P, x_Q, x_R such that

$$(4.1) \quad (x_P, x_Q, x_R) \succ (y_P, x_Q, x_R) .$$

To express the required independence assumptions we use an arbitrary but fixed reference bundle

$$(4.2) \quad x^0 = (x_P^0, x_Q^0, x_R^0)$$

to define six orderings induced by \succsim on subspaces of \mathcal{X} . To be denoted $\succsim_P^0, \succsim_Q^0, \succsim_R^0, \succsim_{P,Q}^0, \succsim_{Q,R}^0, \succsim_{P,R}^0$, all of these orderings are derived from \succsim , along the following lines:

$$(4.3) \quad \begin{cases} x_P \succ_P^0 y_P & \text{means } (x_P, x_Q^0, x_R^0) \succ (y_P, x_Q^0, x_R^0), \\ (x_P, x_Q) \succ_{P,Q}^0 (y_P, y_Q) & \text{means } (x_P, x_Q, x_R^0) \succ (y_P, y_Q, x_R^0), \text{ etc.} \end{cases}$$

In general, these orderings depend on the reference bundle x^0 , in the sense that \succ_P^0 depends on (x_Q^0, x_R^0) , etc., and that $\succ_{P,Q}^0$ depends on x_R^0 , etc. The required independence assumptions will say that in the case of five out of the six orderings this dependence-in-principle is not a dependence-in-fact. We can now state

Proposition 2 [Debreu, 1960, 3 sublists only, modified]

Let \succ be a continuous preference ordering of all commodity bundles

$x = (x_P, x_Q, x_R)$ such that x_P, x_Q, x_R belong to spaces $\mathcal{X}_P, \mathcal{X}_Q, \mathcal{X}_R$,

which are connected subsets of Euclidean spaces of n_P, n_Q, n_R dimensions,

respectively. Let \succ be sensitive in each of P, Q, R , and let

$\succ_P^0, \succ_Q^0, \succ_R^0, \succ_{P,Q}^0, \succ_{Q,R}^0$ (as defined above) be independent of x^0 .

Then there exist functions $u^*(x_P), v^*(x_Q), w^*(x_R)$, defined and continuous on $\mathcal{X}_P, \mathcal{X}_Q, \mathcal{X}_R$, respectively, such that \succ is represented by

$$(4.4) \quad u^*(x) = u^*(x_P) + v^*(x_Q) + w^*(x_R).$$

This representation is unique up to a linear transformation

$$(4.5) \quad \begin{aligned} u^*(x_P) &= \beta_P + \gamma u^*(x_P), \quad v^*(x_Q) = \beta_Q + \gamma v^*(x_Q), \quad w^*(x_R) = \\ &= \beta_R + \gamma w^*(x_R), \quad \gamma > 0. \end{aligned}$$

A similar proposition holds for any partitioning of S into four or more independent sublists. However, for a partitioning into only two sublists the statement of the proposition is not valid.

In principle, the representation (4.4) is still ordinal. That is, any function $u'(x)$ obtained from $u^*(x)$ by (3.3) is likewise a continuous representation of \succsim . However, unless ϕ happens to be linear, the representation $u'(x)$ cannot be written simply as a sum of functions each depending on one of the bundles x_P, x_Q, x_R only, as $u^*(x)$ is written in (4.4). It is only in this limited sense that the representation by $u^*(x)$ can be called cardinal. To argue from this that in the presence of at least three independent subsets of commodities one can meaningfully compare the utility differences between two pairs of prospects would definitely go beyond the assumptions made so far.

4*. Proof of Proposition 2. We shall follow the general ideas of Debreu's beautiful geometrical proof, and of the work of Blaschke and Bol [1938] on which it is based. We modify his reasoning in one respect in order to avoid making the assumption that the sixth induced ordering, $\succsim_{P,R}^0$, is also independent of x^0 . This minor strengthening will be found useful in the application we shall make in Section 7.

Since the Cartesian product

$$\mathcal{X} = \mathcal{X}_P \times \mathcal{X}_Q \times \mathcal{X}_R$$

is a connected subset of a Euclidean space of $n = n_P + n_Q + n_R$ dimensions, the premises of Proposition 1 are satisfied. Hence \succsim is represented by some continuous function

$$(4.6) \quad U(x) = U(x_P, x_Q, x_R)$$

defined on \mathcal{X} . Since an additive constant does not affect the representation, we shall anchor $U(x)$ by requiring

$$(4.7) \quad U(x^0) = 0.$$

The five induced orderings $\succ_P, \succ_Q, \succ_R, \succ_{P,Q}, \succ_{Q,R}$ (superscripts 0 have been dropped because these are now independent of x^0) are therefore represented by the continuous functions

$$(4.8) \quad \begin{cases} u(x_P) \equiv U(x_P, x_Q^0, x_R^0), & v(x_Q) \equiv U(x_P^0, x_Q, x_R^0), & w(x_R) \equiv U(x_P^0, x_Q^0, x_R), \\ W(x_P, x_Q) \equiv U(x_P, x_Q, x_R^0), & U(x_Q, x_R) \equiv U(x_P^0, x_Q, x_R), \end{cases}$$

respectively. Since the domains of all these functions are connected, the range of each is an interval. For three of the ranges we introduce the notations

$$(4.9) \quad \mathcal{U} = u(\mathcal{X}_P), \quad \mathcal{V} = v(\mathcal{X}_Q), \quad \mathcal{W} = w(\mathcal{X}_R).$$

Since \succ is sensitive in each of P, Q, R , none of the five intervals collapses to a point, and, by suitable choice of x^0 , one can ensure that the point

$$(4.10) \quad u(x_P^0) = v(x_Q^0) = w(x_R^0) = W(x_P^0, x_Q^0) = U(x_Q^0, x_R^0) = 0$$

is interior to all five ranges.

Let x, y be bundles such that

$$W(x_P, x_Q) = W(y_P, y_Q), \quad w(x_R) = w(y_R).$$

Then we have

$$(x_P, x_Q, x_R^0) \sim (y_P, y_Q, x_R^0), \quad (x_P^0, x_Q^0, x_R) \sim (x_P^0, x_Q^0, y_R),$$

and, since $\succ_{P,Q}, \succ_R$ are independent of the choice of x^0 ,

$$x = (x_P, x_Q, x_R) \sim (y_P, y_Q, x_R) \sim (y_P, y_Q, y_R) = y,$$

hence $U(x) = U(y)$. Therefore, there exists a function $F(W, w)$ having the

intervals $W(\mathcal{X}_P \times \mathcal{X}_Q)$ and \mathcal{W} as the domains of its variables W, w , and the interval $U(\mathcal{X})$ as its range, such that

$$(4.11) \quad U(x) = F(W(x_P, x_Q), w(x_R)) .$$

By (4.8), using again the independence of $\mathcal{X}_{P,Q}$, \mathcal{X}_R , F is (strictly) increasing in each of its arguments. Then, since $U(x)$ is continuous, F is continuous in each of its arguments and in both jointly.

To avoid repetition in the several similar arguments to follow, we announce in advance that all of the functions G , F^{-1} , g , f , H , h to be introduced below are likewise continuous and increasing on the intervals, or products thereof, over which these arguments range.

Similarly to (4.11), we have from reasoning involving $\mathcal{X}_{Q,R}$, \mathcal{X}_P , that

$$U(x) = G(u(x_P), U(x_Q, x_R))$$

Taking $x_R = x_R^0$ here and in (4.11), we have

$$F(W(x_P, x_Q), 0) = G(u(x_P), v(x_Q))$$

and, if F^{-1} is the inverse of $F(W, 0)$,

$$W(x_P, x_Q) = F^{-1}\left(G(u(x_P), v(x_Q))\right) \equiv g(u(x_P), v(x_Q)),$$

say. Therefore

$$(4.12) \quad U(x) = F\left(g(u(x_P), v(x_Q)), w(x_R)\right)$$

and, symmetrically,

$$(4.13) \quad U(x) = G\left(u(x_P), f(v(x_Q), w(x_R))\right) .$$

We shall study level curves of the function

$$(4.14) \quad F\left(g(u, v), w\right) = G\left(u, f(v, w)\right) \equiv H(u, v, w), \equiv H(t) ,$$

say, defined on the three-dimensional cell $\mathcal{T} \equiv \mathcal{U} \times \mathcal{V} \times \mathcal{W}$, of which the origin $o = (0, 0, 0)$ is an interior point. The ordering \mathcal{Z} on $\mathcal{X}_P \times \mathcal{X}_Q \times \mathcal{X}_R$ represented by $U(x)$ induces an ordering on \mathcal{T} , which we likewise denote by \mathcal{Z} , and which is represented by $H(t)$, where $t = (u, v, w)$.

In the plane $w = 0$ we arbitrarily select (see Figure 1) an indifference curve λ not passing through o , but close enough to o for all the intersection points sought in the following construction to exist. If λ intersects the u - and v -axes in $a = (u', 0, 0)$ and $b = (0, v', 0)$, respectively we have

$$(4.15) \quad a \sim b, \quad \text{implying} \quad g(u', 0) = g(0, v')$$

by taking $w = 0$ in the first member of (4.14). At most one intersection point exists in each case because $g(u, v)$ is increasing in each variable. Precisely one will exist if λ passes close enough to o , because of the continuity of $g(u, v)$.

It will save words to refer to two points s, t of \mathcal{T} as u -congruent if they differ only in their u -coordinate,

$$s = (u^{(1)}, v, w), \quad t = (u^{(2)}, v, w).$$

Similarly we shall speak of v - and w -congruence.

We find $c \equiv (u', v', 0)$, v -congruent to a , u -congruent to b , and draw through c an indifference curve λ in the plane $w = 0$, which intersects the u -axis in $a' \equiv (u'', 0, 0)$, the v -axis in $d \equiv (0, v'', 0)$. In particular,

$$(4.16) \quad c \sim a' \quad \text{implies} \quad g(u', v') = g(u'', 0).$$

Finally we find $c' \equiv (u'', v', 0)$, u -congruent to a' , v -congruent to b , and $d' \equiv (u', v'', 0)$, u -congruent to d , v -congruent to a .

We now wish to prove that $d' \sim c'$. Since Proposition 2 does not hold for a partitioning with only two sublists, we shall need to go into the third dimension to prove this.

On the indifference curve η through d in the plane $u = 0$, we find $b'' = (0, v', w')$, w -congruent to b . Then

$$(4.17) \quad d \sim b'', \quad \text{implying} \quad f(v'', 0) = f(v', w')$$

by the second member of (4.14). Finally we find $o'' = (0, 0, w')$ on the w -axis, v -congruent to b'' , and $a'' = (u', 0, w')$, u -congruent to o'' , w -congruent to a . Then, by taking $w = w'$ in the first member of (4.14), we see that (4.15) in its turn implies $a'' \sim b''$. (In fact the indifference curves \mathcal{K} and \mathcal{K}' are point-by-point w -congruent). Hence $c \sim d \sim b'' \sim a''$, and therefore

$$(4.18) \quad c \sim a'', \quad \text{implying} \quad f(v', 0) = f(0, w') .$$

The second round of the construction is similar to the first. It employs the points $a''' = (u'', 0, w)$, u - and w -congruent to o'' and a' , respectively, and $c'' = b''' = (u', v', w')$, u -, v - and w -congruent to b'' , a'' and $c = b'$, respectively. We have

$$\left. \begin{array}{ll} (4.17) & \text{implies } d' \sim b''' \\ (4.16) & \text{implies } c'' \sim a''' \\ (4.18) & \text{implies } a''' \sim c' \end{array} \right\} \quad \text{so } d' \sim c' .$$

Hence d' and c' are on the same indifference curve μ in the plane $w = 0$.

The rectangle $acc'a'$ has the following characteristics relative to the curves \mathcal{K} , λ , μ :

incidence				congruence type of			
a	c	c'	a'	a, a'	c, c'	a, c	$a'c'$
is on \mathcal{K}	λ	μ	λ	is u	u	v	v

We shall call such a rectangle inscribed in the curves \mathcal{K} , λ , μ . Since the origin could have been chosen anywhere in \mathcal{J} , we have found the following result, illustrated in Figure 2.

Result 1: If three indifference curves \mathcal{K} , λ , μ possess an inscribed rectangle $acc'a'$ then \mathcal{K} , λ , μ possess adjoining inscribed rectangles $bdd'b'$, $b' = c$, and $eff'e'$, $f' = a$, provided only that the intersection points required by their construction exist.

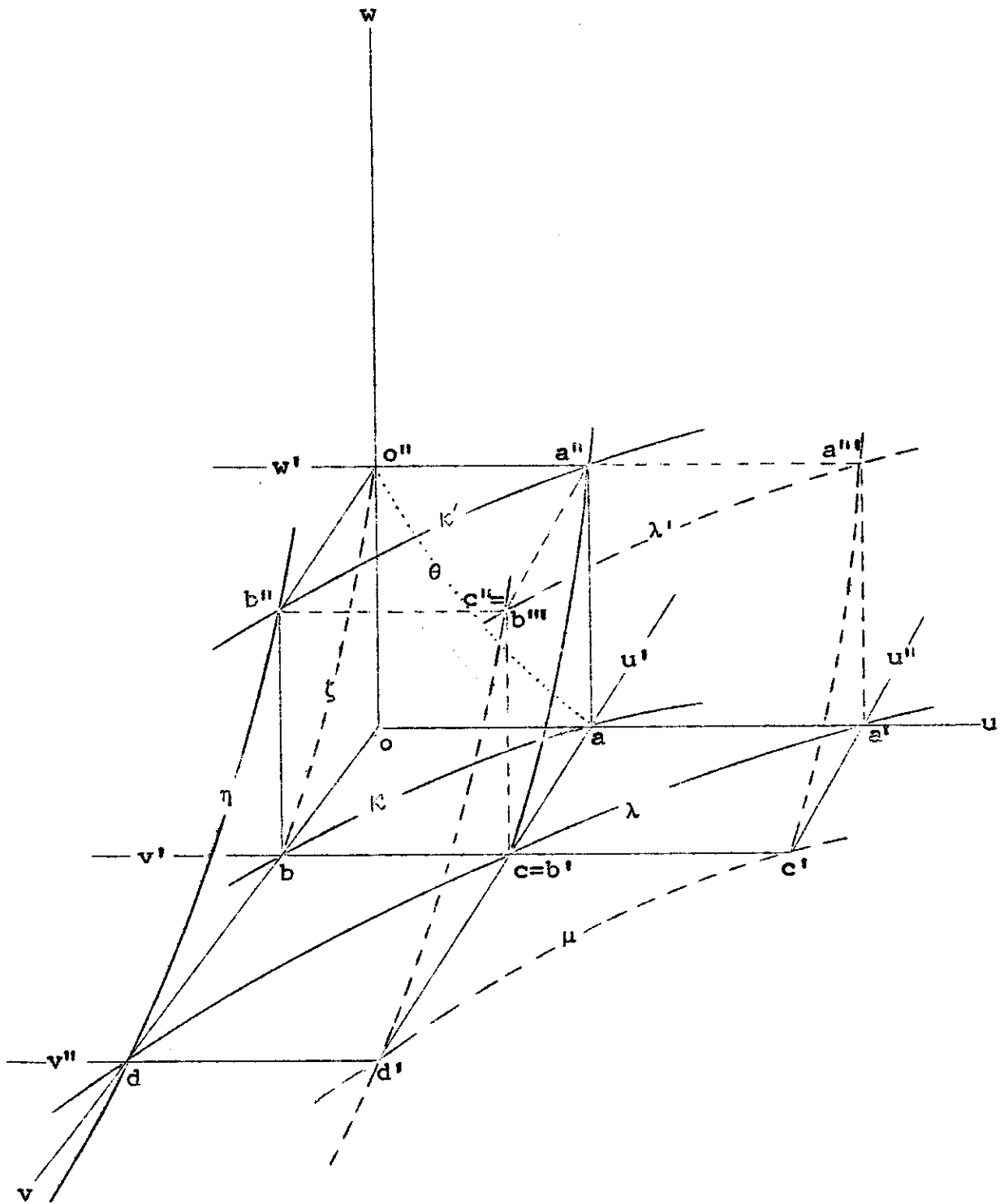


Figure 1

The remainder of the proof is based on the "textile geometry" of Blaschke and Bol. On any three indifference curves χ , λ , μ one can construct a sequence of such rectangles as indicated in Figure 2, going as far in both directions as the intervals \mathcal{U} and \mathcal{V} permit. If there should be an infinite sequence of such rectangles inscribed in χ , λ , μ , such a sequence cannot have a point of accumulation t' in \mathcal{T} , because by the continuity of $H(t)$ such a point would belong to each of χ , λ , μ , which is a contradiction. Hence if \mathcal{U} and \mathcal{V} are bounded, an infinite sequence of inscribed rectangles can only have an accumulation point on the boundary of \mathcal{T} .

A second sequence of rectangles can be inscribed in λ , μ , v if v contains, for instance, the point g , u -congruent to c' and v -congruent to f' . In this way the intersection of \mathcal{T} with the plane $w = 0$ is covered by rectangles inscribed in a sequence of indifference curves $\dots, \chi, \lambda, \mu, v, \dots$, except possibly for uncovered margins near the endpoints (if finite) of \mathcal{U}, \mathcal{V} .

Furthermore, one can interpolate an indifference curve γ "between" χ and λ , say, by choosing p on eh (Figure 2) so that $q \sim r$; and drawing γ through q and r . This construction can be extended over the full length of χ and λ , repeated between λ and μ , etc. and possibly into the uncovered margins, and repeated again between χ and γ , etc.

Let \mathcal{U}' be the set of all u -coordinates $(0, u', u'', \dots)$ of vertices of inscribed rectangles occurring in this construction repeated indefinitely, \mathcal{V}' that of all v -coordinates. Then \mathcal{U}' is dense in $\mathcal{U}, \mathcal{V}'$ in \mathcal{V} . We assign new coordinates (u^*, v^*) to all points of $\mathcal{U}' \times \mathcal{V}'$ in the manner indicated in the margins of Figure 2. Then

$$(4.19) \quad u^* = \pi(u), \quad v^* = \varphi(v)$$

are continuous and increasing functions on \mathcal{U}' and \mathcal{V}' , respectively, for which

$$(4.20) \quad \pi(0) = \varphi(0) = 0$$

These functions are extended to \mathcal{U}, \mathcal{V} , while retaining these properties, by

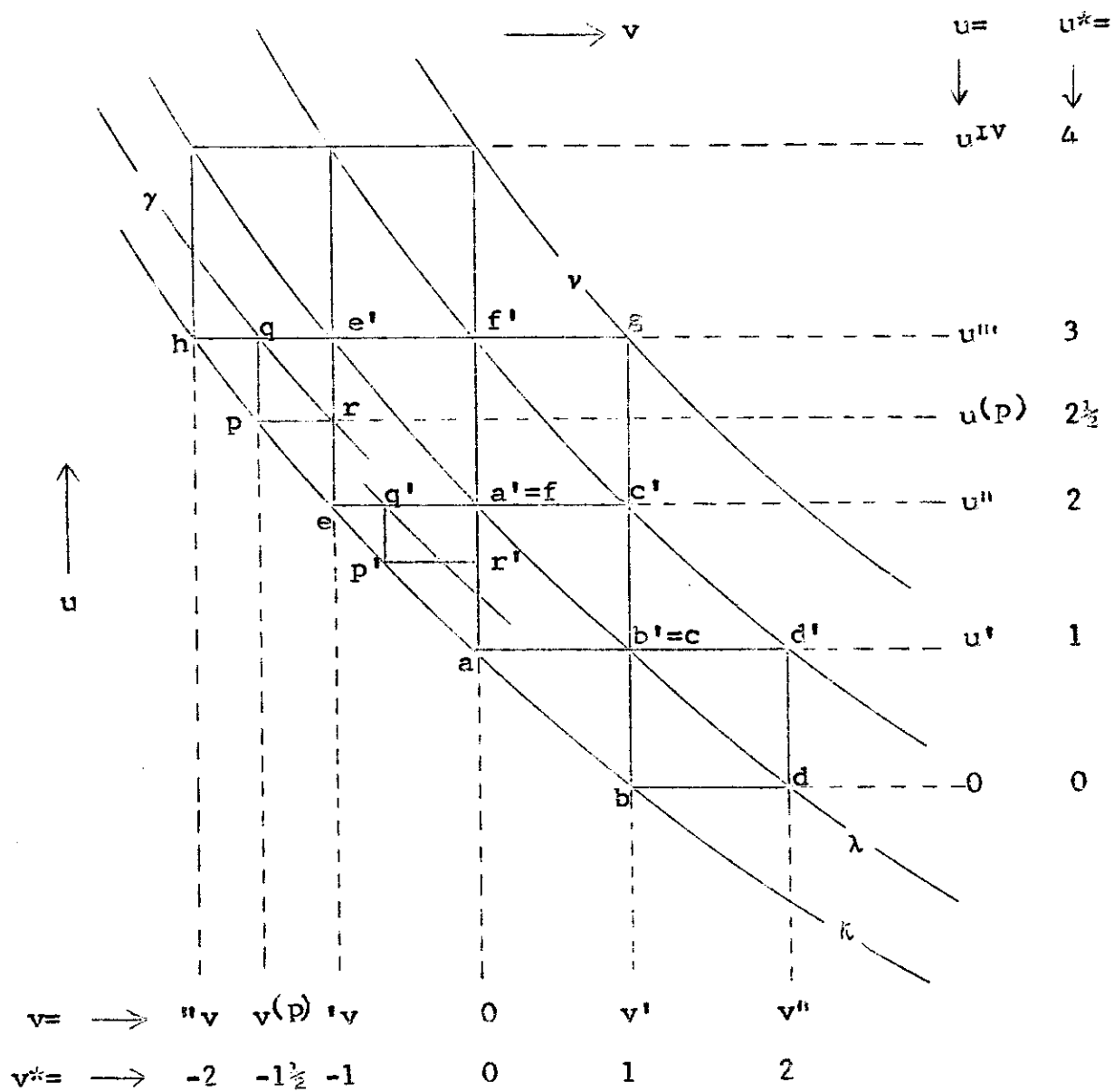


Figure 2

$$\begin{aligned} \pi(u) &\equiv \sup_{\substack{u' \leq u \\ u' \in \mathcal{U}'}} \pi(u') , & \varphi(v) &\equiv \sup_{\substack{v' \leq v \\ v' \in \mathcal{V}'}} \varphi(v') . \end{aligned}$$

It follows from the construction that, for any two equivalent points (u, v) , (u', v') of $\mathcal{U} \times \mathcal{V}$ one has

$$u^* + v^* = \pi(u) + \varphi(v) = \pi(u') + \varphi(v') = u'^* + v'^*$$

By continuity of $H(u, v, 0)$ this property likewise extends to $\mathcal{U} \times \mathcal{V}$.
Therefore, if we now define functions

$$u^*(x_P) = \pi(u(x_P)) , \quad v^*(x_Q) = \varphi(v(x_Q)) ,$$

the ordering \succeq , restricted to points of \mathcal{X} for which $w(x_R) = 0$, is represented by the continuous function

$$(4.21) \quad u^*(x_P) + v^*(x_Q) .$$

To extend this representation to all of \mathcal{X} , we return to Figure 1 to note that (4.18) also implies $b \sim o$. It follows that, had we carried out the preceding construction in the plane $u = 0$ instead of in $w = 0$, starting from ξ instead of from \mathcal{X} , we would have arrived at the same demarcation points o , b , d , ... on the v -axis, the same interpolated points, the same functions $\varphi(v)$, and hence the same function $v^*(x_Q)$, along with a similar function $w^*(x_R)$. It follows that \succeq is continuously represented on those points of \mathcal{X} for which $u^*(x_P) = 0$ by

$$(4.22) \quad v^*(x_Q) + w^*(x_R) .$$

We shall show that \succeq is represented on \mathcal{X} by the continuous function⁵

$$(4.23) \quad U^*(x) = u^*(x_P) + v^*(x_Q) + w^*(x_R) .$$

Consider two bundles $x = (x_P, x_Q, x_R)$, $x' = (x'_P, x'_Q, x'_R)$. By (4.12), (4.19),

⁵ There is an affinity between the following reasoning and a study by Arrow [1952].

their order depends only on the corresponding utility vectors

$$(4.24) \quad (u^*, v^*, w^*), \quad (u'^*, v'^*, w'^*), \quad \text{where } u^* = u^*(x_p), \text{ etc.}$$

Extending the usual notation $[z, z']$ for the interval $z \leq u \leq z'$ to

$$|[z, z']| = \begin{cases} [z, z'] & \text{if } z \leq z' \\ [z', z] & \text{if } z' < z, \end{cases}$$

we consider the set

$$\mathcal{S} = |[u^*, u'^*]| \times |[v^*, v'^*]| \times |[w^*, w'^*]|.$$

This is a block (rectangular parallelepiped) of which each vertex has each coordinate in common with one or the other of the points (4.24), as shown in Figure 3. On the points of each edge of \mathcal{S} the ordering \succ is (strictly) monotonic as indicated by arrows, because of the monotonicity of H in (4.14), and each such edge ordering is represented by the corresponding term in (4.23).

We must show that, for all possible dimensions of the block, the ordering \succ of each of the pairs $(a, h), (b, e), (c, f), (d, h)$ is represented by (4.23). For (a, h) this is already implied in the edge orderings $a \succ b \succ f \succ h$.

Assume first that $v^* \neq v'^*$. Then if either $u^* = u'^*$ or $w^* = w'^*$, the remaining comparisons are likewise settled by edge orderings. Assume therefore that the block \mathcal{S}^* is three dimensional. We shall make use of the equivalences

$$(4.24) \quad (u^*, v^*, w^*) \sim (u^* + p, v^* - p, w^*) \sim (u^* + p, v^* - p + q, w^* - q) \sim \dots,$$

implied in (4.14), (4.21), (4.22), as long as we make sure that all points so compared are in \mathcal{S}^* . This means that all points of any line segment in \mathcal{S}^* parallel to either κ or ζ are equivalent, and these equivalences are represented by (4.23).

As an example, Figure 4 shows the comparison of b and e . We intersect \mathcal{S}^* with a plane \mathcal{P} through b parallel to both κ and ζ . Since a, h are

on opposite sides of \mathcal{P} , the intersection is a two-dimensional convex polygon \mathcal{Q} with sides parallel to χ , ξ or θ . Now \mathcal{P} and hence \mathcal{Q} must intersect the broken line $h e d a$ in precisely one point k . Figure 5, drawn in \mathcal{P} , shows a broken line in \mathcal{Q} with a finite number of segments parallel to χ or ξ , connecting b and k . This establishes the equivalence of b and k , and its representation by (4.23). The comparison of k and e then is made through the edge orderings on $h e d a$, again represented by (4.23). In Figure 4, $b \sim k \succ e$. It is clear from the two-dimensionality of \mathcal{Q} and the condition on the slopes of its sides, that the above reasoning can be carried through regardless of the dimensions of \mathcal{S}^* , and of the pair of opposite vertices compared.

Finally, if $v^* = v'^*$, we first use (4.24) with either $p \neq 0$ or $q \neq 0$ to obtain

$$(u^{**}, v^{**}, w^{**}) \sim (u^*, v^*, w^*), \text{ say, with } v^{**} \neq v^*,$$

and continue from there with the above reasoning. This procedure is unavailable only if

$$\mathcal{I}^* = \mathcal{U}^* \times \mathcal{V}^* \times \mathcal{W}^*, \text{ where } \mathcal{U}^* = u^*(\chi_p) \text{ etc.,}$$

were to be bounded,

$$\mathcal{I}^* = [\underline{u}, \bar{u}] \times [\underline{v}, \bar{v}] \times [\underline{w}, \bar{w}],$$

and if we had

$$(u^*, v^*, w^*) = (u'^*, v'^*, w'^*) = \text{either } (\underline{u}, \underline{v}, \underline{w}) \text{ or } (\bar{u}, \bar{v}, \bar{w}).$$

In either of these cases equality implies equivalence, represented by (4.23).

Finally, to discuss the uniqueness of (4.23), we note first from (4.10), (4.20) that

$$u^*(x_p) = v^*(x_q) = w^*(x_r) = 0.$$

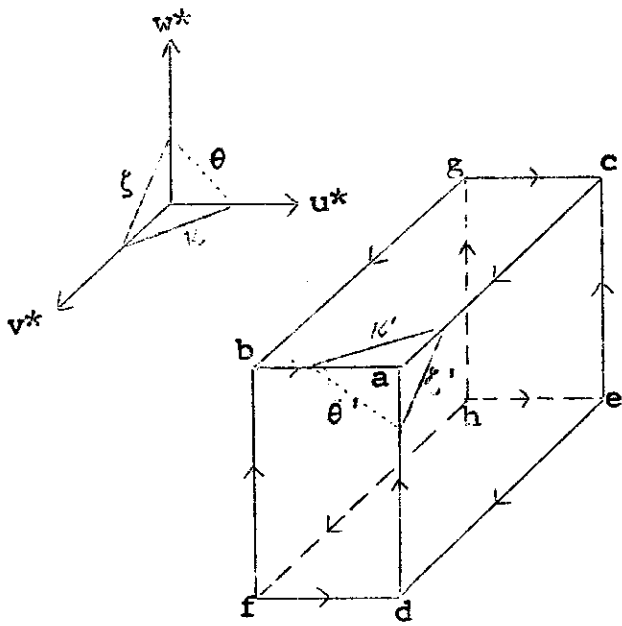


Figure 3

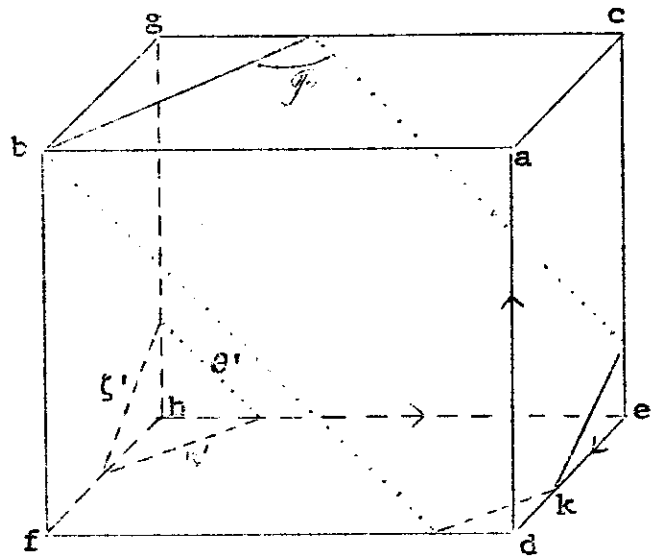


Figure 4

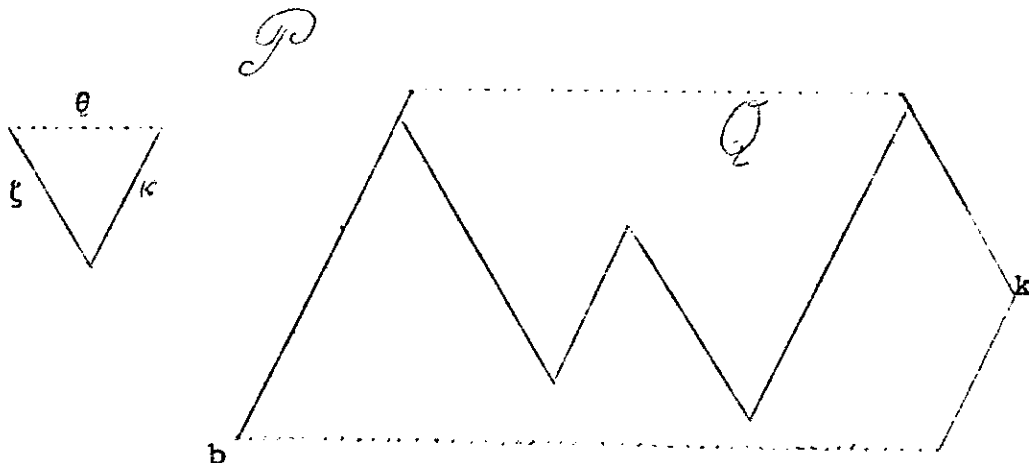


Figure 5

Now assume that γ is also represented by the continuous function

$$U'(x) = u'(x_P) + v'(x_Q) + w'(x_R) .$$

We define

$$\beta_P \equiv u'(x_P^0) , \quad \beta_Q \equiv v'(x_Q^0) , \quad \beta_R \equiv w'(x_R^0) ,$$

$$u''(x_P) \equiv u'(x_P) - \beta_P , \quad v''(x_Q) \equiv v'(x_Q) - \beta_Q , \quad w''(x_R) \equiv w'(x_R) - \beta_R .$$

Then there exists $h(u^*)$ such that, for all x in \mathcal{X} ,

$$u''(x_P) + v''(x_Q) + w''(x_R) = h(u^*(x_P) + v^*(x_Q) + w^*(x_R)) .$$

Inserting $x_R = x_R^0$, and thereafter $x_Q = x_Q^0$ or $x_P = x_P^0$ or both, we have

$$u'' + v'' = h(u^* + v^*) , \quad u'' = h(u^*) , \quad v'' = h(v^*) , \quad 0 = h(0) ,$$

hence

$$h(u^* + v^*) = h(u^*) + h(v^*) , \quad h(0) = 0 ,$$

for all (u^*, v^*) in $\mathcal{U}^* \times \mathcal{V}^*$.

This in turn implies

$$h(n u^*) = n h(u^*)$$

for all integer n and all u^* such that u^* and $n u^*$ are in the interval \mathcal{U}^* . Among continuous functions $h(u^*)$, this property is possessed only by the linear functions

$$h(u^*) = \gamma u^* ,$$

where $\gamma > 0$ because h is increasing. This establishes the transformation (4.5).

The proof of Proposition 2 is now complete. To indicate its extension to a partition of S into k sublists, $k > 3$, consider the case where there is a fourth sublist N in which γ is sensitive, and such that γ_N^0 , γ_{QN}^0

are independent of x^0 . Then by Proposition 2, $z_{P,Q,R}^0$ and $z_{Q,R,N}^0$ can be represented by

$$u^*(x_P) + v^*(x_Q) + w^*(x_R)$$

$$\text{and} \quad v^*(x_Q) = w^*(x_R) + z^*(x_N),$$

respectively (using if necessary a linear transformation to obtain the same functions $v^*(x_Q)$, $w^*(x_R)$ in the representations). We now repeat the part of the above proof beginning after (4.22), replacing

$$x_P, x_Q, x_R \quad \text{by} \quad x_P, (x_Q, x_R), x_N,$$

$$u^*, v^*, w^* \quad \text{by} \quad u^*, v^* + w^*, z^*,$$

a procedure which can be repeated an indefinite number of times.

5. Questions of interpretation and application.

Before going on to the discussion of possible representations of preference orderings over time, let us stop to ask ourselves whose preference is being studied. This question concerns the relevance of the analysis, and is independent of the logical connections between the properties of the ordering and the mathematical form of its representation. In regard to preference over time, the simplest interpretation of the orderings that have been studied most thus far is the normative one considered in Section 1. One looks at various possible preference orderings that may be adopted, by whatever decision process, for the planning of an economy with a constant population size. New problems arise if population is expected to grow indefinitely or to keep changing in other ways.

Another possible interpretation is that one wishes to study descriptively the preference ordering of an individual with regard to his life-time consumption

program, assuming that such an ordering is implicit in his decisions. For this interpretation the bequest motive needs to be considered as well. For applications of such a preference ordering, see Yaari [1964].

Finally -- the ultimate goal of a theory of preference over time for an economy with private wealth -- one may wish to examine whether an aggregate preference ordering over time can be imputed, on an "as if" basis, to a society of individual decision makers each guided by his own preference ordering over time.

In all these interpretations, normative or descriptive, the most intriguing problems arise from the fact that the future has a beginning but no discernible end. In contrast to this central problem, the question whether to use a discrete or a continuous time concept seems in the present state of knowledge primarily a matter of research tactics rather than of substance. So far the indications are that axiomatic analysis is somewhat simpler if one chooses discrete time. On the other hand, the maximization of a utility function of a given form under given technological constraints is often simpler with continuous time. We shall therefore here choose discrete time on the basis of expedience without further excuse or explanation.

6. Postulates concerning a preference ordering \succsim over time

We shall now adopt a set of five postulates about a preference ordering \succsim on a space ${}_1\mathcal{X}$ of programs, that is, of infinite sequences, denoted

$$(6.1) \quad {}_1x = (x_1, x_2, x_3, \dots),$$

of commodity bundles

$$(6.2) \quad x_t = (x_{t1}, x_{t2}, \dots, x_{tn})$$

associated with successive time periods $t = 1, 2, 3, \dots$. The program space ${}_1\mathcal{X}$ is the space of all such sequences, in which each bundle x_t is a point of the same (single-period) commodity space \mathcal{X} , referring to a list of commodities which is the same for all t .

The postulates are modeled after those used in two earlier studies by Koopmans [1960] and by Koopmans, Diamond and Williamson [1964]. The main difference is that the former studies postulated the existence of a continuous representation. In the present study, the postulates refer to a continuous ordering, and the proximate aim of the study is to derive the existence of a continuous representation. A second difference will be noted in connection with the third postulate.

It will be useful to employ brief notations for finite or infinite segments of the program sequence, as follows

$$(6.3) \quad {}_1x = (x_1, {}_2x) = (x_1, \dots, x_{t-1}, t^x) = ({}_1x_{t-1}, t^x)$$

In an infinite-dimensional space such as ${}_1\mathcal{X}$, the choice of the distance functions is crucial for the meaning of the continuity concept

implied in it. We shall adopt the function⁶

$$(6.4) \quad D({}_1x, {}_1y) \equiv \sup_t d(x_t, y_t),$$

where $d(x_t, y_t)$ is the distance between the t -th period installments x_t, y_t of the programs, $x, {}_1y$, according to the definition

$$(6.5) \quad d(x_t, y_t) \equiv \max_i |x_{ti} - y_{ti}|.$$

P_1 (Postulate 1, Continuity). The program space, \mathcal{X} is the space of all programs ${}_1x$ such that x_t is in a metric space \mathcal{X} for all t . The preference ordering \succsim on the program space ${}_1\mathcal{X}$ is complete, and is continuous with reference to the distance function (6.3). The (single-period) commodity space \mathcal{X} on which the program space ${}_1\mathcal{X}$ is based is a connected subset of the n -dimensional Euclidean space R^n .

P_2 (Sensitivity). There exist a program ${}_1x$ in ${}_1\mathcal{X}$ and a bundle y_1 in \mathcal{X} such that

$${}_1x = (x_1, x_2, x_3, \dots) \succ (y_1, x_2, x_3, \dots)$$

The first purpose of P_2 is to exclude the trivial case where all programs in ${}_1\mathcal{X}$ are equivalent. However, P_2 does more than that. It also excludes orderings in which the standing of any program ${}_1x$ relative to other programs is independent of the bundles x_t available in any specific period t

⁶ The symbol $\sup_t d_t$ denotes the largest of the numbers $d_t, t=1, 2, 3, \dots$, if there is a largest, or the smallest number not exceeded by any d_t if there is no largest. Such a number exists whenever \mathcal{X} is bounded (see footnote). If \mathcal{X} is unbounded we admit the possibility of $D({}_1x, {}_1y) = \infty$.

but dependent on the asymptotic behavior of x_t as t tends to infinity.⁷

Next we introduce two independence postulates, $P3'$ and $P3''$, of which the first will be maintained throughout this paper, whereas the second, maintained in most of the paper, will be dropped in Sections . For their formulation and analysis we employ an arbitrary but fixed reference program.

$$(6.6) \quad {}_1x^0 = (x_1^0, {}_2x^0) = (x_1^0, x_2^0, {}_3x^0),$$

to define five orderings, induced by \preceq on subspaces of ${}_1\mathcal{X}$, and denoted \preceq_1^0 , \preceq_2^0 , ${}_1\preceq_2^0$, ${}_3\preceq^0$, \preceq_2^0 , as follows:

$$(6.7) \quad \left\{ \begin{array}{ll} x_1 \preceq_1^0 y_1 & \text{means } (x_1, {}_2x^0) \preceq (y_1, {}_2x^0) \\ {}_2^x \preceq_2^0 {}_2^y & \text{means } (x_1^0, {}_2^x) \preceq (x_1^0, {}_2^y) \\ (x_1, x_2) {}_1\preceq_2^0 (y_1, y_2) & \text{means } (x_1, x_2, {}_3x^0) \preceq (y_1, y_2, {}_3x^0) \\ {}_3^x \preceq_3^0 {}_3^y & \text{means } (x_1^0, x_2^0, {}_3^x) \preceq (x_1^0, x_2^0, {}_3^y) \\ x_2 \preceq_2^0 y_2 & \text{means } (x_1^0, x_2, {}_3x^0) \preceq (x_1^0, y_2, {}_3x^0) \end{array} \right.$$

⁷ A simple example of such an ordering \preceq satisfying all postulates except $P2$ is that in which \mathcal{X} is one-dimensional and \preceq is represented by $\lim_{T \rightarrow \infty} \sup_{t \geq T} x_t$.

P3' (Limited Independence). The two orderings \tilde{x}_1^0 , \tilde{x}_2^0 are independent of the reference program ${}_1x^0$.

P3" (Extended Independence). The ordering ${}_1\tilde{x}_2^0$ is independent of ${}_1x^0$.

For convenient reference, we also introduce

P3 (Independence) Both P3' and P3" hold.

Whenever one or both of P3', P3" are assumed in what follows, the corresponding orderings will be denoted \tilde{x}_1 , \tilde{x}_2 , ${}_1\tilde{x}_2$. Note that ${}_1\tilde{x}_2$ would have been denoted $\tilde{x}_{1,2}$ in Section 4.

In the earlier studies referred to above, the implications of P3' were pursued at length, those of P3" only mentioned briefly. In this study, the emphasis is reversed.

Neither P3' nor P3" can be regarded as realistic. Between them, they will be found to preclude all complementarity between commodities available in different periods. P3' by itself will be seen to permit a limited complementarity among the utility levels to be associated with successive periods, but still no complementarity between individual commodities in different periods. P3 or P3' should therefore be looked upon as first approximations, made to facilitate explorations of the implications of the fourth postulate, the real objective of this paper.

P4 (Stationarity). There exists a first period bundle x_1^* in \mathcal{K} with the property that, whenever the programs

$${}_1x = (x_1^*, {}_2x) = (x_1^*, x_2, x_3, \dots)$$

$${}_1y = (x_1^*, {}_2y) = (x_1^*, y_2, y_3, \dots)$$

are such that ${}_1x \succcurlyeq {}_1y$, then the programs⁸

$${}_1z = (z_1, z_2, z_3, \dots) = (x_2, x_3, x_4, \dots) = {}_2x,$$

$${}_1w = (w_1, w_2, w_3, \dots) = (y_2, y_3, y_4, \dots) = {}_2y,$$

defined by $z_t = x_{t+1}$, $w_t = y_{t+1}$, $t = 1, 2, \dots$, are such that ${}_1z \succcurlyeq {}_1w$.

Before interpreting this postulate in less formal language, we note that, if one particular $x_1 = x_1^*$ in \mathcal{X} has this property, then by P3' every x_1 in \mathcal{X} has this property. Using this, P4 says that if two programs ${}_1x$, ${}_1y$ have a common first period bundle $x_1 = y_1$, then the programs ${}_1z$, ${}_1w$ obtained by deleting x_1 from ${}_1x$ and from ${}_1y$, respectively, and advancing the timing of all subsequent bundles by one period, are ordered in the same way as ${}_1x$, ${}_1y$.

It is worth emphasizing that in this statement, nothing is said or implied about the ordering of "then future" programs ${}_2x$, ${}_2y$ that may be applied after the first period has elapsed. That is, no question of consistency or inconsistency of orderings applicable at different points in time is raised.⁹

⁸ In the notations ${}_2x$, ${}_2y$ as used here, there is no longer a necessary connection between the presubscript of ${}_2x$ and the timing of the first installment x_2 of that program. That is, x_2 simply means the commodity bundle that happened to occur in the second period in the program ${}_1x$. In the program ${}_2x = {}_1z$, that same bundle occurs in the first period. With this understanding, the notations, ${}_1z$, ${}_1w$ will no longer be needed in what follows.

⁹ For a discussion of that question, see Strotz [195].

Only the ordering \succeq applying "now" is under discussion. Applied repeatedly, P4 merely says that the present ordering of two programs $(x_1, \dots, x_{t-1}, x_t)$ and $(x_1, \dots, x_{t-1}, y_t)$ that start to differ in a designated way only from some point t in time onward is independent of where $(t = 1, 2, \dots)$ that point falls.

The fifth and last postulate asserts, roughly, that the end result of an infinite sequence of improvements starting from some given program represents itself an improvement over that program. It will become clear below that, if all but a finite number of the improvements affect the program in only a finite number of future periods, this assertion is already implied in P1, 3', 4. It will therefore suffice to refer only to a sequence of improvements made to successive bundles in the program, taken one at a time.

P5 (Monotonicity). If x_1, y_1 are programs such that for all $t = 1, 2, \dots$,

$(x_1, x_2, \dots, x_{t-1}, y_t, y_{t+1}, y_{t+2}, \dots) \succeq (x_1, x_2, \dots, x_{t-1}, x_t, y_{t+1}, y_{t+2}, \dots)$,
then $y_1 \preceq x_1$. The same statement holds with \succeq replacing \preceq .

It can be shown that, given the other postulates, P5 is implied in a stronger postulate, used in a previous publication [Koopmans, 1960], which simply says that there exist in X_1 a best and a worst program. There is some interest in avoiding that stronger statement.

On the basis of the postulates set out, we wish to construct a representation of \succeq on the entire program space X_1 , or on as large a part of it as we can. Our strategy will be first to find such representations on suitable subspaces of X_1 .

7. Representation of \succsim on any
subspace of ultimately identical programs.

Since the space ${}_1\mathcal{X}$ is infinite-dimensional, Proposition 1 cannot be directly applied to the ordering \succsim given on it. For this reason, we shall in the present section study \succsim on the subspace ${}_1\mathcal{X}_T^0$ of all programs of the form

$$(7.1) \quad {}_1x = ({}_1x_T, {}_{T+1}x^0) .$$

Since programs in this subspace differ only in the segments ${}_1x_T$, the ordering \succsim on ${}_1\mathcal{X}$ restricted to the subspace ${}_1\mathcal{X}_T^0$ becomes an ordering of sequences ${}_1x_T$ of length T , on the space ${}_1\mathcal{X}_T$. We shall denote this ordering by ${}_1\mathcal{X}_T^{0*}$. In Section 7* we shall prove

Result 2. For all T , the ordering ${}_1\mathcal{X}_T^{0*}$ is independent of ${}_{T+1}x^0$, and is represented by a function of the form

$$(7.2) \quad U_T({}_1x_T) = u(x_1) + \alpha u(x_2) + \dots + \alpha^{T-1} u(x_T) , \quad 0 < \alpha < 1 .$$

Here $u(x)$ is a continuous function defined on \mathcal{X} , and both α and $u(x)$ are independent of T . The proof proceeds through a succession of statements which we label (R2a), (R2b), ..., recording in each case the postulates used in the proof, unless all five are used.

(R2a; P3', 4) The ordering ${}_t\mathcal{X}^0$ of sequences ${}_tx$, defined by restricting \succsim to the set of programs $({}_1x_{t-1}^0, {}_tx)$ is independent of ${}_1x^0$ and of t .

(R2b; P3', 4) \succsim_t^0 is independent of ${}_1x^0$ and of t .

(R2c; P3, 4) ${}_{t-1}\tilde{z}_t^0$ is independent of ${}_1x^0$ and of t .

(R2d) ${}_1\tilde{z}_T^0$ is represented by a continuous function of the form

$$(7.3) \quad U_T({}_1x_T) = u_1(x_1) + u_2(x_2) + \dots + u_T(x_T),$$

which is independent of ${}_1x^0$, and unique up to a linear transformation similar to (4.5). The proof of this statement rests on Proposition 2.

(R2e) One can choose the $u_1(x_1)$ in (7.3) in such a way that (7.2) holds with $\alpha > 0$, where α is unique, and where $u(x)$ is unique up to a linear transformation

$$(7.4) \quad u^*(x) = \beta + \gamma u(x).$$

(R2f) $\alpha < 1$.

7* Proof of Result 2. Clearly the continuity of \tilde{z} entails the continuity of all restricted orderings derived from it.

(R2a). P3' allows us to write

$$(7.5) \quad {}_1\tilde{z}_1^0 = {}_1\tilde{z}_1, \quad {}_2\tilde{z}^0 = {}_2\tilde{z}.$$

Using the symbol \Leftrightarrow to denote logical equivalence, these statements are made explicit by

$$(7.6) \quad \text{for all } {}_2x^*, x_1, y_1, (x_1, {}_2x^0) \succeq (y_1, {}_2x^0) \Leftrightarrow (x_1, {}_2x^*) \succeq (y_1, {}_2x^*)$$

$$(7.7) \quad \text{for all } x_1^*, {}_2x, {}_2y, (x_1^0, {}_2x) \succeq (x_1^0, {}_2y) \Leftrightarrow (x_1^*, {}_2x) \succeq (x_1^*, {}_2y)$$

In particular, choosing for x_1^* in (7.7) the x_1^* occurring in P4, we have from P4

(7.8) for all $x_1, 2^x, 2^y, (x_1^0, 2^x) \succeq (x_1^0, 2^y) \Leftrightarrow (x_1, 2^x) \succeq (x_1, 2^y) \Leftrightarrow 2^x \succeq 2^y$,

an implication which can be applied once more to give

$$(x_1^0, x_2^0, 3^x) \succeq (x_1^0, x_2^0, 3^y) \Leftrightarrow (x_2^0, 3^x) \succeq (x_2^0, 3^y) \Leftrightarrow 3^x \sim 3^y, \text{ etc.}$$

These results are summarized in

$$(7.9) \quad {}_t\tilde{\succ}^0 = {}_t\tilde{\succ} = \dots = {}_2\tilde{\succ} = \tilde{\succ}, \quad t = 2, 3, \dots,$$

keeping in mind the notational practice explained in footnote 10.

(R2b). From (7.8) and (7.6),

$$\begin{aligned} (x_1^0, x_2, 3^{x^0}) \succeq (x_1^0, y_2, 3^{x^0}) &\Leftrightarrow (x_2, 3^{x^0}) \succeq (y_2, 3^{x^0}) \Leftrightarrow (x_2, 3^{x^0}) \succeq (x_2, 3^{x^*}) \succeq (y_2, 3^{x^*}) \Leftrightarrow \\ &\Leftrightarrow (x_1^*, x_2, 3^{x^*}) \succeq (x_1^*, y_2, 3^{x^*}). \end{aligned}$$

This reasoning and its repetition yield

$$(7.10) \quad \tilde{\succ}_t^0 = \tilde{\succ}_t = \dots = \tilde{\succ}_2 = \tilde{\succ}_1, \quad t = 2, 3, \dots$$

(R2c) We now bring in $P3''$, written as $1\tilde{\succ}_2^0 = 1\tilde{\succ}_2$.

Together with (7.8), this implies

$$\begin{aligned} (x_1^0, x_2, x_3, 4^{x^0}) \succeq (x_1^0, y_2, y_3, 4^{x^0}) &\Leftrightarrow (x_2, x_3, 4^{x^0}) \succeq (y_2, y_3, 4^{x^0}) \Leftrightarrow \\ &\Leftrightarrow (x_2, x_3, 4^{x^*}) \succeq (y_2, y_3, 4^{x^*}) \Leftrightarrow (x_1^*, x_2, x_3, 4^{x^*}) \succeq (x_1^*, y_2, y_3, 4^{x^*}). \end{aligned}$$

Since this can again be repeated, we have

$$(7.11) \quad {}_{t-1}\tilde{\succ}_t^0 = {}_{t-1}\tilde{\succ}_t = \dots = {}_2\tilde{\succ}_3 = 1\tilde{\succ}_2.$$

(R2d) We consider $1\tilde{\succ}_T^0$, and note that $\tilde{\succ}_t^0$, $t = 1, \dots, T$ and

${}_{t-1}\tilde{\succ}_t^0$, $t = 2, \dots, T$, are all independent of $1\tilde{\succ}^0$. By P2, $\tilde{\succ}_1$ permits $x_1 \tilde{\succ}_1 y_1$.

By (7.10) similar statements apply to λ_t , $t = 2, \dots, T$. Thus the premises of Proposition 2, extended to T independent sublists as described at the end of Subsection 2*, are satisfied, and ${}_1\lambda_T^0$ is represented on ${}_1\mathcal{X}_T$ by a continuous function of the form (7.3), unique up to a linear transformation. Since this function is constructed entirely from orderings λ_t , λ_{t+1} that are independent of ${}_1x^0$, the representation (7.3), and hence the ordering ${}_1\lambda_T^0$, are independent of ${}_{t+1}x^0$. We therefore write ${}_1\lambda_T$ from here on.

(R2e) By (7.8) and (7.3), ${}_2\lambda_T$ is represented on ${}_1\mathcal{X}_T$ by either of the functions

$$\begin{aligned} &u_2(x_2) + u_3(x_3) + \dots + u_T(x_T), \\ &u_1(x_2) + u_2(x_3) + \dots + u_{T-1}(x_T). \end{aligned}$$

It follows, along the lines of the uniqueness proof for Proposition 2, that for all x in \mathcal{X}

$$u_{t+1}(x) = \beta_t + \alpha_t u_t(x), \quad t = 1, \dots, T-1.$$

Since we are free to choose each $u_t(x)$, $t = 2, \dots, T-1$, so as to have $\beta_t = 0$ for all t , (7.2) results, with $u(x) = u_1(x)$.

(R2f) By P2, there exist bundles \underline{x} , \bar{x} in \mathcal{X} such that $\underline{x} \prec \bar{x}$, and hence

$$u(\underline{x}) < u(\bar{x}).$$

Since \mathcal{X} is connected, there exists (see Figure 6) a continuous curve $x(\xi)$ defined for $0 \leq \xi \leq 1$ connecting \underline{x} with \bar{x} ,

$$x(0) = \underline{x}, \quad x(1) = \bar{x}$$

Let ξ_0 be the maximum value of ξ for which $u(x(\xi)) \leq u(\underline{x})$. By the continuity of $u(x)$ and of $x(\xi)$, such a maximum exists, and, if we write $\underline{x} = x(\xi_0)$,

$$(7.12) \quad u(\underline{x}) = u(x) < u(x(\xi)) \quad \text{for all } \xi > \xi_0.$$

Denote by

$$\text{con } \underline{x} = (\underline{x}, \underline{x}, \underline{x}, \dots)$$

the program in which $x_t = \underline{x}$ for all t . Since $\underline{x} \prec_1 \bar{x}$, we have

$$\text{con } \underline{x} = (\underline{x}, \text{con } \underline{x}) \prec (\bar{x}, \text{con } \underline{x}).$$

Hence, by the continuity of γ , there exists $\delta > 0$ such that

$$(7.13) \quad D(\text{con } \underline{x}, \text{con } \underline{x}) \leq \delta \quad \text{implies} \quad \text{con } \underline{x} \prec (\bar{x}, \text{con } \underline{x})$$

By (7.12) and the continuity of $x(\xi)$ one can find $x' = x(\xi')$, $\xi' > \xi_0$, near enough to \underline{x} to have $d(x', \underline{x}) \leq \delta$, hence also

$$(7.14) \quad D(\text{con } x'_T, \text{con } \underline{x}) \leq \delta,$$

if $\text{con } x'_T$ denotes $\text{con } x'_T$ with $x_t = x'$ for $t = 1, \dots, T$.

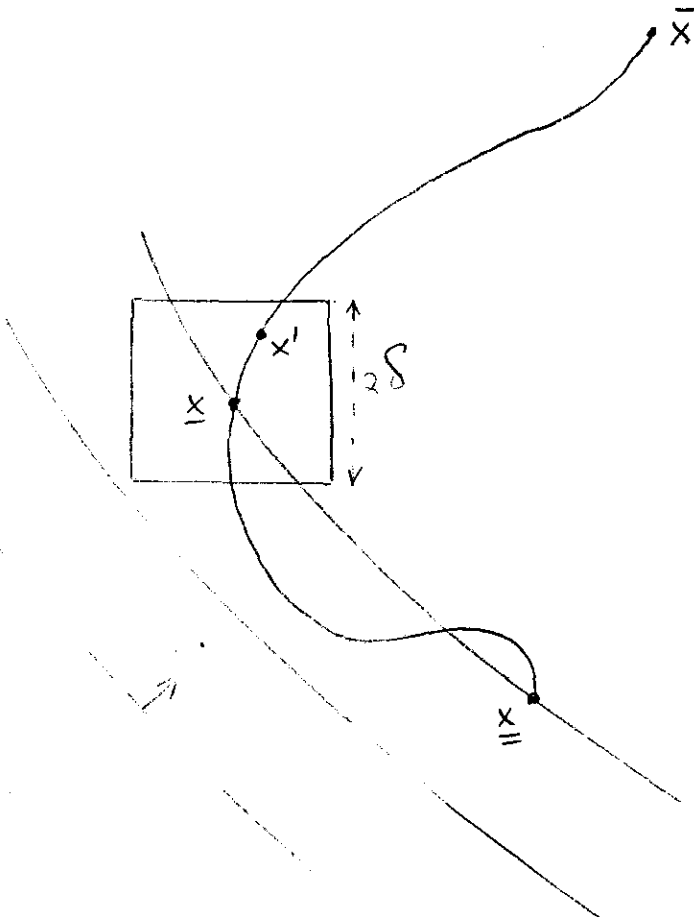


Figure 6

We then have, by (7.2), (7.13), (7.14), for all T ,

$$\sum_{t=1}^T \alpha^{t-1} u(x') < u(\bar{x}) + \sum_{t=2}^T \alpha^{t-1} u(\underline{x}),$$

or

$$u(\bar{x}) > u(\underline{x}) + \sum_{t=1}^T \alpha^{t-1} (u(x') - u(\underline{x})).$$

Since, by (7.12), $u(x') - u(\underline{x}) > 0$, this can be true for all T only if $\alpha < 1$.

8. Representation of Σ on the space of ultimately constant programs.

In this section we choose a favorable ground on which to face the infinite horizon by first restricting ourselves to the space ${}_{\text{con}}\mathcal{X}$ of constant programs

$$(8.1) \quad {}_{\text{con}}x \equiv (x, x, x, \dots),$$

that is, of programs ${}_1x$ for which $x_t = x$ for all t .

The points of ${}_{\text{con}}\mathcal{X}$ are in a one-to-one correspondence

$$(8.2) \quad {}_{\text{con}}x \longleftrightarrow x$$

to those of \mathcal{X} . Because, for all x, x' in \mathcal{X} ,

$$(8.3) \quad D({}_{\text{con}}x, {}_{\text{con}}x') = d(x, x'),$$

this correspondence preserves the distance function, and therewith the continuity concept. Moreover, if x, y are bundles of \mathcal{X} such that $y \preceq_1 x$, then, by R2b and P5,

$$(8.4) \quad {}_{\text{con}}y \preceq (x, {}_{\text{con}}y) \preceq \dots \preceq ({}_{\text{con}}x_T, {}_{\text{con}}y) \preceq \dots \preceq {}_{\text{con}}x.$$

The continuous ordering \preceq_1 on \mathcal{X} is therefore transformed by the correspondence (8.2) into the ordering \preceq restricted to $\text{con } \mathcal{X}$. In particular,

Result 3. Any continuous representation $u(x)$ of \preceq_1 on \mathcal{X} is at the same time a continuous representation of \preceq restricted to $\text{con } \mathcal{X}$.

Next we consider the space \mathcal{X}_{con} of ultimately constant programs, that is, of programs such that, for some $T = 1, 2, \dots$,

$$(8.5) \quad {}_1x = ({}_1x_T, {}_{\text{con}}x) = (x_1, \dots, x_T, x, x, \dots).$$

One readily verifies that the reasoning that led to Result 2 also applies in any subspace $\mathcal{X}_{\text{con}}^{(T)}$ of \mathcal{X}_{con} consisting of programs (8.5) with a fixed T .

The only difference occurs in the last term of (7.2). One now finds for all $T \geq 2$ a continuous representation of \preceq , restricted to $\mathcal{X}_{\text{con}}^T$, by the function

$$(8.6) \quad u(x_1) + \alpha u(x_2) + \dots + \alpha^{T-1} u(x_T) + f_T(u(x)),$$

where $f_T(u)$ is continuous and increasing. From this representation we can derive two representations of \preceq restricted to $\mathcal{X}_{\text{con}}^{T-1}$, one (8.7a) by setting $x_1 = x_1^0$ and applying P4, one (8.7b) by setting $x_T = x$, as follows

$$(8.7) \quad \begin{cases} (a) & U^{(a)}({}_1x) = \alpha u(x_1) + \dots + \alpha^{T-1} u(x_{T-1}) + f_T(u(x)) \\ (b) & U^{(b)}({}_1x) = u(x_1) + \dots + \alpha^{T-2} u(x_{T-1}) + \alpha^{T-1} u(x) + f_T(u(x)) \end{cases}$$

By Proposition 2 these representations are for all $T \geq 3$, unique up to a

linear transformation. Comparison of the first terms shows that

$$U^{(a)}(\mathbf{x}) = \alpha U^{(b)}(\mathbf{x}) + \beta ,$$

which implies that

$$f_T(u) = \alpha^T u + \alpha f_T(u) + \beta , \quad f_T(u) = \frac{\alpha^T}{1-\alpha} u + \frac{\beta}{1-\alpha}$$

Dropping the constant, we have

Result 4. On the space χ_{con} of ultimately constant programs, \succeq is represented by the continuous function

$$(8.8) \quad U(\mathbf{x}) = U(\mathbf{x}_T, \text{con } \mathbf{x}) = u(x_1) + \alpha u(x_2) + \dots + \alpha^{T-1} u(x_T) + \frac{\alpha^T}{1-\alpha} u(x) ,$$

unique up to a linear transformation. Note that in this function T itself depends on the given ultimately constant program \mathbf{x} . For definiteness one can specify that $T+1$ is the earliest time from which onward \mathbf{x} is constant. However, the same value of $U(\mathbf{x})$ is obtained if one allows $T+1$ to be any time, earliest or not, from which on \mathbf{x} is constant. It is for that reason that the function (8.8) represents \succeq on the space χ_{con} of all ultimately constant programs, regardless of the values of their "minimal" T .

9. Representation of \succeq on the space of programs bounded in utility

It is now possible to indicate a large subspace of the program space on which the ordering \succeq is represented by

$$(9.1) \quad U(\mathbf{x}) = \sum_{t=1}^{\infty} \alpha^{t-1} u(x_t) .$$

We shall call a program, x bounded in utility if there exist bundles \underline{x}, \bar{x} in χ with $\underline{x} \prec_1 \bar{x}$ such that

$$(9.2) \quad \underline{x} \prec_1 x_t \prec_1 \bar{x} \quad \text{for all } t = 1, 2, \dots$$

We can then show

Proposition 3. On the space χ^* of all programs bounded in utility,
the ordering \succeq is represented by the continuous function (9.1).

It is to be noted that for ultimately constant programs, the function (9.1) is identical with that in (8.8). Hence Proposition 3 is compatible with Result 4.

9* Proof of Proposition 3. We first note that if ${}_1x$ is bounded in utility, then,

$$u(\underline{x}) \leq u(x_t) \leq u(\bar{x}) \quad \text{for all } t,$$

and, since $0 < \alpha < 1$, the series in (9.1) is absolutely convergent, hence its sum exists and is continuous in ${}_1x$.

Now let ${}_1x$ and ${}_1y$ be two programs bounded in utility, and define bounds applicable to both ${}_1x$ and ${}_1y$ by

$$\underline{z} \equiv \begin{cases} \underline{x} & \text{if } \underline{x} \prec_1 \underline{y} \\ \underline{y} & \text{if } \underline{y} \prec_1 \underline{x}, \end{cases} \quad \bar{z} \equiv \begin{cases} \bar{x} & \text{if } \bar{x} \succeq_1 \bar{y} \\ \bar{y} & \text{if } \bar{y} \succeq_1 \bar{x}, \end{cases}$$

$$\underline{u} \equiv u(\underline{z}), \quad \bar{u} \equiv u(\bar{z}), \quad \text{so } \underline{u} < \bar{u}.$$

Assume first that $U({}_1x) > U({}_1y)$, and write

$$U({}_1x) - U({}_1y) = 3\Delta > 0$$

for the utility difference of ${}_1x$ and ${}_1y$. For comparison purposes we consider two programs

$${}_1x^{(T)} = ({}_1x_T, \text{con}^{\underline{z}}), \quad {}_1y^{(T)} = ({}_1y_T, \text{con}^{\bar{z}}),$$

where T is chosen large enough to have

$$\left(\sum_{t=T+1}^{\infty} \alpha^{t-1} \right) (\bar{u} - \underline{u}) = \alpha^T \cdot \frac{\bar{u} - \underline{u}}{1 - \alpha} \leq \Delta$$

Since then

$$U(\underline{x}) - U(\underline{x}^{(T)}) = \sum_{t=T+1}^{\infty} \alpha^{t-1} (u(x_t) - \underline{u}) \leq \Delta, \quad U(\underline{y}^{(T)}) - U(\underline{y}) \leq \Delta,$$

we must have

$$U(\underline{x}^{(T)}) - U(\underline{y}^{(T)}) \geq \Delta$$

Since $\underline{x}^{(T)}$, $\underline{y}^{(T)}$ are ultimately constant, this implies $\underline{x}^{(T)} \succ \underline{y}^{(T)}$ by Result 4. But then, using P5, $\underline{x} \succ \underline{x}^{(T)} \succ \underline{y}^{(T)} \succ \underline{y}$, which yields

$$(9.3) \quad U(\underline{x}) > U(\underline{y}) \text{ implies } \underline{x} \succ \underline{y},$$

confirming the representation (9.1) in this case.

Assume next that, for two programs \underline{x} , \underline{y} bounded in utility

$$(9.4) \quad U(\underline{x}) = U(\underline{y}) \quad \text{but} \quad \underline{x} \prec \underline{y}.$$

Then there exists t_0 such that

$$(9.5) \quad x_{t_0} \prec_1 y_{t_0}, \quad \text{so} \quad u(x_{t_0}) < u(y_{t_0}),$$

because " $x_t \succeq_1 y_t$ for all t " would contradict " $\underline{x} \prec \underline{y}$ " by P5. On a curve in \mathcal{X} connecting x_{t_0} with y_{t_0} , by a reasoning used in R2f above, there exists a point \underline{x}_{t_0} such that $\underline{x}_{t_0} \sim x_{t_0}$ while there are points x'_{t_0} with $x_{t_0} \prec_1 x'_{t_0}$ arbitrarily close to \underline{x}_{t_0} . Let x'_{t_0} be chosen, using

P1 and (9.3), so that

$${}_1x \prec {}_1x' \equiv ({}_1x_{t-1}, x'_{t_0}, t_0+1x) \prec {}_1y.$$

Then, by (9.5),

$$U({}_1x') > U({}_1x) = U({}_1y) \quad \text{but} \quad {}_1x \prec {}_1y,$$

a contradiction of (9.3). Hence (9.4) is false, and

$$U({}_1x) = U({}_1y) \quad \text{implies} \quad {}_1x \sim {}_1y,$$

confirming (9.1) in this case as well.

[To be continued].

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