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STRUCTURE OF PREFERENCE OVER TIME

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CHAPTER 3

REPRESENTATION OF PREFERENCE ORDERINGS WITH INDEPENDENT COMPONENTS OF CONSUMPTION[1/]

by

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1. Introductory remarks.

A standard model in the theory of consumer's choice assumes that the consumer maximizes a utility function under given budgetary constraints. Even in the case of the individual consumer planning for a single period's consumption, however, the time-honored concept of a utility function is not an entirely satisfactory primary concept. One may wish to look on it as a numerical representation of an underlying preference ordering, a more basic concept to be more fully defined below. Once this step is made, one will also want to know which class of preference orderings permits such a representation. Moreover, one will not want to exclude a priori the consideration of preference orderings that do not permit such a representation.

The present Chapter 3 presents a basic proposition stating sufficient conditions under which a given preference ordering is representable by a continuous function. It goes on to state, and supply

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1/ This Chapter reports on research carried out under a grant from the National Science Foundation. It is a revision of Sections 1-4 of Koopmans [1966].

proof for, a second proposition concerning the implications, for such a representation, of independence of different components of consumption in the given preference ordering. These propositions are presented for their own interest as well as for their application in Chapter 4. In the latter Chapter, both propositions are applied in discussing the choice of a criterion for the evaluation of growth paths, starting from postulates about a preference ordering.

In both Chapters, we aim for the simplest proposition of each type, capable of proof by relatively elementary mathematical methods, rather than for propositions and proofs of greatest generality.

In some sections technical parts of the reasoning are set off in starred subsections bearing the same number, set in smaller type. These can be passed up by readers interested in results rather than proofs. Equality by definition will be denoted by $\equiv$. The numbering of sections, formulae and propositions is consecutive over the two Chapters, and references to these will usually be made without distinguishing the chapters. Separate lists of references to the literature are appended to each chapter.

2. Preference orderings and representations thereof.

We shall now define and describe the mathematical concept of a preference ordering on a prospect space.

The prospect space $\mathcal{X}$ is the set of all alternative prospects between which choice may conceivably arise. The term "space" is a geometric metaphor, and the prospects will sometimes be called "points." In the static model of consumer's choice, the prospects are usually interpreted as bundles of consumption goods imagined used or used up in consumption in a stated period. (A bundle specifies the amount of each good on the list.) Instead of attaching preference to the use of goods, some authors have suggested attaching it to characteristics of goods [Lancaster, 1966a, b], or to the levels of consuming activities each involving either the use or the disappearance of one or more goods [Gale, 1967a, p. 6; 1967b, pp. 4,19]. Everything that follows is compatible with any of these interpretations of the coordinates of the points x of the prospect space. Accordingly, we shall use the term vector to refer either to a bundle of commodities, or to their characteristics, or to a statement of the levels of specified activities.

A complete preference ordering is a relation (to be denoted $\gtrsim$) between the prospects $x, y, \ldots$ in $\mathcal{X}$, compared pairwise, such that

(transitivity) if $x \succsim y$ and $y \succsim z$ then $x \succsim z$,

(completeness) for any pair of prospects x , y of $\mathcal{X}$ either $x \succsim y$ or $y \succsim x$ or both.

The relation $x \succsim y$ is interpreted as " x is at least as good as y ," or synonymously " x is preferred or equivalent to y ." Preference ($\succ$) and equivalence ($\sim$) are again transitive relations, derived from $\succsim$ by

" $x \succ y$ " means " $x \succsim y$ but not $y \succsim x$," and is also denoted " $y \prec x$,"

" $x \sim y$ " means " $x \succsim y$ and also $y \succsim x$."

A partial preference ordering is obtained if we substitute for the completeness requirement above

(reflexivity) for all x of $\mathcal{X}$, $x \succsim x$.

Completeness implies reflexivity (take $x = y$), but the converse is, of course, not true. Hence, in a partially ordered space there may be pairs of prospects that are not comparable.[2]

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[2] What is called a "preference ordering" here is called a "preordering" by Debreu [1959, p. 7]. Arrow [1963, pp. 13, 35] uses "weak ordering" for our "complete preference ordering," and "quasi-ordering" for our "partial preference ordering." In mathematical literature, the term "weak order," or "weak ordering," is used whenever (as here) equivalence $(x \sim y)$ does not necessarily imply equality $(x = y)$.

By a numerical representation of a complete[3/] preference ordering $\succsim$ we mean a function f, defined in all points x of the prospect space $\mathcal{X}$, and whose values $f(x)$ are real numbers, such that

(2.1) $\quad f(x) \geqq f(y) \quad$ if and only if $\quad x \succsim y$.

Using the above definitions of preference and of equivalence, one sees readily that this is logically equivalent to

$$(2.2) \quad \begin{cases} f(x) > f(y) & \text{if and only if } \; x \succ y \text{ , and} \\ f(x) = f(y) & \text{if and only if } \; x \sim y \text{ .} \end{cases}$$

The usefulness of a representation by a continuous function, if one exists, lies primarily in the availability of stronger mathematical techniques in that case. There is a temptation to look on the values, and the differences between values, assumed by a representing "utility function" as numerical measures of satisfaction levels, and of differences thereof, associated with the prospects in question. Such interpretations may have heuristic usefulness because of the brevity of phrasing they make possible. However, their observational basis is not really clear. An observed choice between two prospects reveals at best the fact and the direction of preference, not its strength. A descriptive theory of choice thus stays somewhat closer to what is verifiable by observation

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3/ If the preference ordering is not complete, a numerical representation is a function f such that $f(x) \geqq f(y)$ if $x \succsim y$, together with a specification of the set of pairs (x, y) of prospects x, y in $\mathcal{X}$ which are indeed comparable. Such representations have been considered by Aumann [1964].

if it is built on postulates about the underlying preference ordering. A similar remark applies to normative theory. One can better inspect and appraise a recommendation coached in terms of actual choices in various situations, than one derived from measures of "satisfaction" whose operational significance is unclear.

We shall now describe the results of two postulational studies in the literature, as illustrations of the points just made, and for use in what follows. In Chapters 3, 4, (except for Section 13), we shall discuss only complete preference orderings, without always repeating the adjective.

3. Representation of a continuous preference ordering.

Intuitively, one would call a preference ordering continuous if a small change in any prospect can not drastically change the position of that prospect in the ranking of all other prospects. Starting from a sharp definition of this concept, Debreu [1959, Section 4.6] has shown conditions under which a continuous preference ordering can be represented by a continuous utility function.[4/] In subsection 3* we show that the definition used by Debreu is logically equivalent to the following one.

The notion of a "small" change in a prospect can be made precise by assuming a given distance function in the prospect space.[5/]

4/ See also Wold [1943].

5/ The prospect space thereby becomes a metric space.

This is a function $d(x, y)$, defined for all pairs (x, y) of points in $\mathcal{X}$, with the following properties usually associated with a distance:

$$
(3.1)\quad \begin{cases} d(x, y) = d(y, x) \geqq 0 \quad \text{for all } x, y, \\ d(x, y) = 0 \quad \text{implies } x = y, \\ d(x, z) \leqq d(x, y) + d(y, z) \quad \text{for all } x, y, z . \end{cases}
$$

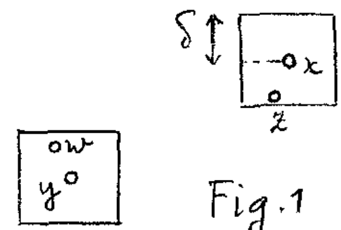

Fig.1

We shall call the preference ordering $\succsim$ continuous on $\mathcal{X}$ if (see Figure 1)

(3.2) for any x, y of $\mathcal{X}$ such that $x \succ y$, there exists a number $\delta > 0$ such that

(a) $z \succ y$ for all z in $\mathcal{X}$ such that $d(z, x) \leqq \delta$, and

(b) $x \succ w$ for all w in $\mathcal{X}$ such that $d(y, w) \leqq \delta$.

(Note that this is vacuously the case if all prospects in $\mathcal{X}$ are equivalent.) The same continuity concept may be obtained from many, but not from all, different choices of the distance function. We now have

Proposition 1 [Debreu, 1959]. *A continuous*[6/] *complete preference ordering $\succsim$ defined on a connected subset $\mathcal{X}$ of n-dimensional Euclidean*

6/ Continuity of $\succsim$ and of $u(x)$ is defined using the same distance function, for instance $d(x, y) = \max |x_i - y_i|$, if x_i, $i = 1, \ldots, n$, are the coordinates of x. While this distance function depends on the units of measurement of the amounts x_i, $i = 1, \ldots, n$, the continuity concept defined by it is independent of these units.

space[7/] $\mathcal{E}^n$ (n finite) can be represented by a utility function $u(x)$ defined and continuous in $\mathcal{X}$.

Not every conceivable preference ordering is continuous. If any increase in this year's food supply, however, small, is deemed preferable to any increase in next year's food supply, however large, we have an example of the discontinuous lexicographic ordering.

If $u(x)$ is a continuous representation of $\succsim$, and if φ is any continuous increasing function defined for all values assumed by $u(x)$ on $\mathcal{X}$, then

$$u^*(x) = \varphi(u(x)) \tag{3.3}$$

is likewise a continuous representation of $\succsim$. Conversely, if $u(x)$ and $u^*(x)$ are two continuous representations of $\succsim$, then such a function φ exists for which (3.3) holds.[8/] Therefore, a remark already made in Section 2 about representations in general applies equally to continuous representations: Only the notion of higher or lower among the levels of $u(x)$ has significance, not the numerical values $u(x)$ themselves or the differences thereof. In particular, even if $\succsim$ should

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7/ Depending on the interpretation, the prospect space $\mathcal{X}$ may be the set of all points x with all coordinates $x_i \geq 0$, or any other representation of the range of alternative prospects suitable in a given problem. $\mathcal{X}$ is called (arcwise) connected if any two points of $\mathcal{X}$ can be connected by a continuous curve contained in $\mathcal{X}$. Debreu credits a paper by Eilenberg [1941] as containing the mathematical essence of Proposition 1. For a stronger theorem establishing existence of a continuous representation without assuming connectedness or finite dimensionality see Debreu [1954] and Rader [1963].

8/ The proof of this statement is implied in the last paragraph of Subsection 4* below: take $x = x_P$ and replace the pair $(U(x), u(x))$ of (4.4) by the pair $(u^*(x), u(x))$ of (3.3).

possess a differentiable representation $u(x)$, there is no intrinsic meaning in the "marginal utility" $\frac{\partial u}{dx_i}$ of any single commodity. This is often expressed by the statement that $u(x)$ is an ordinal, not a cardinal utility. However, even if $u(x)$ is only ordinal, for given units of commodities i , j , the ratio

$$(3.4) \qquad \frac{\partial u^*(x)}{\partial x_i} \Big/ \frac{du^*(x')}{\partial x'_j}$$

of two "marginal utilities" in the same point $(x = x')$, or in two equivalent points $(x \sim x')$, is invariant. That is, the ratio (3.4) is independent of the choice of a differentiable φ in (3.3), hence is a quantity meaningful in terms of the given ordering $\succsim$.

By suitable choice of φ in (3.3) one can make the range $\mathcal{U}^* = u^*(\mathcal{X})$ of $u^*(x)$ coincide with any interval of finite, positive length, that includes the left and/or right endpoint depending on whether $\mathcal{X}$ contains a worst and/or best element of $\succsim$. Thus $u^*(\mathcal{X})$ can be unbounded from below and/or above only if no worst and/or best element exists.

3^* Equivalence of two definitions of continuity of an ordering.

The definitions to be compared are:

D If $\lim_{n\to\infty} y_n = y$ and $x \succsim y_n \succsim z$ for all n , then $x \succsim y \succsim z$.

D' If $y \succ x$ there exists $\delta > 0$ such that

(a) $d(y, w) \leqq \delta$ implies $w \succ x$, and

(b) $d(w, x) \leqq \delta$ implies $y \succ w$

Assume " D' and not D ." Then there exists x, z, y_n with $x \succsim y_n \succsim z$ for all n but either $\lim_{n \to \infty} y_n = y \succ x$ or $z \succ y$. Taking the case $y \succ x$, we choose δ in D' such that $d(y, y_n) \leqq \delta$ implies $y_n \succ x$, and N in the definition of limit such that $d(y, y_N) \leqq \delta$. Then $y_N \succ x \succsim y_N$, a contradiction. The case $z \succ y$ is similar.

Assume next " D and not D'," and take $\delta_n = \frac{1}{2^n}$. Then, for some x, y such that $y \succ x$, there exists either a sequence y_n such that $d(y, y_n) \leqq \delta_n$ but $x \succsim y_n$, or a sequence x_n such that $d(x_n, x) \leqq \delta_n$ but $x_n \succsim y$. By D, both cases imply $x \succsim y$, contradicting $y \succ x$.

Two statements such that the negation of either contradicts the other are equivalent.

4. Separable representation in the presence of two independent components of consumption.

The problem of deriving special forms for a utility function from assumptions about independence among components of consumption has been studied by several authors, including Leontief [1947 a, b] and Samuelson [1948, Ch. VII]. We shall follow Debreu [1960] because he avoids assumptions of differentiability of the utility function that seem unrelated to the essence of the problem.

To illustrate the independence concept in terms of the traditional commodity space, one may wish to assume that preferences between food bundles are independent of the amounts of various articles of clothing and of other commodities consumed, and similarly for preferences between clothing bundles, etc.; furthermore that preferences between food-and-clothing bundles are independent of the amount of other commodities consumed, and so on.

In this section we shall derive a preliminary result for the case of two independent components of consumption. Let $\succsim$ denote a preference ordering on the space

$$\mathcal{X} = \mathcal{X}_P \times \mathcal{X}_Q \tag{4.1}$$

of all vectors $x = (x_P, x_Q)$ such that x_P is in a given space $\mathcal{X}_P$, x_Q in $\mathcal{X}_Q$. In mathematical terminology, $\mathcal{X}$ is called the (Cartesian) <u>product</u> of the spaces $\mathcal{X}_P$, $\mathcal{X}_Q$, the latter the <u>factor spaces</u> of $\mathcal{X}$.

To express the required independence assumption we use an arbitrary but fixed <u>reference vector</u> in $\mathcal{X}$,

$$z = (z_P, z_Q) , \tag{4.2}$$

to define two orderings, $\succsim_P^z$ on $\mathcal{X}_P$ and $\succsim_Q^z$ on $\mathcal{X}_Q$, induced by $\succsim$, as follows,

$$(4.3)\qquad \begin{aligned} x_P \succsim_P^z y_P \quad &\text{means} \quad (x_P, z_Q) \succsim (y_P, z_Q) , \\ x_Q \succsim_Q^z y_Q \quad &\text{means} \quad (z_P, x_Q) \succsim (z_P, y_Q) , \end{aligned}$$

In general, the induced orderings depend on the reference vector, in the sense that $\succsim_P^z$ depends on z_Q, and $\succsim_Q^z$ on z_P. The independence assumption will say that this dependence-in-principle is not a dependence-in-fact. In Subsection 4* we show, following Debreu [1960],

Result A: Let a preference ordering $\succsim$ on a product space $\mathcal{X} = \mathcal{X}_P \times \mathcal{X}_Q$ be representable by a utility function $U(x)$, and let the orderings $\succsim_P^z$, $\succsim_Q^z$ induced by $\succsim$ (as defined above) be independent of the reference vector z. Then $U(x)$ has the form

$$(4.4)\qquad U(x) = F(u(x_P), v(x_Q)) ,$$

where F is a strictly increasing function of both u and v. Moreover, if $\mathcal{X}$ is connected, $U(x)$ continuous, then $u(x_P)$, $v(x_Q)$ and $F(u, v)$ are continuous, and the ranges of $u(x_P)$, $v(x_Q)$ are intervals.

A function of this form has been called a utility tree by Strotz [1957, 1959], and a separable utility function by Gorman [1959a, b]. In the case of two independent components of consumption, therefore, instead of one function U of $n_P + n_Q$ variables (there are a great many such functions!) we have a triple of functions, one (F) of two variables, one (u) of n_P, and one (v) of n_Q variables. In some sense the "number" of such triples forms a much smaller infinity. The utility $U(x)$ of x depends only on the utility levels $u(x_P)$, $v(x_Q)$ associated with x_P, x_Q in their respective spaces, rather than on these vectors in their full detail.

4* Proof of Result A. We define

$$(4.5) \qquad u(x_P) \equiv U(x_P, z_Q) , \qquad v(x_Q) \equiv U(z_P, x_Q) ,$$

and consider two vectors x, y in $\mathcal{X}$ such that

$$u(x_P) = u(y_P) , \qquad v(x_Q) = v(y_Q) .$$

Since U represents $\succsim$, we then have

$$(x_P, z_Q) \sim (y_P, z_Q) , \qquad (z_P, x_Q) \sim (z_P, y_Q) .$$

Since $\succsim_P$ and $\succsim_Q$ are independent of the choice of z, we have further

$$x = (x_P, x_Q) \sim (y_P, x_Q) \sim (y_P, y_Q) = y$$

(choose the alternative reference vector $z' = (y_P, x_Q)$). Hence $x \sim y$, and $U(x) = U(y)$. This means that the value of $U(x)$ for any vector x in $\mathcal{X}$ depends only on the values of $u(x_P)$, $v(x_Q)$ assumed for the subvectors x_P , x_Q of x , respectively, confirming (4.4).

Moreover, from the independence of $\succsim_P^z$ from z , using the definition of $\succ$ in terms of $\succsim$ we have, for all z_Q' , that

$$x_P \succ_P y_P \text{ if and only if } (x_P, z_Q') \succ (y_P, z_Q')$$

It follows that F increases strictly with u , and similarly with v .

Finally, by (4.5), continuity of $U(x)$ implies that of $u(x_P)$, $v(x_Q)$, connectedness of $\mathcal{X}$ that of $\mathcal{X}_P$, $\mathcal{X}_Q$. Hence, for any fixed z in $\mathcal{X}$, the ranges of the functions $U(x)$, $u(x_P)$, $v(x_Q)$, $U(x_P, z_Q)$, $U(z_P, x_Q)$ for all x_P in $\mathcal{X}_P$, x_Q in $\mathcal{X}_Q$ are intervals, nondegenerate unless $u(x_P)$ and/or $v(x_Q)$ is constant. But then $F(u, v(z_Q))$ and $F(u(z_P), v)$ are, for any fixed z in $\mathcal{X}$, increasing functions defined on one interval and taking on all the values in another. This is possible only if $F(u, v)$ is continuous in u for each v , and in v for each u . Since $F(u, v)$ increases in both u , v , it follows that $F(u, v)$ is continuous in u and v jointly.

5. Additively separable representation in the case of three independent components of consumption.

Three independent components of consumption suffice to show the essential traits of the case with $n \geqq 3$ such components. We shall therefore in this section consider a preference ordering $\succsim$ on a product

$$\mathcal{X} = \mathcal{X}_P \times \mathcal{X}_Q \times \mathcal{X}_R \tag{5.1}$$

of three spaces. To make sure that this is really three for the purpose of our reasoning, we shall need a concept of sensitivity of $\succsim$ in a factor space. We shall say that $\succsim$ is <u>sensitive</u> in $\mathcal{X}_P$ if there exist x_P, y_P, z_Q, z_R such that

$$(x_P, z_Q, z_R) \succ (y_P, z_Q, z_R) . \tag{5.2}$$

This will ensure that the induced ordering $\succsim_P^z$ will not declare all vectors x_P equivalent.

Given a reference vector $z = (z_P, z_Q, z_R)$, $\succsim$ now induces six orderings, $\succsim^z_P$, $\succsim^z_Q$, $\succsim^z_R$, $\succsim^z_{P,Q}$, $\succsim^z_{Q,R}$, $\succsim^z_{P,R}$ on various factor spaces, defined along the following lines:

$$(5.3)\begin{cases} x_P \succsim^z_P y_P & \text{means } (x_P, z_Q, z_R) \succsim (y_P, z_Q, z_R) \\ (x_P, x_Q) \succsim^z_{P,Q} (y_P, y_Q) & \text{means } (x_P, x_Q, z_R) \succsim (y_P, y_Q, z_R) , \text{ etc.} \end{cases}$$

Proposition 2 [Debreu, 1960, 3 components only, modified]

Let $\succsim$ be a continuous preference ordering of all consumption vectors $x = (x_P, x_Q, x_R)$ such that x_P , x_Q , x_R belong to spaces $\mathcal{X}_P$, $\mathcal{X}_Q$, $\mathcal{X}_R$, which are connected subsets of Euclidean spaces of n_p , n_Q , n_R dimensions, respectively. Let $\succsim$ be sensitive in each of P , Q , R , and let $\succsim^z_P$, $\succsim^z_Q$, $\succsim^z_R$, $\succsim^z_{P,Q}$, $\succsim^z_{Q,R}$ (as defined above) be independent of z . Then there exist functions $u^*(x_P)$, $v^*(x_Q)$, $w^*(x_R)$, defined and continuous on $\mathcal{X}_P$, $\mathcal{X}_Q$, $\mathcal{X}_R$, respectively, such that $\succsim$ is represented by

$$(5.4) \qquad U^*(x) = u^*(x_P) + v^*(x_Q) + w^*(x_R) .$$

This representation is unique up to a linear transformation

$$(5.5) \qquad u'(x_P) = \beta_P + \gamma u^*(x_P) , \quad v'(x_Q) = \beta_Q + \gamma v^*(x_Q) ,$$

$$w'(x_R) = \beta_R + \gamma w^*(x_R) , \quad \gamma > 0 .$$

A similar proposition holds for any partitioning of x into four or more independently ordered subvectors.

In principle, the representation (5.4) is still ordinal. That is, any function $U'(x)$ obtained from $U^*(x)$ by (3.3) is likewise a continuous representation of $\succsim$. However, unless φ happens to be linear as in (5.5), the representation $U'(x)$ cannot be written simply as a sum of functions each depending on one of the vectors x_P, x_Q, x_R only, as $U^*(x)$ is written in (5.4). It is only in this limited sense that the representation by $U^*(x)$ can be called cardinal.

In the proof of Proposition 2 given in Subsection 5* we shall follow the general ideas of Debreu's beautiful geometrical proof, and of the work of Blaschke and Bol [1938] on which it builds forth. We modify his reasoning in one respect in order to avoid making the assumption that the sixth induced ordering, $\succsim^z_{P,R}$, is also independent[9] of z.

9/ The redundancy of that assumption, as well as the importance of that redundancy for the analysis of utility over time, were perceived and demonstrated by Gorman [1965, 196] for the case of differentiable utility functions. In a recent mimeographed paper, Gorman [1967] has given a complete discussion of the structure of representations with regard to separability and additive separability, without differentiability assumptions. His results imply that, in Proposition 2, the premises that $\succsim^z_P$, $\succsim^z_Q$, $\succsim^z_R$ are independent of z are also implied in those made about $\succsim^z_{P,Q}$ and $\succsim^z_{Q,R}$. This further strengthening, important in itself, turns out to be less crucial to the particular application of Proposition 2 made in Chapter 4 than the dropping of the assumption that $\succsim^z_{P,R}$ does not depend on z.

5* <u>Proof of Proposition</u> 2. Since the Cartesian product

$$\mathcal{X} \equiv \mathcal{X}_P \times \mathcal{X}_Q \times \mathcal{X}_R$$

is a connected subset of a Euclidean space of $n = n_P + n_Q + n_R$ dimensions, the premises of Proposition 1 are satisfied. Hence $\succsim$ is represented by a continuous function

$$U(x) \equiv U(x_P, x_Q, x_R) \tag{5.6}$$

defined on $\mathcal{X}$. Since an additive constant does not affect the representation, we shall anchor $U(x)$ by requiring

$$U(z) = 0 . \tag{5.7}$$

The five induced orderings $\succsim_P$, $\succsim_Q$, $\succsim_R$, $\succsim_{P,Q}$, $\succsim_{Q,R}$ (superscripts z have been dropped because these are now independent of z) are therefore represented by the continuous functions

$$(5.8)\begin{cases} u(x_P) \equiv U(x_P, z_Q, z_R) , \quad v(x_Q) \equiv U(z_P, x_Q, z_R) , \quad w(x_R) \equiv U(z_P, z_Q, x_R) , \\ W(x_P, x_Q) \equiv U(x_P, x_Q, z_R) , \quad \bar{U}(x_Q, x_R) \equiv U(z_P, x_Q, x_R) , \end{cases}$$

respectively. Since the domains of all these functions are connected, the range of each is an interval. For three of the ranges we introduce the notations

$$\mathcal{U} = u(\mathcal{X}_P) , \mathcal{V} = v(\mathcal{X}_Q) , \mathcal{W} = w(\mathcal{X}_R) . \tag{5.9}$$

Since $\succsim$ is sensitive in each of P, Q, R, none of the five intervals collapses to a point, and, by suitable choice of z, one can ensure that the point

$$(5.10) \qquad u(z_P) = v(z_Q) = w(z_R) = W(z_P, z_R) = \bar{U}(z_Q, z_R) = 0$$

is interior to all five ranges.

We now apply Result A twice to $U(x)$, once with the partitioning $x = (x_P, (x_Q, x_R))$, and once with $x = ((x_P, x_Q), x_R)$. With reference to the proof of Result A, this gives us the existence of strictly increasing functions $F(W, w)$ and $G(u; \bar{U})$, such that, for all x in $\mathcal{X}$,

$$(5.11) \qquad U(x) = F(W(x_P, x_Q), w(x_R)) = G(u(x_P), \bar{U}(x_Q, x_R)) .$$

The domains of the arguments W, w of F and u, $\bar{U}$ of G are intervals over which the functions denoted by the same symbols range, respectively. Since these functions as well as $U(x)$ are continuous, F and G are continuous. To avoid repetition of similar reasoning, we announce in advance that the functions F^{-1}, g, f, H, h yet to be introduced are likewise continuous and strictly increasing on the nondegenerate interals, or products thereof, over which their arguments range.

By inserting $x_R = z_R$ in (5.11), using (5.10) and (5.8), one has

$$F(W(x_P, x_Q), 0) = G(u(x_P), v(x_Q)) ,$$

and, if F^{-1} is the inverse of $F(W, 0)$,

$$(5.12) \quad W(x_P, x_Q) = F^{-1}(G(u(x_P), v(x_Q)) = g(u(x_P), v(x_Q)) ,$$

say, and symmetrically

$$(5.13) \qquad \bar{U}(x_Q, x_R) = f(v(x_Q), w(x_R))$$

We can now shed the variables x_P , x_Q , x_R . From (5.11), (5.12) and (5.13) we have

$$(5.14) \quad F(g(u, v), w) = G(u, f(v, w)) \equiv H(u, v, w) = H(t) , \quad \text{say,}$$

where $t \equiv (u, v, w)$. Here $H(t)$ is defined on the three-dimensional cell $\mathcal{T} \equiv \mathcal{U} \times \mathcal{V} \times \mathcal{W}$, of which the origin $0 = (0, 0, 0)$ is an interior point. The ordering $\succsim$ on $\mathcal{X}_P \times \mathcal{X}_Q \times \mathcal{X}_R$ represented by $U(x)$ induces an ordering on $\mathcal{T}$, which we likewise denote by $\succsim$, and which is represented by $H(t)$.

We shall study the level curves of $H(u, v, 0)$ and of $H(0, v, w)$. In the plane $w = 0$ we arbitrarily select (see Figure 2)* an indifference curve κ not passing through $\mathbf{0}$, but close enough to $\mathbf{0}$ for all the intersection points sought in the following construction to exist. If κ intersects the u- and v-axes in $a = (u', 0, 0)$ and $b = (0, v', 0)$, respectively we have

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* On p. 21a.

(5.15) $a \sim b$, implying $g(u', 0) = g(0, v')$

by taking $w = 0$ in the first member of (5.14). At most one intersection point exists in each case because $g(u, v)$ is increasing in each variable. Precisely one will exist if κ passes close enough to 0 , because of the continuity of $g(u, v)$.

It will save words to refer to two points s , t of $\mathcal{J}$ as u-congruent if they differ only in their u-coordinate,

$$s = (u^{(1)}, v, w) , \quad t = (u^{(2)}, v, w) .$$

Similarly we shall speak of v- and w-congruence.

We find $c \equiv (u', v', 0)$, v-congruent to a , u-congruent to b , and draw through c an indifference curve λ in the plane $w = 0$, which intersects the u-axis in $a' \equiv (u'', 0, 0)$, the v-axis in $d \equiv (0, v'', 0)$. In particular,

(5.16) $c \sim a'$ implies $g(u', v') = g(u'', 0)$.

Finally we find $c' \equiv (u'', v', 0)$, v-congruent to a' , u-congruent to b , and $d' \equiv (u', v'', 0)$, u-congruent to d , v-congruent to a .

We now wish to prove that $d' \sim c'$. Since Proposition 2 does not hold for a partitioning of x into only two components,[10] we shall need to go into the third dimension to prove this.

On the indifference curve η through d in the plane $u = 0$, we find $b'' = (0, v', w')$, w-congruent to b . Then

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[10] See Section 7 below.

(5.17) $\quad d \sim b''$, implying $f(v'', 0) = f(v', w')$

by the second member of (5.14). Finally we find $0'' = (0, 0, w')$ on the w-axis, v-congruent to b'' , and $a'' = (u', 0, w')$, u-congruent to $0''$, w-congruent to a . Then by taking $w = w'$ in the first member of (5.14), we see that (5.15) in its turn implies $a'' \sim b''$. (In fact the indifference curves κ and κ' are point-by-point w-congruent.) Hence $c \sim d \sim b'' \sim a''$, and therefore

(5.18) $\quad c \sim a''$, implying $f(v', 0) = f(0, w')$.

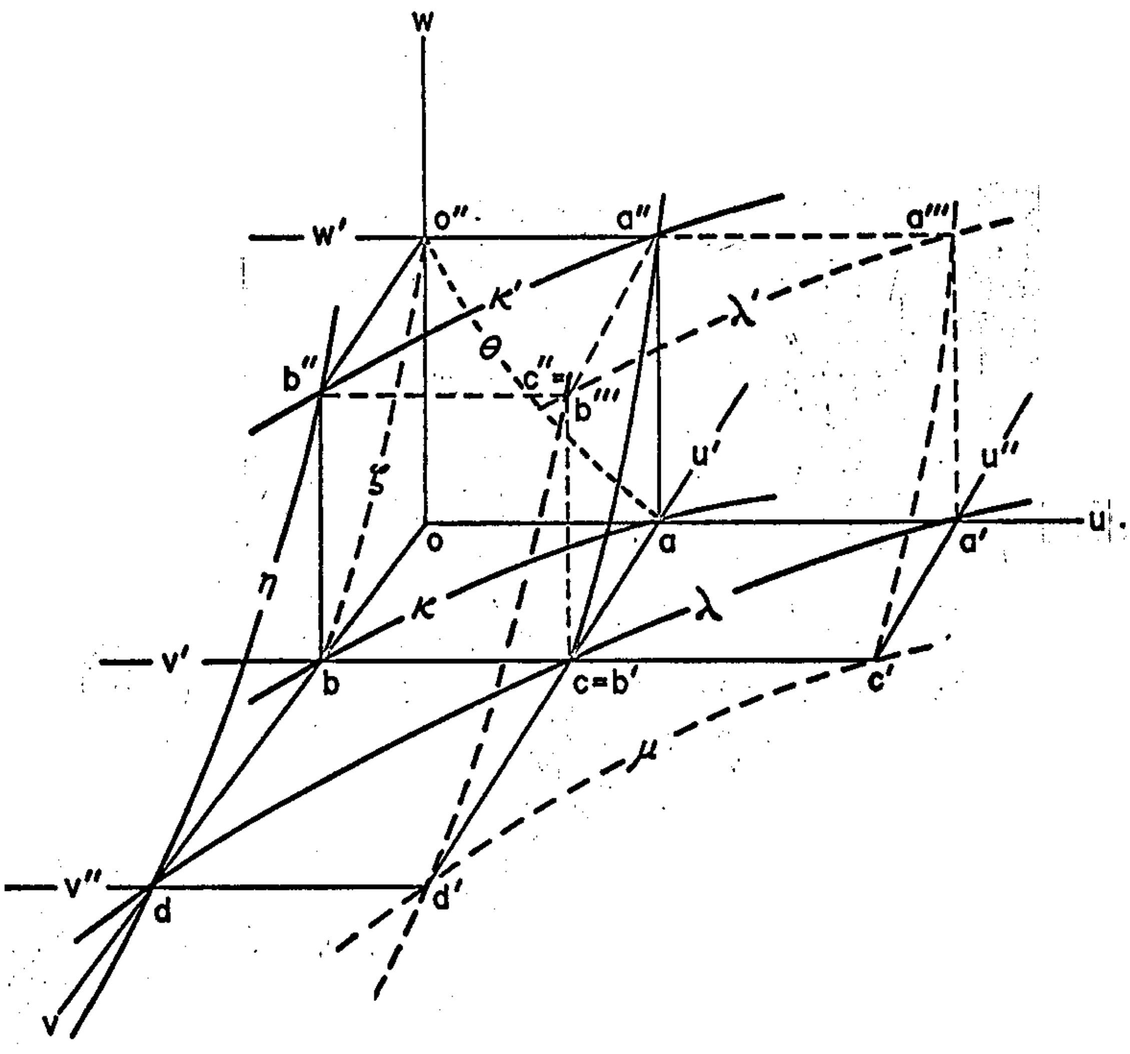
The second round of the construction is similar to the first. It employs the points $a''' \equiv (u'', 0, w')$, u- and w-congruent to $0''$ and a' , respectively, and $c'' = b''' = (u', v', w')$, u- , v- and w-congruent to b'' , a'' and $c = b'$, respectively. We have

$$\left.\begin{array}{lll} (5.17) & \text{implies} & d' \sim b''' \\ (5.16) & \text{implies} & c'' \sim a''' \\ (5.18) & \text{implies} & a''' \sim c' \end{array}\right\} \quad \text{so } d' \sim c' .$$

Hence d' and c' are on the same indifference curve μ in the plane $v = 0$.

The rectangle $acc'a'$ has the following characteristics relative to the indifference curves κ , λ , μ :

incidence ;					congruence type of				
	a	c	c'	a' .		a,a'	c,c' .	a,c	a',c'
is on	κ	λ	μ	λ	is	u	u	v	v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b
c=b'
c'
μ
v''
d
d'
v

We shall call such a rectangle <u>inscribed in the curves</u> κ, λ, μ. Since the origin could have been chosen anywhere in $\mathcal{J}$, we have found the following result, illustrated in Figure 3.

<u>Result B</u>: <u>If three indifference curves κ, λ, μ possess an inscribed rectangle</u> acc'a' <u>then κ, λ, μ possess adjoining inscribed rectangles</u> bdd'b', b' = c, <u>and</u> eff'e', f = a', <u>provided only that the intersection points required by their construction exist.</u>

The remainder of the proof is based on the "textile geometry" of Blaschke and Bol. On any three indifference curves κ, λ, μ one can construct a sequence of such rectangles as indicated in Figure 3,*/ going as far in both directions as the intervals $\mathcal{U}$ and $\mathcal{V}$ permit. If there should be an infinite sequence of such rectangles inscribed in κ, λ, μ, such a sequence cannot have a point of accumulation t' in $\mathcal{J}$, because by the continuity of H(t) such a point would belong to each of κ, λ, μ, which is a contradiction. Hence if $\mathcal{U}$ and $\mathcal{V}$ are bounded, an infinite sequence of inscribed rectangles can only have an accumulation point on the boundary of $\mathcal{J}$.

A second sequence of rectangles can be inscribed in λ, μ, ν if ν contains, for instance, the point g, u-congruent to c' and v-congruent to f'. In this way the intersection of $\mathcal{J}$ with the plane w = 0 is covered by rectangles inscribed in a sequence of indifference curves ..., κ, λ, μ, ν, ..., except possibly for uncovered margins near the endpoints (if finite) of $\mathcal{U}$, $\mathcal{V}$.

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* p. 24

Furthermore, one can interpolate an indifference curve γ "between" κ and λ, say, by choosing p on eh (Figure 3) so that $q \sim r$, and drawing γ through q and r. This construction can be extended over the full length of κ and λ, repeated between λ and μ, etc. and possibly into the uncovered margins, and repeated again between κ and γ, etc.

Let $\mathcal{U}'$ be the set of all u-coordinates $(0, u', u'', \ldots)$ of vertices of inscribed rectangles occurring in this construction repeated indefinitely, $\mathcal{V}'$ that of all v-coordinates. Then $\mathcal{U}'$ is dense in $\mathcal{U}$, $\mathcal{V}'$ in $\mathcal{V}$. We assign new coordinates (u^*, v^*) to all points of $\mathcal{U}' \times \mathcal{V}'$ in the manner indicated in the margins of Figure 3. Then

$$(5.19) \qquad u^* = \pi(u) , \quad v^* = \varphi(v) ,$$

are continuous and increasing functions on $\mathcal{U}'$ and $\mathcal{V}'$, respectively, for which

$$(5.20) \qquad \pi(0) = \varphi(0) = 0$$

These functions are extended to $\mathcal{U}$, $\mathcal{V}$, while retaining these properties, by

$$\pi(u) \equiv \sup_{\substack{u' \leqq u \\ u' \in \mathcal{U}'}} \pi(u') , \qquad \varphi(v) \equiv \sup_{\substack{v' \leqq v \\ v' \in \mathcal{V}'}} \varphi(v') .$$

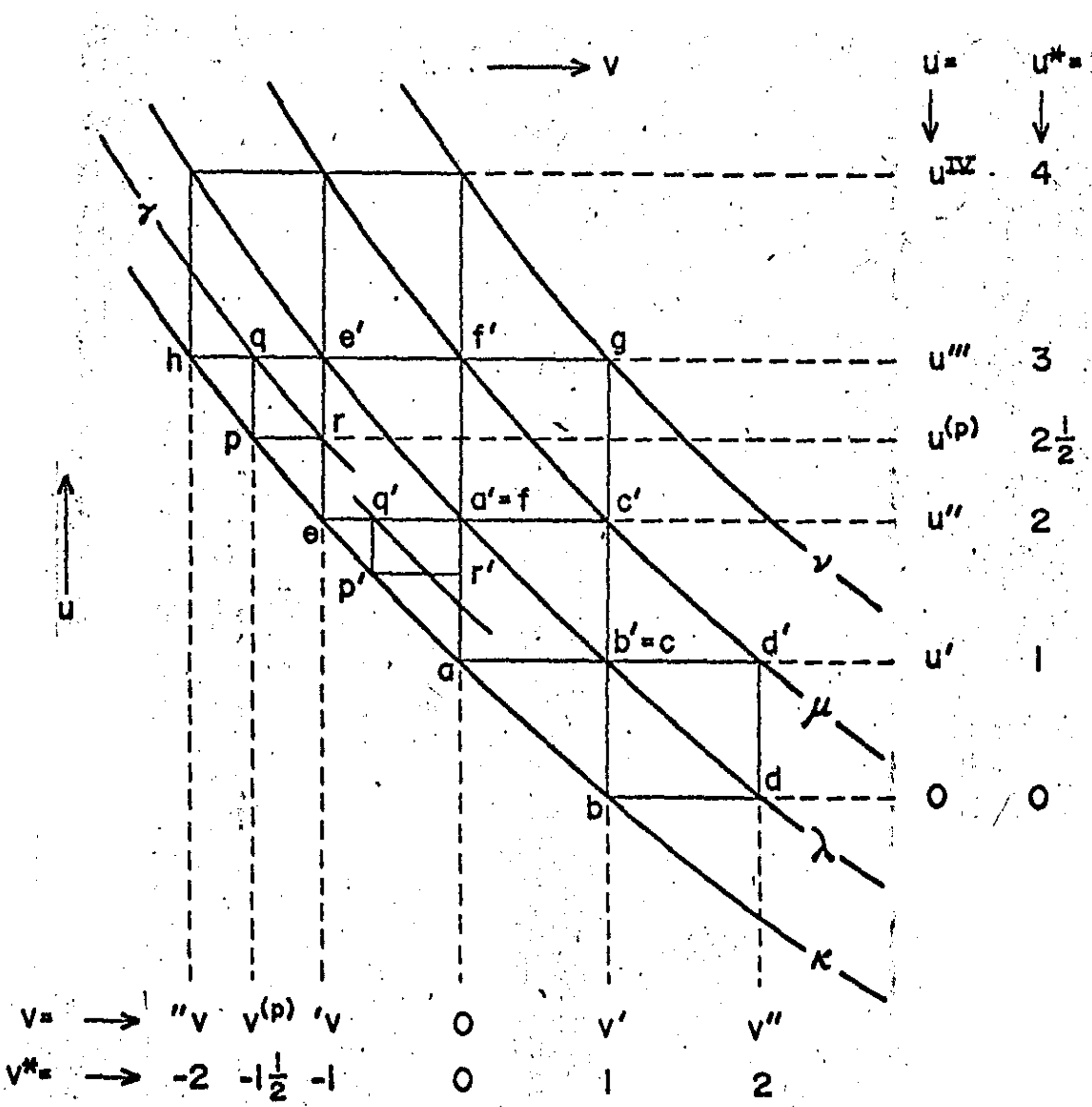
v
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Figure 3

It follows from the construction that, for any two equivalent points (u, v) , (u', v') of $\mathcal{U}' \times \mathcal{V}'$ one has

$$u^* + v^* = \pi(u) + \varphi(v) = \pi(u') + \varphi(v') = u'^* + v'^*$$

By continuity of $H(u, v, 0)$ this property extends to $\mathcal{U} \times \mathcal{V}$. Therefore, if we now define functions

$$u^*(x_P) = \pi\Big(u(x_P)\Big) \quad , \qquad v^*(x_Q) = \varphi\Big(v(x_Q)\Big) \quad ,$$

the ordering $\succsim$, restricted to points of $\mathcal{X}$ for which $w(x_R) = 0$, is represented by the continuous function

(5.21) $$u^*(x_P) + v^*(x_Q) \ .$$

By the independence of $\succsim_{P,Q}$, the same representation applies to any set of points of $\mathcal{X}$ on which $w(x_R)$ takes another constant value.

To extend this representation to all of $\mathcal{X}$, we return to Figure 3 to note that (5.18) also implies $b \sim 0''$. It follows that, had we carried out the preceding construction in the plane $u = 0$ instead of in $w = 0$, starting from ζ instead of from κ , we would have arrived at the same demarcation points 0 , b , d , ... on the v-axis, the same interpolated points, the same function $\varphi(v)$, and hence the same function $v^*(x_Q)$, along with a similar function $w^*(x_R)$. It follows that $\succsim$ is continuously represented, on any set of points of $\mathcal{X}$ for which $u^*(x_P)$ takes a constant value, by

(5.22) $$v^*(x_Q) + w^*(x_R) .$$

We shall finally show that $\succsim$ is represented on $\mathcal{X}$ by the continuous function[11]

(5.23) $$U^*(x) = u^*(x_P) + v^*(x_Q) + w^*(x_R) .$$

Consider two vectors $x = (x_P, x_Q, x_R)$, $x' = (x'_P, x'_Q, x'_R)$. By (5.11), (5.12), (5.13), (5.19) their order depends only on the corresponding utility vectors

(5.24) (u^*, v^*, w^*) , (u'^*, v'^*, w'^*) , where $u^* = u^*(x_P)$, etc.

Extending the usual notation $[m, m']$ for the interval $m \leqq u \leqq m'$ to

$$\left|[m, m']\right| \equiv \begin{cases} [m, m'] & \text{if } m \leqq m' \\ [m', m] & \text{if } m' < m , \end{cases}$$

we consider the set

$$\mathcal{S} \equiv \left|[u^*, u'^*]\right| \times \left|[v^*, v'^*]\right| \times \left|[w^*, w'^*]\right| .$$

This is a block (rectangular parallelepiped) of which each vertex has each coordinate in common with one or the other of the points (5.24), as shown in Figure 4[*] On the points of each edge of $\mathcal{S}$ the ordering

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11/. There is an affinity between the following reasoning and a study by Arrow [1952].

*/ p. 29

$\succsim$ is (strictly) monotonic as indicated by arrows, because of the monotonicity of H in (5.14), and each such <u>edge ordering</u> is represented by the corresponding term in (5.23).

We must show that, for all possible dimensions of the block, the ordering $\succsim$ of each of the pairs (a, h) , (b, e) , (c, f) , (d, g) is represented by (5.23). For (a, h) this is already implied in the edge orderings $a \succ b \succ f \succ h$.

Assume first that $v^* \neq v'^*$. Then if either $u^* = u'^*$ or $w^* = w'^*$, the remaining comparisons are settled by (5.22) or (5.21), respectively. Assume therefore that the block $\mathcal{S}$ is three-dimensional. We shall make use of the equivalences

$$(u^*, v^*, w^*) \sim (u^* + p, v^* - p, w^*) \sim (u^* + p, v^* - p + q, w^* - q) \sim \ldots,$$

implied in (5.21), (5.22), as long as we make sure that all points so compared are in $\mathcal{S}$. This means that all points of any line segment in $\mathcal{S}$ parallel to either κ or ζ are equivalent, and these equivalences are represented by (5.23).

As an example, Figure 5*/ shows the comparison of b and e . We intersect $\mathcal{S}$ with a plane $\mathcal{P}$ through b parallel to both κ and ζ . Since a , h are on opposite sides of $\mathcal{P}$, the intersection is a two-dimensional convex polygon $\mathcal{Q}$ with edges parallel to κ, ζ or θ . Now $\mathcal{P}$ and hence $\mathcal{Q}$ must intersect the broken line h e d a in precisely one point k . Figure 6, drawn in $\mathcal{P}$, shows a broken line in $\mathcal{Q}$ with a finite number of segments parallel

*/ p. 29

to κ and ζ, connecting b and k. This establishes the equivalence of b and k, and its representation by (5.23). The comparison of k and e then is made through the edge orderings on h e d a, again represented by (5.23). In Figure 5*/ $b \sim k \succ e$.
It is clear from the two-dimensionality of $\mathcal{Q}$, from the condition on the slopes of its sides, and from the Archimedian property of real numbers, that the above reasoning can be carried through regardless of the dimensions of $\mathcal{S}$, and of the pair of opposite vertices compared (see Figure 6*/).

On the other hand, if $v^* = v'^*$, we first use (5.24) with either $p \neq 0$ or $q \neq 0$ to obtain

$$(u''^*, v''^*, w''^*) \sim (u^*, v^*, w^*) \ , \quad \text{say, with} \quad v''^* \neq v^* \ ,$$

and continue from there with the above reasoning. This procedure is unavailable with regard to <u>both</u> (u^*, v^*, w^*) and (u'^*, v'^*, w'^*) only if each is either $(\underline{u}, \underline{v}, \underline{w})$ or $(\overline{u}, \overline{v}, \overline{w})$, where $\underline{u}, \underline{v}, \underline{w}$ are finite lower endpoints of $\mathcal{U}^* \equiv u^*(x_P), \mathcal{V}^*, \mathcal{W}^*$, included in $\mathcal{U}^*, \mathcal{V}^*, \mathcal{W}^*$, respectively, and $\overline{u}, \overline{v}, \overline{w}$ are similar upper endpoints. But then $v^* = v'^*$, $\overline{v} > \underline{v}$ forces $(u^*, v^*, w^*) = (u'^*, v'^*, w'^*)$, equality implying equivalence, represented by (5.23).

Finally, to discuss the uniqueness of (5.23), we note first from (5.10), (5.19), (5.20) that

$$u^*(z_P) = v^*(z_Q) = w^*(z_R) = 0 \ .$$

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*/ p. 29

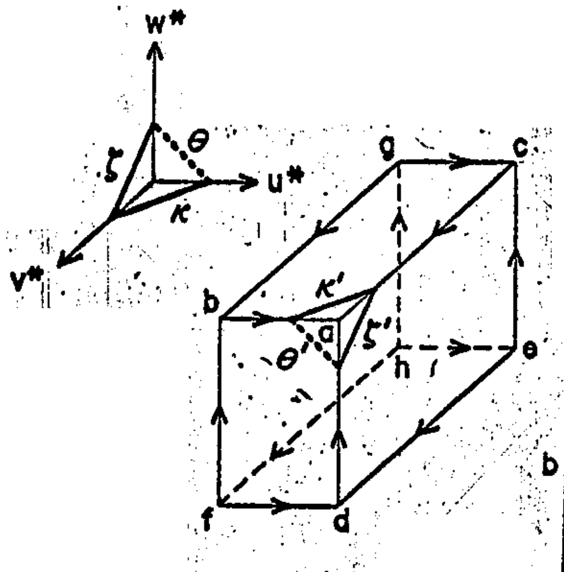

Figure 4

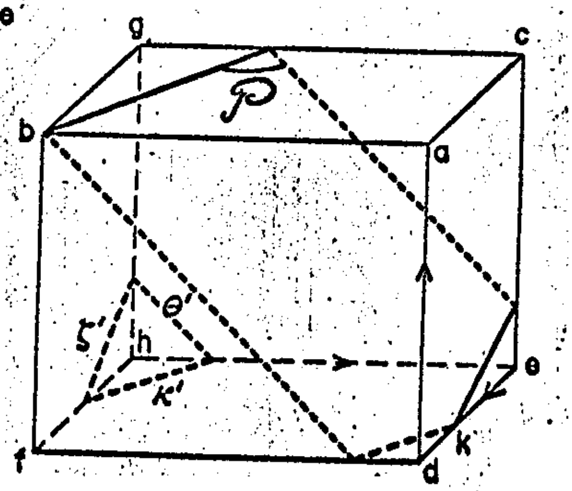

Figure 5

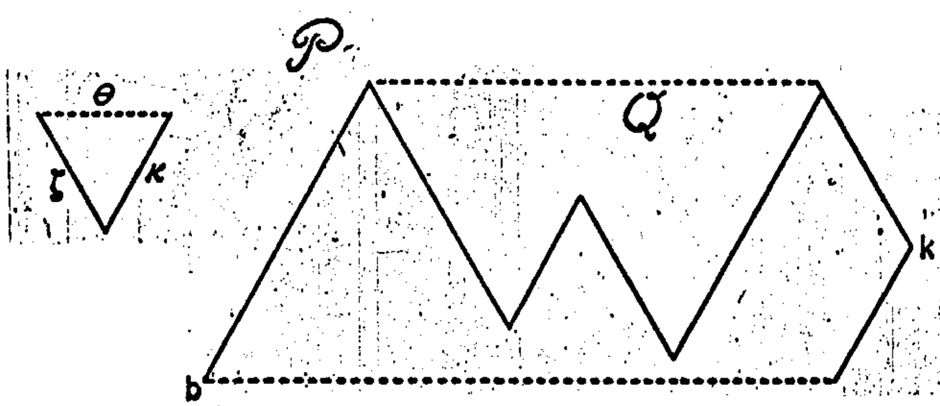

Figure 6

Now assume that $\succsim$ is also represented by the continuous function

$$U'(x) = u'(x_P) + v'(x_Q) + w'(x_R) \ .$$

We define

$$\beta_P \equiv u'(z_P) \ , \quad \text{etc.,} \quad u''(x_P) \equiv u'(x_P) - \beta_P \ , \quad \text{etc.}$$

Then there exists $h(U^*)$ such that, for all x in $\mathcal{X}$,

$$u''(x_P) + v''(x_Q) + w''(x_R) = h(u^*(x_P) + v^*(x_Q) + w^*(x_R)) \ .$$

Inserting $x_R = z_R$, and thereafter $x_Q = z_Q$, or $x_P = z_P$, or both, we have, for all values of the omitted arguments x_P , x_Q , x_R ,

$$u'' + v'' = h(u^* + v^*) \ , \quad u'' = h(u^*) \ , \quad v'' = h(v^*) \ , \quad 0 = h(0) \ ,$$

hence

$$h(u^* + v^*) = h(u^*) + h(v^*) \ , \quad h(0) = 0 \ ,$$

for all (u^*, v^*) in $\mathcal{U}^* \times \mathcal{V}^*$.

This in turn implies

$$h(n\, u^*) = n\, h(u^*)$$

for all integer n and all u^* such that u^* and $n\, u^*$ are in the interval $\mathcal{U}^*$. Among continuous functions $h(u^*)$, this property is possessed only by the linear functions

$$h(u^*) = \gamma u^* ,$$

where $\gamma > 0$ because h is increasing. This establishes the transformation (5.5). The proof of Proposition 2 is now complete.

6. Extensions to the case of more than three independent components of consumption

Debreu has extended Proposition 2 to the case of $k > 3$ independent components of consumption. If we write

$$\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_k \tag{6.1}$$

for the factorization of the prospect space by independent components (with respect to each of which $\succsim$ is sensitive), he has assumed that the orderings induced by $\succsim$ on every product

$$\mathcal{X}_{i_1} \times \mathcal{X}_{i_2} \times \ldots \times \mathcal{X}_{i_j} , \qquad 1 \leqq j \leqq k-1 , \tag{6.2}$$

of j out of the k spaces are independent of the reference vector.

We have already seen that for $k = 3$ independent components only five out of the six such assumptions are needed. As mentioned in footnote 9 above, Gorman [1967] has cut this down further to only two. In the same paper he has given minimal assumptions for the generalization of Proposition 2 to k independent components. To avoid duplication, we mention here only one straightforward extension of Proposition 2 that helps prepare for Chapter 4.

Result C. Let the following orderings, induced on factor spaces by a continuous ordering $\succsim$ on the connected subset (6.1) of a finite-dimensional Euclidean space be independent of the reference vector z,

$$(6.2)\quad \begin{cases} \succsim_i \text{ on } \mathcal{X}_i, & i = 1, 2, \ldots, n, \quad n \geqq 3, \\ \succsim_{i,i+1} \text{ on } \mathcal{X}_i \quad {}_{i+1}, & i = 1, 2, \ldots, k-1. \end{cases}$$

Let $\succsim$ be sensitive in each $\mathcal{X}_i$. Then $\succsim$ is represented on $\mathcal{X}$ by a continuous function of the form

$$(6.3)\qquad U(x) = u_1(x_1) + u_2(x_2) + \ldots + u_n(x_n),$$

unique up to a linear transformation.

6* Proof of Result C. By Proposition 2, the statement is true for $n = 3$. Suppose it is true if n is replaced by $j - 1$, where $4 \leqq j \leqq n$, and consider the ordering $\succsim^z_{1, \ldots, j}$ induced by $\succsim$ on the space

$$\mathcal{X}^{(j)} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_j,$$

using a reference vector z. Then Proposition 2 can be applied to the factorization

$$\mathcal{X}^{(j)} = \mathcal{X}^{(j-2)} \times \mathcal{X}_{j-1} \times \mathcal{X}_j$$

to find a representation of $\succsim^{z}_{1, \ldots, j}$ of the form

$$u_1(x_1) + u_2(x_2) + \ldots + u_{j-1}(x_{j-1}) + u_j(x_j) \, .$$

By induction, $\succsim$ is represented by (6.3). The proof of uniqueness is similar to that for $n = 3$.

7. Reconsideration of the case of two independent components of consumption

To show that the case of $n \geqq 3$ independent components of consumption leads to a more special class of representations than the case $n = 2$, we must show that not every function of the separable form (4.4) can be transformed into the additively separable form

$$(7.1) \qquad U^*(x) = u^*(x_P) + v^*(x_Q) \, .$$

One readily verifies that any ordering representable by (7.1) must satisfy the condition that

$$(7.2) \qquad \left.\begin{array}{l} (x_P, x'_Q) \sim (x'_P, x_Q) \\ \\ (x_P, x''_Q) \sim (x''_P, x_Q) \end{array}\right\} \text{ implies } (x'_P, x''_Q) \sim (x''_P, x'_Q)$$

Given any continuous representation of the separable form (4.4) of an ordering $\succsim$ on $\mathcal{X} = \mathcal{X}_P \times \mathcal{X}_Q$, the test (7.2) can be expressed in terms of the values

$$(u, u', u'') = (u(x_P), u(x'_P), u(x''_P)), \quad (v, v', v'') = (v(x_Q), v(x'_Q), v(x''_Q)) ,$$

assumed by the functions $u(x_P)$, $v(x_Q)$ in the points x , x' , x'' . The configuration of points and indifference curves expressing the test is shown in Figure 7. It is more general than that of the corresponding

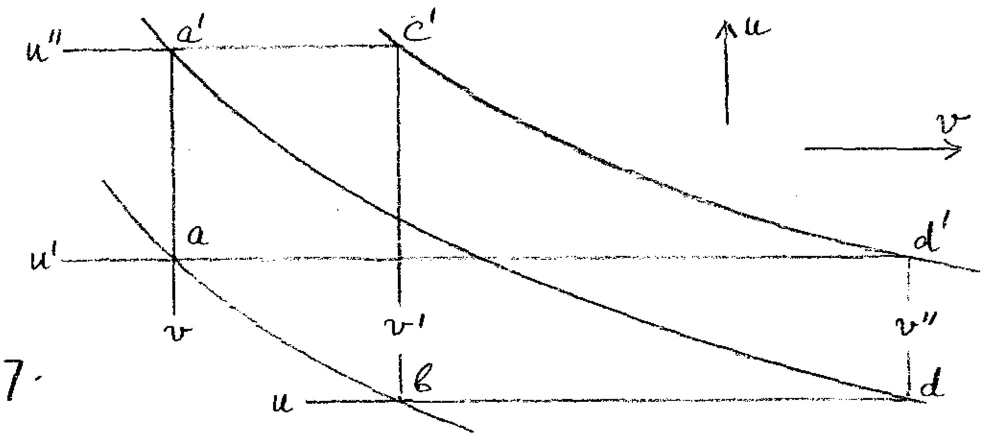

Fig. 7.

points a , b , d , d' , c' , a' in Figure 3, but includes the latter as a special case. Since the latter configuration was already found, in the proof of Proposition 2, to be sufficient for the existence of the representation (7.1), the condition (7.2) is both necessary and sufficient for such representability.

The separable function

$$U(x) = (1 + u)(1 + u + v) , \quad u = u(x_P) \geqq 0 , \quad v = v(x_Q) \geqq 0 ,$$

fails to meet this test for a choice of points x , x', x'' and functions u , v such that

$$(u, u', u'') = (0, 1, 2) , \quad (v, v', v'') = (0, 3, 8) .$$

Hence it cannot be transformed to the form (7.1).

REFERENCES

Arrow, K. J., [1952], "The Determination of Many-Commodity Preferences Scales by Two-Commodity Comparisons," Metroeconomica, Vol. IV, December 1952, pp. 105-115.

__________, [1963], Social Choice and Individual Values, Second Ed., Wiley, 1963.

Aumann, R., [1964], "Subjective Programming," Chapter 12 of Shelly and Brian, Eds., Human Judgments and Optimality, Wiley, 1964, pp. 217-242.

Blaschke, W. and G. Bol, [1938], Geometrie der Gewebe, Springer, Berlin, 1938.

Debreu, G., [1954], "Representation of a Preference Ordering by a Numerical Function," Chapter XI of Decision Processes, Thrall, Coombs and Davis, Eds., Wiley, 1954.

__________, [1959], Theory of Value, Wiley, 1959.

__________, [1960], "Topological Methods in Cardinal Utility Theory," Chapter 2 in Arrow, Karlin, Suppes, Eds., Mathematical Methods in Social Sciences, 1959, Stanford University Press, 1960, pp. 16-26.

Eilenberg, S., "Ordered Topological Spaces," American Journal of Mathematics, Vol. 63, 1941, pp. 39-54.

Goldman, S. M. and H. Uzawa, "A Note on Separability in Demand Analysis," Econometrica, July, 1964, pp. 387-398.

Gorman, W. M., [1959a], "Separable Utility and Aggregation," Econometrica, July, 1959, pp. 469-481.

__________, [1959b], "The Empirical Implications of a Utility Tree: A Further Comment," Econometrica, July 1959, p. 489.

__________, [1965, 196], "Conditions for Additive Preferences," typescript of June, 1965, and "Conditions for Additive Separability," Econometrica, , 196 .

__________, [1967], "The Structure of Utility Functions," Technical Report 151, Inst. for Math. Stud. in the Soc. Sc., Stanford University, August 1, 1967.

Koopmans, T. C. [1966], "Structure of Preference over Time," Cowles Foundation Discussion Paper No. 206, April 15, 1966.

Lancaster, K. T., [1966a], "A New Approach to Consumer Theory," Jn. of Polit. Econ., April 1966.

__________, [1966b], "Change and Innovation in the Technology of Consumption," Am. Econ. Rev., May 1966, pp. 14-23.

Leontief, W., [1947a], "Introduction to a Theory of the Internal Structure of Relationships," Econometrica, October 1947, pp. 361-373.

__________, [1947b], "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives," Bull. Am. Math. Soc., 1947, pp. 343-350.

Rader, T., [1963], "The Existence of a Utility Function to Represent Preferences," Rev. Econ. Stud., XXX, 3,

Samuelson, P. A., [1947], Foundations of Economic Analysis, Harvard University Press, 1947, Chapter VII.

Strotz, R. H., [1957], "The Empirical Implications of a Utility Tree," Econometrica, April 1957, pp. 269-280.

__________, [1959], "The Utility Tree -- A Correction and Further Appraisal," Econometrica, July 1959, pp. 482-488.

Wold, H., [1943], "A Synthesis of Pure Demand Analysis, Part II," Skandinavisk Aktuaritidskrift, 1943, pp. 220-263.

CHAPTER 4

REPRESENTATION OF PREFERENCE ORDERINGS OVER TIME[1]

by

Tjalling C. Koopmans

8. Preference over Time.

In Section 1 of Chapter 3 we have argued the desirability of formalizing the idea of consumers' preference in terms of a preference ordering on a prospect space, before discussing the possibility of representing such an ordering by a utility function. The considerations there adduced have still greater force with regard to problems of evaluative comparison of growth paths for an indefinite future. If one interprets this as an infinite future, neither the concept of a utility function depending on infinitely many variables, nor that of a preference ordering on a space of infinitely many dimensions, have an obvious intuitive understandability about them. To start from the more basic one — the preference ordering — is therefore even more desirable in that case, in that it helps avoid implicit assumptions one is not aware of.

In the present chapter, the propositions of Chapter 3 are applied to the representation of preference orderings over time.

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[1] This chapter reports on research under a grant from the National Science Foundation. It revises and extends Sections 5-9 of Koopmans [1966].

Because of the close connections between the two chapters, the notations are almost identical, and the numbering of sections, propositions and formulae is consecutive over the two chapters. References will be made without distinguishing the chapters. The lists of references to the literature are separate for each chapter.

Before getting into details, a word is in order on the question whose preference is being studied. This question concerns the interpretation and relevance of the analysis, as distinct from the logical connections between the properties of the ordering and the mathematical form of its representation. In regard to preference over time, the simplest interpretation of the orderings that have been studied most thus far is a normative one. One looks at various possible preference orderings that may be adopted, by whatever decision process, for the planning of an economy with a constant population size. New problems arise if population is expected to grow indefinitely or to keep changing in other ways.

Another possible interpretation is that one wishes to study descriptively the preference ordering of an individual with regard to his life-time consumption program, assuming that such an ordering is implicit in his decision. For this interpretation the finite life span and the bequest motive need to be considered as well. For applications of such a preference ordering, see Yaari [1964].

Finally — the ultimate goal of a theory of preference over time for an economy with private wealth — one may wish to examine whether an aggregate preference ordering over time can be imputed, on an "as if" basis, to a society of individual decision makers each guided by his own preference ordering over time.

In all these interpretations, normative or descriptive, the most intriguing problems arise from the fact that the future has a beginning but no discernible end. In contrast to this central problem, the question whether to use a discrete or a continuous time concept seems in the present state of knowledge primarily a matter of research tactics rather than of substance. So far the indications are that axiomatic analysis is somewhat simpler if one chooses discrete time. On the other hand, the maximization of a utility function of a given form under given technological constraints is often simpler with continuous time. We shall therefore here choose discrete time on the basis of expedience without further excuse or explanation.

9. Postulates Concerning a Preference Ordering over Time

We shall adopt a set of five postulates about a preference ordering $\succsim$ on a space ${}_1\mathcal{X}$ of <u>programs</u>, that is, of infinite sequences, denoted

$$(9.1) \qquad {}_1x \equiv (x_1, x_2, x_3, \ldots) ,$$

of vectors

$$(9.2) \qquad x_t \equiv (x_{t1}, x_{t2}, \ldots, x_{tn})$$

associated with successive time periods $t = 1, 2, 3, \ldots$. The <u>program space</u> ${}_1\mathcal{X}$ is the space of all such sequences, in which each vector x_t is a point of the same (single-period) <u>choice space</u> $\mathcal{X}$. Thus the components x_{ti} of x_t refer to a list of commodities, characteristics, or activities (as the case may be), which is the same for all t .

The postulates are modeled after those used in two earlier studies by Koopmans [1960] and by Koopmans, Diamond and Williamson [1964]. The main difference is that the former studies presupposed the existence of a continuous representation. In the present study, the postulates refer to a continuous ordering, and the proximate aim of the study is to derive the existence of a continuous representation. Further differences will be noted in connection with the third and fifth postulates.

The problem of logical independence of the postulates is not investigated. The formulation and sequence of postulates is chosen primarily from the point of view of naturalness of interpretation. One case of recognized dependence between postulates is noted in footnote 4.

It will be usefull occasionally to employ brief notations for finite or infinite segments of the program sequence, as follows

$$(9.3) \qquad {}_1x \equiv (x_1, {}_2x) \equiv (x_1, \ldots, x_{t-1}, {}_tx) \equiv ({}_1x_{t-1}, {}_tx) \text{ .}$$

In an infinite-dimensional space such as ${}_1\mathcal{X}$, the choice of the distance function is crucial for the meaning of the continuity concept implied in it. We shall adopt the function[2/]

$$(9.4) \qquad D({}_1x, {}_1y) \equiv \sup_t d(x_t, y_t)$$

where $d(x_t, y_t)$ is the distance between the t-th period installments x_t, y_t of the programs ${}_1x$, ${}_1y$, according to the definition

$$(9.5) \qquad d(x_t, y_t) \equiv \max_i |x_{ti} - y_{ti}| .$$

P1 (Postulate 1, Continuity). The program space ${}_1\mathcal{X}$ is the space of all programs ${}_1x$ such that, for all t, x_t is in a choice space $\mathcal{X}$, which is a connected subset of n-dimensional Euclidean space. On the program space there exists a complete preference ordering $\succsim$, which is continuous with regard to the distance function (9.4).

P2 (Sensitivity). There exist a program ${}_1x$ in ${}_1\mathcal{X}$ and a vector y_1 in $\mathcal{X}$ such that

$${}_1x = (x_1, x_2, x_3, \ldots.) \succ (y_1, x_2, x_3, \ldots)$$

The first purpose of P2 is to exclude the trivial case where all programs in ${}_1\mathcal{X}$ are equivalent. However, P2 does more than that. It also excludes orderings in which the standing of any

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[2/] The symbol $\sup_t d_t$ denotes the largest of the numbers d_t, $t = 1, 2, 3, \ldots$, if there is a largest, or the smallest number not exceeded by any d_t if there is no largest. Such a number exists whenever $\mathcal{X}$ is bounded, that is, when the range of $d(x,y)$ for all x, y in $\mathcal{X}$ is bounded. If $\mathcal{X}$ is unbounded we admit the possibility that $D({}_1x, {}_1y) = \infty$.

program ${}_1x$ relative to other programs is independent of any vector x_t pertaining to any specific period t, but dependent on the asymptotic behavior of x_t as t tends to infinity.[3/]

Next we introduce two independence postulates, P3' and P3", both of which will be maintained throughout Sections 9-13. In Section 14 we comment briefly on the case where P3" is omitted. In these postulates we employ an arbitrary but fixed reference program,

$$(9.6) \qquad {}_1z = (z_1, {}_2z) = (z_1, z_2, {}_3z) ,$$

to define five orderings, induced by $\succsim$ on factor spaces of ${}_1\mathcal{X}$, and denoted $\succsim_1^z$, ${}_2\succsim^z$, ${}_{1}\succsim_{2}^z$, ${}_3\succsim^z$, $\succsim_2^z$, as follows:

$$(9.7) \begin{cases} x_1 \succsim_1^z y_1 & \text{means} \quad (x_1, {}_2z) \succsim (y_1, {}_2z) \\ {}_2x \; {}_2\succsim^z \; {}_2y & \text{means} \quad (z_1, {}_2x) \succsim (z_1, {}_2y) \\ (x_1, x_2) \; {}_1\succsim_2^z \; (y_1, y_2) & \text{means} \quad (x_1, x_2, {}_3z) \succsim (y_1, y_2, {}_3z) \\ {}_3x \; {}_3\succsim^z \; {}_3y & \text{means} \quad (z_1, z_2, {}_3x) \succsim (z_1, z_2, {}_3y) \\ x_2 \succsim_2^z y_2 & \text{means} \quad (z_1, x_2, {}_3z) \succsim (z_1, y_2, {}_3z) \end{cases}$$

3/ A simple example of such an ordering $\succsim$ satisfying all postulates except P2 is that in which $\mathcal{X}$ is one-dimensional and $\succsim$ is represented by $\lim_{T\to\infty} \sup_{t \geq T} x_t$. This ordering looks only at the highest consumption level that is, ultimately, and again and again thereafter, at least temporarily reached or arbitrarily closely approached. (Note the contrast between succinct mathematical notations and involved equivalent verbal statement!)

P3' (Limited Independence). <u>The two orderings</u> $\succsim_1^z$, $_2\succsim^z$ <u>are independent of the reference program</u> $_1z$.

P3" (Extended Independence). <u>The ordering</u> $_1\succsim_2^z$ <u>is independent of</u> $_1z$.

For convenient reference, we also introduce

P3 (Complete Independence). <u>Both</u> P3' <u>and</u> P3" <u>hold</u>.[4]
Whenever one or both of P3' , P3" are assumed in what follows, the corresponding orderings will be denoted $\succsim_1$, $_2\succsim$, $_1\succsim_2$. Note that $_1\succsim_2$ would have been denoted $\succsim_{1,2}$ in Chapter 3.

In the earlier studies referred to above, the implications of P3' were pursued at length, those of P3 only mentioned briefly. In this study, the emphasis is reversed.

Neither P3' nor P3" can be regarded as realistic. Taken together, they will be found to preclude all complementarity between the consumption of different periods. P3' by itself will be seen to permit a limited complementarity among the utility levels to be associated with consumption in successive periods, but still no complementarity between individual commodities or activities in different periods. P3 or P3' should therefore be looked upon as first approximations, made to facilitate exploration of the implications of the fourth postulate, the real objective of this study:

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[4] By Gorman [1967] (see footnote 9 of Chapter 3), the independence of $_1\succsim_2^z$ and $_2\succsim^z$ implies that of $\succsim_1^z$.

P4 (Stationarity). <u>There exists a first period vector</u> x_1^* <u>in</u> $\mathcal{X}$ <u>with the property that, whenever the programs</u>

$$_1x = (x_1^*, {}_2x) = (x_1^*, x_2, x_3, \ldots.)$$

$$_1y = (x_1^*, {}_2y) = (x_1^*, y_2, y_3, \ldots)$$

<u>are such that</u> $_1x \succsim {}_1y$, <u>then the programs</u>[5/]

$$_1v = (v_1, v_2, v_3, \ldots) = (x_2, x_3, x_4, \ldots) = {}_2x ,$$

$$_1w = (w_1, w_2, w_3, \ldots) = (y_2, y_3, y_4, \ldots) = {}_2y ,$$

<u>defined by</u> $v_t = x_{t+1}$, $w_t = y_{t+1}$, $t = 1, 2, \ldots$, <u>are such that</u> $_1v \succsim {}_1w$.

Before interpreting this postulate in less formal language, we note that, if one particular $x_1 = x_1^*$ in $\mathcal{X}$ has this property, then by P3' every x_1 in $\mathcal{X}$ has this property. Using this, P4 says that if two programs $_1x$, $_1y$ have a common first period vector $x_1 = y_1$, then the programs $_1v$, $_1w$ obtained by deleting x_1 from $_1x$ and from $_1y$, respectively, and advancing the timing of

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5/ In the notations $_2x$, $_2y$ as used here, there is no longer a necessary connection between the presubscript of $_2x$ and the timing of the first installment x_2 of that program. That is, $_2x_2$ simply means the vector that happened to represent second period consumption in the program $_1x$. In the program $_2x = {}_1v$, that same consumption occurs in the first period. With this understanding, the notations $_1v$, $_1w$ will no longer be needed in what follows.

all subsequent vectors by one period, are ordered in the same way as ${}_1x$, ${}_1y$.

It is worth emphasizing that in this statement nothing is said or implied about the ordering of "then future" programs ${}_2x$, ${}_2y$ that may be applied after the first period has elapsed. That is, no question of consistency or inconsistency of orderings adopted at different points in time is raised.[6] Only the ordering $\succsim$ applying "now" is under discussion. Applied repeatedly, P4 implies that the present ordering of two programs $(x_1, \ldots, x_{t-1}, {}_tx) \equiv ({}_1x_{t-1}, {}_tx)$ and $({}_1x_{t-1}, {}_ty)$ that start to differ in a designated way only from some point t in time onward is independent of what that point in time is.

The fifth and last postulate asserts, roughly, that the end result of an infinite sequence of improvements starting from some given program is itself an improvement over that progrom. If all but a finite number of the improvements affect the program in only a finite number of future periods, such an assertion is already implied in P1, P3', P4. For simplicity we will refer only to a sequence of improvements made to successive vectors in the program, taken one at a time. A similar postulate has been used by Diamond [1965]. An alternative postulate in terms of improvements affecting several

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6/ For a discussion of that question, see Strotz [1957].

periods at a time is briefly considered in subsection 13* below.

P5 (Monotonicity). If ${}_1x$, ${}_1y$ are programs such that, for all $t = 1, 2, \ldots$,

$$(x_1, x_2, \ldots, x_{t-1}, y_t, y_{t+1}, y_{t+2}, \ldots) \precsim (x_1, x_2, \ldots x_{t-1}, x_t, y_{t+1}, y_{t+2}, \ldots) ,$$

then ${}_1y \precsim {}_1x$.

It can be shown that, given all other postulates, P5 is implied in the following stronger postulate, used in a previous study [Koopmans, 1960].

P5' (Extreme Programs). There exist in ${}_1\mathcal{X}$ a best and a worst program.

There is some interest in avoiding that stronger statement whereever possible, with a view to problems of optimal growth under continuing technical change.

On the basis of the postulates set out, we seek to construct a representation of $\succsim$ on the entire program space ${}_1\mathcal{X}$, or on as large a subspace of it as we can. Our strategy will be first to find such representations on suitably chosen subspaces of ${}_1\mathcal{X}$.

10. Representation of $\succsim$ on any subspace of ultimately identical programs.

Since the space ${}_1\mathcal{X}$ is infinite-dimensional, Proposition 1 cannot be directly applied to the ordering $\succsim$ given on it. For this reason, we shall in the present section study $\succsim$ on the subspace ${}_1\mathcal{X}_T^z$ of all programs of the form

$$(10.1) \qquad {}_1x = ({}_1x_T,\ {}_{T+1}z) \ ,$$

where ${}_1z$ is again an arbitrary but fixed reference program. Since programs in this subspace differ only in the segments ${}_1x_T$, the ordering $\succsim$ on ${}_1$ restricted to the subspace ${}_1\mathcal{X}_T^z$ induces an ordering of sequences ${}_1x_T$ of length T on the space ${}_1\mathcal{X}_T$. We shall denote this ordering by ${}_1\succsim_T$. In Subsection 10* we shall prove

Result D. For all T, *the ordering* ${}_1\succsim_T$ *is independent of* ${}_1z$, *and is represented by a function of the form*

$$(10.2) \qquad U_T({}_1x_T) = u(x_1) + \alpha u(x_2) + \dots + \alpha^{T-1}u(x_T) \ , \qquad 0 < \alpha < 1 \ .$$

Here $u(x)$ *is a continuous function defined on* $\mathcal{X}$, *and both* α *and* $u(x)$ *are independent of* T.

The proof proceeds through a succession of statements which we label (Da), (Db), ..., recording in each case the postulates and/or previous results used in the proof. The notations for induced orderings extend those of (9.7).

(Da; P3', P4) The ordering ${}_t\succsim^z$ of sequences ${}_t x$, defined by restricting $\succsim$ to the set of programs $({}_1z_{t-1}, {}_tx)$ is independent of ${}_1z$ and of t .

(Db; P3', P4) $\succsim_t^z$ is independent of ${}_1z$ and of t .

(Dc; P3, P4) ${}_{t-1}\succsim_t^z$ is independent of ${}_1z$ and of t .

(Dd; C, Db, Dc) ${}_1\succsim_T^z$ is independent of ${}_1z$, and is represented by a continuous function of the form

$$(10.3) \qquad U_T({}_1x_T) = u_1(x_1) + u_2(x_2) + \dots + u_T(x_T) ,$$

unique up to a linear transformation similar to (5.5).

(De; Dd, P4) One can choose the $u_i(x_i)$ in (10.3) in such a way that (10.2) holds with $\alpha > 0$, where α is unique, and where $u(x)$ is unique up to a linear transformation

$$(10.4) \qquad u^*(x) = \beta + \gamma u(x) .$$

(Df; De, P5) $\alpha < 1$.

10* Proof of Result D. Clearly the continuity of $\succsim$ entails the continuity of all restricted orderings induced by it.

(Da). P3' allows us to write

$$(10.5) \qquad \succsim^z_1 = \succsim_1 , \qquad {}_2\succsim^z = {}_2\succsim .$$

Using the symbol $\Longleftrightarrow$ to denote logical equivalence, these statements are made explicit by

$$(10.6) \quad \text{for all } {}_2x^* ,\ x_1 ,\ y_1 ,\ (x_1, {}_2z) \succsim (y_1, {}_2z) \Longleftrightarrow (x_1, {}_2x^*) \succsim (y_1, {}_2x^*)$$

$$(10.7) \quad \text{for all } x_1^* ,\ {}_2x ,\ {}_2y ,\ (z_1, {}_2x) \succsim (z_1, {}_2y) \Longleftrightarrow (x_1^*, {}_2x) \succsim (x_1^*, {}_2y)$$

In particular, choosing for x_1^* in (10.7) the x_1^* occurring in P4, we have from P4

$$(10.8) \quad \text{for all } {}_2x ,\ {}_2y ,\ (z_1, {}_2x) \succsim (z_1, {}_2y) \Longleftrightarrow {}_2x \succsim {}_2y ,$$

an implication which can be applied once more to give

$$(z_1, z_2, {}_3x) \succsim (z_1, z_2, {}_3y) \Longleftrightarrow (z_2, {}_3x) \succsim (z_2, {}_3y) \Longleftrightarrow {}_3x \succsim {}_3y , \quad \text{etc.}$$

These results are summarized in

$$(10.9) \qquad {}_t\succsim^z = {}_t\succsim = \dots = {}_2\succsim = \succsim , \qquad t = 2, 3, \dots,$$

keeping in mind the notational practice explained in footnote 4 .

(D5). From (10.8) and (10.6), for all ${}_1x^*$,

$$(z_1, x_2, {}_3z) \succsim (z_1, y_2, {}_3z) \Longleftrightarrow (x_2, {}_3z) \succsim (y_2, {}_3z) \Longleftrightarrow$$

$$\Longleftrightarrow (x_2, {}_3x^*) \succsim (y_2, {}_3x^*) \Longleftrightarrow (x_1^*, x_2, {}_3x^*) \succsim (x_1^*, y_2, {}_3x^*) .$$

This reasoning and its repetition yield

$$(10.10)\quad \succsim_t^z = \succsim_t = \dots = \succsim_2 = \succsim_1 , \qquad t = 1, 2, 3, \dots .$$

(Dc) We now bring in P3", written as ${}_1\succsim_2^z = {}_1\succsim_2$. Together with (10.8) this implies, for all ${}_1x^*$,

$$(z_1, x_2, x_3, {}_4z) \succsim (z_1, y_2, y_3, {}_4z) \Longleftrightarrow (x_2, x_3, {}_4z) \succsim (y_2, y_3, {}_4z) \Longleftrightarrow$$

$$\Longleftrightarrow (x_2, x_3, {}_4x^*) \succsim (y_2, y_3, {}_4x^*) \Longleftrightarrow (x_1^*, x_2, x_3, {}_4x^*) \succsim (x_1^*, y_2, y_3, {}_4x^*) .$$

Since this can again be repeated, we have

$$(10.11)\quad {}_{t-1}\succsim_t^z = {}_{t-1}\succsim_t = \dots = {}_2\succsim_3 = {}_1\succsim_2 , \qquad t = 2, 3, \dots .$$

(Dd) We consider ${}_1\succsim_T^z$, and note that $\succsim_t^z$, $t = 1, \dots, T$ and ${}_{t-1}\succsim_t^z$, $t = 2, \dots, T$, are all independent of ${}_1z$. By P2 , $\succsim_1$ permits $x_1 \succ_1 y_1$, and by (10.10) a similar statement holds for $\succsim_t$, $t = 2, 3, \dots$. The premises of Result C of Section 6 are therefore satisfied, and the representation (10.3) follows. Hence ${}_1\succsim_T^z$ is independent of z , and we write ${}_1\succsim_T$ from here on.

(De) By (10.8) and (10.3), ${}_2\succsim_T$ is represented on ${}_2\mathcal{X}_T$ by either of the functions

$$u_2(x_2) + u_3(x_3) + \dots + u_T(x_T) ,$$

$$u_1(x_2) + u_2(x_3) + \dots + u_{T-1}(x_T) .$$

It follows, along the lines of the uniqueness proof for Proposition 2, that, for all x in $\mathcal{X}$,

$$u_{t+1}(x) = \beta_t + \alpha u_t(x) , \qquad t = 1, \ldots, T - 1 , \quad \alpha > 0 .$$

Since we are free to choose each $u_t(x)$, $t = 2, \ldots, T - 1$, so as to have $\beta_t = 0$ for all t , (10.2) results, with $u(x) = u_1(x)$.

(Df) By P2, there exist vectors $\underline{\underline{x}}$, $\overline{\overline{x}}$ in $\mathcal{X}$ such that $\underline{\underline{x}} \prec_1 \overline{\overline{x}}$, and hence

$$u(\underline{\underline{x}}) < u(\overline{\overline{x}}) .$$

Since $\mathcal{X}$ is connected, there exists, by a reasoning illustrated by Figure 8 on page 16, a point $\underline{x}$ in $\mathcal{X}$ with the properties

$$(10.12) \begin{cases} (10.12a) & u(\underline{\underline{x}}) = u(\underline{x}) \\ (10.12b) & \text{for each } \delta > 0 \text{, there exists } x' \text{ in } \mathcal{X} \text{ such that } d(\overline{x}, x') \leqq \delta \text{ and } u(x') > u(\underline{x}) . \end{cases}$$

Denote by

$$_{\text{con}}\, x = (x, x, x, \ldots)$$

the program in which $x_t = x$ for all t . Since $\underline{x} \prec_1 \overline{x}$, we have

$$_{\text{con}}\, \underline{x} = (\underline{x}, {}_{\text{con}}\, \underline{x}) \prec (\overline{x}, {}_{\text{con}}\, \underline{x}) .$$

Hence, by the continuity of $\succsim$, there exists $\delta > 0$ such that

(10.13) $\quad D({}_1x, {}_{con}\underline{x}) \leqq \delta \quad$ implies $\quad {}_1x \prec (\bar{x}, {}_{con}\underline{x})$.

We choose an x' satisfying (10.12b) for that δ . It follows that

(10.14) $D(({}_{con}x'_T, {}_{con}\underline{x}), {}_{con}\underline{x}) \leqq \delta \quad$ for all T ,

if ${}_{con}x'_T$ denotes ${}_1x_T$ with $x_t = x'$ for $t = 1, \ldots, T$. But then we have, by (10.2), (10.13), (10.14), for all T ,

$$\sum_{t=1}^{T} \alpha^{t-1} u(x') < u(\bar{x}) + \sum_{t=2}^{T} \alpha^{t-1} u(\underline{x}) ,$$

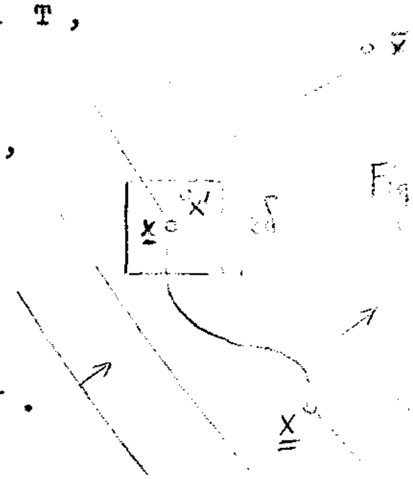

Fig. 8

or

$$u(\bar{x}) > u(\underline{x}) + (u(x') - u(\underline{x})) \sum_{t=1}^{T} \alpha^{t-1} .$$

Since, by (10.12b), $u(x') - u(\underline{x}) > 0$, this can be true for all T only if $\alpha < 1$.

11. Representation of $\succsim$ on the space of ultimately constant programs.

In this section we choose a favorable ground on which to face the infinite horizon by first restricting ourselves to the space ${}_{con}\mathcal{X}$ of <u>constant programs</u>

(11.1) $\qquad {}_{con}x \equiv (x, x, x, \ldots)$, that is, of programs ${}_1x$ for which $x_t = x$ for all t .

The points of ${}_{\text{con}}\mathcal{X}$ are in a one-to-one correspondence

(11.2) $${}_{\text{con}}x \longleftrightarrow x$$

to those of $\mathcal{X}$. Because, for all x , x' in $\mathcal{X}$,

(11.3) $$D({}_{\text{con}}x, {}_{\text{con}}x') = d(x, x') ,$$

this correspondence preserves the distance function, and therewith the continuity concept. Moreover, if x , y are vectors of $\mathcal{X}$ such that $y \precsim_1 x$, then, by Db and P5,

(11.4) $${}_{\text{con}}y \precsim (x, {}_{\text{con}}y) \precsim \dots \precsim ({}_{\text{con}}x_T, {}_{\text{con}}y) \precsim \dots \precsim {}_{\text{con}}x \ .$$

The continuous ordering $\succsim_1$ on $\mathcal{X}$ is therefore transformed by the correspondence (11.2) into the ordering $\succsim$ restricted to ${}_{\text{con}}\mathcal{X}$.

In particular,

<u>Result</u> <u>E</u>. <u>Any continuous representation</u> $u(x)$ <u>of</u> $\succsim_1$ <u>on</u> $\mathcal{X}$ <u>is at the same time a continuous representation of</u> $\succsim$ <u>restricted to</u> ${}_{\text{con}}\mathcal{X}$.

Note that only limited independence (P3') was used in the proof of Result E.

Next we consider the space $\mathcal{X}_{con}$ of <u>ultimately constant programs</u>, that is, of programs such that, for some $T \geqq 0$,

$$(11.5) \qquad {}_1x = ({}_1x_T, {}_{con}x) = (x_1, \ldots, x_T, x, x, \ldots)$$

(for $T = 0$ the term ${}_1x_T$ is absent). One readily verifies that the reasoning that led to Result D also applies in any subspace $\mathcal{X}_{con}^{(T)}$ of $\mathcal{X}_{con}$ consisting of programs (11.5) <u>with a fixed</u> T . The only difference consists in an added term in (10.2). One now finds for all $T \geqq 2$ a continuous representation of $\succsim$, restricted to $\mathcal{X}_{con}^{(T)}$, by the function

$$(11.6) \quad u(x_1) + \alpha u(x_2) + \ldots + \alpha^{T-1}u(x_T) + f_T(u(x)) , \qquad 0 < \alpha < 1 ,$$

where $f_T(u)$ is continuous and increasing. From this representation we can derive two representations of $\succsim$ restricted to $\mathcal{X}_{con}^{(T-1)}$, one (11.7a) by setting $x_1 = x_1^*$ and applying P4, the other (11.7b) by setting $x_T = x$, as follows,

$$(11.7) \begin{cases} (a) \quad U^{(a)}({}_1x) \equiv \alpha u(x_1) + \ldots + \alpha^{T-1}u(x_{T-1}) + f_T(u(x)) \\ (b) \quad U^{(b)}({}_1x) \equiv u(x_1) + \ldots + \alpha^{T-2}u(x_{T-1}) + \alpha^{T-1}u(x) + f_T(u(x)) \end{cases}$$

By Result C these representations are, for all $T \geqq 3$, unique up to a linear transformation. Comparison of the first terms shows that

$$U^{(a)}({}_1x) = \alpha U^{(b)}({}_1x) + \beta \ ,$$

which implies that

$$f_T(u) = \alpha^T u + \alpha f_T(u) + \beta \ , \quad f_T(u) = \frac{\alpha^T}{1-\alpha} u + \frac{\beta}{1-\alpha} \ .$$

Dropping the constant term, we have

<u>Result F. On the space $\mathcal{X}_{con}$ of ultimately constant programs,</u> $\succsim$ <u>is represented by the continuous function</u>

$$(11.8) \quad U({}_1x) = U({}_1x_T, {}_{con}x) \equiv u(x_1) + \alpha u(x_2) + \ldots + \alpha^{T-1} u(x_T) + \frac{\alpha^T}{1-\alpha} u(x) \ ,$$

unique up to a linear transformation. Note that in this function T itself depends on the given ultimately constant program ${}_1x$. For definiteness one can specify that $T + 1$ is the earliest time from which onward ${}_1x$ is constant. However, the same value of $U({}_1x)$ is obtained if one allows $T + 1$ to be any time, earliest or not, from which onward ${}_1x$ is constant. It is for that reason that the function (11.8) represents $\succsim$ on the space $\mathcal{X}_{con}$ for <u>all</u> ultimately constant programs, regardless of the values of their "minimal" T .

12. Representation of $\succsim$ on the space of programs bounded in utility

It is now possible to indicate a large subspace of the program space on which the ordering $\succsim$ is represented by

$$(12.1) \qquad U({}_1x) = \sum_{t=1}^{\infty} \alpha^{t-1} u(x_t) \ , \qquad 0 < \alpha < 1 \ .$$

We shall call a program ${}_1x$ bounded in utility if there exist vectors $\underline{x}$, $\overline{x}$ in $\mathcal{X}$ with $\underline{x} \prec_1 \overline{x}$ such that

$$(12.2) \qquad \underline{x} \precsim_1 x_t \precsim_1 \overline{x} \quad \text{for all} \quad t = 1, 2, \ldots$$

We can then show

<u>Proposition</u> 3. <u>On the space ${}_1\mathcal{X}^*$ of all programs bounded in utility, the ordering $\succsim$ is represented by the continuous function</u> (12.1).

It is to be noted that for ultimately constant programs, the function (12.1) is identical with that in (11.8). Hence Proposition 3 includes Result F.

12* <u>Proof of Proposition 3</u>. We first note that if ${}_1x$ is bounded in utility, then,

$$u(\underline{x}) \leqq u(x_t) \leqq u(\overline{x}) \qquad \text{for all} \quad t \ ,$$

and, since $0 < \alpha < 1$, the series in (12.1) is absolutely convergent, hence its sum exists and is continuous with respect to ${}_1x$.

Now let ${}_1x$ and ${}_1y$ be two programs bounded in utility, and define bounds applicable to both ${}_1x$ and ${}_1y$ by

$$\underline{z} \equiv \begin{cases} \underline{x} & \text{if } \underline{x} \precsim_1 \underline{y} \\ \underline{y} & \text{if } \underline{y} \precsim_1 \underline{x} , \end{cases} \qquad \overline{z} \equiv \begin{cases} \overline{x} & \text{if } \overline{x} \succsim_1 \overline{y} \\ \overline{y} & \text{if } \overline{y} \succsim_1 \overline{x} , \end{cases}$$

$$\underline{u} \equiv u(\underline{z}) , \quad \overline{u} \equiv u(\overline{z}) , \quad \text{so} \quad \underline{u} < \overline{u} .$$

Assume first that $U({}_1x) > U({}_1y)$, and write

$$U({}_1x) - U({}_1y) \equiv 3\Delta > 0$$

for the utility difference of ${}_1x$ and ${}_1y$. For comparison purposes we consider two programs

$${}_1x^{(T)} = ({}_1x_T, {}_{con}\underline{z}) , \quad {}_1y^{(T)} = ({}_1y_T, {}_{con}\overline{z}) ,$$

where T is chosen large enough to have

$$\left(\sum_{t=T+1}^{\infty} \alpha^{t-1} \right) (\overline{u} - \underline{u}) = \alpha^T \cdot \frac{\overline{u} - \underline{u}}{1 - \alpha} \leqq \Delta$$

Since then

$$U({}_1x) - U({}_1x^{(T)}) = \sum_{t=T+1}^{\infty} \alpha^{t-1} (u(x_t) - \underline{u}) \leqq \Delta , \quad U({}_1y^{(T)}) - U({}_1y) \leqq \Delta ,$$

we must have

$$U({}_1x^{(T)}) - U({}_1y^{(T)}) \geqq \Delta$$

Since ${}_1x^{(T)}$, ${}_1y^{(T)}$ are ultimately constant, this implies ${}_1x^{(T)} \succ {}_1y^{(T)}$ by Result F. But then, using P5, ${}_1x \succsim {}_1x^{(T)} \succ {}_1y^{(T)} \succsim {}_1y$, which yields

(12.3) $$U({}_1x) > U({}_1y) \quad \text{implies} \quad {}_1x \succ {}_1y ,$$

confirming the representation (9.1) in this case.

Assume next that, for two programs ${}_1x$, ${}_1y$ bounded in utility,

(12.4) $$U({}_1x) = U({}_1y) \quad \text{but} \quad {}_1x \prec {}_1y .$$

Then there exists t_o such that

(12.5) $$x_{t_o} \prec_1 y_{t_o} , \quad \text{so} \quad u(x_{t_o}) < u(y_{t_o}) ,$$

because " $x_t \succsim_1 y_t$ for all t " would contradict " ${}_1x \prec {}_1y$ " by P5. On a curve in $\mathcal{X}$ connecting x_{t_o} with y_{t_o} , by a reasoning used in (Df) above, there exists a point $\underline{x}_{t_o}$ such that $\underline{x}_{t_o} \sim x_{t_o}$ while there are points x'_{t_o} with $x_{t_o} \prec_1 x'_{t_o}$ arbitrarily close to

$\underline{x}_{t_o}$. Let x'_{t_o} be chosen, using P1 and (12.3), so that

$$_1x \prec {_1x'} \equiv ({_1x_{t_o-1}}, x'_{t_o}, {_{t_o+1}x}) \prec {_1y} .$$

Then, by (12.5),

$$U({_1x'}) > U({_1x}) = U({_1y}) \quad \text{but} \quad {_1x'} \prec {_1y} ,$$

a contradiction of (12.3). Hence (12.4) is false, and

$$U({_1x}) = U({_1y}) \quad \text{implies} \quad {_1x} \sim {_1y} ,$$

confirming (12.1) in this case as well. Since the third case, $U({_1x}) < U({_1y})$, is symmetric to the first, the proof is now complete.

13. Concluding remarks on the representation of $\succsim$.

The representations we have found show unexpectedly strong implications of the postulates used. It turns out that offsetting program changes in future periods can be determined on the basis of just two mathematical data,

(i) the function $u(x)$ which allows the comparison of "utility differences" within the same period, and

(ii) a constant discount factor α which extends that comparison to utility differences in different periods.

The representation may be called cardinal in the sense that only increasing linear transformations, applied simultaneously to $u(x)$ and to $U({}_1x)$, will preserve these simple properties.

Since $\alpha < 1$ the present postulates do not permit expression of the ethical principle of treating all future generations' utilities on a par with present utilities. A way has been found to include that limiting case in models of optimal growth by retreating to the notion of a partial ordering. Von Weizsäcker [1965] has proposed to call a program ${}_1x$ better than a program ${}_1y$ if there exists a $T \geqq 1$ such that

$$\sum_{t=1}^{T'} u(x_t) > \sum_{t=1}^{T'} u(y_t) \quad \text{for all} \quad T' \geqq T .$$

This criterion has been called the overtaking criterion by Gale [1967]. Under appropriate conditions, it has permitted determination of an optimal path which turns out to be comparable with, and better than, every other feasible path [Koopmans,1965,1967a].

Returning to the case of a complete ordering with a discount factor $\alpha < 1$, it is conceivable that the representation (12.1) can be extended on the basis of the present postulates to larger sets of programs not all bounded in utility. In Subsection 13* we allude to a reasoning from a strengthened monotonicity postulate that permits an extension to all programs for which the sum (12.1) exists.

It will be clear that, if $u(x)$ is unbounded on $\mathcal{X}$, then there exist programs for which the sum (12.1) diverges. In such cases the representation (10.2) restricted to a class of ultimately identical programs, all "divergent in utility," may still be valuable. It would permit formulating a partial optimality criterion in which a path is found to stand comparison with all other feasible paths differing from it in a finite number of future periods only. Other considerations would then have to be brought to bear on the choice of the class of ultimately identical programs.

13* One might wish to strengthen P5 to

P5" (Strong Monotonicity). _If_ ${}_1x$, ${}_1y^{(i)}$, $i = 1, 2, \ldots,$ _are programs such that_

$$\left.\begin{array}{l} {}_1y^{(i)} \precsim {}_1y^{(i+1)} , \\ \\ {}_1y^{(i)}_{t_i} = {}_1x_{t_i} , \quad t_i < t_{i+1} \end{array}\right\} \text{for all } i = 1, 2, \ldots$$

then ${}_1y \precsim {}_1x$.

This postulate considers successive improvements each extending over an arbitrary number of periods, but where the set of periods affected by successive improvements becomes more and more remote in time. It allows one, for any program ${}_1x$ for which the sum (12.1) exists, to construct an equivalent constant program ${}_{con}x$ such that $U({}_1x) = U({}_{con}x)$, thus extending the representation (12.1) to all programs for which that sum exists. Conversely, for any program $_1x$ equivalent to a constant program, the sum (12.1) does exist.

14. Limited independence, time perspective and impatience.

If instead of complete independence (P3) we postulate only limited independence (P3'), Proposition 2 is not available, and we must fall back on Result A. A study along these lines was made in two consecutive papers by Koopmans [1960] and by Koopmans, Diamond and Williamson [1964]. The postulates of that study were the analogues of the present postulates of continuity (P1), sensitivity (P2), limited independence (P3'), stationarity (P4) and the existence of extreme programs (P5'), applied to a given utility function $U(_1x)$ rather than to an ordering.[7]

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[7] Apart from this difference, P1 was strengthened to make P1', say, by adding two statements: (a) that the continuity on $\mathcal{X}$ of $U(_1x)$ is uniform on each equivalence set, (b) that $\mathcal{X}$ is bounded and convex. The latter was used in the proof that the range $\mathcal{U}$ of $U(_1x)$ is an interval. Alternatively, that result could have been obtained by adding to P5' that among the extreme programs there are a best and worst constant program, or by deriving that statement in turn from P5 restricted to $\sim$.

A theorem by Diamond [1965, p. 173] now allows us to obtain all the results of the previous study from the present postulates P1' (see footnote 7), P2, P3', P4, P5' as applied to an ordering $\succsim$ on ${}_1\mathcal{X}$. The resulting representation $U({}_1x)$ of $\succsim$ is found to satisfy a <u>recursive relation</u>

$$(14.1) \qquad U({}_1x) = V(u(x_1), U({}_2x)) ,$$

where $V(u, U)$ is a continuous function defined on the product of two nondegenerate intervals, which is increasing in each of its variables. This <u>aggregator function</u> indicates how the single-period utility $u(x_1)$ of the first installment x_1 of ${}_1x$ and the utility $U({}_2x)$ of the sequel ${}_2x$ (were that sequel to start immediately) are combined to form the utility of the entire program ${}_1x$. In particular, if P3 holds, $V(u, U) = u + \alpha U$.

The representation (14.1) is <u>ordinal</u> in the sense that any pair of continuous increasing functions Φ, φ with the appropriate domains will define an alternative representation

$$(14.2) \quad U^*({}_1x) \equiv \Phi(U({}_1x)) = \Phi(V(u(x_1), U({}_2x)) = V^*(u^*(x_1), U^*({}_2x)) ,$$

say, where

$$(14.3) \quad u^*(x) \equiv \varphi(u(x)) , \qquad V^*(u^*, U^*) \equiv \Phi(V(\varphi^{-1}(u^*), \Phi^{-1}(U^*))) .$$

This being so, the question arises what takes the place of the discount factor α, the existence of which was derived in Section 10 from P3. In particular, what corresponds to the inequality $\alpha < 1$ crucial to convergence of the representation (12.1)?

It is readily seen from (14.2) and (14.3) [Koopmans, 1960, Section 14*] that, in the case of a differentiable function $V(u, U)$, the discount factor associated with a constant program

${}_{con}x = (x, x, x, \dots)$,

$$(14.4) \qquad \alpha(x) \equiv \left(\frac{\partial V(u, U)}{\partial U}\right)_{u = u(x), \quad U = U({}_{con}x)} \epsilon [0,1] ,$$

is invariant under differentiable increasing scale changes for u and U. Moreover, as distinct from the representation (12.1), $\alpha(x)$ in (14.4) can vary with x. The limited independence postulate P3' therefore allows scope for the idea already expressed by Irving Fisher [1930, Ch. IV, §3, §6] with regard to individual preferences: that the discount factor may depend on the level of present and prospective income.

As an illustration, let $\mathcal{X}$ be the closed unit interval $\mathcal{I} = [0, 1]$, let $u(x) = x$, and consider the aggregator function

$$(14.5) \qquad V(x, U) = U + (x - U)(a - bx + cU) ,$$

where we require that

$$(14.6) \qquad b,\ c,\ a - 2b,\ a - b - c,\ 1 - a - 2c > 0 .$$

Then, if we assign to U the same range $\mathcal{I}$, $V(x, U)$ is strictly increasing in both variables, and

$$(14.7) \qquad V(0, 0) = 0 , \qquad V(1, 1) = 1 .$$

Finally, since $U({}_{con}x) = x$ is the only root U of $U = V(x, U)$ in the range $\mathcal{I}$,

$$(14.8) \qquad \alpha(x) = 1 - a + (b - c)x \; .$$

Hence the direction of change of $\alpha(x)$ with increasing income x is given by the sign of $(b - c)$. Following Fisher [1930, Ch. IV, §6], most economists I have consulted regard an increasing $\alpha(x)$ as the normal case. This implies that the ratio of the marginal utility of future consumption to that of present consumption increases as the level of the constant consumption flow ${}_{con}x$ is raised. Examples where the sign of $d\alpha(x)/dx$ depends on x can also be constructed.

While $\alpha(x)$ is defined only for constant programs, there is a generalization[8/] of the convergence condition $\alpha < 1$ in (12.1)

8/ This generalization has been derived from statement (a) in footnote 7, here used for the first time. The proof uses the theory of Haar measure.

to the present case that applies in the entire range of $V(u, U)$. It is found that there exists a transformation function Φ (here φ does not play a role) such that the function $V^*(u\;, U^*)$ in (14.3) satisfies (dropping asterisks)

$$(14.9) \quad V(u, U') - V(u, U) \leqq U' - U \quad \text{whenever} \quad U' > U \; .$$

This inequality has been called the (weak) time perspective property of the utility scale resulting from the transformation Φ . It says that the utility difference between two programs, measured in a suitable scale, does not increase (and generally diminishes) if both programs are postponed by one or more periods, while the same consumption or the same sequence of consumptions is inserted in the gaps so created. This inequality between utility differences (though not the ratio of the differences themselves) is satisfied by a class of scales linked by transformations that include nonlinear as well as all linear transformations. For this reason, a representation $U({}_1x)$ satisfying (14.1) where $V(u, U)$ has the property (14.9) has been called quasi-cardinal.

There are indications that the weak inequality sign $(\leq)$ in (14.4) can be strengthened to strict inequality $(<)$, referred to as strong time perspective, without strengthening the postulates. If so, it follows that the function $U({}_1x)$ can be reconstructed from a pair of functions $u(x)$, $V(u, U)$ implied in it. The example (14.5), (14.6) has the strong time perspective property as it stands, without requiring a prior scale change.

Precisely because it compares utility differences between pairs of programs, the time perspective inequality, strong or weak, does not by itself predict the choice within any one pair of programs. However, by elementary steps of reasoning, (14.9) implies a second family of ordinal inequalities, of which the simplest representative is

$$(14.10)\begin{cases}\text{if} \quad u = u(x) < u' = u(x')\ , \quad U({}_{con}x) \leqq U \leqq U({}_{con}x')\ , \\ \text{then} \quad V(u', V(u, U)) \underset{(=)}{>} V(u, V(u', U))\ . \end{cases}$$

This inequality, weak or strong depending on whether the inequality (14.9) is weak or strong, has been called an impatience inequality. It indicates that if the single-period utility of a vector x' exceeds that of a vector x, then any program $(x', x, {}_3x)$, in which ${}_3x$ is selected from a wide class of "continuations," is preferred (or equivalent) to the corresponding program $(x, x', {}_3x)$ in which the better item is moved from first to second place. The class of continuations ${}_3x$ permitted in (14.10) consists of all those which, if started immediately, would be ranked between ${}_{con}x$ and ${}_{con}x'$. This condition should be read in conjunction with Result E of Section 11, which holds also under the present assumptions.

The impatience inequality holds for a wider range of U-values than that indicated in (14.10), and can be generalized to the interchange of two segments ${}_tx_{t'}$, ${}_sx_{s'}$ of a program, that need not be of equal length or contiguous in time.

15. Nonstationary orderings and eventual impatience.

Diamond has studied the implications of postulates similar to those of this chapter, with the main difference that no explicit stationarity postulate corresponding to our P4 is present. However, a certain comparability over time is introduced by assuming, in one interpretation, that there is only a single consumption good ($\mathcal{X}$ is the closed unit interval $\mathcal{J}$), more of which is always better. In another interpretation leading to the same methematical analysis, there is a given single-period utility function $u(x)$ mapping $\mathcal{X}$ onto $\mathcal{J}$, which is the same for all t. For simplicity, we shall adopt the notation of an ordering $\succsim$ of all programs ${}_1x$ on the denumerable product space $\mathcal{J} \times \mathcal{J} \times \ldots = {}_1\mathcal{J}$, say, that corresponds to the first interpretation. The nonstationarity then applies to the way in which the sequences ${}_1x$ of scalars x_t enter into $\succsim$.

Diamond's postulates then can be shown[9] to be equivalent to specializations, to the case $\mathcal{X} = \mathcal{J}$, of our P1, P3, supplemented by a postulate P6 implying similar specializations of P2, P5, P5',

P6 (General Monotonicity) <u>If</u> $x_t \geqq y_t$ <u>for all</u> t, $x_t > y_t$ <u>for some</u> t, <u>then</u> ${}_1x \succ {}_1y$.

From these assumptions he derives the following property of <u>eventual impatience.</u>[10] For any given program ${}_1x$ and any number $\epsilon > 0$, there exists a T such that

$$(15.2) \quad {}_1x \succ (x_t, {}_2x_{t-1}, x_1, {}_{t+1}x) \quad \text{for all } t \geqq T \text{ with } x_1 \geqq x_t + \epsilon .$$

In words, the interchange with x_1 of any x_t which occurs sufficiently far into the future, and which falls short of x_1 by at least ϵ, diminishes the utility of the program ${}_1x$. This subtle result, which appears to miss its aim by a hair's breadth, is both vindicated and complemented

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10/ Diamond [1965], p. 175.

9/ Using the results of Gorman [1967] referred to in footnote 9 of Chapter 3.

by another theorem,[11] attributed to Yaari, which hits the hair on the head. It states that the present specialization of P1 P6 and

taken together are incompatible with the statement

$$\text{for all } t \text{ and all } {}_1x \text{ in } {}_1\mathcal{X}, \quad {}_1x \sim (x_t, {}_2x_{t-1}, x_1, {}_{t+1}x),$$

that expresses "equal treatment of all generations."

Similar but somewhat stronger conclusions are obtained by Diamond by changing the distance function underlying P1 to

$$D^*({}_1x, {}_1y) = \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t d(x_t, y_t),$$

presumably because this modification explicitly reduces the weight attached, in the definition of continuity, to given consumption differences in a more distant future.

16. Concluding remarks.

The main results of the studies reported in this chapter appear to be twofold.

In the first place the studies show a sequence of instances of increasing generality, in which a complete and continuous preference ordering of consumption programs for an infinite future necessarily gives a decreasing, or eventually decreasing, weight to consumption in a more distant future. Somewhat fancifully, one may say that the

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[11] Diamond [1965], p. 176.

real numbers appear to be a sufficiently rich set of labels to accommodate in a continuous manner all infinite sequences of consumption vectors only if one gradually or eventually decreases the weight given to the more distant vectors in the preference ordering to be represented.

Secondly, the studies containing the stationarity postulate P4 have produced interesting special forms for the utility function $U({}_1x)$ in terms of simpler functions $u(x)$, and possibly $V(u, U)$, that facilitate the use of $U({}_1x)$ in models of optimal economic growth, and may perhaps suggest further parametrization or other specialization for econometric studies of individual consumption plans over time.

The use of substantive terms such as "consumption," "preference," "time" in what is essentially a formal mathematical analysis may hinder the perception of other possible applications in which one or more of these terms are inappropriate. The stationarity postulate, however, strongly suggests temporal or other consecutiveness in the vectors x_t, $t = 1, 2, \ldots$, as a condition for meaningful application. In Diamond's study [1965] where stationarity in the aggregation of single-period utilities is dropped, consecutiveness is immaterial in spite of appearances to the contrary in the formulation of some of the postulates. What is interpreted as eventual impatience if t stands for time is therefore also open to the wider interpretation that in any permutation of the vectors in the infinite sequence x_t, $t = 1, 2, \ldots$, the weight given to vectors further up in the sequence must eventually decrease.

REFERENCES

Diamond, P. A., [1965], "The Evaluation of Infinite Utility Streams," Econometrica, January 1965, pp. 170-177.

Fisher, I., [1930], The Theory of Interest, 1930, reprinted by Augustus Kelley, New York, 1961.

Gale, D., [1967a], "On Optimal Development in a Multi-Sector Economy," Review of Economic Studies, XXXIV, 1, January 1967, pp. 1-18.

__________, [1967b], "A Mathematical Theory of Optimal Economic Development," forthcoming in the Bulletin of the American Mathematical Society.

Gorman, W. M., [1967], "The Structure of Utility Functions," Technical Report 151, Inst. for Math. Stud. in the Soc. Sc., Stanford University, August 1, 1967.

Koopmans, T. C., [1964], "On Flexibility of Future Preference," in Shelly and Brian, Eds., Human Judgments and Optimality, Wiley, 1964, pp. 243-254.

__________, [1965], "On the Concept of Optimal Economic Growth," in The Econometric Approach to Development Planning, North Holland Publishing Co. and Rand McNally, 1965 (a reissue of Pontificiae Academiae Scripta Varia, Vol. 28, 1965), pp. 225-300.

__________, [1966], "Structure of Preference over Time," Cowles Foundation Discussion Paper No. 206, April 15, 1966.

__________, [1967a], "Objectives, Constraints and Outcomes in Optimal Growth Models," Econometrica, January 1967, pp. 1-15.

__________, [1967b], "Intertemporal Distribution and 'Optimal' Economic Growth," in Fellner et.al., Ten Economic Studies in the Tradition of Irving Fisher, Wiley, 1967, pp. 95-126.

Von Weizsäcker, C. C., [1965], "Existence of Optimal Programmes of Accumulation for an Infinite Time Horizon," Review of Economic Studies, April 1965, pp. 85-104.

Yaari, M. E., "On the Consumer's Lifetime Allocation Process," Int. Econ. Rev., September, 1964, pp. 304-317.