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**INDIVIDUAL SAVING, AGGREGATE CAPITAL ACCUMULATION,  
AND EFFICIENT GROWTH**

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I. Introduction

The decade-long ground swell of neoclassical growth theory has been devoted, basically, to the study of two questions: At first, the implications for growth of given aggregate saving behavior (e.g., a constant savings ratio) were of principle concern. More recently, with help from Ramsey and from the theory of optimal control, growth theory began to examine the question of what aggregate saving behavior (e.g., a central planner's savings ratio) maximizes some given social welfare function. In contrast, the present discussion adopts the point of view, due essentially to Modigliani and Brumberg [1], that it is possible to explain the existence of positive aggregate saving by the desire of individuals to achieve an optimal lifetime consumption pattern, and attempts to tie the process of capital accumulation to this behavior.

Upon casual examination, the Modigliani-Brumberg life-cycle hypothesis may seem especially well suited for incorporation in a neoclassical growth theory. For this hypothesis implies that, after the economy has "settled down," aggregate saving will depend on such parameters as the rate of population growth and the rate of technological improvement. However, attempts so far

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to construct a dynamic theory on this basis have shown that the task is not as simple as it appears at first sight. The particular studies to which we have reference here are Samuelson's 1958 article on an exact consumption-loan model of interest [8] and Diamond's 1965 article on a neoclassical model of national debt [4]. This paper may very well be thought of as a continuation of the investigation begun by these authors.<sup>1/</sup>

On the production side of the economy we shall assume a single, competitive sector which hires labor and capital, produces a homogeneous output under neoclassical conditions, and sells this output (indifferently) on the capital and consumption goods markets. Our view of aggregate production and distribution is thus essentially that of Solow [9].

On the consumption side of the economy we assume that each individual is born with only a labor endowment and plans his lifetime consumption pattern given the relevant wage rates, interest rates and output prices. Since, in general, a particular individual's desired level of consumption at a moment of time will not coincide with his earnings at that moment, he will be engaged in saving or dissaving. Net aggregate saving, and therefore gross aggregate capital accumulation, is thus simply the sum of all individuals' saving or dissaving. This is, essentially, the Modigliani-Brumberg view.

Given the analogue of perfect information in static equilibrium analysis, perfect foresight, the interaction of the production and consumption sectors determines a competitive equilibrium growth path. Our major concern will be the analysis of this path.

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<sup>1/</sup> Elsewhere [3], we have presented an extensive investigation of Samuelson's model.

## II. The Basic Model

The population at time  $t$  is composed of all individuals born at times  $v$ , for  $t-1 \leq v \leq t$ ; that is, all individuals live a lifetime of length one "year."<sup>2/</sup> The group of individuals born at time  $v$  will be referred to as generation  $v$ . Generation  $v$  consists of  $e^{nv}$  individuals, where  $n$  is thus the constant (relative) rate of population growth. We assume that  $n$  is positive. It is further assumed that generation  $v$  is born into the labor force and continues therein until death.<sup>3/</sup> Hence, the total labor force at time  $t$ , denoted  $L(t)$ , is given by

$$(1) \quad L(t) = \int_{t-1}^t e^{nv} dv = \frac{1 - e^{-n}}{n} e^{nt}.$$

Total output at time  $t$ , denoted  $Y(t)$ , is produced with the cooperation of this labor force and the capital stock at time  $t$ , denoted  $K(t)$ , in many competitive firms whose aggregate activity can be described by a neoclassical production function. More precisely, letting lower case letters stand for per capita quantities, it is assumed that<sup>4/</sup>

$$(2) \quad y(t) = f(k(t)),$$

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<sup>2/</sup> A lifetime of any fixed length could just as well be assumed.

<sup>3/</sup> The implications of modifying this assumption will be explored in Section VI.

<sup>4/</sup> There is no technical progress in this economy. However, as is usual in neoclassical growth theory, labor augmenting or Harrod-neutral technical progress at a constant (relative) rate could easily be assumed. The only difference which this assumption would make is that it would require a re-interpretation of all intensive variables and some parameters of the model.

where  $f$  is a real function which is twice continuously differentiable, strictly concave, and which satisfies the derivative conditions  $f'(0) > n > f'(\infty)$ . It is also assumed that  $f$  is strictly increasing for  $0 \leq k < \tilde{k}$  where  $0 < \tilde{k} \leq \infty$  is the point of capital saturation. The distribution of total output is determined in competitive factor markets, where the competitive real wage and real interest rates are given by the marginal productivities of labor

$$w(t) = f(k(t)) - f'(k(t))k(t)$$

and capital

$$r(t) = f'(k(t)),$$

respectively.

In order to complete the picture of our stylized competitive economy we must describe the behavior of an individual from generation  $v$  <sup>5/</sup>. For such a description we have at our disposal the classical analysis of consumer behavior over time, as introduced, basically, by Irving Fisher [6]. The difficulty is that we do not have much else at our disposal, and using the Fisher analysis involves one assumption which, in the present context, is very drastic indeed. We are referring here to the perfect foresight assumption. To plan his consumption today, the Fisher consumer must know with certainty what he will earn in wages during the balance of his lifetime

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<sup>5/</sup> For the time being we assume that all individuals of generation  $v$  behave indently. This assumption will be relaxed slightly in Section VI.

and what returns can be earned in the future from the ownership of assets. In the model under discussion, however, the effects of consumer decisions on wage and interest rates play a central role, so that the perfect foresight assumption amounts to the assertion that consumers take into account wage and interest rates which accurately reflect their own present and future actions. Clearly, what lies in the background of such an assumption is the notion of a general equilibrium model in which the commodities are dated output, labor and capital, prices (with output price as numeraire) are the wage and interest rates at every moment of time, and the equilibrium conditions are summarized in equation (8) from the sequel.

The existence of such an equilibrium is by no means obvious; in particular, the restrictions on possible trades imposed by the time structure of the model, and therefore not found in the standard static general equilibrium theory, may preclude existence. Only if this equilibrium exists is the perfect foresight assumption legitimate. However, in Section V we shall in fact demonstrate existence, and thereby also establish the basis for this assumption (for a specific utility function).

Notice, in passing, that the alternatives to assuming perfect foresight are either to adopt an arbitrary rule of expectation (and a concomitant arbitrary rule by which to reconcile ex ante with ex post) or to incorporate into the model a full-fledged theory of decision making under uncertainty; the former alternative leads to rather unconvincing results, while the latter lies beyond our present investigative ability.

Given the perfect foresight assumption, one can write down the allocation problem confronting an individual of generation  $v$ . Let  $C(t,v)$

be the rate of consumption of output and  $A(t,v)$  be the (for the present, real capital) asset holdings of an individual of generation  $v$  at time  $t$ , for  $v \leq t \leq v+1$ . We assume that the individual's preferences depend only on his consumption of output <sup>6/</sup> and, furthermore, that they can be represented by a utility function of the form

$$\int_v^{v+1} U(C(t,v)) e^{-\delta(t-v)} dt ,$$

where  $U$  is a nondecreasing, concave and twice differentiable real function, and  $\delta$  is the individual's constant and nonnegative subjective rate of time preference. The allocation problem may now be stated as follows:

$$\text{maximize } \int_v^{v+1} U(C(t,v)) e^{-\delta(t-v)} dt$$

subject to  $C(t,v) = w(t) + r(t) A(t,v) - \frac{\partial A(t,v)}{\partial t}$ ,

$$A(v,v) = A(v+1,v) = 0 , \quad \text{and}$$

$$C(t,v) \geq 0 .$$

The first of the constraint equations is simply the individual's budget identity. The second equation states that an individual is born with no

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<sup>6/</sup> Leisure as a second consumption good can easily be incorporated into the model. In fact, we originally carried out a good deal of the subsequent analysis under this assumption. However, as it is cumbersome, and yields little if any added insight into the central results, we now just mention it as a feasible extension.

assets, i.e., that bequests are absent in this economy, and that an individual dies with no assets. (Actually, the wealth constraint should be written  $A(v+1, v) \geq 0$ , but the monotonicity of the function  $U$  entails that it hold with equality.) Finally, the last of the constraints merely formalizes the fact that a negative rate of consumption of output is meaningless.

By solving the budget equation for  $A(t, v)$ , which yields

$$(3) \quad A(t, v) = e^{\int_v^t r(x) dx} \left\{ \int_v^t [w(s) - C(s, v)] e^{-\int_v^s r(x) dx} ds + A(v, v) \right\},$$

it is possible to rewrite the allocation problem as

$$\begin{aligned} & \text{maximize} \quad \int_v^{v+1} U(C(t, v)) e^{-\delta(t-v)} dv \\ & \text{subject to} \quad \int_v^{v+1} [w(s) - C(s, v)] e^{-\int_v^s r(x) dx} ds = 0 \quad \text{and} \\ & \quad C(t, v) \geq 0. \end{aligned}$$

The solution to this maximization problem is well known from the theory of consumer behavior over time.<sup>7/</sup> For each moment  $t$  at which the nonnegativity constraint is not binding, the optimal consumption plan satisfies the following differential equation:

$$\frac{\frac{\partial C(t, v)}{\partial t}}{C(t, v)} \eta(C(t, v)) = r(t) - \delta,$$

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<sup>7/</sup> See, for example, Yaari [10].



where  $\eta(x) = - \frac{xU''(x)}{U'(x)}$  is the elasticity of marginal utility.

At this point we make the simplifying assumption that  $\eta$  is both constant and equal to one, which reduces the function  $U$  to the logarithm.<sup>8/</sup> Such an assumption is essentially equivalent to postulating a Friedman consumption function, with permanent income elasticities all equal to unity. The optimal consumption plan is now given by the differential equation

$$\frac{\frac{\partial C(t,v)}{\partial t}}{C(t,v)} = r(t) - \delta ,$$

which has the closed solution

$$(4) \quad C(t,v) = C(v,v) e^{\int_v^t [r(x) - \delta] dx}$$

for all  $t$  satisfying  $v \leq t \leq v+1$ . The initial rate of consumption is determined from the wealth constraint after substitution from (4)

$$C(v,v) = \frac{\int_v^{v+1} w(s) e^{-\int_v^s r(x) dx} ds}{\frac{1 - e^{-\delta}}{\delta}} .$$

Finally, the asset holdings corresponding to the optimal consumption plan are derived from (3) after substitution from (4)

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<sup>8/</sup>  $\eta$  equal to any constant could just as well be assumed. Note that this assumption insures that the nonnegativity constraint is never binding (except in the trivial case where the economy produces no output).

$$(5) \quad A(t, v) = e^{-\int_v^t r(x) dx} \int_v^t \left[ w(s) - c(v, v) e^{-\int_v^s [r(x) - \delta] dx} - \int_v^s r(x) dx \right] e^{-\int_v^s r(x) dx} ds$$

for all  $t$  satisfying  $v \leq t \leq v+1$ .

Let  $c(t)$  be the rate of aggregate consumption per capita and  $a(t)$  be aggregate asset holdings per capita at time  $t$ . From (1), (4) and (5) these two quantities are given explicitly by

$$(6) \quad c(t) = \frac{n}{1 - e^{-n}} \frac{\delta}{1 - e^{-\delta}} \int_{t-1}^t \int_v^{v+1} w(s) e^{-\int_v^s r(x) dx} ds e^{-\int_v^t [r(x) - \delta - n] dx} dv$$

and

$$(7) \quad a(t) = \frac{n}{1 - e^{-n}} \frac{\delta}{1 - e^{-\delta}} \int_{t-1}^t \int_v^t \left[ \frac{w(s)}{\delta} - \int_v^{v+1} w(\tau) e^{-\int_v^\tau r(x) dx} d\tau e^{-\int_v^s [r(x) - \delta] dx} \right] e^{-\int_v^s r(x) dx} ds e^{-\int_v^t [r(x) - n] dx} dv.$$

If we now identify the asset holdings of individuals with the capital stock <sup>2/</sup>, then we have the basic equilibrium equation for the economy

$$(8) \quad a(t) = k(t).$$

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<sup>2/</sup> Negative asset holdings (debt) for some consumers could be expressly accounted for by assuming competitive financial intermediaries which hold as assets real capital and consumer-issued bonds and as liabilities intermediary-issued bonds. Equation (8) then represents a netting of the consumer and financial intermediary sectors. We shall have more to say about intermediaries of this sort in Sections VI and VIII.

By differentiating equation (8) and simplifying the result, or, more directly, by equating the rate of net saving per capita to the rate of gross investment per capita, we also have the equilibrium growth equation for the economy

$$(9) \quad \dot{k}(t) = f(k(t)) - nk(t) - c(t) .$$

For some purposes the representation (9) will be more useful than the representation (8); note, however, that (8) implies (9) but not conversely.

### III. Balanced Growth Equilibrium

We initiate the analysis of the equilibrium condition (8) by restricting attention to balanced growth paths, i.e., growth paths on which the capital-labor ratio remains constant over time. The question to be answered is thus: Are there constant capital-labor ratios which satisfy equation (8)?

To begin with, observe that if  $k = \text{const.}$  then  $r = f'(k) = \text{const.}$  and  $w = f(k) - f'(k)k = \text{const.}$  (In fact, because of these constancies, balanced growth in this stylized competitive economy is easily shown to be equivalent to stationary consumer behavior, that is, to the statement that  $C(t,v)$  depends only on the difference  $t-v$  for all  $t \geq 0$ ,  $v \leq t \leq v+1$ .) Hence, utilizing equation (9) with appropriate simplification, it follows that a necessary condition for balanced growth equilibrium is

$$(10) \quad 0 = \left[ f(k) - nk \right] - w \left[ \frac{n}{1 - e^{-n}} - \frac{\delta}{1 - e^{-\delta}} \frac{1 - e^{-r}}{r} \frac{1 - e^{-(n+\delta-r)}}{n + \delta - r} \right] .$$

For convenience, let us rewrite (10) as an equation in the rate of interest  $r$  (which is legitimate by virtue of the strict concavity of the function  $f$ )

$$(11) \quad \psi(r) = \varphi(r) ,$$

where we define the function  $\psi$  as

$$\psi(r) = \frac{f(k) - nk}{f(k) - f'(k)k}$$

and the function  $\varphi$  as

$$\varphi(r) = \frac{n}{1 - e^{-n}} \frac{\delta}{1 - e^{-\delta}} \frac{1 - e^{-r}}{r} \frac{1 - e^{-(n+\delta-r)}}{n + \delta - r} .$$

Properties of the functions  $\psi$  and  $\varphi$ , relevant now and later on, are: For  $\psi$

$$(12a) \quad \psi(r) \leq 1 \text{ for } r \leq n$$

and

$$(12b) \quad \psi'(r) = \frac{1}{f''(k)} \frac{f'(k) - n}{f(k) - f'(k)k} + \psi(r) \frac{k}{f(k) - f'(k)k} ,$$

while for  $\varphi$

$$(12c) \quad \varphi\left(\frac{\delta+n}{2} + x\right) = \varphi\left(\frac{\delta+n}{2} - x\right) \text{ for all } x ,$$

$$(12d) \quad \varphi(\delta) = \varphi(n) = 1 ,$$

$$(12e) \quad \varphi'(r) = [g(r) - g(n+\delta-r)] \varphi(r) \stackrel{<}{>} 0 \text{ for } r \stackrel{<}{>} \frac{n+\delta}{2} ,$$

and

$$(12f) \quad \varphi''(r) = \left[ \left( g'(r) + g'(n+\delta-r) \right) + \left( g(r) - g(n+\delta-r) \right)^2 \right] \varphi(r) > 0 ,$$

where it is useful to introduce the function  $g$  , defined by

$$g(x) = \frac{(1+x)e^{-x} - 1}{x(1 - e^{-x})} \quad \text{for all } x ,$$

and shown in Figure I .

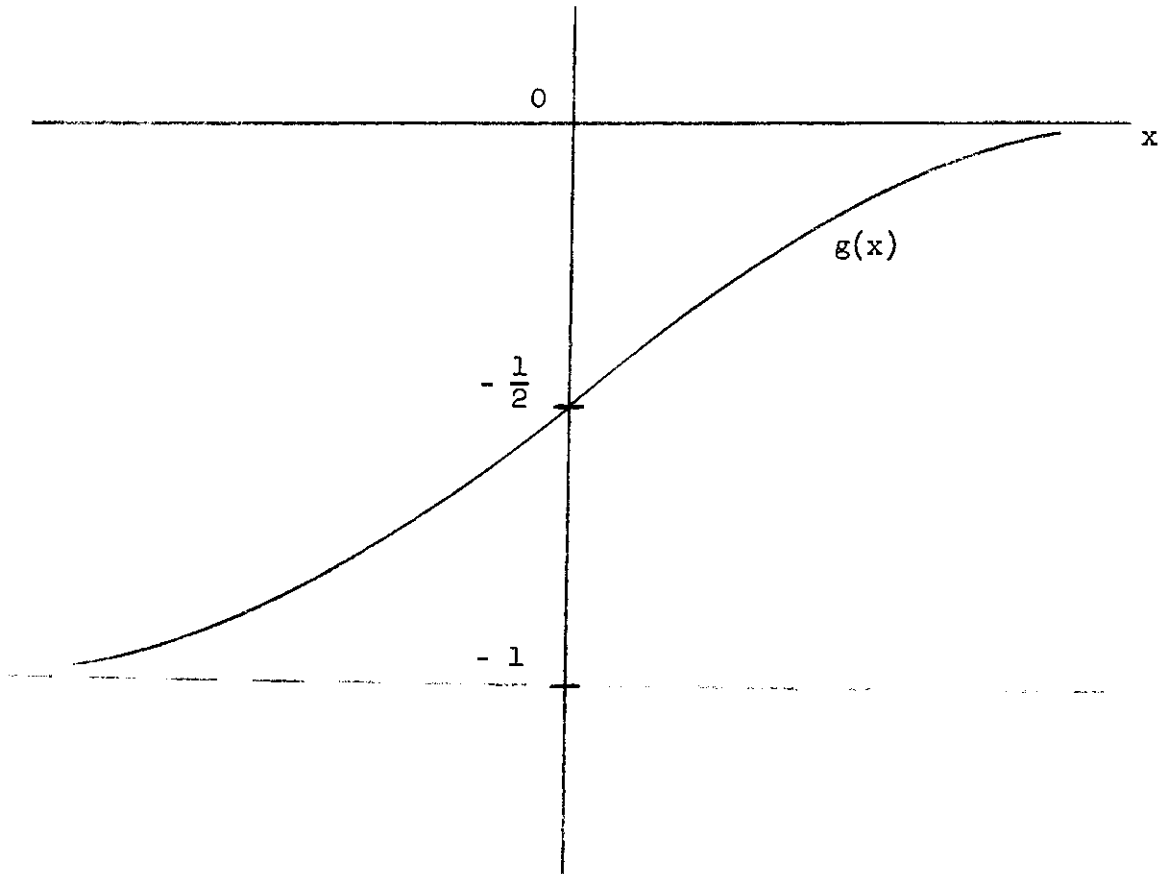


FIGURE I

From the properties listed above it becomes clear that the equation (11) yields at least one candidate,  $r = n$  , and in general more than one candidate,  $r = n$  ,  $r = \bar{r}_1$  ,  $\dots$  ,  $r = \bar{r}_m$  , for balanced growth equilibrium. As an illustration, in Figure II,  $\psi$  ,  $\phi$  and the two solutions to equation (11) are depicted under the assumptions  $0 < \delta < n$  and  $f(k) = Ak^\alpha$  ,  $0 < \frac{\alpha}{1-\alpha} < n [g(n) - g(\delta)]$  . By now examining equation (8)

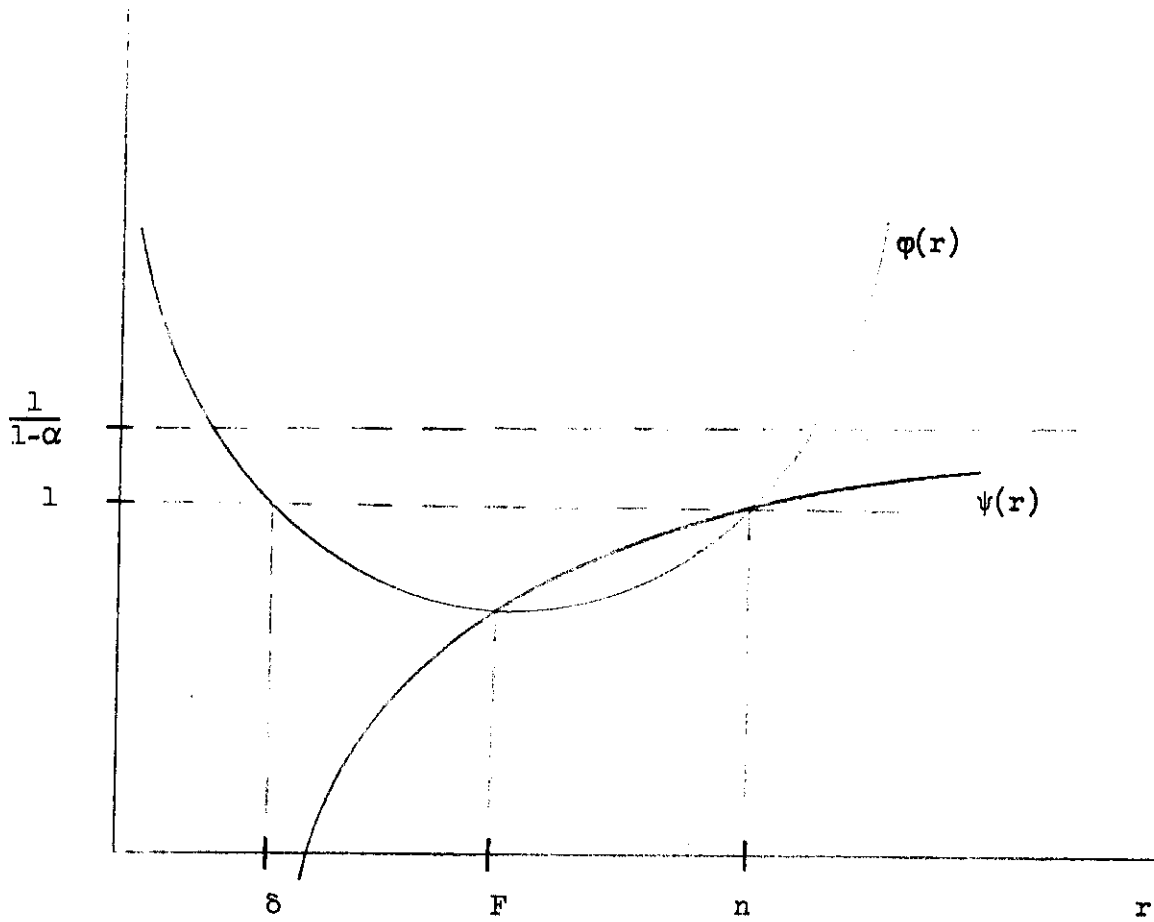


FIGURE II

we can show that  $r = n$  represents a balanced growth equilibrium if and only if  $\psi'(n) = \phi'(n)$  (i.e., if and only if the functions  $\psi$  and  $\phi$  are actually tangent to each other at  $r = n$ ) whereas  $r = \bar{r}_j \neq n$ ,  $j = 1, \dots, m$ , always represents a balanced growth equilibrium.

Let  $k^*$  be the golden rule capital-labor ratio, that is, let  $f'(k^*) = n$ . Then the first part of the assertion follows simply by writing out the expression for assets per capita from (7) when  $r = n$

$$\begin{aligned}
 (13) \quad a^* &= [f(k^*) - f'(k^*)k^*] \left[ \frac{1}{n} \left( \frac{n}{1 - e^{-n}} - 1 \right) - \frac{1}{\delta} \left( \frac{\delta}{1 - e^{-\delta}} - 1 \right) \right] \\
 &= [f(k^*) - f'(k^*)k^*] \varphi'(n)
 \end{aligned}$$

and noting that therefore  $k^* = a^*$  if and only if

$$\frac{k^*}{f(k^*) - f'(k^*)k^*} = \varphi'(n)$$

or, from (12b),  $\psi'(n) = \varphi'(n)$ . On the other hand, let  $\bar{k}$  be the capital-labor ratio corresponding to any other solution to equation (11),  $\bar{r} \neq n$ . Then the second part of the assertion follows by differentiating the expression for assets per capita from (7) when  $r = \bar{r}$

$$0 = [\bar{r} - n] \bar{a} + [f(\bar{k}) - f'(\bar{k})\bar{k}] [1 - \varphi(\bar{r})]$$

or

$$\bar{a} = \frac{[f(\bar{k}) - f'(\bar{k})\bar{k}] [1 - \varphi(\bar{r})]}{n - \bar{r}}$$

and noting that therefore  $\bar{k} = \bar{a}$  if and only if

$$\frac{f(\bar{k}) - n\bar{k}}{f(\bar{k}) - f'(\bar{k})\bar{k}} = \varphi(\bar{r})$$

or  $\psi(\bar{r}) = \varphi(\bar{r})$ .

There are alternative sets of additional conditions on the production function which would insure at least one balanced growth equilibrium, e.g.,

the conditions

$$\psi''(r) \leq 0 \text{ for } r \geq \hat{r},$$

where  $0 < \hat{r} < \infty$ , and

$$\lim_{k \rightarrow 0} r = +\infty.$$

Such conditions are not very transparent. For this reason, in much of the following discussion we will be implicitly assuming that there is at least one balanced growth equilibrium without explicitly assuming any additional conditions on  $f$ .

What conclusions can be drawn from the foregoing? First, that there may be no, one or several balanced growth equilibria. And second, that any feasible, positive rate of interest <sup>10/</sup> may represent a balanced growth equilibrium, depending, in particular, on the specific production technology and rate of time preference prevailing in the economy. Citing a theorem due to Koopmans and elaborated by Phelps [7] -- that if  $r(t) \leq n - \epsilon$  for some  $\epsilon > 0$  and all  $t \geq \hat{t}$ , then the growth path represented by  $r(t)$  is inefficient <sup>11/</sup> -- we can therefore deduce a further important result: The balanced growth equilibrium in our stylized competitive economy need not be an efficient growth path.

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<sup>10/</sup> As we have not forestalled the possibility of capital saturation, for a sufficiently "perverse" rate of time preference, the equation (11) might have a nonpositive solution. This exceptional case is at present excluded by assumption ( $\delta \geq 0$ ), but will receive further attention in Section VI.

<sup>11/</sup> In this context, inefficiency means simply that, given the same initial capital - labor ratio, there is another feasible growth path which provides at least as much total consumption all of the time, and in fact provides more total consumption some of the time.

Readers familiar with Solow's basic diagram, e.g., as employed by Phelps [7, p. 807, Figure 3], can easily grasp the intuitive basis for Koopman's theorem by considering "almost" balanced growth paths satisfying its hypotheses in terms of that diagram.



#### IV. General Growth Equilibrium: An Example

In this section we detail an example which is both interesting in its own right and suggestive of our general treatment of competitive growth equilibrium.

Suppose that production technology in the economy is described by

$$(14) \quad f(k) = k(A + B \log k) , \quad B < 0 ,$$

which is the general representation of the class of production functions having

$$\frac{\pi(k)}{\sigma(k)} \equiv 1 ,$$

the ratio of capital's (competitive) relative share to the elasticity of substitution identically equal to unity. Notice also that the production function (14) exhibits capital saturation when  $k \geq e^{-(1+A/B)}$ . Some further properties we shall use are

$$(15a) \quad r = A + B + B \log k ,$$

$$(15b) \quad w = - Bk$$

and

$$(15c) \quad \psi(r) = \left(1 + \frac{n}{B}\right) + \frac{1}{-B} r .$$

The significant attribute of an economy having a production function (14) is stated in the following proposition: For every  $B < 0$  there exists a unique constant  $\bar{\xi}$ , with  $\varphi\left(\frac{n+\delta}{2}\right) \leq \bar{\xi}$  and  $\bar{\xi} \gtrless 1$  according as  $-\frac{1}{B} \lesseqgtr g(n) - g(\delta)$ , such that the growth path on which

$$(16) \quad c(t) = \bar{\xi} w(t)$$

is an equilibrium growth path. In words, such an economy will exhibit a competitive equilibrium in which total consumption is simply a constant proportion of labor income.

That the competitive growth equilibrium need not be efficient is thus an immediate corollary (as  $\bar{\xi} \geq 1$  implies  $\lim_{t \rightarrow \infty} r(t) \leq n$ ).

By way of demonstration we observe initially that for any neoclassical production function (no longer denoting dependence on time unless necessary for clarity)

$$\frac{\dot{w}}{w} = \frac{\pi}{\sigma} \frac{\dot{k}}{k} ,$$

and therefore that given the assumption (14), if (16) holds, then

$$(17) \quad \begin{aligned} \frac{\dot{w}}{w} - r &= \left[ (A + B \log k) - n + \bar{\xi} B \right] - r \\ &= - \left[ n + (1 - \bar{\xi})B \right] . \end{aligned}$$

Utilizing (17), from the earlier definition of consumption per capita in (6), we have

$$c(t) = w(t) \varphi \left( n + (1 - \bar{\xi})B \right) .$$

Consequently, to verify that (16) defines an equilibrium growth path we need only to show that the relevant solution to the equation

$$(18) \quad \xi = \varphi \left( n + (1 - \xi)B \right)$$

is unique.

But, letting  $r = n + (1 - \xi)B$ , it is easily seen that equation (18) is simply equation (11) of Section III, where the function  $\psi$  now has the linear form given in (15c). It follows directly that  $\bar{r}$ , the rate of interest representing balanced growth equilibrium, and therefore also  $\bar{\xi}$  are uniquely determined -- as is illustrated in Figure III under the assumptions  $0 < \delta < n$  and  $\psi'(n) = \frac{1}{-B} > g(n) - g(\delta) = \varphi'(n)$ .

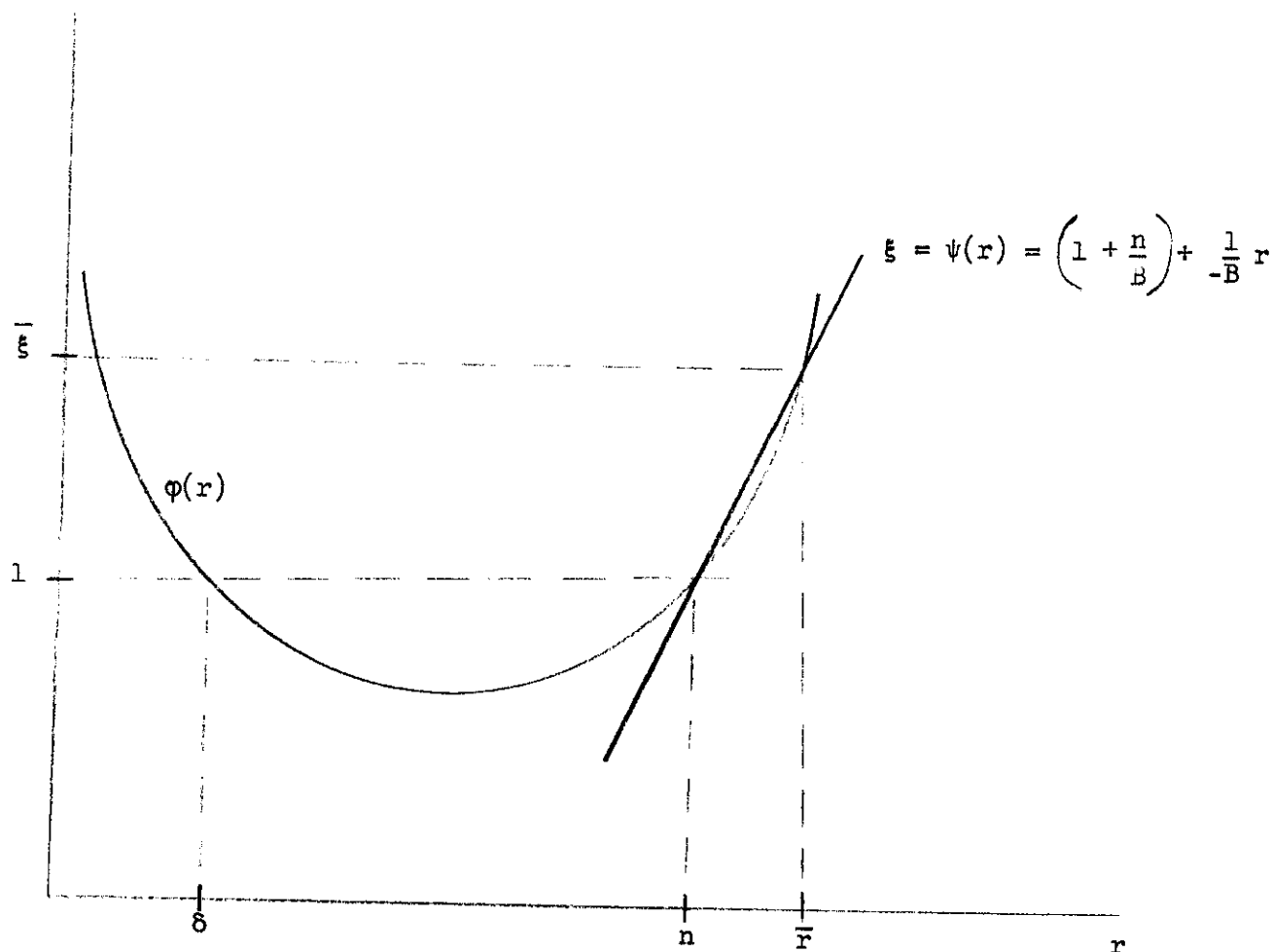


FIGURE III

The equality of assets per capita to the capital stock per head if (16) holds is also straightforward to prove. For, from the earlier definition of assets per capita in (7), again utilizing (17), we have

$$(19) \quad a(t) = w(t) \left\{ \frac{n}{1-e^{-n}} \left[ \frac{1}{\bar{r}} \left( \frac{1-e^{-(n-\bar{r})}}{n-\bar{r}} - \frac{1-e^{-n}}{n} \right) - \frac{1}{\delta} \frac{1-e^{-\bar{r}}}{\bar{r}} \frac{\delta}{1-e^{-\delta}} \left( \frac{1-e^{-(n-\bar{r})}}{n-\bar{r}} - \frac{1-e^{-(n+\delta-\bar{r})}}{n+\delta-\bar{r}} \right) \right] \right\},$$

where  $\bar{r} = n + (1 - \xi)B$ . Asymptotically  $a(t) \rightarrow \bar{a} = \bar{k}$  so that the constant factor on the RHS of (19) must equal one, or  $a(t)$  must equal  $k(t)$  for all  $0 \leq t$ .

The competitive growth equilibrium (16) has an alternative characterization which is quite instructive. Specifically, if  $\bar{r} > n$ , then that growth path maximizes

$$(20) \quad \int_{-1}^{\infty} \int_{\max(0,v)}^{v+1} \log C(t,v) e^{-\delta(t-v)} dt e^{-\rho v} dv =$$

$$\int_0^{\infty} \int_{t-1}^t \log C(t,v) e^{(\rho-\delta)(t-v)} dv e^{-\rho t} dt,$$

where  $\rho = \bar{r} - n$ , the social welfare function which gives each generation's total welfare a weight  $e^{-\bar{r}v}$ .<sup>12/</sup> This assertion follows from the facts, elaborated in the next section, first, that the expression

$$(21) \quad \log c(t) + \left\{ \log \frac{1-e^{-n}}{n} - (\rho-\delta+n) \frac{1-e^{\rho-\delta}}{\rho-\delta} g(\delta-\rho) \right\}$$

<sup>12/</sup> Furthermore, if  $\bar{r} \leq n$ , then the competitive growth equilibrium (16) "attempts" to maximize the same social welfare function, in the sense that it is the unique asymptotically balanced growth path which satisfies the Euler equation for the calculus of variations problem (23) below, a problem which in fact has no solution when  $\rho \leq 0$ . If  $\bar{r} < n$ , then the "attempt" is clearly a failure; the competitive growth equilibrium is in fact inefficient. If  $\bar{r} = n$ , then the "attempt" is a failure, but may be rectified by redefining social welfare as the sum of the amounts by which each generation's total welfare exceeds its (hypothetical) golden rule welfare.

is the value of the solution to the calculus of variations problem

$$\begin{aligned}
 (22) \quad & \text{maximize } \int_{t-1}^t \log C(t,v) e^{(\rho-\delta)(t-v)} dv \\
 & \text{subject to } \frac{n}{1 - e^{-n}} \int_{t-1}^t C(t,v) e^{-n(t-v)} dv = c(t) \quad \text{and} \\
 & C(t,v) \geq 0,
 \end{aligned}$$

and second, that the growth path (16) is the solution to the calculus of variations problem

$$\begin{aligned}
 (23) \quad & \text{maximize } \int_0^{\infty} \log c(t) e^{-\rho t} dt \\
 & \text{subject to } \dot{k}(t) = (A - n + B \log k(t)) k(t) - c(t), \quad k(0) > 0 \quad \text{and} \\
 & c(t) \geq 0.
 \end{aligned}$$

Hence, the asserted conclusion is obtained by solving the problem (22) and substituting the value of its solution (21) into the social welfare function (20), and then solving the problem (23).

This latter characterization has a crucial feature which may not be readily apparent, namely that, by virtue of (17), the generational weights  $e^{-\bar{r}v}$  are proportional to the present value of the labor endowment

of an individual of generation  $v$  along the competitive growth equilibrium (16)  
itself

$$(24) \quad e^{-\bar{r}v} = \frac{1}{w(0) \frac{1 - e^{-\bar{r}}}{\bar{r}}} \int_v^{v+1} \bar{w}(s) e^{-\int_0^s \bar{r}(x) dx} ds .$$

The results outlined in the two preceding paragraphs suggest that, for a general competitive growth equilibrium, we might well look for a growth path having a similar property, that is, a growth path that is (or "tries" to be) socially optimal in terms of a social welfare function like that in (20) with generational weights like those in (24). The purpose of the next section is to carry out this suggestion. Along the way we demonstrate the existence of an asymptotically balanced growth equilibrium for the model of Section II starting from an arbitrary capital-labor ratio  $k^0 > 0$  at time  $t = 0$ .

#### V. General Growth Equilibrium: Existence, Asymptotic Balance, and Social Optimality

For any feasible growth path  $\hat{k}(t)$ ,  $-\infty \leq t \leq \infty$ , satisfying the time zero condition

$$(25) \quad \hat{k}(0) = k^0 ,$$

where  $k^0$  is a given capital-labor ratio, and the terminal condition

$$(26) \quad \lim_{t \rightarrow \infty} \hat{k}(t) = \bar{k} ,$$

where  $\bar{k}$  is a given competitive balanced growth equilibrium, define the function

$$\hat{D}(v) = \int_v^{v+1} \hat{w}(s) e^{-\int_0^s \hat{r}(x) dx} ds$$

$$= e^{-\int_0^v \hat{r}(x) dx} \int_v^{v+1} \hat{w}(s) e^{-\int_v^s \hat{r}(x) dx} ds, \quad -\infty \leq v \leq \infty,$$

which represents the present value of the labor endowment of an individual of generation  $v$  along this particular growth path. Further, define the social welfare generated by any feasible growth path  $k(t)$ ,  $0 \leq t \leq \infty$ , satisfying the initial condition

$$(27) \quad k(0) = k^0,$$

where  $k^0$  is the same given capital-labor ratio as in (25), as follows:

Using the function  $\hat{D}(v)$  as the weight attached to each generation's total welfare after time zero, let the social welfare function be denoted  $\hat{W}$  and written

$$(28) \quad \hat{W} = \int_{-1}^{\infty} \left( \int_{\max(0,v)}^{v+1} \log C(t,v) e^{-\int_0^t \delta(t-v) dt} e^{nv} \right) \hat{D}(v) dv =$$

$$\int_0^{\infty} \left( \int_{t-1}^t \log C(t,v) e^v \int_v^{v+1} \hat{w}(s) e^{-\int_v^s \hat{r}(x) dx} ds dv \right) e^{-\int_0^t [\hat{r}(x) - n] dx} dt.$$

Now consider the variational problem

$$\begin{aligned}
 (29) \quad & \text{maximize } \hat{W} \\
 & \text{subject to } \frac{n}{1 - e^{-n}} \int_{t-1}^t c(t,v) e^{-n(t-v)} dv = c(t) , \\
 & \dot{k}(t) = f(k(t)) - nk(t) - c(t) , \quad k(0) = k^0 , \quad \text{and} \\
 & c(t,v) \geq 0 , \quad c(t) \geq 0 .
 \end{aligned}$$

If the problem (29) has a solution, denoted  $\hat{k}^1(t)$  ,  $0 \leq t \leq \infty$  , and referred to as the true solution, then that true solution will satisfy

$$(30) \quad \lim_{t \rightarrow \infty} \hat{k}^1(t) = \bar{k} .$$

On the other hand, if the problem (29) does not have a true solution, which will be the case if and only if

$$\lim_{T \rightarrow \infty} \int_{-1}^T e^{nv} \hat{D}(v) dv = \infty$$

or

$$\lim_{T \rightarrow \infty} \int_{-1}^T e^{-\int_0^v [\hat{r}(x) - n] dx} dv = \infty ,$$



then define as the, say, misguided solution that growth path which satisfies the feasible growth and Euler equations for (29) (see equations (32) and (33) below) and the terminal condition (30), and denote it once again  $\hat{k}^1(t)$ ,  $0 \leq t \leq \infty$ .

It can be shown that the class of all growth paths  $\hat{k}(t)$  is weakly compact in the space  $L_\infty$ . Furthermore, the operator which carries the growth path  $\hat{k}(t)$  into the growth path  $\hat{k}^1(t)$ , with the latter then being extended backwards to  $-\infty$  by means of the feasible growth and Euler equations for (29) (themselves appropriately extended backwards with the use of the function  $\hat{D}(v)$ )<sup>13/</sup>, is a weakly continuous operator. Therefore, the Schauder-Tychonoff fixed point theorem (see [5], p. 456) is applicable, and enables us to conclude that this operator has a fixed point in the class of all growth paths  $\hat{k}(t)$ . Let such a fixed point be denoted  $\hat{k}^*(t)$ ,  $-\infty \leq t \leq \infty$ . Then, it is straightforward to prove that this growth path is a competitive equilibrium: Given the generational weights  $\hat{D}^*(v)$ , decompose (29) into two parts, first, the problem of choosing individual distribution on an arbitrary feasible growth path having consumption per capita  $c(t) \geq 0$ ,

$$(29I) \quad \text{maximize } \bar{V}^* = \int_{t-1}^t \log C(t,v) e^{\int_v^t [\hat{i}^*(x) - \delta - n] dx} \int_v^{v+1} \hat{w}^*(s) e^{-\int_v^s \hat{r}^*(x) dx} ds dv$$

$$\text{subject to } \frac{n}{1 - e^{-n}} \int_{t-1}^t C(t,v) e^{-n(t-v)} dv = c(t) \quad \text{and}$$

$$C(t,v) \geq 0,$$

<sup>13/</sup> Undoubtedly the reader has noticed our studied avoidance, prior to this section, of reference to initial conditions for the model of Section II. Here, however in discussing the general question of existence we are forced to face the fact that if a competitive growth equilibrium exists today, then it always has existed and always will exist.

which yields an optimum value  $\hat{V}^*(c(t), t)$  for all  $0 \leq t \leq \infty$ , and second, that of choosing a feasible growth path,

$$(29 \text{ II}) \quad \text{maximize} \quad \hat{W}^* = \int_0^{\infty} \hat{V}^*(c(t), t) e^{-\int_0^t [\hat{r}^*(x) - n] dx} dt$$

$$\text{subject to} \quad \dot{k}(t) = f(k(t)) - nk(t) - c(t), \quad k(0) = k^0, \quad \text{and} \\ c(t) \geq 0. \quad \underline{14/}$$

Consider (29 I). Its solution is easily derived in a manner similar to that of the above-mentioned consumer lifetime allocation problem, and is given by

$$C(t, v) = c(t) \frac{\int_v^{v+1} \hat{w}^*(s) e^{-\int_v^s \hat{r}^*(x) dx} ds e^{\int_t^s [\hat{r}^*(x) - \delta] dx}}{\frac{n}{1 - e^{-n}} \int_{t-1}^t \int_v^{v+1} \hat{w}^*(s) e^{-\int_v^s \hat{r}^*(x) dx} ds e^{\int_v^t [\hat{r}^*(x) - \delta - n] dx} dv},$$

which yields

$$(31) \quad \hat{V}^*(c(t), t) = \log c(t) \int_{t-1}^t \int_v^{v+1} \hat{w}^*(s) e^{-\int_v^s \hat{r}^*(x) dx} ds e^{\int_v^t [\hat{r}^*(x) - \delta - n] dx} dv + \hat{F}^*(t),$$

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14/ (29 I) is essentially the same as the consumer lifetime allocation problem of Section II, while (29 II) is essentially the same (but for a variable discount rate) as the optimum growth problem analyzed by, for example, Cass [2].

where  $\hat{F}^*(t)$  is a function depending only on the path  $\hat{k}^*(t)$  (compare with the result (21), recalling the special property of that example expressed in (24)). Now, consider (29 II) after substitution from (31). Its true or misguided solution is easily derived in a manner similar to that of the above-mentioned optimum growth problem, and is represented by the feasible growth constraint

$$(32) \quad \dot{\hat{k}}^1(t) = f(\hat{k}^1(t)) - n\hat{k}^1(t) - \hat{c}^1(t), \quad \hat{k}^1(0) = k^0,$$

the Euler equation

$$(33) \quad \frac{\dot{\hat{c}}^1(t)}{\hat{c}^1(t)} = \hat{r}^1(t) - \left\{ \hat{r}^*(t) - \frac{d \log \left[ \frac{n}{1 - e^{-n}} \int_{t-1}^t \int_v^{v+1} \hat{w}^*(s) e^{\int_v^s \hat{r}^*(x) dx} \int_v^t [\hat{r}^*(x) - \delta - n] dx dv \right]}{dt} \right\},$$

and the terminal condition (30). But, recalling the fixed point property of the growth path  $\hat{k}^*(t)$ ,

$$\hat{k}^1(t) \equiv \hat{k}^*(t), \quad 0 \leq t \leq \infty,$$

we have from equation (33) that

$$(34) \quad \frac{\dot{\hat{c}}^1(t)}{\hat{c}^1(t)} = \frac{\dot{\hat{c}}^*(t)}{\hat{c}^*(t)} = \frac{d \log \left[ \frac{n}{1 - e^{-n}} \int_{t-1}^t \int_v^{v+1} \hat{w}^*(s) e^{\int_v^s \hat{r}^*(x) dx} \int_v^t [\hat{r}^*(x) - \delta - n] dx dv \right]}{dt},$$

and from equation (30), as well as our particular choice of  $\bar{k}$  that

$$(35) \quad \lim_{t \rightarrow \infty} \hat{c}^1(t) = \lim_{t \rightarrow \infty} \hat{c}^*(t) = \bar{c} ,$$

where  $\bar{c}$  is

$$\bar{c} = \lim_{t \rightarrow \infty} \frac{n}{1 - e^{-n}} \int_{t-1}^t \int_v^{v+1} \hat{w}^*(s) e^{-\int_v^s \hat{r}^*(x) dx} \frac{\int_v^t [\hat{r}^*(x) - \delta - n] dx}{ds} e^{-\int_v^t [\hat{r}^*(x) - \delta - n] dx} dv ,$$

consumption per capita on the competitive balanced growth equilibrium  $\bar{k}$ .

Hence, by virtue of (6), (34) and (35) together imply

$$\hat{c}^*(t) = \text{competitive consumption per capita on the growth path } \hat{k}^*(t) .$$

And, by virtue of (7), a similar argument concludes

$$(36) \quad \hat{k}^*(t) = \text{competitive assets per capita on the growth path } \hat{k}^*(t) .$$

Let us summarize the foregoing: Given any positive capital-labor ratio,  $k^0$ , and a particular competitive balanced growth equilibrium,  $\bar{k}$ , there exists a competitive general growth equilibrium which passes through  $k^0$  at time  $t = 0$  and then asymptotically approaches  $\bar{k}$  as  $t \rightarrow \infty$ . Each of these equilibria maximizes, or attempts to maximize, the social welfare function which weights each generation's total welfare by the present value of its typical individual's labor endowment along the competitive general growth equilibrium.

Two additional remarks are in order: First, nothing we have said rules out the existence of other competitive general growth equilibria which do not exhibit such special properties. Moreover, the prospects for a uniqueness argument do not seem to us very promising. Second, in general the golden rule

path can be utilized in the terminal condition (25) to yield all the results of this section except for the equality of the capital stock per head and competitive assets per capita (36) <sup>15/</sup>. And, of course, if  $\psi'(n) = \phi'(n)$ , then it can be used even to derive (36).

## VI. An Interpretation

In this section, for simplicity, we will concentrate on balanced growth equilibrium. We also make the convenient assumption that, whatever the production technology, it satisfies the condition

$$\psi''(r) \leq 0,$$

which guarantees a unique balanced growth equilibrium. The central question we will be addressing is: Why may competitive capital accumulation go astray? <sup>16/</sup>

The answer to the latter question is, in fact, rather simple: At efficient rates of interest consumers may want to hold more real assets than are available in the existing capital stock. (More precisely, as was implicitly demonstrated earlier in the discussion of assets per capita on balanced growth paths, if  $\psi''(r) \leq 0$  and  $\psi'(n) < \phi'(n)$ , then  $\bar{r} < n$  and  $a \geq k$  for  $r \geq \bar{r}$ , or  $a > k$  for  $r \geq n$ .) An extreme example points up this difficulty especially well:

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<sup>15/</sup> This point will be important in our discussion of the effects stemming from the existence of a monetary system; or Marxian entrepreneurs; see Sections VII and VIII below.

<sup>16/</sup> There is some overlap between our discussion here and the argument presented in [3]. We are guilty of this redundancy principally for the purpose of completeness.

Suppose, in a more classical tradition, that population is stationary ( $n = 0$ ) and that capital saturation is possible ( $f'(\tilde{k}) = 0$  for some  $0 < \tilde{k} < \infty$ ). Also suppose that consumers exhibit "perverse" time preference ( $\delta < 0$ ). Finally, suppose that production technology entails that  $\psi'(0) < \phi'(0)$ . Then, there is a unique stationary solution to equation (8) at some  $\bar{r} < 0$  (depicted in Figure I by shifting the vertical axis to the point  $n$  on the horizontal axis), which obviously represents a grossly inefficient situation; merely by utilizing less than the whole capital stock, total output could be increased. However, for this example, because consumers can costlessly carry inventories or hold capital as assets, the rate of interest would never fall below zero, and equation (8) is an incomplete statement of the equilibrium condition, which instead should be

$$(8') \quad a(t) = k(t) + x(t) ,$$

with

$$k(t) \leq \tilde{k}$$

and

$$x(t) \geq 0 , \text{ equality for } k(t) < \tilde{k} ,$$

where  $x(t)$  stands for inventories of consumption goods per head at time  $t$ . Therefore, it is easily seen that the true stationary state for this stylized, classical competitive economy must occur at the rate of interest  $r = n = 0$ , which, from the production side, is clearly efficient. On the other hand, from the consumption side, this stationary state is just as clearly inefficient: As we have the relationship  $a \geq k$  for  $r \geq \bar{r}$ , it follows that  $x > 0$  when  $r = 0$ , or, that in the stationary state, consumers desire to carry an

inventory of consumption goods over and above their holdings of capital, an inventory which will in actuality never be consumed. We recall for emphasis that from each consumer's viewpoint, carrying such a dead weight is quite sensible; he is doing the best he can given the wage rate  $w = f(\tilde{k})$  and interest rate  $r = 0$  prevailing.

Given our interpretation, the question naturally arises: Have we overlooked something, say, some aspect of consumer behavior, or perhaps some institution, which will in fact insure the "right" relationship between real asset preferences and opportunities? Let us consider these possibilities in turn.

One point, purposely left in the background until now, is that a sufficiently high rate of time preference would guarantee (at least) efficient balanced growth equilibrium under all conceivable (neoclassical) production technologies. That is, if  $\delta \geq n$ , then

$$(37) \quad \phi'(n) = g(n) - g(\delta) \leq 0 ,$$

which rules out the possibility of a solution to equation (11) for  $r < n$ . Intuitively this is a rather plausible result; if a consumer strongly prefers consumption today to consumption tomorrow, then, even at relatively high rates of interest, his total savings will be relatively small (algebraically) at every instant during his lifetime (cf. equation (5)). However, the requisite balancing of parameters cannot be depended on in theory, and indeed, loses a degree of support if we admit some variation among consumers' rates of time preference.<sup>17/</sup>

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<sup>17/</sup> Some support is regained in the concluding section, where we analyze an economy in which uncertainty about the end of the world contributes to positive time preference.

Suppose now that each generation has a proportion  $\lambda_j > 0$  of individuals with rate of time preference  $\delta_j \geq 0$ ,  $j = 1, \dots, m$  and

$\sum_{j=1}^m \lambda_j = 1$ . Then we have

$$\varphi(r) = \sum_{j=1}^m \lambda_j \varphi_{\delta_j}(r),$$

where

$$\varphi_{\delta_j}(r) = \frac{n}{1 - e^{-n}} \frac{\delta_j}{1 - e^{-\delta_j}} \frac{1 - e^{-r}}{r} \frac{1 - e^{-(n+\delta_j)r}}{n + \delta_j - r},$$

and the stylized competitive economy will exhibit efficient balanced growth equilibrium for every (neoclassical) production technology if

$$(38) \quad \varphi'(n) = \sum_{j=1}^m \lambda_j \varphi'_{\delta_j}(n) = g(n) - \sum_{j=1}^m \lambda_j g(\delta_j) \leq 0.$$

If we now associate the rate of time preference introduced in Section II,  $\delta$ , with the average rate of time preference here,  $\sum_{j=1}^m \lambda_j \delta_j$ , then by the strict concavity of  $g(x)$  for  $x > 0$ , it follows that

$$-g(\delta) < -\sum_{j=1}^m \lambda_j g(\delta_j),$$

which means that the inequality in (38) might not be satisfied even if that in (37) were. While not placing too much weight on this example, we also remark that it illustrates that diversity in consumer behavior cannot be expected, a priori, to reduce the likelihood that consumers may want to hold more real assets than are available in the capital stock at efficient rates of interest.



Another aspect of consumer behavior whose effects we have investigated is biological or sociological restrictions on the length of and efficiency during the work life. <sup>18/</sup> As a gross approximation to these restrictions, suppose that the individuals of each generation enter the labor force at age  $m_1$ , (because they are maturing and undergoing education from age 0 to age  $m_1$ ,) and leave it at age  $m_2$  (because they are no longer able to work), where  $0 \leq m_1 \leq m_2 \leq 1$ . In the spirit of the two preceding paragraphs, it is easily shown that under these additional assumptions the function  $\varphi$  remains strictly convex, while  $\varphi'(n) \leq 0$  if and only if

$$(39) \quad -g(\delta) \leq -(m_2 - m_1) g(n(m_2 - m_1)) + m_1.$$

The RHS of inequality (39) is depicted in Figure IV under the further assumption that the growing-up and retirement periods are of equal length,  $m_1 = \epsilon$  and  $m_2 = 1 - \epsilon$  for  $0 \leq \epsilon < \frac{1}{2}$ . Again, though one can certainly derive conditions

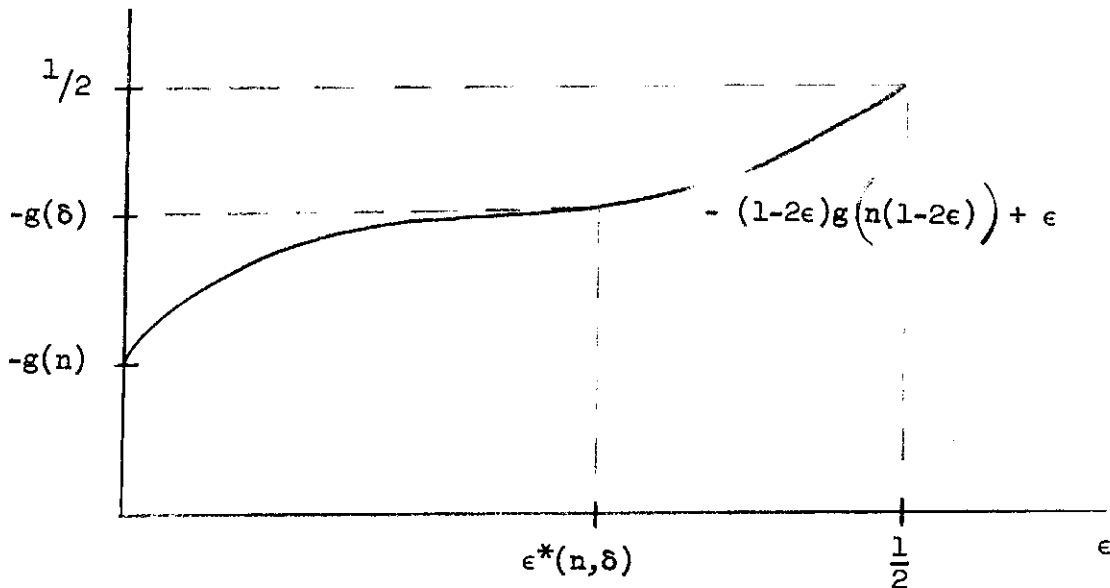


FIGURE IV

<sup>18/</sup> Both Diamond [4] and Samuelson [8] adopt one such restriction, forced retirement, as an integral part of their models. Unlike them, we prefer to treat these restrictions as less fundamental, partly because their particular formulation seems somewhat more arbitrary than that of the other elements in the basic model, but mostly because we feel that their introduction at the outset tends to obscure the basic issue being discussed presently.

from such restrictions which would insure the efficiency of competitive behavior (e.g.,  $\epsilon \geq \epsilon^*(n_1 \delta)$  in Figure IV), a presumption that they would be satisfied is not justified.

It is clear, at least to us, after analyzing these and other aspects of consumer behavior, that further complications along similar lines do not basically alter the general conclusion stated at the beginning of this section. What, then, about an overlooked institution, one which might (naturally?) exist in a potentially inefficient economy <sup>19/</sup> -- giving all the appearances of having "room for a deal" -- and whose existence might forestall the possibility of inefficiency?

That any such institution would have as its essential function the provision of assets to be held by consumers in lieu of capital goods should be apparent after a moment's reflection. The logical candidate is thus a financial intermediary, (possibly) holding as assets capital goods, while issuing as liabilities various types of financial instruments. We deduce immediately, <sup>20/</sup> however, that in a potentially inefficient economy, private ownership of the intermediary sector and efficient balanced growth equilibrium

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<sup>19/</sup> Hereafter this term signifies a stylized competitive economy in which the prevailing production technology and rate of time preference entail  $\psi'(n) < \phi'(n)$ .

<sup>20/</sup> And also somewhat reluctantly. Notice, especially, that this conclusion is independent of the type or behavior of private intermediaries postulated. In an earlier stage of this research we thought that a sector of private intermediaries with existence in their own right (i.e., owning capital goods purchased from internal funds) would set matters aright. Though this conjecture proved wrong, such a private intermediary sector does generate effects of some interest, and therefore will be detailed in Section VIII.

are mutually exclusive. This conclusion follows from the fact that the only efficient balanced growth path consistent with a competitive consumer sector is the golden rule path (i.e., the rate of interest  $r = n$  is the only efficient solution to equation (11) for a potentially inefficient economy). Hence, assuming for simplicity that consumers hold only financial assets, if an intermediary sector were to provide the real assets consumers desire on this balanced growth path,  $L(t)a^*$ , then it would have negative and, indeed, continually decreasing net worth,  $L(t)(k^* - a^*) < 0$ .

But, public ownership of the intermediary sector -- for instance in the form of a Social Saving System or, like in Diamond's model [4], a Fiscal Authority, or, as suggested by Samuelson [8], a Monetary Authority -- is certainly possible. And, in contrast to the emphasized statement of the preceding paragraph, given proper policies a public intermediary sector could insure efficient balanced growth equilibrium -- precisely because it need not back its liabilities with specific assets, but rather with the general fiscal and monetary authority vested in government, and can therefore supply assets to the consumer sector (in part) independently of the existing capital stock.

To elucidate the last statement, in the next section we take up Samuelson's suggestion, and present a rigorous treatment of the effects stemming from the existence of a monetary system. Before that, however, let us recapitulate: First, in the stylized competitive economy of Section II, inefficiency may occur because consumers desire to hold more real assets than are available in the capital stock at efficient rates of interest. Second, there appears to be no aspect of consumer behavior which would, nor any

feasible private financial intermediary which could automatically correct this deficiency. And third, a public financial intermediary could correct this deficiency, though not necessarily automatically, for the reason that a public financial intermediary can issue liabilities without itself holding commensurate assets.

#### VII. A Passive Monetary Authority

Suppose that into a potentially inefficient economy <sup>21/</sup> we introduce a Monetary Authority which has nominal liabilities equal to the nominal money supply,  $M(t) > 0$ , and which pays a nominal rate of interest,  $-\infty < \rho(t) < \infty$ . Then the basic model of Section II must be modified accordingly: First, there is an additional growth equation for  $m(t)$ , the money supply per capita at time  $t$ ,

$$(40) \quad \frac{\dot{m}(t)}{m(t)} = \rho(t) - n, \quad m(0) = m^0 > 0.$$

Second, the money price level,  $p(t)$ , i.e., the price of output in terms of money at time  $t$ , must be accounted for. Third, individual holdings of real assets may now be composed of both real money balances,  $\frac{M(t,v)}{p(t)}$ , and real capital assets (including, perhaps, the liabilities of private, competitive financial intermediaries),  $B(t,v)$ ,

$$(41) \quad A(t,v) = \frac{M(t,v)}{p(t)} + B(t,v).$$

Fourth, the equilibrium condition (8) becomes that requiring equality of real assets per capita to the sum of the real money supply per capita and the capital stock per head

$$(8'') \quad a(t) = \frac{m(t)}{p(t)} + k(t).$$

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<sup>21/</sup> In our frictionless, perfect foresight world, money is not required to carry out transactions or guard against unforeseen contingencies, and is likely to be irrelevant except in the context of a potentially inefficient economy.

And finally, observing that individuals will hold real capital assets if and only if the real rate of interest is at least as large as the real rate of return on money balances, while they will hold many balances (i.e., the existence of money will be "nonneutral") if and only if the real rate of return on money balances is at least as large as the real rate of interest, we deduce a second equilibrium condition to determine the money price level.

$$(42) \quad \frac{\dot{p}(\hat{t})}{p(\hat{t})} = \rho(t) - r(t) , \quad p(\hat{t}) = \frac{m(\hat{t})}{a(\hat{t}) - k(\hat{t})} > 0 \text{ for some } 0 \leq \hat{t} < \infty .$$

It should be clear that (9) is still a necessary condition for competitive growth equilibrium. Moreover, if we have found a solution to (9) for which  $a(\hat{t}) - k(\hat{t}) > 0$  for some  $0 \leq \hat{t} < \infty$ , then from (7), (9), (40) and (42) it follows that

$$\begin{aligned} \frac{\dot{a} - \dot{k}}{a - k} &= \frac{[(r-n)a + (w-c)] - [f(k) - nk - c]}{a - k} \\ &= \frac{(r-n)a + (w-f(k) - nk)}{a - k} \\ &= r - n \\ &= \frac{\dot{m}}{m} - \frac{\dot{p}}{p} , \end{aligned}$$

equation (8'') is satisfied. Hence, as  $r = n$  is the only solution to equation (11) for which  $a > k$ , the balanced growth equilibrium in this potentially inefficient economy, after the introduction of the Monetary Authority, must be the golden rule path. And from this we can deduce, on the basis of the result remarked at the end of Section V, that, given a "passive" Monetary Authority in a

potentially inefficient economy, there is always a competitive general growth equilibrium which asymptotically approaches the golden rule path.

One might speculate, on the basis of the last proposition, that the existence of such a monetary system would always entail at least one efficient, competitive growth equilibrium. Without more specific knowledge of the growth paths  $\bar{k}^*(t)$ , where  $\bar{k} = k^*$ , this conjecture cannot be verified in general; indeed, a growth path asymptotic to the golden rule path may be inefficient.<sup>22/</sup> Notice, however, that given the special production function (14) with  $\frac{1}{-B} < g(n) - g(\delta)$ , an efficient, competitive growth equilibrium described by equation (16) with  $\bar{\xi} = 1$  (capital's share invested, labor's share consumed) always exists in the situation under discussion.

Observe finally that the central result of this section is true regardless of the particular monetary policy pursued, i.e., nominal rate of interest promised, even if that nominal rate of interest should at times be negative.

#### VIII. A Mixed Neoclassical-Marxian Model

One of the characteristics of neoclassical (in the broad sense of the term) economics is the notion that consumers are the sole source of independent decision making in the decentralized economy. Firms and financial institutions act as agents of their share holders, who are consumers. At another extreme, we find the Marxian view, which holds that in a private economy all the relevant decision making is concentrated in the hands of capitalists whose sole objective is accumulation, and consumption only enters the picture to the extent that the labor force must be kept at subsistence.

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<sup>22/</sup> An example suffices to demonstrate this claim:

$$\dot{k} = f(k) - nk - c^*, \quad k(0) > k^* .$$

In the foregoing discussion we have seen that a purely consumer-oriented economy may be inefficient. Consumer decisions were not sufficient to guarantee an efficiently operating system. Indeed, in order to guarantee efficiency we had to introduce into the model an institution which could not be thought of as a privately owned (i.e., consumer-owned) firm whose actions represent the interest of its owners. In this section, we shall take a brief look at the role which a similar institution might play in the efficient economy. Our model will turn out to be a blend of the neoclassical (consumer oriented) and the Marxian (accumulation oriented) points of view.

Let us assume that individuals cannot hold real capital as an instrument of saving but must, rather, save by holding corporate bonds. These bonds are issued by a multitude of competitive firms which, in turn, hold real capital as assets. We now depart from the neoclassical tradition by assuming that the ownership of these firms is not located in the consumer sector. In other words, firms behave according to certain independent objectives which are not reducible to consumer decisions. In particular, we shall assume that firms act so as to maximize the rate of increase in net worth at every moment of time. This is indeed a Marxian postulate. Firms must of course repay their debt to consumers (at competitive interest rates) but with this repayment their commitment to the consumer sector ends.

Let  $B(t)$  be the total number of corporate bonds <sup>23/</sup> outstanding

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<sup>23/</sup> By a "bond" we mean a debt instrument which is traded by whoever issues it for one unit of output, and which is recontracted at every instant at the current rate of interest.

at time  $t$  , and let  $S(t)$  be the aggregate net worth of firms at time  $t$  .  
Then, the consolidated balance sheet equation for all firms is

$$K(t) = B(t) + S(t)$$

which is equivalent to

$$(43) \quad k(t) = b(t) + s(t)$$

where  $b(t) = B(t)/L(t)$  and  $s(t) = S(t)/L(t)$  . At time  $t$  , firms hire labor and issue bonds so as to maximize the rate of increase of net worth. To find this rate of increase, we must write down the profit and loss statement of the firm at time  $t$  , under the assumption that profits are never distributed:

$$\dot{S}(t) = L(t)f(k(t)) - L(t)w(t) - r(t)B(t)$$

which, we note for later reference, reduces immediately to

$$\dot{s}(t) = f(k(t)) - ns(t) - w(t) - r(t)b(t) .$$

It is the quantity  $\dot{S}(t)$  which the firm is assumed to maximize, given the wage rate and the rate of interest. This maximization leads, once again, to the conditions

$$f(k(t)) - k(t)f'(k(t)) = w(t)$$

and

$$f'(k(t)) = r(t) .$$



The first of these equations may be regarded as a demand-for-labor equation and the second as a supply-of-bonds equation. These two equations may now be used in our equation for  $\dot{s}(t)$ , to obtain the following result:

$$(44) \quad \frac{\dot{s}(t)}{s(t)} = r(t) - n .$$

The dynamic behavior of this system is described by equations (43) and (44), together with the equilibrium condition for the bond market, namely,

$$(45) \quad a(t) = b(t) .$$

Again, (43), (44) and (45) entail the feasible growth equation

$$(9) \quad \dot{k}(t) = f(k(t)) - nk(t) - c(t) .$$

Let us now consider balanced growth equilibrium. Along a balanced growth path,  $k$  is constant by definition and  $a$  is constant by virtue of the stationarity of consumer decisions. Thus, from (45),  $b$  must also be constant and, since  $s = k - b$ , we conclude that along a balanced growth path  $s$  must be constant. It now follows from equation (44) that  $r = n$ , i.e., the only possible balanced growth path in our new model is the golden rule path.

Let  $k^*$ ,  $a^*$ ,  $b^*$  and  $s^*$  denote the values of  $k$ ,  $a$ ,  $b$  and  $s$ , respectively, along the golden rule path. We have already seen (equation (13)) that

$$a^* = [f(k^*) - nk^*] \varphi'(n)$$

and since  $a^* = b^*$  and  $s^* = k^* - b^*$ , we get

$$s^* = k^* - [f(k^*) - nk^*] \varphi'(n) .$$

Thus, using equation (12b) , we obtain

$$s^* \geq 0 \text{ if and only if } \psi'(n) \geq \phi'(n) .$$

In other words, the statement  $s^* \geq 0$  is equivalent to the statement that the economy is efficient. By the same token,  $s^* < 0$  means that the economy is potentially inefficient, so that in the inefficient case our firms reduce to the negative net worth intermediaries which are discussed in Sections VI and VII.<sup>24/</sup> Indeed, by identifying the quantity  $s(t)$  with the quantity  $-m(t)/p(t)$  of Section VII, we see immediately that our present model and the model of Section VII are really the same model, except that now we are concentrating on the efficient case whereas in Section VII we concentrated on the inefficient case.

We turn now to a brief comment about the actual equilibrium path along which the economy will travel, a path which will not, in general, be the balanced growth path. We shall restrict our attention to the efficient case, i.e., to the case  $s^* \geq 0$  . The first thing to notice is that if  $s(0) = 0$  then it follows from equation (43) that  $s(t) = 0$  for all  $t \geq 0$  . In other words, if our firms start out with zero net worth, there is no way for them to get to a state of positive net worth (since payments to laborers and creditors always exhaust the firms' receipts). Thus, if  $s(0) = 0$  we find ourselves back in the original model of Section II. However, if  $s(0) > 0$  , then an analysis similar to that of Section V may be used to show that there exists a general equilibrium path which asymptotically approaches the golden rule path. In

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<sup>24/</sup> Of course, in the inefficient case our firms can no longer be viewed as maximizing the rate of growth of net worth, because by shutting down they can guarantee themselves a net worth of zero.

other words, given any initial conditions (so long as  $s(0) > 0$ ) there exists an equilibrium path satisfying

$$\lim_{t \rightarrow \infty} k(t) = k^* \quad \text{and} \quad \lim_{t \rightarrow \infty} s(t) = s^* .$$

To summarize: In our mixed neoclassical-Marxian model economic forces will operate so as to bring the consumer sector and the capitalist sector into eventual balance relative to each other, a balance which permits the economy to proceed to the golden rule equilibrium.

#### IX. The End of the World

Many of the properties of the economic system which we have been discussing in this study depend upon the assumption that civilization will survive forever. This fact is duely emphasized (indeed, sometimes over-emphasized) by both Diamond and Samuelson. Before bringing our discussion to a close, let us, therefore, indicate briefly how the grim prospect of the end of the world might be incorporated in the foregoing discussion. We confess that this remark will be made somewhat tongue-in-cheek.

The end of the world only matters in the foregoing discussion if it enters into the expectations of decision makers. Now, it seems to us somewhat extreme to assume that all decision makers expect with certainty that the world will come to an end at some definite time, say  $T$ . It is more plausible that if the end of the world enters decision makers' expectations at all, it enters in a probabilistic fashion. In other words, the date on which the world will end,  $T$ , is not a fixed number but a random variable with a subjective probability distribution. To make things simple, suppose that the subjective

density function of this random variable is identical for all individuals and given by the exponential density with parameter  $\lambda$ . The subjective probability for each individual of the event that the world will end when he is  $\tau$  years old is thus given by

$$(46) \quad \begin{aligned} \Pi(\tau) &= \lambda e^{-\lambda\tau}, \quad 0 \leq \tau < 1 \\ &= e^{-\lambda} \quad \tau = 1. \end{aligned}$$

From (46) it follows that each consumer believes he will be alive at age  $\tau$  with probability

$$(47) \quad \begin{aligned} 1 - \Pi(\tau) &= e^{-\lambda\tau}, \quad 0 \leq \tau < 1 \\ &= 0 \quad \tau = 1. \end{aligned}$$

Finally, suppose that individuals behave according to the expected utility hypothesis, which simply means that the lifetime allocation problem of Section II now becomes, from (47),

$$\begin{aligned} \text{maximize} \quad & \int_v^{v+1} [1 - \Pi(t-v)] \log C(t,v) e^{-\delta(t-v)} dt = \int_v^{v+1} \log C(t,v) e^{-\delta'(t-v)} dt \\ \text{subject to} \quad & \int_v^{v+1} [w(t) - C(t,v)] e^{-\int_v^t r(x) dx} dt = 0 \quad \text{and} \\ & C(t,v) \geq 0, \end{aligned}$$

where  $\delta' = \delta + \lambda$ , i.e., the consumer's rate of time preference is the sum of his "pure" rate of time preference and the reciprocal of his expectation of the world's end (measured from his birth date). It is now obvious that the stylized competitive economy in which people have probabilistic expectations with regard to the world's end behaves just like the stylized competitive economy which is infinite with certainty, except that in the former consumers have a higher rate of time preference. Indeed, this may serve as one rationalization for the existence of a positive rate of time preference.

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