COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut

COWLES FOUNDATION DISCUSSION PAPER NO. 199

QUASI AND GENERALIZED GOLDEN RULE PATHS: A STUDY OF COMMANDING GROWTH PATHS

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November 30, 1965

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The Golden Rule path is the golden-age growth path which gives higher consumption at every point in time than any other golden age path. These other golden age paths are all "parallel" to the Golden Rule path. The parallelism may be defined in terms of any of several variables. For example, since the capital-augmented labor ratio on any golden age path is a constant (different for each golden age), there is a constant difference (positive or negative) between the capital-augmented labor ratio on the Golden Rule and that on any other particular golden age path; the two paths therefore are equidistant or "absolutely" parallel in terms of this ratio. All golden age paths are parallel to the Golden Rule path in terms of the capital-output ratio and the rate of interest in precisely the same way. With respect to the capital stock, the parallelism is slightly different: On any particular golden age path, the ratio of the capital stock to the Golden Rule capital stock is a constant. Equivalently, the difference between the logarithm of the capital stock on the golden age path and the logarithm of the Golden Rule capital stock is a constant. This may be called "relative" or "logarithmic" parallelism. (There is also such logarithic parallelism in terms of the capital-output ratio and the rate of interest.)

Hence, the Golden Rule path is a growth path which gives uniformly higher consumption than any path which is parallel in one of the above respects.

It is tempting to say that the Golden Rule path is a "dominating" path, that it "dominates" (with respect to consumption) all other parallel paths. But probably such use of the term "dominate" would be unfortunate.

I shall use the term "dominate" only in the following, I believe customary, sense. A growth path is said to <u>dominate</u> another growth path if and only if both paths are feasible from the same initial conditions (capital good endowments) and if the former gives more consumption at least some of the time and never less consumption. Provided that consumption is the only desideratum and that more consumption is always preferred to less, any path which is dominated (in this sense) is dynamically inefficient (even if the statical efficiency conditions for maximum output are satisfied at every point in time) and therefore cannot be optimal.

It will be convenient expositionally, and it will help to reinforce an important distinction, to have another term to characterize a growth path which gives more consumption some of the time and never less consumption than another path, regardless of difference in initial state. I shall say that a growth path <u>commands</u> another path if, beginning at the present time and forever after, it gives higher consumption at least some of the time and never less consumption, <u>whether or not initial conditions are the same on the two paths</u>. The path giving the higher consumption (at least some of the time) will be said to be the <u>commanding</u> path.

This property will be said to be the property of command.[1]

It is clear that command is a wider concept than dominance, since the former concept is not restricted to paths originating from the same initial state. Not every path which commands another path also dominates that path; only if the two paths start from the same initial state is the commanding path also a dominating path. On the other hand, every dominating path is a commanding path. Hence, of two paths one of which commands the other, the commanding path is a dominating path if and only if the two paths start from the same initial state.

In these terms, the Golden Rule path is a commanding growth path: it commands all paths which are parallel to it in any of the aforementioned respects, i.e., it commands all other golden age paths. It does not dominate these paths - any of them - because each of them starts with an initial capital stock different from the Golden Rule initial capital stock.[2]

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1. I have chosen this term only after much thought, research and consultation. "Command" is exactly synonymous with "dominate" in the non-technical sense of the latter term which corresponds to the technical use of that term. According to Webster's New International Dictionary (Second edition), "to dominate" means "to have a commanding position over" (used transitively) or "to occupy a superior position" (used intransitively). "To command" means "to dominate in situation, as by height; also, to overlook" (used transitively) or "to dominate or overlook as from a superior position" (used intransitively).

2. It is sometimes said, contrariwise, that any golden age path which keeps the capital stock in excess of the Golden Rule capital stock is truly dominated by the Golden Rule path. What is intended by this statement, though it does not say it, is that such a golden age path is dominated, for example, by a path which (necessarily) starts from the same initial (and excessive) capital stock and on which the "excess" capital is immediately consumed (or thrown away) and the Golden Rule path is followed thereafter. (Golden age paths with always less capital than the Golden Rule path are not dominated because society cannot make up the initial "deficiency" of capital without giving up initial consumption.) Analysis of these topics is contained in the next essay.

(However, some of these golden age paths are dominated by certain _other_ paths, paths which bear a close connection to the Golden Rule path, as the preceding footnote indicates.)

The immediate and ostensible purpose of this essay is to discover paths, other than the Golden Rule path, which command all paths parallel to them. Even in the Golden Rule model, in which pure labor augmentation and labor force growth are exponential, it will be seen that the Golden Rule path is not the only path commanding all paths parallel (in some respect) to it. (But it appears that every such commanding path which is efficient, i. e., not dominated by another path, is asymptotic to the Golden Rule path.) The search for commanding paths is especially interesting in models not having those special features of exponential labor augmentation and labor force growth, since, in such a model, no Golden Rule path exists (in the standard sense of a consumption - maximizing golden age path).

Some of these commanding growth paths I call Quasi-Golden Rule Paths and others Generalized Golden Rule paths. Any path which commands all paths parallel to it might well be called a Quasi-Golden Rule path since it is like the Golden Rule path in having this command property. But I elevate such a path to the status of a Generalized Golden Rule path if both of the following conditions are satisfied. First, the path is a commanding path in a quite _general_ (aggregative) model, i. e., a model in which there may be any number of primary inputs (land and labor of various varieties) and in which (exogenous) technical progress is arbitrary with respect to rate and factor saving bias. Second, in the special case where the general model is equivalent to the Golden Rule model (with exponentially growing augmented labor), the commanding path must reduce to the pure and simple Golden Rule path for suitable initial conditions. (This requires that the kind of parallelism be similar in terms of some variable to the parallelism between the Golden Rule and golden age paths.)

Why do I devote an essay to the study of commanding paths? The study of command in this essay serves as a prelude to the study of dominance, and hence of dynamical efficiency, contained in the next essay. I assign a separate essay to this study primarily for expositional convenience. In addition, the notion of a commanding path, like the Golden Rule notion, seems to possess interest or curiosity value independently of its applications to efficient and to optimal growth.

I. Quasi-Golden Rule Paths

It is postulated throughout this section that there is just one non-constant primary resource, labor; it is a continuously differentiable function of time. The aggregate production function is posited to be homogeneous of degree one in capital and labor. (The existence and essential characterizations of the Quasi-Golden Rule paths in this section will continue to hold for production functions which are homogeneous of of any positive degree.) The function is twice differentiable with everywhere positive marginal products and diminishing marginal productivity of capital and of labor. There is positive technical progress but the standard Golden Rule postulate that progress is purely labor augmenting and that augmented labor grows exponentially (so that there is a "natural" or golden-age growth rate) is abandoned. Hence the production function

(1) $$Q = F(K, L; t)$$

satifies the relations

(2) $$\begin{gathered} F_K , F_L , F_t > 0 , \\ F_{KK} < 0 , F_{LL} < 0 . \\ F_K K + F_L L = Q \end{gathered}$$

In the early analysis, however, we place some restrictions on the factor saving character of technical progress.

A. Pure Labor Augmentation

As a first step, let us continue to suppose that technical progress is purely labor augmenting but let us relax the postulate that augmented labor grows at a constant exponential rate. Then our production function takes the form

(3) $$Q = G(K, AL) , A > 0$$

where A is a function of time only - an increasing continuously differentiable function of time. It will be supposed that, for all t ,

$$(4) \qquad \frac{\dot{L}}{L} + \frac{\dot{A}}{A} > 0$$

where a variable with a dot over it denotes the absolute time rate of increase of the variable (its first total derivation with respect to time). The left-hand side of (4), the (proportionate) rate of growth of augmented labor, need not be constant.

We now develop an equation for consumption in terms of the ratio of capital to augmented labor, say k , and its absolute rate of change, $\dot{k}$. Taking depreciation to be zero, we have

$$(5) \qquad C = G(K, AL) - \dot{K}$$

or

$$(5a) \qquad C = \left[\frac{G(K, AL)}{AL} - \frac{\dot{K}}{AL} \right] AL .$$

The first term in the brackets, output per unit augmented labor, is a function of capital per unit augmented labor on our assumption of linear homogeneity:

$$(6) \qquad \frac{G(K, AL)}{AL} = G\left(\frac{K}{AL}, 1\right) \equiv g(k) , \quad k \equiv \frac{K}{AL} .$$

The derivative, $g'(k)$, is G_K , the marginal product of capital. Diminishing marginal productivity implies $g''(k) < 0$.

By differentiation of k with respect to time we obtain

$$(7) \qquad \dot{k} = \frac{\dot{K}}{AL} - \left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A} \right) k .$$

This states that the absolute rate of increase of k is equal to the excess of investment per augmented labor over that amount necessary to keep k constant.

Substitution of (6) and (7) into (5a) yields

$$(8) \qquad C = \left[g(k) - \left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A} \right) k - \dot{k} \right] AL .$$

The following propositions will now be shown. If there exists a path, say $\hat{k}(t)$, which commands all other paths absolutely parallel to it in terms of the capital-augmented labor ratio, then this path is uniquely characterized by equality between the marginal product of capital and the (proportionate) rate of growth of augmented labor:

$$(9) \qquad g'\left(\hat{k}(t)\right) = \frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} .$$

Conversely, if there exists a path satisfying (9), this path commands all others absolutely parallel to it in terms of the capital - augmented labor ratio. In other words, equality of the marginal product of capital and the rate of growth of augmented labor is a necessary and sufficient condition for a path to be a commanding path in the above sense. All this assumes that $\hat{k}(t) > 0$ for all t.[3]

3. If $\hat{k}(t) = 0$ for some t, the equality sign in (9) is replaced by a less-than inequality sign for those values of t.

Consider an arbitrary path $\bar{k}(t)$, feasible or not feasible. The paths absolutely parallel to it are defined by

$$(10) \qquad k(t) = \bar{k}(t) + \delta$$

where δ is a constant parameter, positive or negative.[4] Noting that $\dot{k}(t) = \dot{\bar{k}}(t)$ for all t , we find, from (8), that the consumption path corresponding to any given δ - positive, zero, or negative - is given by

$$(11) \qquad C(t) = \left\{ g\left(\bar{k}(t) + \delta \right) - \left[\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} \right] \left(\bar{k}(t) + \delta \right) - \dot{\bar{k}}(t) \right\} A(t)\, L(t)$$

Now if there is a path, $\hat{k}(t)$, which commands all the parallel paths, it must make the derivative of $C(t)$ with respect to δ equal to zero for all $\underline{t}$, assuming that $\hat{k}(t) > 0$ for all $\underline{t}$. Taking this derivative and equating it to zero yields

$$(12) \qquad g'\left(\bar{k}(t) + \delta \right) - \left[\frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} \right] = 0$$

This indicates that a necessary condition for a $k(t)$ path to command all others absolutely parallel to it is that it continuously equate the marginal product of capital to the rate of growth of augumented labor. This is also a sufficient condition since (12) describes a unique path and the second-order condition that (12) describe a maximum is satisfied -

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4. Clearly, these parallel paths will be infeasible - i. e., require negative $k(t)$ or else investment in excess of output - for sufficiently large absolute values of δ . Also, $\bar{k}(t)$ itself may be infeasible.

both by virtue of the strict concavity of $g(k)$, i. e., diminishing marginal productivity of capital. Of course, (12) uniquely determines $\bar{k}(t)$ and hence the particular class of parallel paths of which one member commands all others.

I call this commanding path a Quasi-Golden Rule path for reasons indicated earlier. However this particular path is, in a certain sense, a generalization of the Golden Rule path in that the standard Golden Rule path is a special case of the Quasi-Golden Rule path - arising if the rate of growth of augmented labor is constant.[5]

The conditions for the existence of this Quasi-Golden Rule path with $\hat{k}(t) > 0$ are entirely analogous to the conditions for the existence of an interior Golden Rule path. For (9) defines a path with $\hat{k}(t) > 0$ if and only if, for all t ,

$$(13) \qquad g'(\infty) < \frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} < g'(0).$$

If the lefthand inequality is not satisfied at some t , then no finite $\hat{k}(t)$ exists. Failure of the righthand side inequality to be satisfied means only that the maximum is a corner maximum at $\hat{k}(t) = 0$.

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5. It may be of interest to note that, on the Quasi-Golden Rule path, investment is less than competitive profits if the augmented labor growth rate is rising, greater than profits if that growth rate is falling, and equal to profits if the augmented labor growth rate is constant. This can be shown from differentiation of (9).

Existence should not be confused with feasibility: It could be that the rate of growth of augmented labor, to which the marginal product of capital must be equated, would, over some interval of time, fall so fast as to require investment in excess of output. (This feasibility problem does not arise with the standard Golden Rule because, in that model, the rate of growth of augmented labor is constant.) But even if infeasible, this Quasi-Golden Rule path can be used to prove a theorem concerning dynamical efficiency, as will be seen in the next essay.

We have considered only absolute parallelism thus far. It is natural, at this point, to consider relative parallelism in terms of the capital-augmented labor ratio. But such parallelism is equivalent to relative parallelism in terms of the capital stock. It is economical to defer study of such parallelism to the point at which we reach the most general model considered in this paper; otherwise, we would find ourselves proving the same proposition with each extension of the model.

For the same reason, I shall not consider at this stage the analysis of commanding paths in terms of other variables such as the capital-output ratio and the rate of interest. That analysis will be conducted in the context of a much more general model.

B. Factor Augmentation

We suppose now that technical progress is factor augmenting; positive capital augmentation is allowed. The production function takes the form

(14) $$Q = G(BK, AL) \; , \quad B,A > 0 \; ,$$

where B and A are both increasing, continuously differentiable functions of time only. It will be supposed that, for all <u>t</u> ,

(15) $$\frac{\dot{L}}{L} + \frac{\dot{A}}{A} - \frac{\dot{B}}{B} > 0 .$$

From the equation for consumption,

(16) $$C = G(BK, AL) - \dot{K} \; ,$$

we have

(17) $$C = \left[\frac{BG(BK, AL)}{AL} - \frac{B\dot{K}}{AL} \right] \frac{AL}{B} .$$

Under constant returns to scale, output per augmented labor can be expressed as a function of <u>k</u> , now defined as the ratio of augmented capital to augmented labor:

(18) $$\frac{G(BK, AL)}{AL} = G\left(\frac{BK}{AL} , 1\right) = G(k,1) \equiv g(k) \; , \; k \equiv \frac{BK}{AL} .$$

In these terms, the marginal product of capital is

(19) $$\frac{\partial G(BK, AL)}{\partial K} = \frac{\partial}{\partial K}\left[AL \; G\left(\frac{BK}{AL}, 1\right)\right] = BG_1\left(\frac{BK}{AL} , 1\right) = Bg'(k).$$

Diminishing marginal productivity implies $g''(k) < 0$.

By differentiation of <u>k</u> with respect to time we have

(20) $$\dot{k} = \frac{B\dot{K}}{AL} - \left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) k .$$

From (17), (18) and (20) we then obtain

$$C = \left[Bg(k) - \left(\frac{\dot{L}}{L} + \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) k - \dot{k} \right] \frac{AL}{B} . \tag{21}$$

Using this equation, it can be shown, in precisely the same manner as the analogous proposition was demonstrated in the previous section, that a necessary and sufficient condition that a $k(t)$ path command all other paths absolutely parallel to it in terms of the augmented capital-augmented labor ratio is that, on this path, the marginal product of capital equal the rate of growth of the labor force plus the proportionate rate of "net" labor augmentation. That is, the commanding path, $\hat{k}(t)$, is defined by

$$B(t)\, g'\left(\hat{k}(t)\right) = \frac{\dot{L}(t)}{L(t)} + \frac{\dot{A}(t)}{A(t)} - \frac{\dot{B}(t)}{B(t)} \tag{22}$$

This path is a Quasi-Golden Rule path in the factor augmenting case.

The analysis of existence parallels that of the previous case. An interesting wrinkle here is that if $g'(\infty) > 0$ and $B(t) \to \infty$ as $t \to \infty$, then the lefthand side of (22) will eventually (if not immediately) come to exceed the righthand side for all finite $\underline{k}$, in which case there is no path defined (for all t) by (22). The inequality $g'(\infty) > 0$ occurs if the limit of the substitution elasticity as $k \to \infty$ exceeds unity.

As for feasibility, some interesting results can be obtained if we suppose that

$$\frac{\dot{L}}{L} = \gamma , \quad \frac{\dot{A}}{A} = \lambda , \quad \frac{\dot{B}}{B} = \mu , \quad \gamma + \lambda - \mu > 0 . \tag{23}$$

Then, by differentiation of (22) and using (23), the definition of the investment-output ratio, $\underline{s}$, (namely $\dot{K}/Q$) and (20), we obtain

$$(24) \qquad \hat{s} = \hat{a} + \frac{\mu \hat{\sigma}}{1-\hat{a}} \hat{x} ,$$

where $\hat{\underline{s}}$ is the investment ratio, $\hat{\underline{a}}$ is capital's share, $\hat{\sigma}$ is the elasticity of substitution, and $\hat{\underline{x}}$ is the capital-output ratio along the Quasi-Golden Rule path.[6] We see immediately that, given (23), investment exceeds profits along this path.

Now suppose that $\sigma < 1$ for all $\underline{k}$. Since $\hat{k}(t)$ is increasing, by (22), and $\sigma < 1$, $\hat{\underline{a}}$ will be decreasing. Since there is positive capital augmentation and $\sigma < 1$, progress is capital saving in Harrod's sense; therefore, $\hat{\underline{x}}$ will be falling along this constant interest-rate path. If σ is bounded <u>below</u> one, then $\hat{\underline{a}}$ and $\hat{\underline{x}}$ will approach zero as $t \to \infty$ so that $\hat{\underline{s}}$ will approach zero. If σ is not so bounded but rather $\sigma \to 1$ as $k \to \infty$, then $\hat{s}$ will approach a constant between zero and one. In either case, it is feasible, either immediately or eventually, to get onto the Quasi-Golden Rule path.

The Quasi-Golden Rule path, defined in (22) is unsatisfying on two counts. First, the righthand side of (22) could be negative, in which case this Quasi-Golden Rule path will not exist (unless $g'(k)$ turned negative for large $\underline{k}$, contrary to our postulate). Second, in some cases at least,

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6. Use is made here of the identity $\mathcal{E} = -(1-a)/\sigma$, where $\mathcal{E}$ is the k - elasticity of $g'(k)$, i.e., $g''(k)\ k/g'(k)$, a negative number.

it is an inefficient path: The interest rate is too low, as will be seen in the next essay. We need, then, to look for other generalizations or analogues of the Golden Rule path. But, as was indicated earlier, these can best be derived in more general models.

C. Arbitrary Technical Progress

We adopt now the most general production function used in this part of the essay, that given in (1):

$$(1) \qquad Q = F(K, L; t)$$

Of course, such arbitrary technical progress includes factor augmentation as a special case so that the results of this section apply to that special case as well as other cases.

Now consumption can be expressed as

$$(25) \qquad C = \left[\frac{F(K, L; t)}{L} - \frac{\dot{K}}{L}\right] L$$

The first term in the brackets, output per unit labor, may be written as a function of the capital-labor ratio, denoted k , and time:

$$(26) \qquad \frac{F(K, L; t)}{L} = F\left(\frac{K}{L}, 1; t\right) \equiv f(k; t), \quad k \equiv \frac{K}{L} .$$

The marginal product of capital, F_K , is equal to f_k . We also have

$$(27) \qquad \dot{k} = \frac{\dot{K}}{L} - \frac{\dot{L}}{L} k$$

whence, from (25), (26) and (27),

$$C = \left[f(k;\ t) - \frac{\dot{L}}{L}\ k - \dot{k} \right] L \tag{28}$$

By analysis precisely like the foregoing, it can be shown that a necessary and sufficient condition for a growth path to command all others absolutely parallel to it in terms of the capital-labor ratio is that the path, $\hat{k}(t)$, satisfy

$$f_k\left(\hat{k}(t);\ t\right) = \frac{\dot{L}(t)}{L(t)} \tag{29}$$

That is, the marginal product of capital must equal the rate of growth of the labor force.

The analysis of the existence of this new Quasi-Golden Rule path is similar to the analysis of existence in the factor augmentation case: If $f_k(k, t)$ increases without limit as $t \to \infty$ for every $\underline{k}$, then we require $f_k(\infty, t) = 0$ for the existence of this path.

I shall say very little about feasibility. Suppose that $\dot{L}(t)/L(t) = \gamma > 0$. Differentiating (29) and using (27), (29) and the definition of $\underline{s}$, the investment-output ratio, we have

$$\hat{s} = \hat{a} + (\hat{f}_{kt}/\hat{f}_k)\ \frac{\hat{\sigma}}{1-\hat{a}}\ \ \hat{x} \tag{30}$$

Hence, if $f_{kt} > 0$, which is reasonable to suppose, then investment exceeds profits on this path. If, further, technical progress is capital-saving for all σ , then $\dot{x}$ and $\dot{a}$ will be falling so there is, in that case, some presumption of feasibility, at least eventually.

The reader will note that this Quasi-Golden Rule path is not a true generalization of the Golden Rule path: Even if progress is purely labor augmenting and augmented labor grows exponentially, this Quasi-Golden Rule path does not reduce to the Golden Rule path. The reason is that in the Golden Rule model the paths commanded by the Golden Rule path - the other golden age paths - exhibit <u>relative</u> parallelism in terms of the capital-(unaugmented) labor ratio, not the absolute parallelism studied here. Relative parallelism will be examined in Part II of this essay.

This Quasi-Golden Rule path, like the previous one, is inefficient in at least some cases when there is positive technical progress: it is a path on which there is excessive capital deepening. Nevertheless it is a useful tool for showing certain other paths to be inefficient, as the next essay will show.

Thus far we have been examining paths which are (absolutely) parallel in terms of the ratio of capital to labor, or augmented capital to augmented labor. It is possible to characterize paths which command all others parallel to them in terms of the <u>output</u> - labor ratio or in terms of the wage rate. But these characterizations are complex and I doubt that such analysis would be useful. Further, such concepts are necessarily restricted to models with a single kind of variable primary input (labor). We turn now to a more general model in which we study parallelism in terms of the capital stock, the marginal product of capital and the average product of capital.

II. Generalized Golden Rule Paths

Let us now permit the existence of many variable primary resources or kinds of labor, L_1, L_2, ..., L_n, which are taken to be continuously differentiable, nondecreasing functions of time. Our aggregative production function,

$$(31) \qquad Q = \psi(K, L_1, L_2, \ldots, L_n; t) ,$$

is no longer posited to be linear homogeneous in the capital and labor inputs or even homogeneous of any position degree. But we continue to suppose that the function is twice differentiable with positive marginal products and diminishing marginal productivity of capital everywhere. There is positive technical progress for all t .[7]

Since the labor inputs are exogeneous, being a function only of time, we may put the production function in the form

$$(32) \qquad Q = P(K, t)$$

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7. Note that in this model with many primary inputs, there exists generally a Golden Rule path only if (31) is linear homogeneous in the inputs, progress is purely primary-input augmenting, and all augmented primary inputs grow exponentially at the same rate.

The above postulates imply that, for all K and $\underline{t}$,

(33) $$P_K > 0, \quad P_{KK} < 0 , \quad P_t > 0 .$$

I turn first to relative parallelism in terms of the capital stock.[8] I owe to Christian von Weizsäcker the suggestion that relative parallelism should be considered. Indeed, he correctly indicated that any path on which the investment ratio equals the capital elasticity of output would command all paths relatively parallel to it in terms of the capital stock. That is the "sufficiency" part of the proposition demonstrated below.

The proposition is the following. A necessary and sufficient condition that a growth path command all others relatively parallel to it in terms of the capital stock is that the investment ratio equal the capital elasticity of output (capital's competitive share under constant returns to scale), or equivalently, that the rate of growth of the capital stock equal the marginal product of capital. That is,

(34) $$\frac{\dot{\hat{K}}}{\hat{Q}} = \frac{\hat{P}_K \hat{K}}{\hat{Q}}$$

8. Note that relative parallelism in terms of the capital-labor ratio or in terms of the augmented capital-augmented labor ratio is equivalent to relative parallelism in terms of the capital stock. For if, as an example, the proportionate rate of change of the capital-labor ratio is the same on two paths (relative parallelism in terms of the capital-labor ratio), the proportionate rate of change of the capital stock on the two paths must also be the same (relative parallelism in terms of the capital stock), since the latter rate equals the former plus the rate of change of labor and the labor growth rate is the same on the two paths.

or

$$(34a) \qquad \frac{\hat{\dot{K}}}{\hat{K}} = \hat{P}_K$$

characterizes the commanding path. This will now be shown.

Consider a class of logarithmically parallel paths defined by

$$(35) \qquad K(t) = \pi \bar{K}(t) , \quad \pi > 0 , \quad \bar{K}(t) > 0 \text{ for all } t ,$$

where $\bar{K}(t)$ is an arbitrary path, feasible or infeasible, and where π is a constant parameter greater than, equal to, or less than one. Upon noting that $\dot{K}(t) = \pi \dot{\bar{K}}(t)$ we obtain the path of consumption that corresponds to any value of π:

$$(36) \qquad C(t) = P(\pi \bar{K}(t) , t) - \pi \dot{\bar{K}}(t)$$

Now if there is a path, say $K(t)$, which commands all the others in the class of parallel paths, the derivative of $C(t)$ with respect to π must be equal to zero for all $\underline{t}$ at the value of π corresponding to that path. Hence

$$(37) \qquad \bar{K}(t) \; P_K\left(\pi\bar{K}(t) , t\right) - \dot{\bar{K}}(t) = 0$$

By virtue of (35), which implies $\dot{K}/K = \dot{\bar{K}}/\bar{K}$, (37) can be written

$$(38) \qquad \frac{\dot{K}(t)}{K(t)} = P_K \left(K(t) , t\right) .$$

This indicates that equality between the rate of growth of capital and the marginal product of capital is a necessary condition for a path to command all others logarithmically parallel to it. Further, this equality is a sufficient condition that the path described by (38) be a commanding path since, by virtue of diminishing marginal productivity of capital ($P_{KK} < 0$), the second-order condition that the stationary value given by (37) is a maximum is satisfied. (It should be clear that only if $\bar{K}(t)$ is logarithmically parallel to a path described by (38) does the class of logarithmically parallel paths possess a commanding path.)

It will be observed that (34) - and its equivalent, (34a) - is a differential equation in $\hat{K}(t)$: It determines a unique path if and only if the initial state, $\hat{K}(0) = K_0$, is specified. Thus (34) defines a whole class of commanding paths, one for each initial state.

Under what conditions does there exist a path described by (34)? Given any positive initial capital stock, K_0, there necessarily exists a path satisfying (34).

Further, these commanding paths are necessarily feasible on our assumption of everywhere diminishing marginal product of capital. For that implies that average product of capital is falling; hence marginal product is less than average product; therefore the capital elasticity of output, to which the investment-output ratio is equated on the commanding paths, is less than unity, and so investment never exceeds output on these paths. (Also, capital never becomes negative since, by virtue of a positive marginal product of capital everywhere, capital grows from its initial positive level on any commanding path.)

I shall call these commanding paths Generalized Golden Rule paths for two reasons. First, they emerge from a fairly general model in which neither pure labor augmentation nor the existence of exactly one kind of labor nor constant returns to scale is assumed. Second, if we specialize the production function (31) in such a way that a Golden Rule path exists, meaning the assumption of pure labor augmentation and all the rest, then the Golden Rule path is one of these Generalized Golden Rule paths. This follows immediately from the fact that the Golden Rule path is a path on which the investment-output ratio equals the capital elasticity of output or, equivalently, the rate of growth of capital equals the marginal product of capital. A further, though inessential, property of these paths is that, as already noted, they exhibit the mystical equality between investment and competitive profits (under constant returns to scale).

It will be illuminating to express in terms of marginal and of average product of capital the differential equation that characterizes these Generalized Golden Rule paths. First, I shall let $\underline{r}$ denote the marginal product of capital; it is equal to the competitive rate of interest under constant returns to scale.

$$r = P_K(K, t) \tag{39}$$

Upon differentiating this totally with respect to time, we obtain

(40) $$\frac{\dot{r}}{r} = \mathcal{E} \frac{\dot{K}}{K} + h$$

where $\mathcal{E} \equiv \frac{P_{KK}K}{P_K} < 0$ [by (33)] and $h \equiv \frac{P_{Kt}}{P_K}$.

On any Generalized Golden Rule path we have, from (34a),

(41) $$r = \frac{\dot{K}}{K}.$$

From (40) and (41) we obtain the following differential equation in r that characterizes the Generalized Golden Rule paths:

(42) $$\frac{\dot{r}}{r} = \mathcal{E} \left(r - \frac{h}{\mathcal{E}} \right)$$

Suppose that $h > 0$; a sufficient condition for this is that the various kinds of labor are all substitutes for capital and are non-decreasing over time and that technical change is not "very" capital saving in the Hicksian sense. Suppose further that $P_K(\infty, t) = 0$, which is a frequent assumption in growth theory; then $\mathcal{E}$, which can be expressed as a function of r and t , could approach zero only as r approaches zero (for every t). On these assumptions, r will "track" or "chase" the variable $(h/-\mathcal{E})$ from every initial r in the sense that r will be rising if $r < (h/-\mathcal{E})$ and will be falling if $r > (h/-\mathcal{E})$. However, if $P_K(\infty, t) > 0$ and $(h/-\mathcal{E})$ should be below the lower bound on r , then r will merely approach its lower bound. Similarly, if $h < 0$, r will approach its lower bound.

Of special interest is the case in which $h/-\mathcal{E}$ is a positive constant, independent of r and t . Suppose that $P_K(\infty, t) < h/-\mathcal{E}$.

Then one r path satisfying (42) is the constant r path:

(43) $$r = h/\mathcal{E}$$

Further, if the initial r fails to satisfy (43) then r will approach $(h/-\mathcal{E})$ asymptotically.

It will now be shown that (43) describes the Golden Rule path if that path exists. Assume constant returns to scale, factor augmentation and just one labor input:

(44) $$P(K, t) = G\left[B(t)\,K(t),\ A(t)\,L(t)\right]$$

Then

(45) $$P_K\,(K, t) = B(t)\,G_1\left[B(t)\,K(t),\ A(t)\,L(t)\right]$$

and, it can be shown,

(46) $$\frac{P_{Kt}}{P_K} = \frac{\dot{B}}{B} + \mathcal{E}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}\right)$$

whence

(47) $$\frac{h}{-\mathcal{E}} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} + \left(\frac{1}{-\mathcal{E}} - 1\right)\frac{\dot{B}}{B}$$

or, using $-\mathcal{E} = b/\sigma$ where b is labor's share and σ is the substitution elasticity,

(48) $$\frac{h}{-\mathcal{E}} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} + \left(\frac{\sigma-b}{b}\right)\frac{\dot{B}}{B}\ .$$

Now in the Golden Rule model,

$$(49) \qquad \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = \text{constant}, \quad \frac{\dot{B}}{B} = 0 ,$$

so we see that $(h/-\mathcal{E})$ is a constant, independent of $\underline{r}$ and $\underline{t}$, in the Golden Rule model, and that $(h/-\mathcal{E})$ equals the rate of growth of augmented labor in that model. Hence (43) is the Golden Rule path in the Golden Rule model. Of course, in the Cobb-Douglas case, existence of the Golden Rule path does not require $\dot{B}/B = 0$ but progress is necessarily purely labor augmenting (<u>and</u> purely capital augmenting) in the Cobb-Douglas case. In this case, (48) gives

$$(50) \qquad \frac{h}{-\mathcal{E}} = \frac{\dot{L}}{L} + \frac{\dot{A}}{A} + \frac{\alpha}{1-\alpha} \quad \frac{\dot{B}}{B}$$

where $\alpha \equiv 1 - b$, the capital exponent of the Cobb-Douglas function. Manipulation of this function shows that the sum of the last two terms in (50) can be regarded as the "true" rate of labor augmentation.

It should not be inferred that if $(h/-\mathcal{E})$ is a constant, independent or $\underline{r}$ and $\underline{t}$, then (43) is the Golden Rule path. From (47) we see that if

$$(51) \qquad \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = \text{constant}, \quad \frac{\dot{B}}{B} = \text{constant}, \quad \mathcal{E} = \text{constant}$$

then $(h/-\mathcal{E})$ is a constant. Yet if $\dot{B}/B \neq 0$ and the production function is not Cobb-Douglas, then no Golden Rule path exists. [9]

9. In the constant-returns to scale, two-input, factor-augmenting case, the constant - $\mathcal{E}$ production function is of the form

$$Q = a_1 AL + a_2 (BK)^{1-\beta} (AL)^{\beta} , \qquad a_1 \geq 0, \ a_2 > 0, \ 0 < \beta < 1 .$$

The differential equation, (34), that describes the class of Generalized Golden Rule paths under discussion can also be expressed in terms of the capital-output ratio, $\underline{x}$. First, write the production function (32) in the form

$$P(K, t) = \eta\,(x, t) \tag{52}$$

This requires only that the capital elasticity of output be less than one, which follows from everywhere diminishing returns to capital. Differentiation of

$$x = \frac{K}{Q} \tag{53}$$

yields

$$\dot{x} = s - \frac{K}{Q}\;\frac{\dot{Q}}{Q} \tag{54}$$

where $s \equiv \dot{K}/Q$, the investment-output ratio. It can be shown that

$$\frac{\dot{Q}}{Q} = \frac{a}{1-a}\;\frac{\dot{x}}{x} + \frac{\eta_t}{\eta} \tag{55}$$

where $\underline{a}$ denotes the capital elasticity of output. Further, according to (34), on any Generalized Golden Rule path

$$s = a \tag{56}$$

From (54), (55), and (56) we obtain

$$\dot{x} = \frac{(1-a)\,\eta_t}{\eta}\left(\frac{a}{\eta_t/\eta} - x\right) \tag{57}$$

The analysis of this differential equation characterizing the class of Generalized Golden Rule paths is somewhat similar to the previous analysis of the differential equation in <u>r</u> . On certain assumptions, <u>x</u> will "track" the quantity $(a\eta/\eta_t)$.

Any constant <u>x</u> path satisfying (57) is one on which

(58) $$x = \frac{a}{\eta_t/\eta}$$

or equivalently

(59) $$r = \eta_t/\eta$$

The Golden Rule path is such a path. For in the Golden Rule model, η_t/η, the rate of growth of output when the capital-output ratio is constant, equals the rate of growth of augmented labor.

I have discussed thus far only paths which command all others logarithmically parallel to them. Absolute parallelism deserves brief mention. On our assumption that the marginal product of capital is everywhere positive, there exists no path which commands all path absolutely parallel to it in terms of the capital stock. This can be seen if we let

(60) $$K(t) = \bar{K}(t) + \delta$$

define the class of absolutely parallel paths, so that consumption is given by

(61) $$C(t) = P\left[\bar{K}(t) + \delta, t\right] - \dot{\bar{K}}(t)$$

with $P_K > 0$ for all K, the greater δ the greater is the consumption path, so there exists no path, $\bar{K}(t) + \delta$, which commands all the other parallel paths.

If the marginal product of capital equals zero on some path $\tilde{K}(t)$ and is negative for all $K(t) > \tilde{K}(t)$, then the path $\tilde{K}(t)$ commands all paths absolutely parallel to it in terms of the capital stock.

In the earlier model having a single primary input, we measured capital intensity in terms of the capital-labor ratio or some variant of that ratio. In the present model we have thus far worked in terms of the capital stock. Two other measures of capital intensity in the present model are the marginal product of capital and the average product of capital (or its reciprocal, the capital-output ratio). For the sake of completeness, I now analyze growth paths which command in terms of these measures of capital intensity.

Consider first absolute parallelism in terms of the marginal product of capital. Again, let r denote the marginal product of capital. The class of absolutely parallel paths is defined by

$$r(t) = \bar{r}(t) + \delta \tag{62}$$

Hence the $K(t)$ path corresponding to any $r(t)$ path in this class of paths is determined by

$$\bar{r} + \delta = P_K(K, t) \tag{63}$$

Differentiating totally with respect to time, we obtain

$$(64) \qquad \dot{K} = \frac{\dot{\bar{r}}}{P_{KK}} - K\frac{P_{Kt}}{P_K} \cdot \frac{P_K}{P_{KK}K}$$

or

$$(64a) \qquad \dot{K} = \frac{\dot{\bar{r}}}{P_{KK}} - \frac{Kh}{\mathcal{E}}$$

Partial differentiation of (63) with respect to δ yields

$$(65) \qquad \frac{\partial \dot{K}}{\partial \delta} = \frac{1}{P_{KK}} < 0$$

Upon differentiating (64a), partially with respect to δ we obtain

$$(66) \qquad \frac{\partial \dot{K}}{\partial \delta} = \left\{ -\frac{P_{KKK}\,\dot{\bar{r}}}{(P_{KK})^2} - \frac{h}{\mathcal{E}} - K\left[\frac{(\partial h/\partial K)\mathcal{E} - (\partial \mathcal{E}/\partial K)\,h}{\mathcal{E}^2}\right]\right\} \frac{\partial K}{\partial \delta}$$

or equivalently,

$$(66a) \qquad \frac{\partial \dot{K}}{\partial \delta} = \left\{ -E(P_{KK})\frac{1}{\mathcal{E}}\;\frac{\dot{\bar{r}}}{\bar{r}+\delta} - \frac{h}{\mathcal{E}} - \frac{h}{\mathcal{E}}\left[E(h) - E(\mathcal{E})\right]\right\} \frac{\partial K}{\partial \delta}$$

where $E(y)$ denotes the K-elasticity of any variable y :

$$E(y) = \frac{(\partial y/\partial K)K}{y}$$

Using the consumption relation

$$(67) \qquad C = P(K,t) - \dot{K}$$

we have

(68) $$\frac{\partial C}{\partial \delta} = \frac{\partial P}{\partial K} \quad \frac{\partial K}{\partial \delta} - \frac{\partial \dot{K}}{\partial \delta}$$

Now if there exists a path $\bar{r}(t)+\delta$ which commands all the other parallel paths, that path must make $\partial C/\partial \delta = 0$, assuming that an interior maximum occurs. Substituting (63) for $\partial P/\partial K$ and (66a) for $\partial \dot{K}/\partial \delta$ we obtain

(69) $$\frac{\partial C}{\partial \delta} = \left\{ (\bar{r}+\delta)+ \frac{h}{\mathcal{E}} + \frac{h}{\mathcal{E}} \left[E(h)-E(\mathcal{E}) \right] + E\left(P_{KK}\right) \frac{1}{\mathcal{E}} \quad \frac{\dot{\bar{r}}}{\bar{r}+\delta} \right\} \frac{\partial K}{\partial \delta}$$

Setting this derivative equal to zero, noting that $\partial K/\partial \delta < 0$ for all K and t, and writing $\hat{r}(t)$ for $\bar{r}(t)+\delta$, the commanding path, yields

(70) $$\frac{\dot{\hat{r}}}{\hat{r}} = \frac{\mathcal{E}}{-E(P_{KK})} \left\{ \hat{r} - \frac{h}{-\mathcal{E}} - \frac{h}{-\mathcal{E}} \left[E(h) - E(\mathcal{E}) \right] \right\}$$

Any path which commands all others absolutely parallel to it in terms of the marginal product of capital must satisfy (70). Hence (70) is a necessary condition of command in this respect. However (70) is a sufficient condition only if the second-order condition is satisfied; even then there may be local maxima so that not every path satisfying (70) commands <u>all</u> other parallel paths. In fact, the second-order condition, $\partial^2 C/\partial \delta^2 < 0$ is not necessarily satisfied; the algebraic sign of this derivative depends upon the signs of such quantities as $E(h)$ and $\partial E(h)/\partial \delta$ which can be of either sign.

Of course, the Golden Rule path (if the model is such as to permit its existence) satisfies (70). In the Golden Rule model, as was shown earlier, $(h/-\mathcal{E})$ is the rate of growth of augmented labor; it is a constant, independent of K and t. Hence, in that model, $E(h) - E(\mathcal{E}) = 0$, so that the Golden Rule path $r = (h/-\mathcal{E}) =$ constant satisfies (70).

Consider now relative parallelism in terms of the marginal product of capital. The class of logarithmically parallel paths is defined by

$$(71) \qquad r(t) = \pi\bar{r}(t)\ , \quad \bar{r}(t) > 0 \ \text{ for all } t\ ,$$

so that

$$(72) \qquad \pi\bar{r}(t) = P_K\,(K,\ t)$$

determines the $K(t)$ path associated with any of these parallel paths.

Then, by differentiation of (72)

$$(73) \qquad \dot{K} = \frac{\pi\dot{\bar{r}}}{P_{KK}} - \frac{Kh}{\mathcal{E}}$$

and

$$(74) \qquad \frac{\partial K}{\partial \pi} = \frac{\bar{r}}{P_{KK}} < 0$$

From (73),

$$(75) \qquad \frac{\partial \dot{K}}{\partial \pi} = \frac{\dot{\bar{r}}}{P_{KK}} \cdot \frac{\partial \pi}{\partial K} \cdot \frac{\partial K}{\partial \pi} + \left\{ - \frac{P_{KKK}\ \pi\dot{\bar{r}}}{(P_{KK})^2} - \frac{h}{\mathcal{E}} - \frac{h}{\mathcal{E}}\left[E(h)-E(\mathcal{E})\right] \right\} \frac{\partial K}{\partial \pi}$$

$$= \left\{ \frac{\dot{\bar{r}}}{\bar{r}} \left[1 - \frac{E(P_{KK})}{\mathcal{E}} \right] - \frac{h}{\mathcal{E}} - \frac{h}{\mathcal{E}} \left[E(h)-E(\mathcal{E}) \right] \right\} \frac{\partial K}{\partial \pi}\ .$$

From the consumption relation we then obtain

(76) $$\frac{\partial C}{\partial \pi} = \left\{ \pi\bar{r} + \frac{h}{\mathcal{E}} + \frac{h}{\mathcal{E}}\left[E(h)-E(\mathcal{E})\right] + \left[\frac{E(P_{KK})}{\mathcal{E}} - 1\right]\frac{\dot{\bar{r}}}{\bar{r}} \right\} \frac{\partial K}{\partial \pi}$$

Equating this derivative to zero and writing $\hat{r}(t)$ for $\pi\bar{r}(t)$, the commanding path, we have

(77) $$\frac{\dot{\hat{r}}}{\hat{r}} = \frac{\mathcal{E}}{\mathcal{E}-E(P_{KK})} \left\{ \hat{r} - \frac{h}{-\mathcal{E}} - \frac{h}{-\mathcal{E}}\left[E(h)-E(\mathcal{E})\right] \right\}$$

This is only a necessary condition. Once again, the second-order conditions for a maximum may not be satisfied.

Let us turn finally to parallelism in terms of the capital-output ratio. From the relations

(78) $$Q = \eta\,(x, t)$$

and

(79) $$K = x\,Q$$

we obtain

(80) $$\dot{K} = \frac{Q}{b}\,\dot{x} + HK$$

where $H \equiv \dfrac{\eta_t\,(x, t)}{\eta(x, t)}$, $b \equiv 1-a$.

H is the rate of growth of output when the capital-output ratio is constant.

Now consider the class of absolutely parallel paths

(81) $$x(t) = \bar{x}(t) + \delta$$

Regarding b and H as functions of K and t, we obtain from (80)

$$(82)\qquad \frac{\partial \dot{K}}{\partial \delta} = \left[\frac{1}{b} \left(\frac{P_K K}{P} - \frac{b_K K}{b} \right) \frac{\dot{\bar{x}}}{\bar{x}+\delta} + H \left(1 + \frac{H_K K}{H} \right) \right] \frac{\partial K}{\partial \delta}$$

$$= \left[\frac{1}{b} \left(a - E(b) \right) \frac{\dot{\bar{x}}}{\bar{x}+\delta} + H \left(1 + E(H) \right) \right] \frac{\partial K}{\partial \delta}$$

where $\partial K/\partial \delta = Q/b > 0$. Hence, using the consumption relation, we have

$$(83)\qquad \frac{\partial C}{\partial \delta} = \left\{ P_K - \left[H \left(1+E(H) \right) + \frac{1}{b} \left(a - E(b) \right) \right] \right\} \frac{\partial K}{\partial \delta}$$

Equating this derivative to zero and letting $\hat{x}(t)$ denote $\bar{x}(t)+\delta$, we have

$$(84)\qquad \frac{\mathring{\hat{x}}}{\hat{x}} = \frac{b}{a-E(b)} \left\{ P_K - H \left[1 + E(H) \right] \right\}$$

This is a necessary condition that the path $\hat{x}(t)$ be commanding. It is not a sufficient condition on our previous assumptions; the second-order conditions for a maximum need not be satisfied. Note that (84) is somewhat symmetrical to (70).

The Golden Rule path satisfies (84). In the Golden Rule model, H measures the growth rate of augmented labor and $E(H) = 0$. On the Golden Rule path, $P_K = H$ and $\mathring{\hat{x}} = 0$.

Analysis of relative parallelism in terms of the capital-output ratio will yield an equation somewhat like (84). Again, the second-order conditions are not necessarily satisfied on present assumptions.

We have, in Part II of this essay, considered a number of paths which, on certain assumptions, can be regarded as generalizations of the Golden Rule path. In each case, the Golden Rule path is a special case of the Generalized Golden Rule path. The commanding paths in terms of the marginal and average product of capital cannot be characterized simply; in addition, the second-order conditions are cumbersome and difficult to interpret. It will not be surprising, therefore, that the first of the Generalized Golden Rule paths - that on which the investment ratio equals the capital elasticity of output - is the most useful. It is time - high time, the reader will no doubt think - that we examine the uses to which the Golden Rule path and its generalizations can be put.