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The purpose of this paper is to describe the use to which the Golden Rule notion can be put in solving the Ramsey problem and to describe the relation that the Golden Rule path bears to the "optimal" growth path in the Ramsey problem. By the "Ramsey problem" I mean the problem of choosing a capital accumulation program over infinite time, given a known social utility function and certain production and consumption constraints. The term is especially descriptive here for I confine myself to a one-sector model and use an additive utility function, as did Ramsey [14] in his path-breaking investigation of the problem nearly forty years ago.

The first part of this paper presents a somewhat informal and slightly simplified version of the original Ramsey model in which the population and technology are constant. In the second part I introduce an exponentially growing population and show how the Golden Rule path has been used to find a solution to the Ramsey problem in this case. The third part introduces exponential labor augmentation (technical progress which is everywhere Harrod neutral) and again uses the Golden

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Rule path in analyzing the Ramsey problem. This section contains some new results on the conditions for the existence of an optimum when there is no discounting of individual utility rates. Some remarks on two difficulties faced by the Ramsey approach to the national saving decision conclude the paper.

I. Stationary Population and Technology

Ramsey postulated a stationary population size for all time. While leisure was one of the variables to be optimized, simultaneously with saving, in Ramsey's formulation, I shall take per capita leisure to be constant, as most recent analyses do, so that labor at time t, $L(t)$, is proportional to population and therefore constant:

$$L(t) = L_0, \qquad L_0 > 0 \tag{1.1}$$

The technology is also stationary and is summarized by an aggregate production function which makes aggregate output, $Q(t)$, a function of capital, $K(t)$, labor, and possibly other fixed factors:

$$Q(t) = F[K(t), L(t)] \tag{1.2}$$

or, since labor is fixed,

$$Q(t) = G[K(t)], \qquad G(K) > 0 \text{ for } K > 0 . \tag{1.3}$$

It is unnecessary to postulate everywhere diminishing marginal productivities or even constant returns to scale. But in the absence of a certain restriction on the utility function to be introduced, it will be necessary to make a capital saturation assumption to be specified shortly.

Supposing for simplicity that there is no depreciation, we can interpret $Q(t)$ as net income. Net income is divided between consumption, $C(t)$, and net investment, $\dot{K}(t) \equiv d\,K(t)/dt$.

$$(1.4) \qquad C(t) + \dot{K}(t) = Q(t)\,, \qquad C(t) \geq 0\,.$$

Consumption must be non-negative.

From (1.3) and (1.4), therefore,

$$(1.5) \qquad C(t) + \dot{K}(t) = G\,[K(t)]\,, \qquad C(t) \geq 0\,.$$

Also we have the initial condition

$$(1.6) \qquad K(0) = K_o\,, \qquad K_o > 0$$

Equations (1.5) and (1.6) constitute the production and consumption constraints in the optimization problem.

Turning to the preference side of the model, Ramsey postulated a social utility function of the form

$$(1.7) \qquad U = \int_0^{\infty} u[C(t)]\, dt \;, \qquad u'(C) > 0,\ u''(C) < 0 \;,$$

where <u>u</u> is called the <u>rate</u> of (social) utility.

Such a utility function is additive. It follows from work by Debreu [5], as Koopmans [7] has pointed out, that if, in addition to certain postulates guaranteeing that preferences can be represented by some utility function, one postulates "non-complementarity between periods" in the sense that the preferences among consumption paths in any series of periods are independent of what is consumed in other periods, then (and only then) the preferences can be represented by an additive utility function

$$U = u_1\,(C_1) + u_2\,(C_2) + \ldots .$$

Further, the utility function (7) is "stationary" in the sense that calendar time has no effects on the utility differences associated with different consumption programs. Koopmans [7] indicates that if, in addition to the non-complementarity postulate, one makes the "stationarity" postulate that preferences among consumption programs beginning next period would be unchanged if these programs were to begin this period, we obtain the utility function

$$U = \sum_{t=1}^{\infty} \alpha^{t-1}\, u(C_t) \qquad 0 < \alpha < 1$$

where α is the "discount factor".

Beside the detail that the above function is a sum of discrete utilities rather than an integral, the two functions differ only in that Koopmans discounted future utilities while Ramsey did not, regarding such discounting as "ethically indefensible". Koopmans shows discounting to be a necessary logical consequence of the postulates of his study if one requires the utility function to give a complete preference ordering of consumption programs over an infinite time horizon. (Incidentally, these postulates include a weakened substitute for the non-complementarity postulate which permits him to derive a utility function with a variable discount factor.) Ramsey's unwillingness to discount presented him with a difficulty from which he sought ingeniously to escape.

Ramsey's objective was the maximization of the social utility function in (1.7) subject to the constraints (1.5) and (1.6). The difficulty that is immediately encountered is that the postulated technology may permit infinite utility (U) to be achieved by more than one feasible consumption program. If there is some sustainable rate of consumption for which the utility rate is positive, i.e., $u[G(K^{o})] > 0$ for some K^{o} attainable in finite time, then any policy which eventually sustains that consumption rate will cause the utility integral to diverge to plus infinity. When such divergence arises the problem is not determinate.

Nevertheless Ramsey devised a trick to yield a determinate optimization problem. He postulated that either $G(K)$ is bounded from above or that $u(C)$ is bounded from above (or both):

$$G(K) \leq \hat{G} \quad \text{for all } K ,$$

<u>or</u> $$u(C) \leq \bar{u} \quad \text{for all } C .$$

The idea was that on either of these restrictions there would be a maximum sustainable rate of utility -- $\hat{u} = u(\hat{C}) = u(\hat{G})$ in the first case, $\bar{u}$ in the second case, the smaller of the two if both $G(K)$ and $u(C)$ are bounded. Ramsey then <u>minimized</u> the integral of the <u>shortfall</u> of the actual rate of utility, $u(C)$, from the maximum sustainable rate (which he called "bliss"), arguing that there would be at least one feasible consumption program that would make this integral converge and that the optimal consumption program is that program (among those which make the integral converge) which yields the smallest value of the integral.

In fact this restriction is not sufficient for the existence of programs that make the integral converge. If $G(K)$ or $u(C)$ approaches its upper bound asymptotically at too slow a rate, the integral to be minimized will not converge. To simplify matters most contemporary analysts, e.g., Samuelson and Solow [16], make the overly strong postulate that either $G(K)$ or $u(C)$ can attain its upper bound at a finite K or C, respectively:

(1.8) $$G(K) = \hat{G} , \quad K \geq \hat{K} ; \quad G'(K) > 0 \text{ for } K < \hat{K} ; \quad 0 < \hat{K} < \infty ;$$

<u>or</u> $$u(C) = \bar{u} , \quad C \geq \bar{C} ; \quad u'(C) > 0 , u''(C) < 0 \text{ for } C < \bar{C} ; \quad 0 < \bar{C} < \infty .$$

Thus there is capital saturation at $\hat{K}$ or utility satiation at $\bar{C}$, or both. Again, the maximum sustainable rate of utility is $\hat{u} = u(\hat{G})$ or $\bar{u} = u(\bar{C})$ or whichever is smaller if both $\hat{K}$ and $\bar{C}$ exist. In what follows, I suppose that capital saturation is binding i.e., $\hat{u}$ is the maximum sustainable rate of utility.

Now, with Ramsey, one may minimize the integral of the shortfall of the rate of utility from the bliss rate, $\hat{u}$, or, equivalently, one may maximize, subject to the constraints,

$$(1.9) \qquad V = \int_0^\infty (u[C(t)] - \hat{u})\, dt$$

The constant $\hat{u}$ plays the role of a "subtractor" in the integrand.

Supposing always that $K_0 < \hat{K}$, meaning that the economy is not initially saturated with capital, there will be many feasible consumption programs which cause the integral in (1.9) to diverge the minus infinity. But since, by saving, it is feasible in finite time to equate $K(t)$ to $\hat{K}$, there must be some programs which cause the integral to converge (to a finite negative number); these are the programs that equate $u[C(t)]$ to $\hat{u}$ in finite time or asymptotically at a sufficiently fast rate. Further, it can be shown that no feasible consumption program can make the integral diverge to plus infinity, or indeed to converge to any positive number. The consumption program which makes the integral converge to the algebraically largest number is designated the "optimal" program, all others giving a smaller V , some of them a V of minus infinity.

Ramsey and some latter-day writers have evidently regarded this "optimal" consumption program to be the solution to the <u>original problem</u> of maximizing the social utility integral in (1.7). But the maximization of V in (1.9) is a different problem from maximizing U in (1.7). As was seen earlier, U fails to discriminate among a certain class of consumption programs that V does discriminate among; and V fails to discriminate among some consumption programs that U discriminates among. More formally, V is not a monotonically increasing function of U , since it is possible for U to be undefined where V is defined and conversely, so that the two problems are not equivalent.

This does not imply that the solution to the V maximization problem cannot be regarded as "optimal". But a new criterion of optimality is required.

The modern approach to the divergence problem, which may be found in von Weizsäcker [20], Atsumi [1], and somewhat implicitly in Koopmans [8], is the following. A consumption program (over infinite time), $C_1(t)$, is said to be preferred or indifferent to another consumption program, $C_2(t)$, if there exists a T^o such that for all $T \geq T^o$

$$\int_0^T u[C_1(t)]\, dt \geq \int_0^T u[C_2(t)]\, dt$$

$C_1(t)$ is strictly preferred if the strong inequality holds.

A feasible consumption program $C^*(t)$ is said to be optimal if for every other feasible consumption program $C(t)$, $C(t) \neq C^*(t)$ for some $\underline{t}$, there exists a T^o (not necessarily the same for each alternative program) such that for all $T \geq T^o$

$$\int_0^T u[C^*(t)]\, dt \geq \int_0^T u[C(t)]\, dt$$

The optimum is unique if there exists a T_o such that the strong inequality holds.

No discounting of future utility rates is necessary here (although one could introduce utility discounting). The price paid for this luxury is that the proposed preference criterion may fail to order many pairs of feasible consumption programs. Consider a pair of consumption programs $C_0(t)$ and $C_1(t)$ and a sequence of values $T_1, T_2, \ldots$ converging to plus infinity such that

$$\int_0^{T_i} u[C_0(t)] < \int_0^{T_i} u[C_1(t)]\, dt\,, \qquad i = 1, 2, \ldots$$

Then $C_0(t)$ is <u>not</u> preferred or indifferent to $C_1(t)$. But for the same pair of consumption programs there may exist a different sequence of values $T_1', T_2', \ldots$ converging to plus infinity such that

$$\int_0^{T_i'} u[C_1(t)]\, dt < \int_0^{T_i'} u[C_0(t)]\, dt\,, \qquad i = 1, 2, \ldots$$

Then $C_1(t)$ is <u>not</u> preferred or indifferent to $C_0(t)$. Since neither program is preferred or indifferent to the other on the proposed criterion, the criterion fails to give a preference ordering of such pairs of consumption programs. If discounting were introduced, one could order all programs by comparing the limits of the integrals as $T \to \infty$.

This weakness in the preference criterion is of no importance if there exists an optimal program in the above sense. However, no optimum need exist. Case 1: No optimum exists if for every feasible consumption program $C_1(t)$ there exists a feasible consumption program

$C_2(t)$ and some T^o such that for all $T \geq T^o$

$$\int_0^T u[C_1(t)]dt \quad < \quad \int_0^T u[C_2(t)]dt$$

In this case, there being no best feasible program, the optimization problem is insoluble. Case 2: No optimum (in the above sense) exists if there is a class of consumption programs for which no preference ordering is given (as with the pair $C_0(t)$ and $C_1(t)$ discussed earlier) all of which are preferred or indifferent to all programs outside this class. Then the programs in this class are "good" -- there is no better path -- but there exists no optimal path in the sense of the present criterion. Nevertheless, if there are certain restrictions on the technology (or on $u(C)$) such as were made above, an optimum in this sense will exist.

This optimality criterion justifies Ramsey's identication of the V-maximizing consumption policy as the optimal consumption policy. For if there exists a consumption path $\tilde{C}(t)$ (which is feasible for some initial K , not necessarily the given K_0) and hence some corresponding path of the rate of utility $u[\tilde{C}(t)]$ such that maximization of

$$\int_0^\infty (u[C(t)] - u[\tilde{C}(t)])dt$$

yields a solution, say $C^{**}(t)$, then $C^{**}(t)$ is an optimal path $C^*(t)$ in the sense of the new optimality criterion. If the maximum is unique, the optimum is also unique.[1] In the Ramsey model, the path $C(t) = \hat{C} = G(\hat{K})$ is the path $\tilde{C}(t)$ and $\hat{u} = u(\hat{C})$ is the subtractor, $u[\tilde{C}(t)]$. Thus we have described the modern basis for Ramsey's trick of maximizing V instead of U .

When Ramsey maximized (1.9) subject to the constraints (1.5) and (1.6) (given the existence of a $\hat{K}$) he obtained the following remarkably simple formula for the optimal saving policy:

$$\dot{K} = \frac{\hat{u} - u(C)}{u'(C)} \tag{1.10}$$

Before discussing some features of this solution, we shall present two methods by which it can be derived.

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1. If $C^{**}(t)$ is a unique maximizing path, then

$$\int_0^\infty [u(C^{**}) - u(\tilde{C})]dt \;>\; \int_0^\infty [u(C) - u(\tilde{C})]dt$$

for all other C paths, $C \neq C^{**}$ for some t . Then

$$\int_0^\infty [u(C^{**}) - u(\tilde{C})]dt - \int_0^\infty [u(C) - u(\tilde{C})]dt > 0$$

and hence

$$\int_0^\infty [u(C^{**}) - u(C)]dt > 0$$

If the following integrals are continuous in T , it follows that, for some T^o ,

$$\int_0^T u(C^{**})dt \;>\; \int_0^T u(C)dt \,, \qquad T \geq T^o \,,$$

so that $C^{**}(t)$ is the unique optimum.

Ramsey reported that Keynes produced an ingenious proof of this formula using the following argument. (See Ramsey's statement of the argument or Meade's more detailed presentation [9].) Suppose that it is decided to do an extra "day's worth" of saving. Then the whole time schedule of progress towards "bliss" will be advanced by one day (if the economy reverts to its previous consumption policy after today). Therefore there will be a gain of one extra day at bliss[1], hence a gain of $\hat{u}$. But the utility, u , that would have been enjoyed tomorrow will be forever lost (since it is the utility rate that would otherwise have been enjoyed on the following day that is enjoyed tomorrow). Hence the true gain from advancing the schedule is $\hat{u} - u$. The cost of doing so is the extra saving multiplied by the marginal utility of consumption, so that if S is the amount that was being saved per day the cost is $S\,u'(C)$. Now if the original consumption program was optimal there will be no gain or loss from departing (infinitesimally) from the original program, so that $S\,u'(C) = \hat{u} - u$, which is (1.10). (Of course, these formulae for the cost and gain are only approximations in the discrete-time context of Keynes' argument but it is possible to formulate a somewhat analogous argument in continuous time.)

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1. Actually, bliss is approached only asymptotically which indicates that Keynes' argument requires some modification to be valid.

There are a variety of more formal derivations of the Ramsey result. We choose here to use the technique of dynamic programming developed by Bellman [2, especially pp. 249-250].

First, we define $w(K_o)$ to be the maximum value of V when the initial stock of capital is K_o. That is,

$$w(K_o) = \max \int_0^{\infty} (u[C(t)] - \hat{u})dt \tag{1.11}$$

$$\text{subject to} \quad C(t) + \dot{K}(t) = G(K)$$
$$K(0) = K_o$$

Now Bellman's "principle of optimality" states that whatever the initial state and initial policy, the remaining decisions must be optimal with regard to the state resulting from the first decision if the overall policy is to be optimal. The approach is to divide time into an initial small interval of length Δ, over which the initial policy is to be made, and the remaining open-ended interval beginning at $t = \Delta$, over which it is assumed that an optimal policy is followed. If the average rate of consumption over the initial interval is C then the amount of utility (V) earned is approximately $[u(C) - \hat{u}]\Delta$ and the capital stock at $t = \Delta$ will be approximately $K(\Delta) = K_o + \Delta[G(K_o) - C]$. Hence we obtain the following approximate relation

$$w(K_o) = \max_C \left\{[u(C) - \hat{u}]\Delta + w(K_o + \Delta[G(K_o) - C])\right\} \tag{1.12}$$

Using an approximation for $w(K_o + \Delta[G(K_o) - C])$ gives

$$(1.13) \qquad w(K_o) = \max_C \left\{[u(C) - \hat{u}]\,\Delta + w(K_o) + w'(K_o)\,\Delta[G(K_o) - C]\right\}$$

Subtracting the constant $w(K_o)$ from both sides then gives

$$(1.14) \qquad 0 = \max_C \left\{[u(C) - \hat{u}]\,\Delta + w'(K_o)\,\Delta[G(K_o) - C]\right\}$$

Dividing by Δ and letting $\Delta \to 0$ yields

$$(1.15) \qquad 0 = \max_C \left\{u(C) - \hat{u} + w'(K_o)\,[G(K_o) - C]\right\}$$

where now C represents the initial rate of consumption.

From (1.15) we see that if C is optimal, so that the expression in braces is maximized, then we have the following equation in the optimal C :

$$(1.16) \qquad u(C) - \hat{u} + w'(K_o)\,[G(K_o) - C] = 0$$

Further, if the maximum is an interior one, then the derivative with respect to C of the expression in braces in (1.15) must equal zero, which yields another equation in optimal C :

$$(1.17) \qquad u'(C) - w'(K_o) = 0$$

From (1.16) and (1.17) we obtain

$$(1.18) \qquad G(K_o) - C = \frac{\hat{u} - u(C)}{u'(C)}$$

which is (1.10), the Ramsey-Keynes formula.

Some features of Ramsey's solution can be brought out by the geometric representation of the formula in Figure 1. To obtain the optimal rate of consumption one needs to know only initial income, $G(K_o)$, the shape of the function $u(C)$ and $\hat{u}$. Hence variations of the production function which leave initial income and $\hat{u}$ unchanged have no effect upon the optimal rate of consumption; in a sense, therefore, C^* is independent of the marginal product of capital and the functional distribution of income. Further the optimal policy is "myopic" in that the present value of future income or other wealth-like variables play no part. This rather simple dependence of optimal consumption on income alone, given $u(C)$ and $\hat{u}$, may have suggested to Keynes the consumption function of the <u>General Theory</u>. (The optimal marginal propensity to consume in Ramsey's model is positive but it need not be less than unity as Keynes postulated.)

The diagram shows that if $K_o < \hat{K}$, then $C < G(K_o)$ so that $\dot{K} > 0$; thus there is saving as long as the economy is short of capital saturation. Further, C approaches $G(K)$ only as the latter approaches

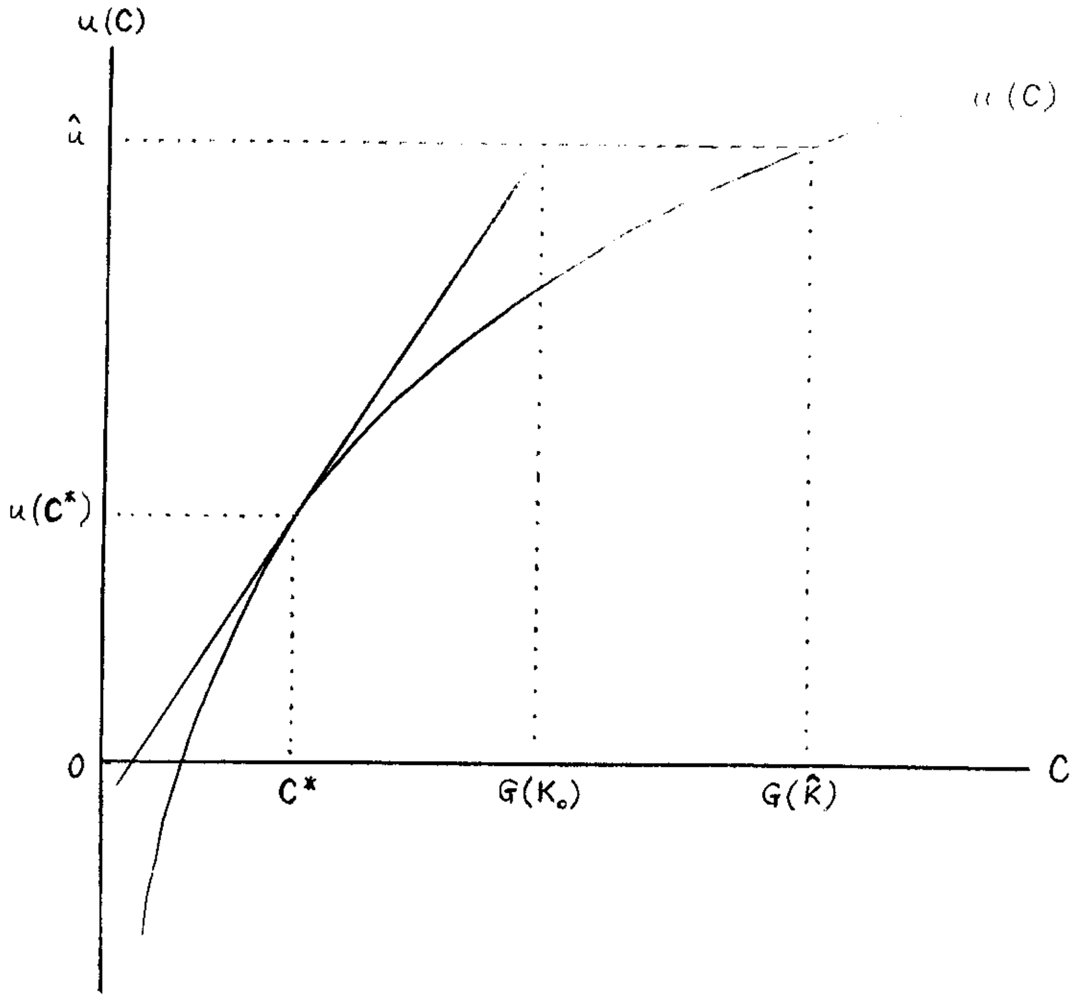

Figure 1

$G(\hat{K})$. Hence, K will asymptotically approach $\hat{K}$ and C will asymptotically approach $G(\hat{K})$. At $K = \hat{K}$, $C = G(\hat{K})$. (It is not a new result that $\hat{K}$ will be approached since we already knew that all programs "eligible" to be optimal had to make $u(C)$ approach $\hat{u}$ so that V would converge; it is a new result that $\hat{K}$ will be approached only asymptotically.) Note finally that $\dot{K}$ is invariant to a linear transformation of $u(C)$ which is as it should be since the underlying preferences determine the function $u(C)$ only up to a linear transformation.

In closing this section we note that Samuelson and Solow [16] have extended the Ramsey analysis to the case of many heterogeneous capital goods. Phelps [11] has investigated the consequences of capital risk for the optimal rate of consumption also in an infinite time-horizon model. But the major extensions needed are those to cope with the facts of growing population and technical progress.

II. Exponentially Increasing Population and Stationary Technology

Let us first develop the production side of this model. Since labor is increasing exponentially, the counterpart of (1.1) in the previous model is

$$L(t) = L_o \, e^{\gamma t} , \qquad \gamma > 0 \tag{2.1}$$

The technology is still supposed to be stationary so that the production function is independent of time:

(2.2) $$Q(t) = F[K(t), L(t)] .$$

Now, however, we posit constant return to scale, twice differentiability, diminishing marginal productivities, and that both factors are essential to production, so that one may write

(2.3) $$Q(t) = L(t)\, f[k(t)]$$

where $k(t) = K(t)/L(t)$, $f[k(t)] = F[k(t), 1]$,

and $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$, $f'(\infty) = 0$.

Again supposing there is no depreciation, we have that consumption per unit labor, $\underline{c}$, (which is proportional to consumption per head since we are fixing leisure per head) and investment per unit labor sum to output per unit labor:

(2.4) $$c(t) + \dot{K}(t)/L(t) = f[k(t)] , \qquad c(t) \geq 0 ,$$

where $c(t) = C(t)/L(t)$.

Since $$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$$

$$\dot{k} = \frac{\dot{K}}{L} - \gamma k$$

we therefore have

(2.5) $$c(t) + \dot{k}(t) = g[k(t)] , \qquad c(t) \geq 0$$

where $g[k(t)] = f[k(t)] - \gamma k(t)$.

Finally we have the initial condition

$$(2.6) \qquad k(0) = k_o \, , \qquad k_o > 0 \, .$$

Equations (2.5) and (2.6) constitute the constraints in the present problem; they are analogous to (1.5) and (1.6) in the previous problem. The equations are identical but for the fact that (2.5) and (2.6) are in per capita terms and that, as we shall see, $g(k)$ is somewhat different from $G(K)$. I shall later place restrictions on $f(k)$ such that $g(k)$ reaches a maximum at some finite $\hat{k}$, as we supposed $G(K)$ to do. But I first consider the preference side of the model.

In specifying our criterion of optimality I wish to be unspecific about the form of the Ramsey-like utility integral. Let us therefore write $u = u[c(t), t]$ which is as general as one can be with respect to the dependence of u on consumption, population (which is an exogenously given function of time) and time. Then we say that a feasible consumption program $c^*(t)$ is optimal if and only if for every other feasible path $c(t)$, $c(t) \neq c^*(t)$ for some t, there exists a T^o such that for all $T \geq T^o$

$$\int_0^T u[c^*(t), t]dt \geq \int_0^T u[c(t), t]dt \, .$$

Now to the matter of preferences. Pearce [10], Srinivasan [17], Uzawa [19], Koopmans [8], von Weizsäcker [20], Inagaki [6], Atsumi [1], Samuelson [15], Cass [3] and no doubt others have produced a variety of utility functions for consideration in solving the present problem and related problems. Probably the single most popular function, used extensively (though not exclusively) by Koopmans and von Weizsäcker, is the one which makes the rate of social utility, u, an increasing, concave and unbounded function only of *per capita* consumption; that is,

$$(2.7) \qquad U(T) = \int_0^T u[c(t)]dt \; , \quad u'(c) > 0, \quad u''(c) < 0,$$

gives the social utility accumulated up to $t = T$ of a path $c(t)$. The quantity $u[c(t)]$ is sometimes called the rate of *per capita* utility; I shall simply designate it as the rate of utility.

It is sometimes said that a utility function like (2.7) involves no "discounting" of future "utilities". But this is somewhat misleading for as Koopmans has pointed out, while (2.7) treats "generations" alike (to make somewhat figurative use of the term "generation"), it does not treat "individuals" alike (again somewhat figuratively). Suppose, in calculating social utility at $t = T$, $U(T)$, one integrated over the sum of the social utility rates assigned to the living individuals at each moment of time. Then, if $v_i[c_i(t), t]$ denoted the individual social utility rate of the i th individual and $c_i(t)$ his consumption rate at time t, one would have

$$U(T) = \int_0^T \sum_{i=1}^{L(t)} v_i[c_i(t), t]dt$$

where $L(t)$ (an integer) is the size of the population at $\underline{t}$. Now if, at every $\underline{t}$, consumption is equalized and the individual social utility rate functions are identical, so that $v_i[c_i(t), t] = v[c(t), t]$, then, treating population size as a continuous variable given by (2.1), one would have (letting $L_0 = 1$)

$$U(T) = \int_0^T e^{\gamma t} v[c(t), t]dt$$

Now if "equal treatment of individuals" demands, as Koopmans suggests, that we do not "discount" individual social utility rates, so that $v[c(t), t]$ is, say, $u[c(t)]$, then our social utility integral would be of the form

$$U(T) = \int_0^T e^{\gamma t} u[c(t)]dt$$

which differs from the proposed (2.7). It becomes clear, therefore, that (2.7) can be interpreted as <u>discounting</u> <u>individual</u> <u>utility</u> <u>rates</u> by the population growth rate γ :

$$U(T) = \int_0^T e^{\gamma t} v[c(t), t]dt = \int_0^T e^{\gamma t} (e^{-\gamma t} u[c(t)])dt = \int_0^T u[c(t)]dt$$

While I shall later consider the implications of postulating the equal-treatment-of-individuals function, let us return to (2.7) and examine the Koopmans-von Weizsäcker method of solution. (Incidentally, Atsumi, who worked with a discrete-time, two-sector model, apparently is also a discoverer of this approach. Srinivasan, Uzawa and Cass use a different utility function, and correspondingly, a different approach.)

If one attempts to maximize the limit of $U(T)$ in (2.7) as T goes to infinity, subject to the constraints (2.5) and (2.6), one may encounter, as did Ramsey in the stationary population problem, the difficulty that more than one feasible consumption program will cause the integral to diverge. The problem will arise if there are sustainable rates of <u>per capita</u> consumption for which $u(c) > 0$. To overcome this difficulty Koopmans and von Weizsäcker employ a trick analogous to Ramsey's trick in the same difficulty: they use the Golden Rule notion to establish the existence of a maximum sustainable rate of utility, $\hat{u}$, which can be introduced to make convergent the limit of the utility integral.

Suppose that $f'(0) > \gamma$, hence $g'(0) > 0$. Since $f'(\infty) = 0$ (labor required for production) and $f''(k) < 0$ (diminishing returns) the function $g(k)$ will then achieve a unique maximum, $\hat{c} > 0$, at some $\hat{k} > 0$:

$$(2.8) \qquad g'(\hat{k}) = 0 \quad \text{or} \quad f'(\hat{k}) = \gamma, \qquad \hat{k} > 0$$
$$\hat{c} = g(\hat{k}) > 0, \ \hat{u} = u(\hat{c}) \ .$$

$\hat{c}$ is the maximum sustainable rate of <u>per capita</u> consumption and $\hat{u}$ is the maximum sustainable rate of utility. The path, $k(t) = \hat{k}$, is, of course, the Golden Rule path; it is the consumption-maximizing golden age path. On this path, the marginal product of capital, $f'(k)$, equals the population growth rate and investment equals competitive profits. Thus our postulates imply that a Golden Rule path exists. The next step is to assign to the Golden Rule path the same rôle played by the capital saturation path in the stationary population model. The utility rate corresponding to the Golden Rule path, $\hat{u}$, is subtracted from the actual rate of utility, $u[c(t)]$, to form the new integral

$$(2.9) \qquad V = \int_0^{\infty} (u[c(t)] - \hat{u}) \, dt$$

analogous to (1.9).

Once again, if there exists a feasible path which maximizes V , this is an optimal path in the sense of the above optimality criterion; if the maximum is unique, so is the optimum. Koopmans shows that a unique maximum exists. In particular, there is no divergence problem: V is bounded from above for all feasible consumption paths and all initial capital stocks. If the economy starts below the Golden Rule path, $k_0 < \hat{k}$, the integral is negative for all feasible paths; but since the Golden Rule path can be reached in finite time, there are necessarily some paths which cause the integral to converge (to a finite negative number). If the economy starts above the Golden Rule path, the integral, while positive, will still converge (for all paths of interest). Thus there are no paths which are "infinitely better" than the Golden Rule path.

Since the problem of maximizing (2.9) subject to (2.5) and (2.6) is mathematically identical, in every essential respect, to the previous problem of maximizing (1.9) subject to (1.5) and (1.6), one necessarily obtains for the optimal rate of consumption per head the Ramsey-like formula

$$\text{(2.10)} \qquad \dot{k} = \frac{\hat{u} - u(c)}{u'(c)} \quad .$$

Recalling that $\dot{k} = \dot{K}/L - \gamma k$, one can write (2.10) in the form

$$\text{(2.10a)} \qquad \dot{K} = \left[\frac{\hat{u} - u(c)}{u'(c)} + \gamma k \right] L_o \, e^{\gamma t}$$

Setting $L_o = 1$ (which we are free to do), we see that (2.10a) gives (1.10) of the previous model if and only if $\gamma = 0$ (since, if $L_o \, e^{\gamma t} = 1$, the $\underline{C}$ of the previous model is equal to consumption per head, $\underline{c}$, of the present model).

The features of this solution are, of course, the same as in the previous model. If $k_o = \hat{k}$, we have $c = g(\hat{k})$, $\dot{k} = 0$ and the Golden Rule path will be followed. If $k_o < \hat{k}$, we have $c < g(k)$ for all $\underline{t}$ and $k(t)$ will approach $\hat{k}$ asymptotically and monotonically. Likewise, if $k_o > \hat{k}$, $c > g(k)$ for all $\underline{t}$ and $\hat{k}$ will be approached asymptotically and monotonically.

A question that may occur to many readers is: How, if at all, does the optimal rate of consumption depend upon the population growth rate? We note first that in the long run, $\dot{K} \to \gamma\hat{k}\, e^{\gamma t} > 0$ if $\gamma > 0$ (setting $L_o = 1$) while $\dot{K} \to 0$ if $\gamma = 0$ and a finite $\hat{K}$ exists (which means in the present context that $\hat{k}$ does not go to infinity as γ goes to zero), so that, in the long run, there is more saving when there is population growth, both absolutely and as a ratio to income. Since income per head will, of course, be smaller in the long run when there is population growth (there being less capital per head asymptotically), consumption per head will also be smaller in the long run.

As for <u>initial</u> saving, given income per head $G(K_o) = f(k_o)$, and labor force $L_o = 1$, there are two conflicting influences: On the one hand, the greater γ, the smaller will be $g(k_o)$, the amount available to be divided between consumption per head and the increase of capital per head, and this decreases optimal consumption per head as manipulation of a diagram like Figure 1 will show; on the other hand, the greater γ, the smaller will be $\hat{k}$, $\hat{c}$ and hence $\hat{u}$ and this increases the optimal consumption per head as diagrammatics will easily show. As for global comparisons of the sort $\gamma = 0$ vs. $\gamma > 0$, it is pretty clear that one can devise g and u functions that will make the former or the latter effect decisive, whichever is desired. It might be of interest, however, to verify that the derivative $dc/d\gamma$ is of indeterminate sign without new restrictions on $g(k)$ and $u(c)$. First, write (2.10) in the following form to obtain initially optimal consumption per head, c_o:

$$[g(k_o) - c_o]\, u'(c_o) = u[g(\hat{k})] - u(c_o) \tag{2.10b}$$

Taking the total differential we have

$$u'(c_o) \frac{dg(k_o)}{d\gamma} d\gamma + \left\{ - u'(c_o) + u''(c_o)\, [g(k_o) - c_o] \right\} dc_o = u'[g(\hat{k})] \frac{dg(\hat{k})}{d\gamma} d\gamma - u'(c_o)\, dc_o , \tag{2.11}$$

where $\frac{dg(k_o)}{d\gamma} = - k_o , \quad \frac{dg(\hat{k})}{d\gamma} = - \hat{k} .$

Hence

$$\frac{dc_o}{d\gamma} = \frac{u'(c_o)\, k_o - u'[g(\hat{k})]\, \hat{k}}{u''(c_o)[g(k_o) - c_o]} \tag{2.12}$$

While the denominator is unambiguously negative, the numerator may be of either sign. The first term there (taken as a ratio to the denominator) represents the first aforementioned influence of γ on $\underline{c}$ through its effect upon $g(k_o)$ (which influence is negative as we indicated earlier) while the second term represents the second influence of γ on $\underline{c}$ through its effect upon $g(\hat{k})$ and hence $\hat{u}$ (which influence is positive). Curiously, if $k_o = \hat{k}$, so that initial $c_o = g(\hat{k})$, then $dc_o/d\gamma = 0$ (initially) but there is otherwise no presumption of such invariance. (This last result does not mean that Golden Rule $\hat{c}$ is invariant to γ; in fact, $\frac{dg(\hat{k})}{d\gamma} < 0$.)

In this connection I note finally that if there were "utility satiation" at some $\bar{c}$ for which $\bar{c} < \hat{c}$ then $\bar{u} = u(\bar{c})$ would take the place of $\hat{u}$ in (2.10) and, since $\bar{u}$ would presumably be invariant to γ, the second of our two influences would be absent so that an increase of γ would unambiguously decrease optimal consumption.

As we have seen, the optimal accumulation policy drives the economy toward the Golden Rule path and hence drives the marginal product of capital or real interest rate toward the population growth rate. It may seem puzzling to some readers that the economy should stop at the Golden Rule path. Should not society deepen capital further as long as the interest rate exceeds the rate of pure time preference (the utility discount rate) which has been taken to be zero? There are a number of answers to this question. First, an accumulation policy which permanently drove the interest rate finitely below the population growth rate could not be optimal since growth paths which violate the Golden Rule in this manner are dynamically inefficient. Second, as we saw earlier, there is in reality implicit discounting of individual utility rates at the rate γ so it should perhaps be expected that capital deepening would cease as the interest rate approached the population growth rate. However, the following heuristic exercise may make the "optimality" of the Golden Rule path especially clear.

Consider the discrete-time analogue of our present model and suppose that society contemplates a departure from the path it originally intended to follow: c_0, c_1, ... In particular suppose that the L_0 people in period zero each save an extra unit in period zero with the intention of consuming the extra capital plus the interest on it in period one, thus permitting c_2, c_3, ... to be unchanged. The initial loss of utility will be $u'(c_0)$ since consumption per head has fallen by one unit. The increase in total consumption next period will be $L_0(1 + r)$ where r is the rate of interest or marginal product of capital. If the number of people next period, L_1, is equal to $(1 + \gamma) L_0$ then the increase in consumption per head in period one will be only $(1 + r)(1 + \gamma)^{-1}$ and hence the gain in utility will be only $(1 + r)(1 + \gamma)^{-1} u'(c_1)$. Now if the original path is optimal, the net gain from such proposed alterations will be approximately zero; hence, for every t,

$$u'(c_t) = (1 + r_t)(1 + \gamma)^{-1} u'(c_{t+1}) \tag{2.13}$$

We see that a stationary program, $c_0 = c_1 = c_2 = \ldots$, $r_t = r$, will satisfy this necessary optimality condition only if $u'(c) = 0$, which the assumption of an unbounded utility function excludes, or if $r = \gamma$, which is the Golden Rule path. And if $r = \gamma$ initially, the stationary program $c_1 = c_2 = \ldots = \hat{c}$, that is, obedience to the Golden Rule, will satisfy this condition and hence be optimal.

Equation (2.13) is the discrete-time version of the necessary Euler condition for a maximum V in (2.9) subject to (2.5) and (2.6):

$$(2.14) \qquad \left[\frac{d}{dt} u'(c)\right] / u'(c) = -(r - \gamma) , \quad r = f'(k) .$$

This equation tells the same story as (2.13).

Cass [3] and Koopmans [8] have also studied the Ramsey problem (with $\gamma \geq 0$) when future utility rates are discounted at a positive rate ρ . Then the problem is one of maximizing

$$(2.15) \qquad U = \int_0^\infty e^{-\rho t} u[c(t)]dt , \quad \rho > 0 ,$$

subject to (2.5) and (2.6). While ρ is called the discount rate, we should remember that $\rho + \gamma$ is implicitly the rate at which <u>individual</u> utility rates are being discounted. Cass and Koopmans postulate, as usual, that $u'(c) > 0$, $u''(c) < 0$ and $u(c)$ unbounded. Note that since

$\rho > 0$, the utility integral will necessarily converge (by virtue of the concavity of $\underline{u}$ and $\underline{f}$), so there is no need for the Golden Rule device. (It should be mentioned that Srinivasan [17] and Uzawa [19] had earlier studied a similar problem, where $u(c) = c$, in a two-sector model. The results in these two papers resemble those obtained in the one-sector model considered here. See also the multi-sector analysis by Radner [13], especially of the "linear logarithmic" case.)

In this new problem, the analogue of (2.13) is

$$u'(c_t) = (1 + r_t)\,(1 + \gamma)^{-1}\,(1 + \rho)^{-1}\,u'(c_{t+1}) \tag{2.16}$$

if one replaces $e^{-\rho t}$ by $(1 + \rho)^{-t}$. The Euler equation is now

$$\left[\frac{d}{dt}\,u'(c)\right]\Big/u'(c) = -\,(r - \gamma - \rho) \tag{2.17}$$

Like (2.14), this states that the proportionate rate of decrease of the marginal utility of per capita consumption, in this case $(du'/dt)/u' - \rho$, must equal the excess of the rate of interest over the population growth rate.

These two equations indicate that, since $u'(c) > 0$ for all c , the only stationary equilibrium that can be optimal is that path on which $r = \rho + \gamma$. If initially $r = \rho + \gamma$, a stationary path with constant per capita and constant r will satisfy the Euler equation. Such a path is (like the Golden Rule path) a particular golden age path. Following Inagaki [6], I shall call this path the Golden Utility path. Clearly it coincides with the Golden Rule path if and only if $\rho = 0$; if $\rho > 0$, the Golden Utility path gives a lower capital intensity, $\underline{k}$, lower income per head and, of course, smaller consumption per head.

Cass and Koopmans have shown that if the economy starts from a position off the Golden Utility path, the latter will be approached asymptotically and monotonically. When $\gamma = 0$, this coincides with Ramsey's result that if there is a constant pure time preference rate, $\rho > 0$, the rate of interest will approach that rate asymptotically.

It might be thought that if $\rho < 0$ there still exists a Golden Utility path with $r = \rho + \gamma < \gamma$ but this is not so. Such a stationary path satisfies the necessary Euler condition but that path, and any path asymptotic to it, cannot be optimal in the sense of our optimality criterion for it is dynamically inefficient (i.e., dominated by another path). In fact, as Koopmans has shown, there is no optimal consumption program when $\rho < 0$.

First, we recall, letting $\sigma \equiv - \rho > 0$, that an optimum exists if and only if there is at least one feasible path, $c^*(t)$, and some T^0 such that, for every other feasible path $c_0(t)$, $c_0(t) \neq c^*(t)$ for some $\underline{t}$,

$$\int_0^T e^{\sigma t} u[c^*(t)]dt \geq \int_0^T e^{\sigma t} u[c_0(t)]dt \qquad \text{for all } T \geq T^0 .$$

Hence, if an optimum exists (for the given k_0), there must be a path $\tilde{c}(t)$ and some T^o such that for every other feasible path $c(t)$, $c(t) \neq \tilde{c}(t)$,

$$W(T) = \int_0^T e^{\sigma t} (u[c(t)] - u[\tilde{c}(t)])\, dt \leqq 0 \,, \quad \text{for all} \quad T \geq T^o \,.$$

In particular, any optimal path, $c^*(t)$, is such a path (i.e., will fill the role of $\tilde{c}(t)$.)

Now Koopmans proved that if $\rho < 0$, i.e., $\sigma > 0$, then for every feasible $\tilde{c}(t)$ path and every number $N > 0$ there exists another feasible path $c(t)$ and a number T^o such that[1]

$$W(T) > N \qquad \text{for all} \quad T \geq T^o$$

Hence there exists no optimum when $\rho < 0$.

The intuitive explanation offered by Koopmans is that if we start on the Golden Rule path, where $u'(c) = u'(\hat{c})$ for all t , then a sacrifice of one unit of per capita consumption in any short initial interval Δ will permit an equal gain of per capita consumption in any subsequent interval of equal duration; the utility initially sacrificed will be $u'(\hat{c})\,\Delta$ and the utility gained will be $e^{\sigma t}\, u'(\hat{c})\,\Delta$ which is greater than the sacrifice if $\sigma > 0$. But it will always pay to delay indefinitely the date t at which

1. The proof is overly strong since $N = 0$ would be sufficient.

the fruit of the initial sacrifice is reaped, which suggests that there is no optimum. (The example certainly shows that the Golden Rule path is not optimal, as does the Euler equation (2.17) when $\rho \neq 0$.)

To close this part of the paper, we briefly mention another result which is really somewhat out of the present context. Suppose that all the equations on the production side (2.1) to (2.6), continue to apply but that the world is expected with certainty to come to an end at some $t = T$, or at least that only the period, $0 \leq t \leq T$, is of interest. A terminal capital constraint of the form $k(T) \geq k_T \geq 0$ is stipulated.

Samuelson [15] and Cass [4] then investigated the optimal accumulation program in the interval [0, T]. Samuelson, working with a utility function like (2.7),

$$U(T) = \int_0^T u[c(t)]dt \,, \tag{2.18}$$

showed that the optimal $k(t)$ path would "arch" (in catenary fashion) toward the Golden Rule path, $\hat{k}$, and that, as T becomes sufficiently large, the optimal path $k(t)$ will spend an arbitrarily large portion of the time arbitrarily near the Golden Rule path. Thus, the Golden Rule path is a kind of "turnpike" quite similar to the von Neumann ray in models of a different character. (See the references in [15] for the literature on the Turnpike Theorem.) Most of the time the rate of interest, $\underline{r}$, will be close to γ , its Golden Rule value.

Cass, working with the more general utility function

$$(2.19) \qquad U(T) = \int_0^T e^{-\rho t} u[c(t)]dt \,, \qquad \rho \geq -\gamma \,,$$

showed that the path on which $r = \rho + \gamma \geq 0$ possesses the identical turnpike property.

III. Technical Progress and Exponential Population Growth

I first develop the production side of a dynamic economy in which population grows exponentially or is constant, technical progress is labor augmenting and the rate of labor augmentation is constant for all time.

The rate of growth of population and labor, γ , is non-negative and constant:

$$(3.1) \qquad L(t) = L_o e^{\gamma t} \,, \qquad \gamma \geq 0$$

The production function differs from (2.2) only in that there is a constant rate of labor augmentation, λ :

$$(3.2) \qquad Q(t) = F[K(t), e^{\lambda t} L(t)]$$

whence, if we continue to denote by $k(t)$ the capital-labor ratio (<u>not</u> the capital-augmented labor ratio as it is so frequently convenient to do),

$$(3.3) \qquad Q(t) = e^{\lambda t} L(t)\ f[k(t)/e^{\lambda t}] = e^{\lambda t} L(t)\ f[e^{-\lambda t} k(t)]$$

where $\qquad k(t) = K(t)/L(t),\ f[k(t)/e^{\lambda t}] = F[k(t)/e^{\lambda t}\ , 1]$

and $\qquad f(0) = 0,\ f'(k) > 0,\ f''(k) < 0,\ f'(\infty) = 0$

Since consumption plus investment equal output, we have

$$(3.4) \qquad c(t) + \dot{K}(t)/L(t) = e^{\lambda t} f[e^{-\lambda t} k(t)]$$

where $\quad c(t) = C(t)/L(t)$.

Since $\dot{k} = \dot{K}/L - \gamma k$, we therefore have

$$(3.5) \qquad c(t) + \dot{k}(t) = e^{\lambda t} f[e^{-\lambda t} k(t)] - \gamma k(t),\ c(t) \geq 0$$

In any golden age, $c(t)$, $k(t)$ and hence $\dot{k}(t)$ grow like $e^{\lambda t}$. Noting this, it can easily be shown, using (3.5), that on the (interior) Golden Rule path (if it exists), $k(t) = \hat{k}(t) = \hat{k}(0)\ e^{\lambda t} > 0$ where $f'[e^{-\lambda t}\ \hat{k}(t)] = \lambda + \gamma$, that is, the marginal product of capital (interest rate), $f'[e^{-\lambda t}\ k(t)]$, equals the golden age growth rate, $\lambda + \gamma$. We assume hereafter that a Golden Rule path exists.

In addition to the constraint (3.5), we have the initial condition

$$(3.6) \qquad k(0) = k_o \,, \qquad k_o > 0$$

Equations (3.5) and (3.6) are constraints in the problem of optimal saving.

Under what conditions on preferences, given the above model of production, will it be optimal, when $k_o = \hat{k}(0)$, to follow the Golden Rule of Accumulation or, when $k_o \neq \hat{k}(0)$, to approach the Golden Rule path asymptotically? Pearce [10], in response to my Golden Rule essay, asked the related question: if the economy happened initially to be on the Golden Rule path, i.e., $k_o = \hat{k}(0)$, and bound itself to end on the Golden Rule path at some $T > 0$, i.e., $k(T) = \hat{k}(T)$, thus possibly fulfilling some obligation to future generations, would society find it optimal to maintain the economy on the Golden Rule path throughout the intervening time, i.e., equate $k(t)$ to $\hat{k}(t)$ for all t, $0 \leq t \leq T$? Pearce then produced a utility function such that society, to maximize utility in the interval, would have to depart from the Golden Rule path -- such that obedience to the Golden Rule would not be optimal, despite the favorable end-point conditions. Specifically he showed that if the social utility function were

$$(3.7) \qquad U(T) = \int_0^T c(t)\, dt \,,$$

implying constant marginal utility of per capita consumption, then society would want (until T) to deepen capital in excess of the Golden Rule path, driving the interest rate down to γ (below the Golden Rule level $\lambda + \gamma$) as quickly as possible and remaining on that (different) golden age path until T at which point the capital in excess of the Golden Rule level, $L(T)\,\hat{k}(T)$, is instantaneously consumed.[1] (Note that as T is increased sufficiently, the economy will spend an arbitrarily large fraction of the time on this path, so that it constitutes the "turnpike", given (3.7)).

Hence, if the utility function is that in (3.7), it cannot be optimal to follow the Golden Rule for infinite time since it cannot be optimal to follow that path for any finite interval of time. (In fact I believe it can be shown that, in the untruncated problem where we let $T \to \infty$, there exists no optimum policy at all, when (3.7) is the basis for choosing among consumption programs.)

Pearce's analysis suggests the question, under what condition on the utility-rate function will it be optimal to follow the Golden Rule for a finite interval, given that the economy begins and ends on the Golden Rule path? Consider the following class of utility functions:

$$(3.8) \qquad U(T) = \int_0^T e^{-\rho t}\, u[c(t)]dt, \; u'(c) > 0, \; u''(c) < 0 .$$

Under what conditions on ρ and $\underline{u}$ will maximization of (3.8) subject to (3.5) and (3.6) make $k(t) = \hat{k}(t),\ 0 \le t \le T$, when $k_0 = \hat{k}(0),\ k(T) = \hat{k}(T)$?

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1. Why not drive $\underline{r}$ down to zero? When $r = \gamma$, the sacrifice of a unit of *per capita* consumption initially can permit just a one-unit increase of *per capita* consumption later so that it does not pay to make the sacrifice, thus driving $\underline{r}$ below γ . (Recall also that the implicit discount rate on individual utility rates implied by a function like (3.7) is equal to γ .)

The Euler condition (which is sufficient as well as necessary for a maximum since u is a strictly concave function of the variables $k(t)$ and $\dot{k}(t)$) is

(3.9) $$\left[\frac{d}{dt}\ u'(c)\right] / u'(c) = -(r_t - \gamma - \rho)$$

where $r_t = f'[e^{-\lambda t}\ k(t)]$.

Differentiating $u'(c)$ with respect to time, we find that this equation may be written

(3.10) $$E(c)\ \frac{\dot{c}}{c} = -(r_t - \gamma - \rho)$$

where $E(c) = \frac{u''(c)c}{u'(c)}$ = elasticity of marginal utility < 0 .

Now on the Golden Rule path

(3.11) $$r_t = \hat{r} = \lambda + \gamma$$

and

(3.12) $$\frac{\dot{c}}{c} = \frac{\dot{\hat{c}}}{\hat{c}} = \lambda$$

Putting (3.10), (3.11) and (3.12) together we obtain

(3.13) $$E(c) \cdot \lambda = -\lambda + \rho$$

Since $\lambda > 0$ here, it follows that $E(c)$ must be a constant, say E. Hence for the optimality of the Golden Rule path in the present end-point problem it is required that $u(c)$ and ρ satisfy

$$(3.14) \qquad \rho = \lambda(1 + E) \quad \text{or} \quad E = -1 + \frac{\rho}{\lambda}$$

(Were $\lambda = 0$ one would require $\rho = 0$ and no restrictions on $E(c)$, as we learned in the previous section.)

It has been shown that, should the economy start on the Golden Rule and be constrained to and on it, it would be optimal to follow the Golden Rule throughout the interval [0, T] if and only (3.14) is satisfied. Let us now consider the standard Ramsey problem of finding an optimum when there is an infinite time horizon and when the economy starts from an arbitrary initial capital stock.

I shall first study this standard problem for the class of $u(c)$ functions such that the elasticity of marginal utility is constant: $E(c) = E = \text{constant} < 0$. Then

$$(3.15) \qquad u'(c) = \alpha\, c^{E}, \qquad E < 0, \alpha > 0 .$$

The following propositions will be developed:

A. If $\rho \geq \lambda(1 + E)$ (or equivalently $E \leq -1 + \rho/\lambda$) an optimum exists. Hence, if we should wish to "treat individuals equally", meaning $\rho = -\gamma$, an optimum exists if $E \leq -1 - \gamma/\lambda$. It appears that no optimum exists if $\rho < \lambda(1 + E)$.

B. If $\rho = \lambda\,(1 + E)$, i.e., (3.14) is satisfied, the optimal path will approach (or coincide with) the golden age path on which $r = \lambda + \gamma$, i.e., the Golden Rule path.

C. If $\rho > \lambda\,(1 + E)$, the optimal path will approach (or coincide with) the golden age path on which $r = \lambda + \gamma + \rho - \lambda\,(1 + E)$, which is the Golden Utility path.

First we shall cast our constraint equations (3.5) and (3.6) in terms of consumption and capital per unit augmented labor $(e^{\lambda t}\, L(t))$. Let $\bar{c}(t)$ and $\bar{k}(t)$ denote these respective variables. Then

$$\bar{c}(t) = c(t)\, e^{-\lambda t} , \quad \bar{k}(t) = k(t)\, e^{-\lambda t} \tag{3.16}$$

The Golden Rule relations in these variables are

$$f'[\hat{\bar{k}}] = f'[\hat{k}(t)\, e^{-\lambda t}] = \lambda + \gamma , \tag{3.17}$$

$$\hat{\bar{c}} = \hat{c}(t)\, e^{-\lambda t} = f[\hat{\bar{k}}] - f'[\hat{\bar{k}}]\, \hat{\bar{k}} . \tag{3.18}$$

From (3.5) one easily derives the new constraint relation

$$\bar{c}(t) + \dot{\bar{k}}(t) = g[\bar{k}(t)] \equiv f[\bar{k}(t)] - (\lambda + \gamma)\, \bar{k}(t) , \quad \bar{c}(t) \geq 0 , \tag{3.19}$$

and from (3.6)

$$(3.20) \qquad \bar{k}(0) = \bar{k}_o = k_o\, e^{-\lambda t} = k_o \ .$$

Concerning our utility function, there are two cases to be considered: $E = -1$ and $-1 \neq E < 0$. Suppose first that $E = -1$. Then, given (3.15), $u(c)$ is logarithmic:

$$(3.21) \qquad u(c) = \ln c \ ,$$

and our utility function is

$$(3.22) \qquad U(T) = \int_0^T e^{-\rho t} \ln c(t)\, dt \ .$$

Now if $\rho \leq 0$, maximization of the limit of (3.22) as $T \to \infty$ will raise the familiar divergence problem. Let us instead form the integral V , to be maximized, in which the rate of utility corresponding to the Golden Rule path, $e^{-\rho t} \ln (\hat{c}(t))$ is subtracted from the actual rate of utility, $e^{-\rho t} \ln (c(t))$. Noting that $c(t) = \bar{c}(t)\, e^{\lambda t}$ by (3.16) we have

$$(3.23) \qquad V = \int_0^\infty [e^{-\rho t} \ln c(t) - e^{-\rho t} \ln \hat{c}(t)]\, dt$$

$$= \int_0^\infty \left\{ \ln [\bar{c}(t)\, e^{\lambda t}] - \ln [\hat{\bar{c}}\, e^{\lambda t}] \right\} e^{-\rho t}\, dt$$

$$= \int_0^\infty \left\{ \ln \bar{c}(t) - \ln \hat{\bar{c}} + \lambda t - \lambda t \right\} e^{-\rho t}\, dt$$

$$= \int_0^\infty [\ln \bar{c}(t) - \ln \hat{\bar{c}}]\, e^{-\rho t}\, dt$$

First, if $\rho = 0$ (in which case (3.14) is satisfied for $E = -1$) then the problem of maximizing V in (3.23) subject to (3.19) and (3.20) is identical in form to maximizing V in (2.9) subject to (2.5) and (2.6): $\bar{c}(t)$ replaces $c(t)$ of the previous section, $\ln \bar{c}(t)$ replaces $u[c(t)]$ and $\ln \hat{\bar{c}}$ replaces $\hat{u} = u(\hat{c})$. The integral converges and an optimum exists. By analogy to that problem, the optimal path, when $\rho = 0$ and $E = -1$, approaches the Golden Rule path, $\bar{k}(t) = \hat{\bar{k}}$, if $\bar{k}_0 \neq \hat{\bar{k}}$, and follows it continuously if $\bar{k}_0 = \hat{\bar{k}}$.[1] Since $E = -1$ here, one sees that this result is consistent with proposition B above.

Before turning to the possibilities $\rho > 0$ and $\rho < 0$ let us consider further the merits of the time-independent logarithmic function

$$(3.24) \qquad U(T) = \int_0^T \ln c(t)\, dt$$

which we have just shown causes the Golden Rule path to be approached. Can it be defended as a reasonable social utility function? This function is implied by the social indifference curve map proposed by Tobin to be

1. That the logarithmic function with $\rho = 0$ would give this result was suggested by Robert Solow in a lecture at Yale University in 1963.

"intertemporally impartial", given a technology like the one under discussion [18; see especially his Figure 2, p. 8, and pp. 15-16].

Tobin proposed that a representative social indifference curve between _per capita_ consumption at $t = 0$, $c(0)$, and _per capita_ consumption at any future date $\underline{t}$, $c(t)$, should have the property that the marginal rate of substitution (MRS) should equal the ratio of $c(t)$ to $c(0)$. Thus, on the 45° line from the origin, where $c(0) = c(t)$, the MRS equals one or the marginal rate of time preference (MRS minus one) equals zero; where $c(t) = e^{\lambda t} c(0)$, the MRS equals $e^{\lambda t}$, meaning that the sacrifice of a unit of _per capita_ consumption at $t = 0$ requires an increase of $c(t)$ equal to $e^{\lambda t}$ to keep social utility constant. Since the "noncomplementarity" and "stationarity" axioms are implicit in Tobin's proposed indifference map, we have an additive Ramsey-like utility integral like (3.8); since MRS = 1 when $c(t) = c(0)$, we have $\rho = 0$; and since MRS = $u'[c(0)]/u'[c(t)]$ $= c(t)/c(0)$, we have the logarithmic function (3.24).

Despite the appeal of the logarithmic function, we should remember that $\rho = 0$ is a controversial postulate for it implies that _individual_ future utility rates are being discounted at the rate γ , the population growth rate. Let us proceed then to the possibilities $\rho > 0$ and $\rho < 0$.

If $\rho > 0$, then the V integral again converges and there is an optimum. By analogy to standard results obtained for the case $\lambda = 0$, we can infer that the optimal path approaches a golden age path on which

$r = \lambda + \gamma + \rho$. Since $E = -1$, $\rho = \rho - \lambda(1 + E)$, and we see that this result is consistent with proposition C : The Golden Utility path, on which $r = \lambda + \gamma + \rho - \lambda(1 + E)$, is approached asymptotically (or followed continuously if the economy should start on it).

These results for $\rho = 0$ and $\rho > 0$ confirm (for the case $E = -1$) the proposition A , namely that an optimum exists if $\rho \geq \lambda(1 + E)$. If $\rho < \lambda(1 + E)$, i.e., $\rho < 0$ in this case, we encounter the divergence of integral V . This strongly suggests that no optimum exists when $\rho < \lambda(1 + E)$ although I have not demonstrated this.

Consider now the other case, $-1 \neq E < 0$. Integrating (3.15) in this case we obtain

$$u(c) = \frac{\alpha}{1 + E} c^{1+E} + \beta \tag{3.25}$$

$$\text{where} \quad \beta = \begin{cases} u(0) & \text{if } 1 + E > 0 \ (E > -1) , \\ \bar{u} & \text{if } 1 + E < 0 \ (E < -1) . \end{cases}$$

In other words, if $E > -1$, the utility rate function is bounded from below by β and is unbounded from above; if $E < -1$, the function is unbounded from below and bounded from above, $\bar{u}$ being the upper bound, approached asymptotically as $c \to \infty$.

Now we perform again the familiar trick of subtracting from the actual rate of utility, $e^{-\rho t} u[c(t)]$, the (discounted) rate of utility corresponding to the Golden Rule path, thus forming the integral V which is to be maximized subject to (3.19) and (3.20):

$$V = \int_0^\infty \left\{ e^{-\rho t} u[c(t)] - e^{-\rho t} u[\hat{c}(t)] \right\} dt \tag{3.26}$$

$$= \int_0^\infty \left\{ \frac{\alpha}{1 + E} c(t)^{1+E} + \beta - \frac{\alpha}{1 + E} \hat{c}(t)^{1+E} - \beta \right\} e^{-\rho t} dt$$

$$= \int_0^\infty \left\{ \frac{\alpha}{1 + E} [\bar{c}(t) e^{\lambda t}]^{1+E} - \frac{\alpha}{1 + E} [\hat{\bar{c}} e^{\lambda t}]^{1+E} \right\} e^{-\rho t} dt$$

$$= \int_0^\infty \left\{ \frac{\alpha}{1 + E} \bar{c}(t)^{1+E} - \frac{\alpha}{1 + E} \hat{\bar{c}}^{1+E} \right\} e^{-[\rho - \lambda(1+E)]t} dt .$$

If $\rho = \lambda(1 + E)$, so that we are doing no "effective" discounting of the utility of consumption per <u>augmented</u> labor, and (3.14) is satisfied (the condition for the optimality of the Golden Rule path in the two-point problem) then we have again the standard problem of the previous section. The integral converges and an optimum exists. The Golden Rule path is approached asymptotically if $\bar{k}_0 \neq \hat{\bar{k}}$ and is followed continuously if $\bar{k}_0 = \hat{\bar{k}}$. This confirms proposition B .

If $\rho > \lambda(1 + E)$ or equivalently $E \leq - 1 + \rho/\gamma$, the integral clearly converges and there is again an optimum. By analogy to the standard result for the corresponding problem without technical progress, the optimal path is either coincident with or asymptotic to a golden age path on which $r = \lambda + \gamma + \rho - \lambda(1 + E)$, where $\rho - \lambda(1 + E)$ (> 0) is the effective rate of discount of utility of consumption per augmented labor. This is the Golden Utility path. It entails a higher interest rate and smaller capital intensity than does the Golden Rule path, when $\rho > \lambda(1 + E)$. When $\rho = \lambda(1 + E)$, the two paths coincide. Proposition C is confirmed.

Finally, as we have just seen, an optimum exists when $\rho \geq \lambda(1 + E)$, i.e., $E \leq - 1 + \rho/\lambda$, which confirms proposition A.

Proposition A is quite interesting for it indicates that <u>an optimum will exist even if there is "equal treatment of individuals", provided that E is algebraically sufficiently small</u>. As was indicated in the previous section, non-discounting of <u>individual</u> utility rates when the population grows at rate γ implies discounting <u>per capita</u> utility, $u[c(t)]$, at the rate $-\gamma$, i.e., $\rho = -\gamma$. As proposition A indicates, an optimum will exist, even if $\gamma > 0$, provided $E \leq - 1 - \gamma/\lambda$, for then $\rho = \gamma \geq \lambda(1 + E)$. Further, if $E = - 1 - \gamma/\lambda$, then $\rho = \gamma = \lambda(1 + E)$ and the Golden Rule path will be approached asymptotically. If $E < - 1 - \gamma/\lambda$, then $\rho = \gamma > \lambda(1 + E)$, and the Golden Utility path will be approached. In any case, it is <u>not</u> true that "equal treatment of individuals" $(\rho = -\gamma)$ necessarily precludes the existence of an optimal accumulation policy when $\lambda > 0$. Koopmans' theorem on this subject postulated that $\lambda = 0$. Note that the requirement $E \leq - 1 - \gamma/\lambda$ implies that the required $E \to -\infty$ as $\lambda \to 0$ when $\gamma > 0$; this supports Koopmans' theorem.

Attention has been confined thusfar to the class of utility rate functions, $e^{-\rho t} u[c(t)]$, for which the marginal utility elasticity, $E(c)$, is a constant (E). I shall now describe the results of some work by von Weizsäcker [20] and Inagaki [6] and draw from this some rather general conclusions.

Proposition A indicates that when $\rho = 0$, $E \leq 1$ is sufficient for the existence of an optimum. Von Weizsäcker has proved that, when $\rho = 0$ and $E(c)$ is allowed to vary with $\underline{c}$, $E(c) \leq 1$ for all $\underline{c}$ is a sufficient condition for the existence of an optimum, given the present production model. Since proposition A states, more generally, that $E \leq -1 + \rho/\lambda$ is sufficient for the existence of an optimum, it is a reasonable conjecture, by analogy to von Weizsäcker's theorem, that, when $E(c)$ is variable, $E(c) \leq -1 + \rho/\lambda$ for all $\underline{c}$ is a sufficient condition for the existence of an optimum. But it is to be doubted that this is a necessary condition.

Presumably, what matters for the existence and asymptotic properties of an optimal accumulation program is the limiting behavior of $E(c)$ as $c \to \infty$. Inagaki studied the present model (specializing unnecessarily to the Cobb-Douglas function), employing two utility functions: one of them having the property that the limit $E(\infty) = 0$ and the other the property that $E(\infty) = -\nu$, $0 < \nu < 1$, with $dE(c)/dc > 0$ in both cases. He purported to show that $\rho > \lambda(1 + E(\infty))$ is necessary and sufficient for the existence of an optimum; but the analysis in the present paper strongly suggests that $\rho = \lambda(1 + E(\infty))$ would also admit an optimum, given that $E(\infty)$ is approached from below, for it was shown here that $\rho = \lambda(1 + E)$ is sufficient for an optimum when E is constant. (This is a reiteration of our conjecture in the previous paragraph, given that $dE(c)/dc > 0$.) If, however, $dE(c)/dc < 0$ then it would be reasonable to expect that $\rho > \lambda(1 + E(\infty))$ is necessary and sufficient when $E(\infty)$ exists.

Concerning asymptotic properties of the optimal path, Inagaki showed, given either of his utility functions, that the golden age path on which $r = \lambda + \gamma + \rho - \lambda(1 + E(\infty))$ will be approached asymptotically. This is the Golden Utility path again with this difference: since $E(c)$ is not a constant in Inagaki's model, the economy, once placed on the Golden Utility path would depart from it, returning to it asymptotically. This result by Inagaki, together with the previous analysis, suggests the following, final conjecture: If $E(\infty)$ exists and if an optimum exists, the optimal path will be asymptotic to the Golden Rule path if and only if $\rho = \lambda(1 + E(\infty))$.

IV. Concluding Remarks

There are two difficulties associated with the Ramsey approach to optimal economic growth on which I shall comment. One is the possible non-existence of an optimum. The second is the problem of how the social utility function is to be obtained. (Another difficulty -- that the utility functions and production models thusfar considered are unrealistic -- needs no discussion.)

As was shown, an optimum may fail to exist in a variety of models, even in the model with stationary population and technology. The possibility that no optimum will exist is heightened if society wishes to accord "equal treatment to individuals" when the population grows without bound. Suppose there exists no optimum when individual utility rates are not discounted. What then?

Koopmans believes that one might reasonably abandon such a utility function in view of that consequence. " ... the problem of optimal economic growth is too complicated, or at least too unfamiliar, for one to feel comfortable in making an <u>entirely</u> a priori choice of an optimality criterion before one knows the implications of alternative choices. One may wish to choose between principles on the basis of the results of their application." [8, p.2] An issue which this view raises is: should social preferences be invariant to the demographic and technological environment? Koopmans apparently believes that the environment should be allowed to influence preferences. But there will surely be many who disagree.

If one insists upon "equal treatment," despite the nonexistence of an optimum, what program of growth should be adopted? With some reservation, I suggest the following. The V-integral formed by subtracting the Golden Rule utility rate from the actual rate of utility will sort consumption programs into three classes when no optimum exists: those paths which cause the integral to diverge to minus infinity, those paths which cause the integral to converge, and those paths which cause the integral to diverge to plus infinity. The latter class of paths is "infinitely better" than the Golden Rule path and paths in the other classes. My suggestion is to choose arbitrarily a path from this class. My reservation is that if some path other than the Golden Rule path had been used as a reference path in forming a V-integral, the class of "infinitely better" paths would be changed.

The other problem that deserves some discussion is the matter of the social utility function. I feel that, in a democratic society, this function must represent the preferences of those in the body politic, hence only those living at the present time. If this is correct, then there is first the problem of obtaining a social utility function from the (living) individuals' utility functions. Samuelson's social indifference curves require centralized information about individual utility functions; it must be assumed that the government has such information. Second, concerning those individual utility functions, the present generation must know the preferences of future people if it gives weight to their welfare or if it cares about their decisions; even its own consumption path will be affected by future governments representing itself and new generations. Hence this problem is quite a complex one in itself. And the information requirements make such analysis of little value to policy makers.

Is there for policy purposes, an alternative to the Ramsey approach? That is, must an optimal consumption program be computed? Recently, I considered [12] an alternative in which the government follows certain rules of taxation, rules which do not require centralized utility information, leaving total saving ultimately in the hands of the consumer in the market. Would growth then be Pareto optimal for the present generation? Unfortunately, there are still considerable information requirements and, in the presence of market imperfections, externalities and overlap of generations, the fiscal principle studied cannot be defended except as a very crude approximation to a Pareto-optimal policy toward growth. So we are still very far from a solution of the problem of optimal growth policy.

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