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THE CORE OF AN N PERSON GAME

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I. Introduction

The problems of distribution in an economic system may be analysed either by means of the behavioral assumptions of a competitive model or by the more flexible techniques of n person game theory. In the competitive model consumers are assumed to respond to a set of prices by maximizing utility subject to a budget constraint and producers by maximizing profit. Consistent production decisions and an allocation of commodities are obtained by the determination of a set of prices at which all markets are in equilibrium.

The analysis of these problems by means of n person game theory requires us to specify the production and distribution activities that are available to an arbitrary coalition of economic agents. It is frequently sufficient to summarize the detailed strategic possibilities open to a coalition by the set of possible utility levels that can be achieved by the coalition. For example, in a pure exchange economy each coalition will have associated with it the collection of all utility levels that can be obtained by arbitrary redistributions of the resources of that coalition.

The core of an n person game is a generalization of Edgeworth's contract curve. A vector of utility levels is suggested which is feasible for all of the players acting collectively, and an arbitary coalition is examined to see whether it can provide uniformly higher utility levels for

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all of its members. If this is possible, the utility vector which was originally suggested is said to be blocked by the coalition. The core of the n person game consists of those utility vectors which are feasible for the entire group of players and which can be blocked by no coalition.

As we have seen during the last several years, there is an intimate connection between these two methods of analysis. If the conventional assumptions of the competitive model are made, such as convexity of preferences and convexity and constant returns to scale for the production set, then there will be a price system at which all markets are in equilibrium and a resulting assignment of commodity bundles to consumers. The utility vector associated with this competitive equilibrium may be shown to be in the core. Even further, if the number of consumers tends to infinity in a suitable way, the set of possible utility vectors in the core becomes smaller and tends, in the limit, to those utility vectors associated with competitive equilibria [3].

We do not, of course, expect a competitive equilibrium if the classical assumptions of the competitive model are not made. On the other hand the formulation of the problem of distribution by means of n person game theory is sufficiently flexible to accommodate any number of departures from the classical model. The set of possible utility levels achievable by a coalition can be discussed in the presence of increasing returns to scale in production, public ownership of some commodities, and social rather than exclusively private goods, to name only a few departures from the classical model. This raises the question of determining conditions which are sufficient to guarantee the existence of a utility vector in the core, and which are described directly in terms of the structure of an n person game rather than appealing indirectly to the existence of competitive equilibria.

In order to see the form that such conditions might take, let us begin by examining a game with three players. In this case there are seven possible coalitions; the three one player coalitions, the three two player coalitions, and the coalition of all three players. Each such coalition will be able to obtain a set of utility vectors depending on the strategies available to its members. It will be useful to denote by V_S the set of those vectors achievable by the coalition S. $V_{(123)}$ will be represented geometrically by a set of vectors in three space, $V_{(12)}$ will lie in the plane determined by the coordinate axes 1 and 2, and in general V_S will lie in that linear subspace of three space whose coordinates correspond to the members of S. The sets V_S will be assumed to have several technical properties such as closedness and free disposal.

For this game to have a core which is not empty $V_{(123)}$ must be sufficiently large so as to contain a vector which cannot be blocked by any coalition. One meaning of the term "sufficiently large" can be obtained by assuming that it is to the advantage of a disjoint collection of coalitions to combine. For example if $u_1 \in V_{(1)}$ and $(u_2, u_3) \in V_{(23)}$ then I will assume that $(u_1, u_2, u_3) \in V_{(123)}$, and similarly for all other partitions of the set of three players. The assumption that the game is superadditive, in this sense, is quite natural for most economic models. It is however not sufficient to guarantee the existence of a vector in the core, and one additional relationship is required.

Let us assume for a moment that the game derives from a market model in which the three players exchange the commodities which they initially own. The preferences of the i^{th} player will be represented by a utility function $u_i(x^i)$, with x^i the commodity bundle received by this player. The commodity bundle initially owned by the i^{th} player will be denoted by ω^i. With this notation the set $V_{(123)}$ is described by

$$V_{(123)} = \{(u_1, u_2, u_3) \mid u_j \leq u_j (x^j) \text{ for some } (x^1, x^2, x^3)$$

$$\text{with } x^1 + x^2 + x^3 = \omega^1 + \omega^2 + \omega^3\},$$

the set $V_{(12)}$ by

$$V_{(12)} = \{(u_1, u_2) \mid u_j \leq u_j (x^j) \text{ for some } (x^1, x^2) \text{ with}$$

$$x^1 + x^2 = \omega^1 + \omega^2\},$$

with a similar definition for every set V_S.

This game is clearly superadditive in the sense given above, even in the absence of convex preferences. We know, however, that a market game without convex preferences need not have a core [3], and we should therefore look for some way of translating the convexity of preferences into a relationship that can be stated solely in terms of the sets V_S, in order to find the missing condition. Let us proceed in the following way. Assume that we are given a vector (u_1, u_2, u_3) which is arbitrary except that it satisfies the following three conditions

$$(u_1, u_2) \in V_{(1,2)}$$
$$(u_2, u_3) \in V_{(2,3)}$$
$$(u_1, u_3) \in V_{(1,3)} \quad .$$

In the market economy this means that there are commodity bundles (x^1, x^2), (y^2, y^3) and (z^1, z^3) with

$$\begin{aligned} x^1 + x^2 &= \omega^1 + \omega^2 \\ y^2 + y^3 &= \omega^2 + \omega^3 \\ z^1 + z^3 &= \omega^1 + \omega^3 \end{aligned}$$

and

$$\begin{aligned} u_1(x^1) \geq u_1 \quad &, \quad u_1(z^1) \geq u_1 \quad , \\ u_2(x^2) \geq u_2 \quad &, \quad u_2(y^2) \geq u_2 \quad , \\ u_3(y^3) \geq u_3 \quad &, \quad u_3(z^3) \geq u_3 \quad . \end{aligned}$$

But then

$$\frac{x^1 + z^1}{2} \quad , \quad \frac{x^2 + y^2}{2} \quad , \quad \frac{y^3 + z^3}{2}$$

represents a feasible trade for all three players since these vectors total to $\omega^1 + \omega^2 + \omega^3$. If the preferences of the three consumers are convex then the utility levels associated with this trade can be described quite easily, since convexity implies that

$$u_1\left(\frac{x^1+z^1}{2}\right) \geq \min\left[u_1(x^1), u_1(z^1)\right] \geq u_1$$

$$u_2\left(\frac{x^2+y^2}{2}\right) \geq \min\left[u_2(x^2), u_2(y^2)\right] \geq u_2$$

$$u_3\left(\frac{y^3+z^3}{2}\right) \geq \min\left[u_3(y^3), u_3(z^3)\right] \geq u_3 .$$

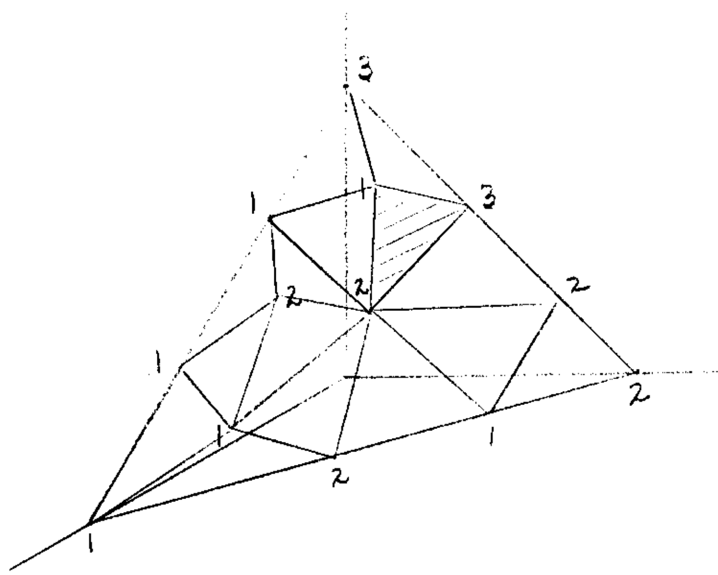
In other words the vector (u_1, u_2, u_3) is obtainable by the three player coalition and is therefore in $V_{(1,2,3)}$.

This rather curious translation of convexity connecting the three two player coalitions to the coalition of all players is in fact the missing condition for the existence of a core in a three person game. The condition requires that a vector (u_1, u_2, u_3) must be in $V_{(1,2,3)}$ if its three projections onto the coordinate planes are achievable for the two element sets corresponding to those planes. In the general case of an n person game we will impose a variety of conditions that require a vector to be achievable for all players if its projections onto a number of coordinate subspaces are feasible for the coalitions corresponding to those subspaces. Each such condition will refer to a particular collection of coalitions, corresponding to the collection of all two person coalitions in the three person game. These conditions, which are immediately satisfied in an exchange economy with convex preferences, will be shown to be sufficient for the existence of a vector in the core of a general n person game.

The first arguments that I obtained for this theorem made a rather elaborate use of fixed point theorems, and it was some time before I realized that an alternative proof could be given, based on the remarkable algorithm recently discovered by Lemke and Howson [4,5] for the solution of two person non zero sum games. The Lemke Howson argument may be adapted to obtain a direct proof of the theorem on the existence of the core in which no fixed point arguments appear. By letting the number of consumers tend to infinity in an n person game arising from an economic model with classical convexity assumptions we obtain a proof of the existence of equilibrium prices with no reference to fixed point theorems. An alternative procedure, discovered by Mr. Rolf Mantel is to apply a modification of the Lemke Howson argument directly to the existence of equilibrium prices without any reference to n person game theory. Mr. Mantel's work appears in his thesis [6].

I think that there are essentially two reasons for attempting to avoid fixed point theorems. The first reason involves the possibility of calculating either equilibrium prices or a vector in the core. Of course a prohibitive amount of information would be required at the present time, to calculate equilibrium prices for an actual economy, and in all probability this would continue to be true even if the capabilities of electronic computers were to increase at a fanciful rate in the future. On the other hand experimental calculations on small economic models may be quite useful, for example in determining the size of the core as a function of the number of consumers, or the degree of monoply power conferred by a privileged position.

To be sure, fixed point theorems do provide a method of calculating equilibrium prices in the following sense. Let $f_j(\pi)$ be the excess demand for the j^{th} commodity at prices π. These functions are assumed to be continuous, satisfy the Walras law and be homogeneous of degree zero. At an equilibrium system of prices each excess demand is less than or equal to zero, and the property of homogeneity permits us to restrict our attention to prices lying on the simplex $\pi_j \geq 0$, $\Sigma \pi_j = 1$. Let the price simplex be divided into a barycentric subdivision and label each vertex of the subdivision with some commodity whose excess demand is less than or equal to zero at that system of prices. Then Sperner's lemma, the central argument in the proof of fixed point theorems, tells us that there will be at least one small simplex all of whose vertices are labeled differently.



If the barycentric subdivision is sufficiently fine, the price system at the center of the distinguished simplex will have excess demands which are less than or equal to zero or if positive the excess demands will be quite small. The price obtained this way will be close to an equilibrium price in a functional sense though not necessarily in terms of Euclidean distance. The functional sense of distance in which each market is approximately in equilibrium is probably the relevant one, and since the excess demands for a finite number of price vectors can be calculated if consumer preferences and the production set are known, we do have a computing procedure based on fixed point theorems for calculating equilibrium prices with a given degree of accuracy.

When the Lemke Howson procedure for calculating Nash equilibrium points is adapted to our problem it provides an algorithm for calculating, in a similar functional sense, a point arbitrarily close to the core. For some examples a point actually in the core is found in a finite number of steps. Its advantage over Spermer's lemma in calculating seems to me to be that the latter technique is essentially an exhaustive search, whereas our algorithm proceeds systematically from step to step. This is admittedly a vague distinction which can only be clarified by computational experience, but the reader should compare the situation with that arising in linear programing. An exhaustive search of all vertices of a convex polyhedron in order to determine the maximum of a linear function is much less efficient as a computational device than the orderly sequence of steps prescribed by the simplex method.

The second virtue of the method to be discussed later in this paper, is in the possibility of its economic interpretation. Since the conventional Walrasian adjustment of prices does not always converge to equilibrium, we have at present no uniformly valid computational procedure for problems involving the consumer side of an economy which is at the same time economically suggestive. An exhaustive search based on Sperner's lemma has no economic interpretation as a possible adjustment mechanism for arriving at an equilibrium state. On the other hand the method of this paper may be capable of such an interpretation. It is sequential and proposes at each stage a utility vector based on a certain collection of coalitions and which cannot be blocked by any set of players. If the utility vector is feasible for the set of all players the problem is finished. If not a new collection of coalitions is determined by what is essentially a pivot step, and a new utility vector proposed which differs from the old vector for only two players, one of whom receives more and the other less. The reader will have to judge for himself whether the examples have any economic interpretation as an adjustment mechanism, but my feeling is that the possibility of interpretation is distinctly higher than that arising from an exhaustive search.

Uzawa has demonstrated [9] that the existence of equilibrium prices for an arbitrary set of market demand functions $f_j(\pi)$ which are continuous, homogeneous of degree one and satisfy the Walras law is, in fact, equivalent to the Brouwer fixed point theorem. Since the algorithm of this paper leads to a proof of the existence of equilibrium prices, the reader may wonder whether the algorithm is basically a disguised proof of the Brouwer theorem. But there is a gap in this argument which could only be completed by showing that a set of arbitrary market demand functions of the sort described above, can be obtained by the summation of individual demand functions arising from utility maximization. This has never been demonstrated and is probably incorrect.

With these remarks I shall turn to a more formal definition of an n person game following Aumann and Peleg [1]. The set of n players will be denoted by N and an arbitrary coalition by S For each set S, E^S will mean the Euclidean space of dimension equal to the number of players in S and whose coordinates have as subscripts the players in S. If u is a vector in E^N then u^S will be its projection onto E^S.

We shall associate with each coalition S a set V_S, in E^S, which represents the set of possible utility levels that can be obtained by that coalition. The members of S may have to engage in a variety of activities, depending on the nature of the n person game, in order to obtain a particular vector in V_S. For our purposes, however, all that is required is a summary of the utility vectors achievable by each coalition.

It will be useful to make the following assumptions about the sets V_S .

1. For each S , V_S is a closed, nonempty set.

2. If $u \in V_S$ and $y \in E^S$ with $y \leqq u$, then $y \in V_S$.

3. The set of vectors in V_N in which each player receives no less than he could obtain by himself is a nonempty, bounded set.

These conditions are all quite mild and need no particular comment. They are slightly different from the conditions assumed by Aumann and Peleg; in particular the sets V_S are not assumed to be convex.

We have already seen how a three person exchange model gives rise to a game in this form. In an exchange economy with n consumers, where the i^{th} player's utility function is given by $u_i(x^i)$ and his initial holdings by ω^i , a vector $u \in E^S$ will be in V_S if we can find commodity bundles x^i with $\sum_{i \in S} x^i = \sum_{i \in S} \omega^i$ and $u_i(x^i) \geq u_i$ for all $i \in S$. Production may be introduced by assuming that each coalition has the ability to transform commodities according to some production set, though this is by no means the only way to incorporate production into an n person game theory model.

As another example consider the classical case of an n person game with transferable utility described by a number f_S associated with each coalition. In this case a vector $u \in E^S$ may be obtained by the coalition S if $\sum_{i \in S} u_i \leq f_S$, so that the sets V_S consist of hyperplanes whose normal vectors have components either zero or one.

Let u be a point in V_N and u^S its projection onto E^S. The vector u is blocked by the set S if we can find a point $y \in V_S$ with $y > u^S$, or in other words if the coalition S can obtain a higher utility level for each of its members than that given by the vector u. A point $u \in V_N$ will be in the core if it cannot be blocked by any set S.

Some conditions must be placed on the sets V_S in order to guarantee the existence of at least one vector in the core. In the three person case discussed above it was sufficient to assume, in addition to a form of superadditivity, that a vector $(u_1, u_2, u_3) \in V_{(123)}$ if the following three conditions are satisfied:

$$(u_1, u_2) \in V_{(12)} \quad , \quad (u_1, u_3) \in V_{(13)} \text{ and } (u_2, u_3) \in V_{(23)} .$$

In order to determine the appropriate generalization of this condition to an arbitrary n person game, we must have recourse to the concept of a balanced collection of coalitions studied by Shapley [8], Peleg [7] and Bondareva [2] in the context of a game with transferable utility. Let $T = \{S\}$ be a collection of coalitions in an n person game. T is said to be a balanced collection if it is possible to find nonnegative weights δ_S, for each coalition in T, such that

$$\sum_{\substack{S \in T \\ S \ni i}} \delta_S = 1 \qquad \text{for each } i .$$

In other words, the weights δ_S are to have the property that if any individual is selected, the sum of the weights corresponding to those coalitions in T which contain the individual, must be equal to one. Another way to phrase the definition is by saying that the characteristic function of the set of all players is a nonnegative linear combination of the characteristic functions of the coalitions in a balanced collection.

Balanced collections of coalitions do represent a generalization of the collection of all two player coalitions studied in the three person game, since $\delta_{(12)} = \delta_{(13)} = \delta_{(23)} = 1/2$ will serve as an appropriate system of weights. It is somewhat unfortunate, given the importance of balanced collections in the study of the core, that we have no really intuitive definition for determining when a given collection is balanced.

This concept permits us to extend to an n person game the additional requirement imposed in the three person case. We say that an n person game is balanced if for every balanced collection T, a vector u must be in V_N if $u^S \in V_S$ for all $S \in T$. Our main result is

THEOREM 1.

A balanced n person game always has a nonempty core.

The condition that a game be balanced is undoubtedly quite obscure and it will be useful to examine some examples. A market game with convex preferences will always be balanced, for let T be an arbitrary balanced collection and u a vector with $u^S \in V_S$ for each S in T. This means that for each such coalition there is a way of redistributing its assets so as to obtain the vector u^S. If the redistribution gives player i (assuming that he is a member of S) the commodity bundle x_S^i, then

$$\sum_{i \in S} x_S^i = \sum_{i \in S} \omega^i \quad \text{and} \quad u_i\,(x_S^i) \geq u_i \ .$$

In order to show that the game is balanced we need to construct an allocation $x^1, \ldots x^n$ with $\sum_1^n x^i = \sum_1^n \omega^i$ and $u_i(x^i) \geq u_i$ for all i . This allocation may be constructed in terms of the weights δ_S used in the definition of a balanced collection.

For each player i , let us define x^i as $\sum_{\substack{S \in T \\ S \ni i}} \delta_S x_S^i$. By the definition of δ_S , each x^i is a <u>convex</u> combination of x_S^i with S ranging over those sets in T which contain the i^{th} player. If the preferences are assumed to be convex then $u_i(x^i)$ is greater than or equal to the smallest of the numbers $u_i(x_S^i)$, and is therefore greater than or equal to u_i . We have, by this device, constructed an assignment of commodity bundles which provides a utility level for each player no less than his corresponding component of u . In order to show that $u \in V_N$ we need only verify that $\sum_1^n x^i = \sum_1^n \omega^i$. But

$$\begin{aligned}
\sum_1^n x^i &= \sum_1^n \sum_{S \ni i} \delta_S x_S^i \\
&= \sum_{S \in T} \delta_S \sum_{i \in S} x_S^i \\
&= \sum_{S \in T} \delta_S \sum_{i \in S} \omega^i \\
&= \sum_1^n \omega^i \sum_{\substack{S \in T \\ S \ni i}} \delta_S = \sum_1^n \omega^i .
\end{aligned}$$

This argument demonstrates that an exchange economy with convex preferences will always give rise to a balanced n person game, and assuming the validity of the main result of this paper, such a game will always have a nonempty core. It is interesting that no additional assumptions are required such as strict monotonicity of the preferences, or strict positivity of the initial holdings.

In our second example, an n person game with transferable utility, it is quite easy to verify that a balanced game has a nonempty core without using the more subtle techniques to be developed later. If the sets V_S consist of those vectors in E^S with $\sum_{i \in S} u_i \leq f_S$, the vector $(u_1, \ldots u_n)$ will be in the core if

$$\sum_{i=1}^{n} u_i \leq f_N \quad \text{and}$$

$$\sum_{i \in S} u_i \geq f_S \quad \text{for all subsets } S .$$

The first inequality implies that $u \in V_N$ and the second set that u cannot be blocked by any coalition S. In other words the game will have a core if the linear programing problem

$$\min \sum_{1}^{n} u_i$$

$$\sum_{i \in S} u_i \geq f_S \quad , \qquad \text{for all } S \quad ,$$

has a solution in which the objective function is equal to f_N. The dual variables may be denoted by δ_S, one for each subset, and the dual linear programing problem is

$$\max \sum_S \delta_S f_S$$

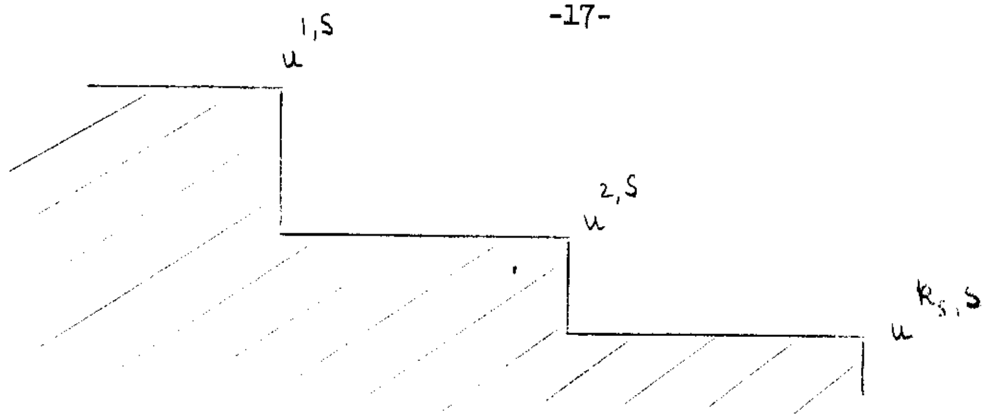
$$\sum_{S \supset j} \delta_S = 1 \ .$$

Let $\{\hat{\delta}_S\}$ be a solution of the dual problem and $\hat{u}_i$ a solution to the primal problem. Then the collection T of those coalitions for which $\hat{\delta}_S > 0$ is, by definition, a balanced collection, and the solution of the primal problem provides us with a vector $\hat{u}$ such that $\sum_{i \in S} \hat{u} = f_s$ for all $S \in T$. But then $\hat{u}^S \in V_S$ for all $S \in T$ and if the game is balanced, this implies that $\hat{u} \in V_N$ or $\sum_1^n \hat{u}_i \leq f_N$. This shows that a balanced game has a nonempty core in the case of transferable utility. As we shall see, the general n person game, without the assumption of transferable utility, requires more elaborate techniques than those of linear programing.

2. The Algorithm

The algorithm for calculating a point in the core will converge in a finite number of steps if each set V_S is determined by a finite number of vectors $u^{1,S}$, $u^{2,S}$, ... $u^{k_S,S}$, in the sense that V_S is the union of the sets

$$\{ u \in E^S \mid u \leqq u^{j,S} \} \ .$$



The details involved in approximating an arbitrary game by a game of this sort will be discussed later.

It is useful to summarize the data of a finite game by means of a matrix C with n rows, the number of players in the game, and $\sum_S k_S$ columns, one column for each of the vectors involved in defining the game. The rows of C will be indexed by i and the columns will require a pair of subscripts (j,S), so that a typical entry in C will be denoted by c_{ijS}. If player i is contained in the coalition S, then c_{ijS} is defined to be the i^{th} component of the vector $u^{j,S}$. It will be useful to define c_{ijS} to be equal to some large number M if player i is not a member of S. The particular choice of M is irrelevant to the actual calculation as long as it is selected to be larger than any of the components of the vectors $u^{j,S}$.

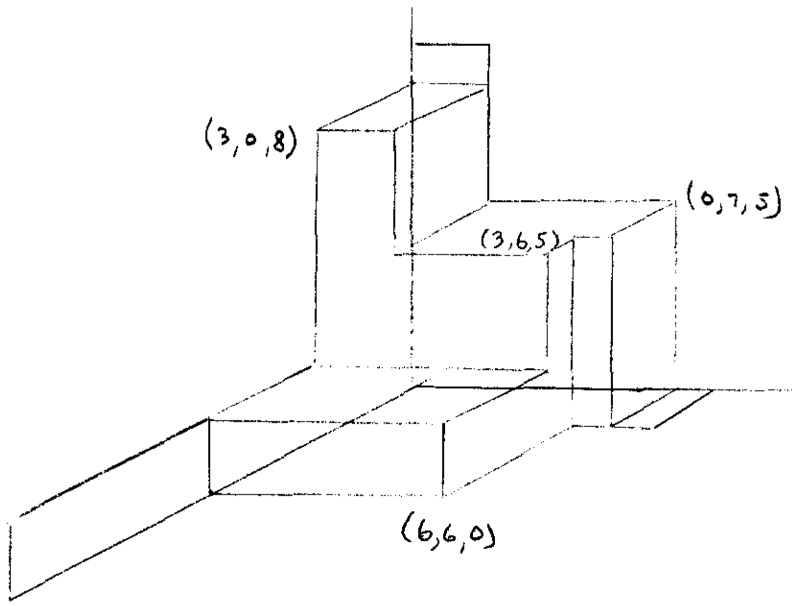
Let us also define a matrix A with n rows and $\sum k_S$ columns by $a_{ijS} = 1$ if player i is in coalition S, and zero otherwise. A is the incidence matrix of players versus sets, with the column representing S appearing as many times as there are corners in V_S.

Let me begin with an example of a three person game in order to illustrate the problem. In this example, the set V_S for a typical two player coalition will be assumed to have two corners. The matrix C is given by

(1)	(2)	(3)	$(12)_1$	$(12)_2$	$(13)_1$	$(13)_2$	$(23)_1$	$(23)_2$
0	M	M	6	2	12	3	M	M
M	0	M	6	8	M	M	7	2
M	M	0	M	M	2	8	5	9

In general, information about V_N need not be included in the matrix C .

It is perhaps useful to begin by examining the problem from a geometric point of view. I have drawn, in the accompaning figure, the set, which may be called V , of those points which are necessarily contained in $V_{(123)}$ if the game is balanced. V contains those points on the coordinate planes which are achievable by the two player coalitions, since a two player coalition and its complementary one player coalition form a balanced collection. V also contains those vectors (u_1, u_2, u_3) with $(u_1, u_2) \in V_{(12)}$, $(u_1, u_3) \in V_{(13)}$ and $(u_2, u_3) \in V_{(23)}$. From the assumption that the game is balanced we know only that $V_{(123)}$ contains V ; it may be considerably larger.



No point in the non-negative orthant can be blocked by a one player coalition. Is there a point in V which can be blocked by no two player coalition? In other words is there a point in V whose three projections are on the boundaries of the sets of utility levels achievable by the two player coalitions? The reader may verify that there is only one such point, $u = (3, 6, 5)$. Of course this vector need not be in the core if it is not Pareto optimal, but then any Pareto optimal point in $V_{(123)}$ which is greater than or equal to u , is in the core.

The vector (3, 6, 5) is generated by the points (6, 6, 0), (3, 0, 8) and (0, 7, 5) in the sense that if we form the square submatrix of C , corresponding to these points,

$$\begin{bmatrix} 6 & 3 & M \\ 6 & M & 7 \\ M & 8 & 5 \end{bmatrix},$$

and define u_i to be the minumum of the i^{th} row of this square submatrix, then $(u_1, u_2, u_3) = (3, 6, 5)$. The fact that u cannot be blocked is clear from the C matrix, for if u were blocked by S , then there would be a column j,S with $c_{ijS} > u_i$ for all i , and the reader may verify that no such column exists.

Analytically the argument that u is in $V_{(123)}$, if the game is balanced, depends on the observation that

$$(3, 6, 5)^{(12)} = (3, 6) \leq (6, 6) \in V_{(12)}$$

$$(3, 6, 5)^{(13)} = (3, 5) \leq (3, 8) \in V_{(13)}$$

$$(3, 6, 5)^{(23)} = (6, 5) \leq (7, 5) \in V_{(23)} ,$$

and that the three two element sets form a balanced collection.

In the general case we also consider a square submatrix of C , and define u_i to be the minimum of the entries in the i^{th} row of this submatrix. For the vector u to lead us to a point in the core two properties are required. First of all we want u to be blocked by no coalition, and this means that for every column in the C matrix at least one entry must be less than or equal to the corresponding entry in the u vector. Of course not every square submatrix of C will produce a u vector with this property, and part of the algorithm will be concerned with determining submatrices of this sort.

In order to conclude that the vector u is in V_N a second condition will have to be imposed on the submatrix of C. The columns of the submatrix correspond to a collection of coalitions in the n person game, and for each coalition S in this collection the vector u^S is surely in V_S, since it is less than or equal to one of the corners appearing in V_S. In order to conclude that $u \in V_N$ it is sufficient that the collection of coalitions be balanced, and this is the case if the equations $Ax = e$

(with e the vector all of whose components are 1) have a nonnegative solution, with $x_{jS} = 0$ for any column not appearing in the square submatrix of C .

In other words we look for a feasible basis for the linear equations $Ax = e$. The n columns of this feasible basis give rise to a square submatrix of C , and u_i is defined to be the minimum of the i^{th} row of this submatrix. The feasible basis is to be selected so that for every column in the C matrix at least one entry is less than or equal to the corresponding entry in the u vector.

The problem may be generalized by considering an arbitrary matrix A , rather than a repeated incidence matrix, a C matrix of the same dimensions as A , and an arbitrary vector b . In this more general case the columns of both the A and C matrix will have the subscript j rather than the more cumbersome subscript (j, S) appropriate to a repeated incidence matrix. We look for a feasible basis for the equations $Ax = b$, and for each such basis we define $u_i = \min \{c_{ij} \mid \text{for all } j \text{ appearing in the feasible basis}\}$. Will there be a basis, so that for every column k , there is at least one i with $u_i \geq c_{ik}$? When the problem is cast in this form the answer may be in the negative, in the sense that slack variables are required to accommodate a general matrix A . When slack variables are introduced, the following theorem applies.

THEOREM 2.

Let A be a matrix such that the convex set $\{x \mid x \geqq 0 \text{ and } A\, x \leqq b\}$ is bounded where b is a nonnegative vector, and let C be an arbitrary matrix of the same dimensions. Then there is a feasible basis for the equations $A x \leq b$ so that if we define

$$u_i = \min c_{ij} \quad , \quad \text{for all columns } j \text{ in this basis,}$$

then for every column k there is an index i , corresponding to a zero slack vector and such that $u_i \geq c_{ik}$.

It is easy to show that the additional slack variables do not enter in the case where A is an incidence matrix with repeated columns, and all of the components of b are 1 . For in this case we obtain a nonnegative vector x_{jS} corresponding to a feasible basis with $\sum_{S \ni i} (\sum_j x_{jS}) \leq 1$, and u is defined by $u_i = \min c_{ijS}$ for all columns (j,S) in this feasible basis. Let i^* be a row with $\sum_{S \ni i^*} (\sum_j x_{jS}) < 1$. Consider a column of the C matrix in which the set S is $\{i^*\}$, so that the entries in this column are all equal to M except for the number in row i^* . But by the conclusion of the theorem there is an index i corresponding to a zero slack variable (so that $i \neq i^*$) , with $u_i \geq M$. If i corresponds to a zero slack variable, however, we must have $u_i < M$, since for each such row, $x_{j,S}$ must be in the feasible basis for some S containing i , and for such an S , $c_{ijS} < M$. This contradiction shows that all of the rows do in fact have zero slacks and that the collection of coalitions S for which x_{jS} is in the feasible basis for some j, is balanced. The general theorem applies therefore to the case where A is a repeated incidence matrix and demonstrates the existence of a point in the core.

The determination of a point in the core has been reduced to the problem of finding a vector satisfying the properties outlined in Theorem 2.

This is not remotely a linear programing problem even though only linear inequalities are involved. Any attempt to cast this problem in a linear programing form would run into the difficulty that not all of the relevant inequalities are to be satisfied simultaneously. An attempt to use integer programing methods would neither provide an existence theorem nor take advantage of the special structure of the problem. The algorithm of this paper, which is based on the ingenious procedure discovered by Lemke and Howson for the solution of a two person nonzero sum game, provides a method for calculating a solution to our problem, and since the algorithm terminates in a finite number of steps, the existence of at least one solution is guaranteed.

In order to discuss the algorithm for the general theorem it is useful to enlarge the A matrix by including columns referring to slack variables and the constant column, so that A is written as

$$A = \begin{bmatrix} 1 & 0 \cdots\cdots 0 & a_{1,n+1} \cdots\cdots a_{1,m} & b_1 \\ 0 & 1 \cdots\cdots 0 & a_{2,n+1} \cdots\cdots a_{2,m} & b_2 \\ \vdots & \ddots & & \\ 0 & \quad 1 & a_{n,m+1} \cdots\cdots a_{n,n} & b_n \end{bmatrix},$$

with $m > n$. The C matrix will also be enlarged by including elements corresponding to the slack variables, so that C becomes

$$C = \begin{bmatrix} L & M_2 \cdots\cdots M_n & c_{1,n+1} \cdots\cdots c_{1,m} \\ M_1 & L \qquad M_n & c_{2,n+1} \cdots\cdots c_{2,m} \\ \vdots & \vdots \qquad \vdots & \vdots \\ M_1 & M_2 \qquad L & c_{n,n+1} \cdots\cdots c_{n,n} \end{bmatrix}.$$

The numbers L and M_i are arbitrary except for the conditions $L < c_{ij}$ for all $j > n$ and all i, and $M_n > M_{n-1} \dots > M_1 > c_{ij}$ for all $j > n$ and all i. These constants are introduced only to avoid a certain lack of symmetry in one step of the algorithm; their specific values are not at all relevant, and any other ordering of the M's would be just as appropriate.

We make the standard nondegeneracy assumption on the matrix A, namely that all of the variables associated with the n columns of a feasible basis are strictly positive. The nondegeneracy assumption for C takes the rather novel form that no two elements in the same row are equal. Both of these assumption can be brought about by perturbations of the corresponding matrices.

DEFINITION: A "feasible basis" for the enlarged C matrix consists of a set of n columns $j_1, j_2, \dots j_n$ so that if

$$u_i = \min\,(c_{ij_1}, c_{ij_2}, \dots c_{ij_n}),$$

then for every column k, there is at least one i with $u_i \geq c_{ik}$.

The term "feasible basis" used in this definition is meant to be suggestive of an analogy with linear programing, as we shall see from some of the properties described below. But first of all it should be noticed that our theorem will be demonstrated if we can exhibit a feasible basis for A which is <u>simultaneously</u> a feasible basis for C. For if this were possible we would obtain a vector of nonnegative variables x corresponding to a feasible basis of $A\,x = b$, and a vector u determined from the extended C matrix by $u_i = \min c_{ij}$ over all columns j in this feasible basis. If $\sum_{j=n+1}^{m} a_{ij} x_j < b_i$, then the i^{th} column of the extended C matrix will contribute to the formation of u, so that $u_i = L$. On the other hand if $\sum_{j=n+1}^{m} a_{ij} x_j = b_i$, then $u_i = \min c_{ij}$ over all columns in the feasible basis other than the slack variables, and corresponds to the vector described in Theorem 2. Then if k is in any column, a row i for which $u_i \geq c_{ik}$ must have zero slack.

In a feasible basis for C, the column k may be selected as one of the columns actually in the basis, say the column j_1. But then we must have $u_i \geq c_{ij_1}$ for some i, and this means that $u_i = c_{ij_1}$ for some i. In other words every column in the basis has at least one element which is a minimizer of its row. If the nondegeneracy assumption is used we see that each column of a feasible basis has precisely one row minimizer to be used in forming the vector u.

Something very much like taking a pivot step may be applied to a feasible basis of C. If a feasible basis for A is given and the constraint set is bounded as in our case, then an arbitrary column outside of this basis may be introduced and one of the columns in the basis dropped so that the resulting basis is feasible. Only one column can be dropped if the nondegeneracy assumption holds. With a feasible basis for a nondegenerate C matrix an arbitrary column in the basis may be dropped and precisely one column outside the basis added so that the new basis is feasible.

An example may be useful before the general discussion. Let C be given by

$$\begin{bmatrix} L & M_2 & M_3 & 6 & 1 & 9 & 4 & 5 \\ M_1 & L & M_3 & 2 & 8 & 1 & 7 & 3 \\ M_1 & M_2 & L & 4 & 3 & 2 & 6 & 8 \end{bmatrix},$$

where $L < 1$ and $M_3 > M_2 > M_1 > 9$. Consider the 3^{rd}, 7^{th} and 8^{th} columns. I have encircled the three row minimizers, which appear in different columns. The u vector is given by $(4, 3, L)$ and the reader may easily verify that for any column k, $u_i \geq c_{ik}$ for at least one i.

Let us attempt to eliminate column 8 from this basis, so that the new basis is to be made up of columns 3, 7 and some column other than 8, say column j^*. The submatrix corresponding to these three columns is

$$\begin{bmatrix} M_3 & 4 & c_{1j^*} \\ M_3 & 7 & c_{2j^*} \\ L & 6 & c_{3j^*} \end{bmatrix},$$

and for this basis to be feasible we must have the three row minimizers appearing in different columns, L surely minimizes its row and so the first column can contain no other row minimizers. The row minimizers for rows 1 and 2 must therefore be selected from columns 2 and 3. There are two possible cases, either

$$u = \begin{pmatrix} 4 \\ c_{2j^*} \\ L \end{pmatrix} \text{ or } \quad u = \begin{pmatrix} c_{1j^*} \\ 7 \\ L \end{pmatrix} \quad .$$

Consider the first case in which $c_{1j^*} > 4$, $c_{3j^*} > L$. For this to be a basis, then for any column k not in the basis we must have $u_i \geq c_{ik}$ for at least one i . But there are several columns k with $c_{1k} > u_1$ and $c_{3k} > u_3$, namely columns 2, 4, 6 and 8 and for every one of these four columns we must therefore have $u_2 = c_{2j^*} \geq c_{2k}$. j^* must be one of these four columns, so that to obtain j^* , we examine all columns with $c_{1k} > u_1$ and $c_{3k} > u_3$, and select the one which maximizes c_{2k} . But this gives column 8 and we are back where we started. This will always happen in the case in which the row minima occur in the same places as in the original basis.

Let us try the second case in which

$$u = \begin{pmatrix} c_{1j^*} \\ 7 \\ L \end{pmatrix}$$

so that $c_{2j^*} > 7$ and $c_{3j^*} > L$. But then for any column k with $c_{2k} > 7$ and $c_{3k} > L$, we must have $u_1 = c_{1j^*} \geq c_{1k}$. Only two columns meet this test, the first and the fifth, and of these two the fifth column gives the highest value of c_{1k} . Column 5 is, therefore the only column that can be brought into the basis. The submatrix corresponding to this new basis is

$$\begin{bmatrix} M_3 & 1 & 4 \\ M_3 & 8 & 7 \\ L & \boxed{3} & 6 \end{bmatrix}$$

and the u vector is $(1, 7, L)$.

We shall try another pivot step, this time removing the column (M_3, M_3, L) from this new basis. In the remaining two columns 3 appears as the new row minimizer for the 3rd row and I have put a square around it. Regardless of the column to be brought in, 4 cannot be the minimizer of the first row, nor 6 the minimizer of the 3rd. Since each column is to have precisely one row minimizer, 7 will have to be the minimum for the 2nd row. As far as the 1st and 3rd rows there are two cases only one of which will lead to a new basis, that is 3 must be the minimum for the 3rd row. As before we examine all columns with $c_{2k} \geq 7$ and $c_{3k} \geq 3$, and select the one which maximizes c_{1k} . There is only one such column, the first, and the new basis is therefore

$$\begin{bmatrix} L & 1 & 4 \\ M_1 & 8 & 7 \\ M_1 & 3 & 6 \end{bmatrix}$$

with $u = (L, 7, 3)$.

Pivot steps are no more difficult to perform, in the general case. Let the columns $j_1, j_2, \dots j_n$ constitute a feasible basis for the C matrix, and consider the square matrix formed by these columns.

$$\begin{bmatrix} c_{1j_1} & \boxed{c_{1j_2}} & c_{1j_3} & \cdots & c_{1j_n} \\ c_{2j_1} & c_{2j_2} & c_{2j_3} & \cdots & c_{2j_n} \\ c_{3j_1} & c_{3j_2} & c_{3j_3} & \cdots & c_{3j_n} \\ \vdots & \vdots & \vdots & & \\ c_{nj_1} & c_{nj_2} & c_{nj_3} & \cdots & c_{nj_n} \end{bmatrix}$$

There is no loss in generality in assuming that the row minimizers occur along the main diagonal, since this can be brought about by a relabeling of the rows. The u vector is given by

$$\begin{pmatrix} c_{1j_1} \\ c_{2j_2} \\ \vdots \\ c_{nj_n} \end{pmatrix} ,$$

and since the basis is feasible, for every column k there is at least one i with $u_i \geq c_{ik}$. Let us attempt to remove column 1 from this basis. I will assume, to be definite, that the smallest of the remaining (n-1) elements in the first row is given by c_{1j_2}, the element which is enclosed in a box.

When a new column is brought in to replace the first column, c_{3j_3} must still be the row minimum for the $3^{\underline{rd}}$ row, since otherwise there would be no row minimum in the $3^{\underline{rd}}$ column, and similarly for

$c_{4j_4}, \dots c_{nj_n}$. We know therefore that if column j^* is brought into the basis, then $c_{3j^*} > c_{3j_3}$, $c_{4j^*} > c_{4j_4}$, ... $c_{nj^*} > c_{nj_n}$, and that the new u vector, say u' , will equal the old u vector in all components other than the first two.

Two cases occur for the row minima of the first two rows, either $c_{1j^*} < c_{1j_2}$ and $c_{2j^*} > c_{2j_2}$ or the reverse. The first case leads back to the original basis. To see this we notice that if the first case does take place, the new u vector will be given by

$$\begin{pmatrix} c_{1j^*} \\ c_{2j_2} \\ \vdots \\ c_{nj_n} \end{pmatrix} ,$$

and if the basis is to be feasible then for any column k we must have $u_i \geq c_{ik}$ for at least one k . But if k is the column j_1 , this means $c_{1j^*} \geq c_{1j_1}$. On the other hand the old basis was assumed to be feasible, and so for any k $c_{ij_i} \geq c_{ik}$ for some i . But here we may take $k = j^*$ and we see that $c_{1j_1} \geq c_{1j^*}$, so that $c_{1j_1} = c_{1j^*}$. But by the nondegeneracy assumption, no two elements of the same row are equal and therefore $j^* = j_1$ and we are back to the original basis.

It is in the second variant, in which the minimizing elements in the first two rows are reversed, that we move to a new feasible basis. For in this case we look for a column j^* in which $c_{1j^*} > c_{1j_2}$, $c_{3j^*} > c_{3j_3}$, ... $c_{nj^*} > c_{nj_n}$. Assuming for the moment that there is such a column, the new u vector is

$$u' = \begin{pmatrix} c_{1j_2} \\ c_{2j^*} \\ c_{3j_3} \\ \vdots \\ c_{nj_n} \end{pmatrix} ,$$

and for the basis to be feasible this means that j^* is selected so as to maximize c_{2k} for all columns k with $c_{1k} > c_{1j_2}$, $c_{3k} > c_{3j_3}$, ... $c_{nk} > c_{nj_n}$. If there are such columns this gives a unique determination for the column to be brought into the basis.

Are there any columns satisfying the required inequalities? The column

$$\begin{pmatrix} M_2 \\ L \\ M_2 \\ \vdots \\ M_2 \end{pmatrix}$$

will surely do this unless $M_2 \le c_{1j_2}$ or $M_2 \le c_{ij_i}$ for some $i > 2$.

But c_{ij_i} is the minimum of its row and if any of the latter inequalities were to hold it would mean that the i^{th} row of the original basis had all of its elements $\geq M_2$. This is surely impossible even if M_2 were the smallest of the M's, since there are at most $(n-1)$ M's in any row of the **C** matrix. The only case that can occur is $M_2 \leq c_{1j_2}$, but since c_{1j_2} is the smallest element in the first row of the original basis after column j_1 is removed, this means that the original basis must have had the form

$$\begin{bmatrix} c_{1j_1} & \boxed{M_2} & M_3 & \cdots & M_n \\ c_{2j_1} & L & M_3 & \cdots & M_n \\ c_{3j_1} & M_2 & L & \cdots & M_n \\ \vdots & \vdots & \vdots & & \vdots \\ c_{nj_1} & M_2 & M_3 & & L \end{bmatrix}$$

with j_1 selected so as to maximize c_{1k} for all k with $c_{2k} > L$, ... $c_{nk} > L$. This is the <u>only</u> pivot step which cannot be carried out. It is an attempt to move from one of the n feasible C bases containing $(n-1)$ slack variables to one consisting only of slack variables, and since the slack variables do not form a feasible basis ($u_i = L$ for all i), the pivot step cannot be carried out. From any other feasible basis, an arbitrary column may be eliminated and then a unique column will be brought in to the basis.

Since a pivot step involves only ordinal comparisons of elements in the same row, the entries in the C matrix can be selected from arbitrary ordered sets, one for each row, rather than being real numbers. For example the entries in a row may be commodity bundles ordered by a preference ordering.

The details of a pivot step may be summarized as follows. If a column is to be eliminated from a feasible basis, then precisely one of the remaining $(n-1)$ columns will contain two row minimizers, one of which is new and the other a row minimizer for the original basis. Let the latter have an index i^* , and examine all columns k for which

$$c_{ik} > \min \left\{ c_{ij} \;\middle|\; \begin{array}{l} \text{with } j \text{ ranging over the remaining} \\ n-1 \text{ columns of the basis} \end{array} \right\}$$

holds for all $i \neq i^*$. Let j^* be selected from this set of columns so as to maximize c_{i^*k} . The column j^* is then introduced into the basis. It is useful to note that pivot steps are reversible; if j_1 is eliminated from a basis and j^* brought in, then j^* may be eliminated from the new basis and the original basis will be obtained.

With this concept of a pivot step, we are ready to discuss the algorithm for determining a basis which is simultaneously feasible for the A matrix and for the C matrix. It is quite easy to find a pair of feasible bases, one for the A matrix and the other for the C which, while not identical, are quite close, and we shall use such a pair of bases as a starting point in the algorithm. The columns corresponding to slack variables $(1, 2, \ldots n)$ form a feasible basis for the A matrix, and as we have seen, the columns

$$\begin{bmatrix} c_{1j} & M_2 & M_3 & \cdots & M_n \\ c_{2j} & L & M_3 & \cdots & M_n \\ c_{3j} & M_2 & L & \cdots & M_n \\ \vdots & \vdots & \vdots & & \vdots \\ c_{nj} & M_2 & M_3 & & L \end{bmatrix}$$

form a feasible basis for the C matrix if j is selected from all of the non slack columns k so as to maximize c_{1k}. The columns in the C basis are given by $(j, 2, \ldots n)$. The relationship between the two bases may be described by saying that the A basis contains column 1 and n-1 remaining columns. The n-1 remaining columns are also contained in the C basis along with one additional column other than the first. The ingenious idea introduced by Lemke and Howson is to insist that this relationship be maintained between the two bases.

In other words we will always be in a position where the feasible A basis can be described by the columns $(1, j_2, j_3, \ldots j_n)$ and the feasible C basis by $(j_1, j_2, j_3, \ldots j_n)$, with $j_1 \neq 1$. What pivot steps can be taken so as to preserve this property? There are only two possible pivot steps, one for the A matrix and the other for the C matrix.

A pivot step on the A matrix will only retain this form if column j_1 is introduced into the A basis. It is of course possible that column 1 will be eliminated from the A basis when column j_1 is brought in; the problem would be solved if this were to occur since the basis $(j_1, j_2, \ldots j_n)$ would then be feasible for both matrices. If column 1 is not eliminated by the pivot step then some other column, say j_i, will be. The two bases will still stand in the same relation with j_i being the column in the C basis which does not appear in the A basis.

The other possible continuation is to do a pivot step on the C matrix, eliminating one of its columns. The mutual relation between the A and C bases will only be retained if column j_1 is eliminated from the C basis. If j_1 is eliminated and column 1 is introduced into the C basis, the problem is solved since the columns $(1, j_2, \ldots j_n)$ will then be feasible for both matrices. On the other hand if column $j^* \neq 1$ is brought into the C basis when j_1 is eliminated, the two bases again stand in the same relationship with j^* being the column in the C basis which does not appear in the A basis.

As we have seen, the pivot step on the C matrix can always be carried out, except in the case where j_1 is the only non slack variable in the feasible C basis. For this case to occur the A basis must be given by $(1, 2, \ldots n)$ and the C basis by $(j, 2, \ldots n)$ so that we are in the starting position described above. From the starting position there is only one pivot step to be taken, namely to introduce column j into the A basis. From all other positions in which the A and C basis stand in the correct relationship, two pivot steps are available.

These considerations suggest the following algorithm. Start with the bases described above and take the one pivot step that is available. At any other point one of the two possible pivot steps will have been used in reaching that position, so that the only continuation is by means of the one remaining pivot step. There is a unique continuation at each step and the process can only terminate when we pivot into a solution of the problem. There are a finite number of possible positions, and if we never return to the same position the process will inevitably terminate at a basis which is feasible for both matrices.

Cycling is impossible, for if the first position to be repeated is the initial position, this would imply that there are two possible pivot steps from the initial position, which we have seen to be false. On the other hand if the process first repeats at some point other than the initial position, then there would be <u>three</u> possible pivot steps proceeding from that point, which is again impossible. The algorithm must therefore terminate in a finite number of steps with a solution to the problem.

There is more than a superficial similarity between this technique and that proposed by Lemke and Howson, for the determination of an equilibrium point in a two person game. If the Lemke Howson technique is applied to a game with payoff matrices (a_{ij}) and $(c_{ij}^{-\eta})$, our algorithm is obtained by letting the parameter η tend to infinity.

Let us consider an example of the algorithm applied to a balanced three person game. I will assume that each player by himself can achieve a zero utility level and nothing larger. Each two player V_S will be assumed to have three corners, and since the game is balanced $V_{(123)}$ need not be explicably considered. The C matrix will consist of twelve columns, three for each two player coalition, and one column for each single player. Each column will have a large arbitrary number entered in those rows corresponding to players not in the coalition, and in order to avoid degeneracy these numbers will be different for different columns.

$$C = \begin{array}{c} \begin{array}{cccccccccccc} (1) & (2) & (3) & & (12) & & & (13) & & & (23) & \end{array} \\ \left[\begin{array}{cccccccccccc} 0 & M_2 & M_3 & \overbrace{12 \quad 3 \quad 2} & & & \overbrace{9 \quad 5 \quad 4} & & & \overbrace{M_{10} \quad M_{11} \quad M_{12}} & & \\ M_1 & 0 & M_3 & 1 & 7 & 9 & M_7 & M_8 & M_9 & \boxed{6} & 2 & 8 \\ M_1 & M_2 & 0 & M_4 & M_5 & M_6 & 3 & 8 & 10 & 6 & 9 & 4 \end{array}\right] \end{array}$$

The M's are arbitrary other than satisfying the unequalities $M_1 > M_2 > \ldots > M_{12} > 12$. The A matrix is the incidence matrix of players versus sets suitably repeated.

$$A = \left[\begin{array}{cccccccccccc|c} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 + \epsilon_1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 + \epsilon_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 + \epsilon_3 \end{array}\right]$$

In order to avoid degeneracy in the A matrix, the last column has been perturbed by small ϵ's, subject to $0 < \epsilon_1 < \epsilon_2 < \epsilon_3$. There is no need to add additional slack variables, since the single player sets play this role if all of the other columns in the C matrix are strictly larger than zero.

Step 1. We begin with a basis for the A matrix consisting of columns (1, 2, 3) and for the C matrix columns (10, 2, 3), so that the u vector associated with this C basis is $u = (M_{10}, 0, 0)$. The first step is to bring the $10^{\underline{th}}$ column into the A basis, and the pivot element for this step is encircled. The resulting A matrix is

$$\left[\begin{array}{cccccccccccc|c} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1+\epsilon_1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1+\epsilon_2 \\ 0 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & \epsilon_3-\epsilon_2 \end{array}\right] .$$

Column 2 has been removed from the A basis, and therefore it must be removed from the C basis. In the remaining two columns, column 10 has two row minimizers, with the new one enclosed in a box. We therefore examine all columns k , with $c_{2k} > 6$, $c_{3k} > 0$ and select the one which maximizes c_{1k} . This is column (12).

Step 2. The A basis is (1, 3, 10) and the C basis (12, 3, 10) , with a u vector given by $u = (M_{12}, 6, 0)$. We continue by bringing column (12) into the A basis. No calculation is required since column (10) is to be eliminated. Column (10) must therefore be eliminated from the C basis. If we consider the submatrix of columns (12, 3, 10)

$$\begin{bmatrix} M_{12} & M_3 & M_{10} \\ \boxed{8} & M_3 & 6 \\ 4 & 0 & 6 \end{bmatrix},$$

we see that when column (10) is eliminated, column (12) has two row minima, with the new one enclosed by a box. We therefore maximize c_{1k} subject to $c_{2k} > 8$, $c_{3k} > 0$, and obtain column 7.

Step 3. The A basis is (1, 3, 12) and the C basis (7, 3, 12) with $u = (9,8,0)$. Column 7 must be brought into the A basis. We shall write the A matrix in the condensed form

$$\begin{array}{cccccc|c} (1) & (2) & (3) & (12) & (13) & (23) & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 1+\epsilon_1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1+\epsilon_2 \\ 0 & -1 & 1 & -1 & 1 & 0 & \epsilon_3-\epsilon_2 \end{array}\right],$$

rather than repeating the columns for the two player sets. When column 7 is brought in, column 3 is eliminated from the A basis, and therefore column 3 must be eliminated from the C basis. The submatrix of columns (7, 3, 12) is

$$\begin{bmatrix} 9 & M_3 & M_{12} \\ M_7 & M_3 & 8 \\ \boxed{3} & 0 & 4 \end{bmatrix},$$

with the new row minima enclosed in a box. We therefore maximize c_{1k} subject to $c_{2k} > 8$, $c_{3k} > 3$, and obtain column 8.

Step 4. The A basis is (1, 7, 12), the C basis (8, 7, 12) and $u = (5, 8, 3)$. The A matrix is

$$\begin{array}{cccccc|c} (1) & (2) & (3) & (12) & (13) & (23) & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & 1 & -1 & 2 & 0 & 0 & 1+\epsilon_1+\epsilon_2-\epsilon_3 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1+\epsilon_2 \\ 0 & -1 & 1 & -1 & 1 & 0 & \epsilon_3-\epsilon_2 \end{array}\right],$$

and when column 8 is introduced, column 7 is eliminated without any calculation. Column 7 must therefore be eliminated from the C basis. The submatrix of columns (8, 7, 12) is

$$\begin{bmatrix} 5 & 9 & M_{12} \\ M_8 & M_7 & 8 \\ 8 & 3 & \boxed{4} \end{bmatrix}.$$

We then maximize c_{2k} subject to $c_{1k} > 5$, $c_{3k} > 4$, which gives column (10).

Step 5. The A basis is (1, 8, 12), the C basis (10, 8, 12) and $u = (5, 6, 4)$. Column (10) is introduced into the A basis, and column (12) eliminated, again with no calculation. The submatrix of columns (10, 8, 12) is

$$\begin{bmatrix} M_{10} & 5 & M_{12} \\ 6 & M_8 & 8 \\ \boxed{6} & 8 & 4 \end{bmatrix}$$

so that we maximize c_{2k} subject to $c_{1k} > 5$, $c_{3k} > 6$, obtaining column 11.

Step 6. The A basis is (1,8,10), the C basis (11,8,10) and $u = (5,2,6)$. Column 11 enters the A basis and column 10 is dropped. The submatrix of columns (11,8,10) is

$$\begin{bmatrix} M_{11} & 5 & M_{10} \\ 2 & M_8 & 6 \\ 9 & \boxed{8} & 6 \end{bmatrix},$$

and we maximize c_{1k} subject to $c_{2k} > 2$, $c_{3k} > 8$, giving column 9.

Step 7. The A basis is (1,8,11), the C basis (9,8,11) and $u = (4,2,8)$. Column 9 enters the A basis and column 8 is eliminated. The submatrix (9,8,11) is

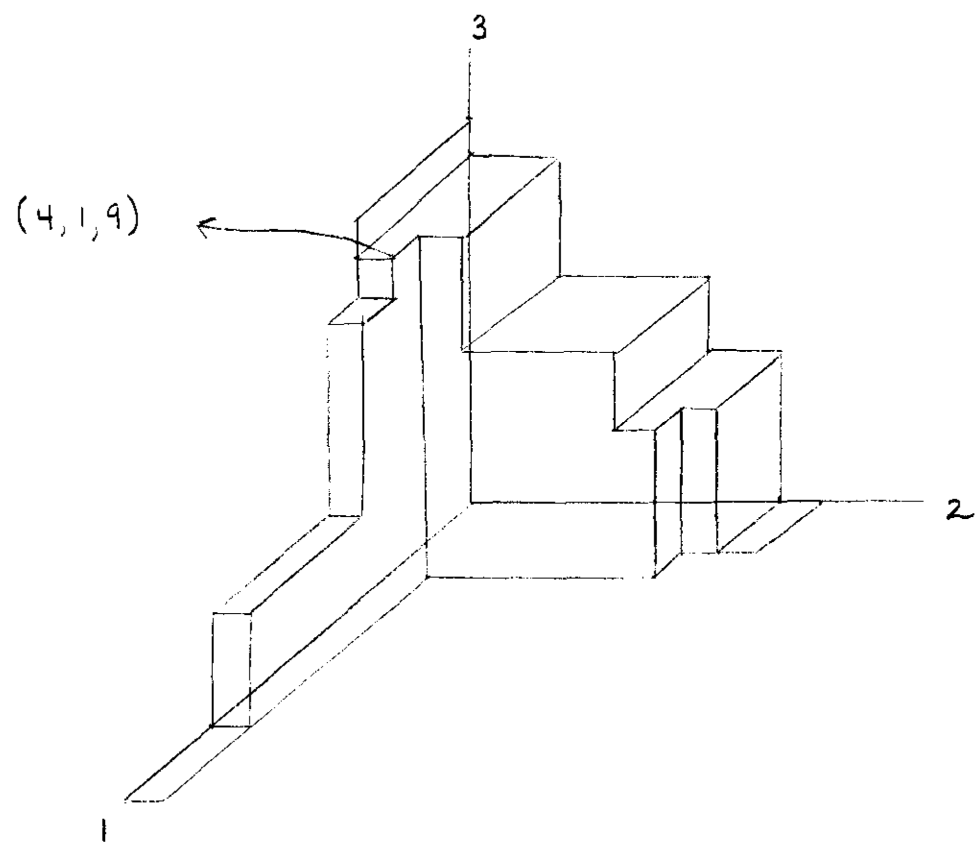
$$\begin{bmatrix} 4 & 5 & M_{11} \\ M_9 & M_8 & 2 \\ 10 & 8 & \boxed{9} \end{bmatrix}$$

and we maximize c_{2k} subject to $c_{1k} > 4$, $c_{3k} > 9$, giving column 4.

Step 8. The A basis is (1,9,11), the C basis (4,9,11) and $u = (4,1,9)$. Column 4 must be introduced into A basis, but then column 1 is eliminated and the solution is obtained. Columns (4,9,11) are feasible for both the A and C matrix. The utility vector

$$u = (4,1,9) \text{ cannot be blocked}$$

by any two or one player set, and if the game is balanced this vector <u>must</u> be in $V_{(123)}$. Any Pareto optimum point in $V_{(123)}$ which is greater than or equal to (4,1,9) must be in the core.



THE SET V FOR THIS EXAMPLE

The algorithm has provided the following sequence of u vectors:

$$
\begin{array}{ll}
u^1 = (M_{10}, 0, 0) & (2), \ (3), \ (23) \\
u^2 = (M_{12}, 6, 0) & (3), \ (23) \\
u^3 = (9, 8, 0) & (3), \ (13), (23) \\
u^4 = (5, 8, 3) & (13), (23) \\
u^5 = (5, 6, 4) & (13), (23) \\
u^6 = (5, 2, 4) & (13), (23) \\
u^7 = (4, 2, 8) & (13), (23) \\
u^8 = (4, 1, 9) \quad , & (12), (13), (23)
\end{array}
$$

each one obtained from the previous one by modifying two of the components of the vector. Each vector is generated by a collection of coalitions, and cannot be blocked by any proper coalition in the game. The generating coalitions are, however, not balanced until the last step and it is only then that a vector in V_N is obtained.

There is a substantial amount of arbitrariness in initiating the algorithm; in the ordering of the M's, in the lexicographical ordering used in the A matrix and in the determination of a pair of bases which differ in at most one column. The question of which points in the core will be determined by these variations is an intriguing one but one which I would prefer to postpone.

4. The General Case

In general the sets V_S are smooth rather than having a finite number of corners. The algorithm may still be used to demonstrate that a balanced n person game has a point in the core. If a finite number of vectors are selected in each V_S, the algorithm will produce a point in V_N which cannot be blocked by any of these vectors. The number of vectors in each V_S may be systematically increased so that in the limit an everywhere dense subset of V_S is obtained. This will produce a sequence of points in V_N which cannot be blocked by more and more vectors. Any limit point of this sequence will be blocked by no coaltion, so that a balanced game will have a point in the core. A similar technique can be applied to the case of a general matrix A, and a mathematical theorem of some interest may be obtained.

The appeal to a convergent subsequence is unfortunate from a computational point of view. Two approaches seem to me to be possible in avoiding this computational difficulty. The first is to be content with an approximation to each V_S by means of a set with a large number of corners. This would provide us with a point in V_N which is close to being in the core in the functional sense that if it is blocked it cannot be blocked by very much.

A second approach is to attempt to deal directly with the sets V_S. For example in an exchange economy with piece wise linear concave utility functions, the sets V_S will be convex polyhedra, and we may imagine the algorithm directly applied to A and C matrices with an infinite number of columns, one for each point in each set V_S. It would not be necessary to list all of the columns at the beginning of the algorithm; only those involved in the current basis would be required. In a pivot step on the C matrix the column to be brought into the basis is determined by maximizing a particular entry subject to constraints on the remaining entries of the column. When the sets V_S are convex polyhedra this can be done by means of a finite number of linear programing problems reminiscent of the way in which pivot steps are carried out in the decomposition principle. Degeneracy in the C matrix becomes a serious problem, however, but my guess is that this is capable of being resolved so that a direct algorithm exists for polyhedral V_S .

REFERENCES

[1] Aumann, R.J. and B. Peleg, "von Neumann - Morgenstern Solutions to Cooperative Games without Side Payments", Bul. Amer. Math. Society, 66, (1960) pp. 173-179.

[2] Bondareva, O., "The Core of an N Person Game", Vestnik Leningrad Univ., 17, (1962) No. 13, pp. 141-142.

[3] Debreu, G. and H. Scarf, "A Limit Theorem on the Core of an Economy", Inter. Econ. Review, Vol. 4, No. 3, Sept., 1963.

[4] Lemke, C.E., "Bimatrix Equilibrium Points and Mathematical Programming", Management Science, Vol. 11, No. 7, May, 1965.

[5] Lemke, C.E. and J. T. Howson, "Equilibrium Points of Bi-matrix Games", SIAM Journal, Vol. 12, July, 1964.

[6] Mantel, Rolf, Toward a Constructive Proof of the Existence of Equilibrium in a Competitive Economy, submitted as a thesis to the Department of Economics, Yale University, 1965.

[7] Peleg, B., An Inductive Method for Constructing Minimal Balanced Collections of Finite Sets, Research Program in Game Theory and Mathematical Economics, Memorandum No. 3, February, 1964, Dept. of Math., Hebrew University, Jerusalem.

[8] Shapley, L.S., On Balanced Sets and Cores, RAND Corp. Memorandum, RM-4601-PR, June, 1965.

[9] Uzawa, H., "Walras' Existence Theorem and Brouwer's Fixed-Point Theorem", Economic Studies Quarterly, 8, No. 1, pp. 59-62.