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THE EXISTENCE OF A COMPETITIVE EQUILIBRIUM

IN A MONETARY ECONOMY

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By

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I. INTRODUCTION

1.1. In this paper we consider an intertemporal general equilibrium model of a competitive economy, in which economic decisions are taken repeatedly in a sequence of successive market conventions. In our economy production takes time and the economic horizon taken into account in each convention is finite (and rather short). During each market convention the participants are influenced by the knowledge that subsequent conventions will be held in which different circumstances may prevail.

This type of a model, which may be called the recursive intertemporal model, is an extension of the Walrasian model of capital formation, credit, and money (Walras [16, Lessons 20-30, 36]). The Walrasian model examined the operation of a competitive economy at one point of time. Although the importance of the existence of subsequent conventions on current economic decisions was not fully recognized, the Walrasian model is the first formulation of an intertemporal competitive equilibrium model. The Walrasian model has been substantially extended (with respect to the process of intertemporal equilibria) in the epoch-making 'Value and Capital' of Hicks [7], and in its monetary aspects by Patinkin [14]. The present model is an attempt of a mathematical formulation of the Walras-Hicks-Patinkin model.

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On the other hand most of the recent work on models of capital accumulation has been centered on models in which perfect foresight is assumed for the whole period covered by the model. In essence, all economic decisions are taken once, in one market convention held in the first time period. These models may be called planning models and they include e.g. those examined by Malinvaud [10] and Dorfman, Samuelson and Solow [5].

1.2. Our economy is composed of production and consumption units. The various existing goods include capital goods yielding services over a number of periods, and commodities which last for one period only. Paper money is the medium of transactions in each market convention. All assets in the form of capital goods, commodities, or paper money are owned by the consumption units.

The first part of this paper contains a systematic description of the recursive intertemporal model, whose distinguishing characteristic is the succession of market conventions. We examine in detail the implications of this on the behavior of the consumption units in each convention. One implication is the emergence of speculation, since economic actions taken in one convention can be reversed in the succeeding ones. The other is the possibility open to the consumption units to devote a part of their wealth to the purchase of commodities, capital goods, or paper money not for direct consumption purposes but for the purchasing power which is thus made available in the succeeding market conventions. Our immediate purpose is to extend the Walrasian model so that the problems created by production lags and transactions in futures, are properly analyzed. The essential features, however, of the Walrasian model are still preserved. This of course implies that the present

model is not closer to reality than the Walrasian one.

1.3. In the latter part of the paper we prove the existence of a competitive equilibrium in each market convention. This is an application of the existence theorems of Arrow and Debreu [1], McKenzie [12], and Debreu [3], [4], to a more complex model, where paper money and savings are considered. The detailed specification of the model to the point where one can actually apply an existence theorem must always form an integral part of the formulation of any general equilibrium model. For only then the completeness and internal consistency of the model is firmly assured, and all needed conditions have been introduced.

1.4. A detailed examination of the monetary aspects of our model is reserved for a following paper. The rigorous formulation of a model in which paper money and other assets available at different time periods exist will enable us to answer several questions which are very important in current monetary theory. The distinction between paper money itself and services of paper money, the importance of the 'pre-trade' transaction demand for paper money, the place of Walras' Law in a monetary economy, the relation between the loanable funds and the liquidity preference theories of the determination of the money interest rate, and the (in-)consistency of the classical and neo-classical monetary theory, are among the problems which will be considered there.

2. A BROAD OUTLINE OF THE MODEL

2.1. We consider an economy as it evolves through time by holding market conventions at regular time intervals, say at t , $t + 1$,

In this economy there exists a finite number of distinct goods, which are distinguished into k capital goods and n commodities, subdivided into k capital services, l primary factors, and m final commodities. The capital goods, D_h ($h=1, \dots, k$), which are available at t , have been produced in various periods before t , and they yield at t capital services, C_h ($h=1, \dots, k$).^{1/} The primary factors, C_i ($i=1, \dots, l$), become available in the beginning of each period, and they are not considered as result of previously completed production. The final commodities, C_j ($j=1, \dots, m$), available at t have been produced during the $t - 1$ period.

All n commodities, C_g ($g=1, \dots, n$), may be used both for production and for consumption purposes during each period. The production of capital goods and final commodities requires one time period and is completed in the beginning of the next period.

2.2. The economy is composed of production and consumption units. In each period a market convention is held, in which the quantities and prices

^{1/} Units are so chosen that one unit of a capital good available at t yields one unit of capital services at t .

of all goods available within the economic horizon are determined under a purely competitive regime. We assume that the market operates through a tâtonnement process; i.e., the formation of prices and the determination of the quantities of the goods exchanged precedes the process of exchange itself. The market convention is closed and contracts between the participants become binding only when an equilibrium has been reached.

Immediately after the market convention the actual exchange of the various goods begins. We are not interested in the timing of these exchanges which are supposed to be made in any manner appropriate to the nature of the goods concerned and the purpose of their use. On the other hand, we will make special assumptions about the timing of the payments for the exchanged quantities of goods. All payments are made in paper money. Money serves as the actual medium of transactions ^{2/} and as a "store of wealth." ^{3/} The quantity of money is exogenously given and it is carried from period to period. We will examine in more detail the functions of money in our model in § 2.8. For the time being, we assume that the market participants at t demand services of paper money available at t and paper money available at $t+1$. ^{4/}

^{2/} A medium of transactions is an object which intervenes in an indirect exchange. See Wickseil [17, Vol. II, pp. 15-24].

^{3/} Any commodity (or capital good) available in the future can be used as a "store of wealth," i.e., it can be bought at t with the explicit purpose to be sold again in a future market and thus make then available the proceeds for further transactions.

^{4/} We will see that paper money can be considered as a capital good. Consequently, we can distinguish between the services of paper money at t , and paper money itself available at $t+1$.

2.3. In each market convention the economic horizon extends over two periods only. Thus in the convention at t all decisions refer to goods available at t and at $t+1$. In particular, the market decides, on the one hand, upon the quantities of the commodities to be currently used for the production of capital goods and final commodities which will become available at $t+1$, and on the other hand, upon the quantities of the commodities to be consumed at t , and the quantities of the commodities and capital goods available at $t+1$ to be "held" by the consumers from the current to the next convention.

The prices of all goods available within the horizon are also determined in the market convention. The price associated with a good corresponds to an amount of paper money available at t which is paid during the t^{th} period by the participants for the availability at t or $t+1$ of one unit of the good concerned. All prices are expressed in terms of an abstract unit of account at t .

2.4. The technological knowledge existing in the economy at the time of each market convention is exogenously given. It consists of blueprints describing the technical properties of various production processes.

There is a finite number of production units denoted by the superscript r . Each production unit determines according to appropriate decision rules the levels at which it will operate its production processes in the current period.

We assume that each production unit considers the market prices as given and selects a production plan which maximizes its profits. Accordingly, each

production unit on the one hand demands commodities available at t to be used as inputs during the t^{th} period, and on the other hand supplies final commodities and capital goods to become available at $t+1$.

The breadth of the economic decisions which are classified as entrepreneurial decisions in our model is extremely limited. The production units participate in each convention with nothing else but the blueprints of the production processes at their disposal.

2.5. In our economy there exists a finite number of consumption units denoted by the superscript s . The consumption units are the owners of all the wealth of the economy, which exists in the beginning of the period t . Namely, they own all commodities available at t and all capital goods available at $t+1$, resulting from all previous periods.

Their decisions during the market convention refer not only to commodities available at t , which are demanded for consumption at t , but also to commodities which are to be produced at t and become available at $t+1$, to capital goods available at $t+2$ ^{5/}, to services of paper money at t , and to paper money available at $t+1$.

The object of our consumers' demand for commodities available at t is the satisfaction of their consumption needs by the direct use of these commodities.

^{5/} We will see in § 2.7 that we prefer to consider separately the capital services provided by the capital goods within the economic horizon in each market and the corresponding capital goods themselves. The capital services for these periods are included among the commodities available in the corresponding periods. Consequently, the existing capital goods are considered as available from the subsequent period and on.

Services of paper money available at t are demanded because paper money is the medium of transactions in each market convention. However, the real object of the consumers' demand at t for (a) commodities available at $t+1$, (b) capital goods available at $t+2$, and (c) paper money available at $t+1$, is the reservation of purchasing power, which will be made available in future market conventions by the ownership of these goods at t . These goods are not demanded at t for their direct use for consumption purposes. They may very well be offered back to the market in the subsequent convention, if the consumption units so wish.

2.6. The proper consideration of the consumers' demand at t for commodities available at $t+1$, capital goods available at $t+2$, and paper money available at $t+1$, is the distinguishing characteristic of the present intertemporal model of a competitive economy, in which market conventions take place in every time period. In such an economy with successive conventions the possibility of economic actions taken by the consumption units with the explicit purpose of their reversal in a later market convention so as to insure economic benefits from possible changes in the environment has to be, and can be, recognized.

In fact, the introduction of some speculative transactions of this kind is indispensable in an intertemporal model such as ours. The need for consideration of speculative transactions is specifically created by the introduction of an explicit production lag and the unique position of the commodities (and especially the final commodities) available at the end of the

horizon (i.e., at $t+1$). A portion of these commodities will be devoted to consumption at $t+1$, but a large part of them can only be used as inputs for production beyond the period t . Consequently, in the absence of any speculative demand for these commodities in the market at t , their price would be "abnormally low" since there is no entrepreneurial demand for them at t . Under the circumstances it is reasonable to assume that some consumption units will demand at t these commodities with the explicit purpose to "hold" them and offer them back to the market at $t+1$. At $t+1$ entrepreneurial demand will be forthcoming and this may insure a profit for the consumption units.

This particular situation with respect to the commodities available at the end of the horizon can lead us to a general consideration of speculative demand in an intertemporal model. However, our immediate objectives are very limited in this direction. The present treatment only attempts to provide a unified way for the introduction of savings^{6/} in the analysis of the behavior of the consumption units.

2.7. Let us now indicate briefly the main features of our model concerning capital goods and their services. All capital goods existing at t are owned by the consumption units. Similarly all decisions concerning the acquisition of capital goods and the disposal of their services are consumers' decisions.

^{6/} The economy's savings at t is defined to be the net change in the accounting value of the purchasing power reserved by the consumption units at t and at $t-1$. As we saw in § 2.5, consumption units reserve at t purchasing power available at $t+1$ by demanding commodities available at $t+1$, capital goods available at $t+2$, or paper money available at $t+1$.

Capital goods are subject to depreciation at constant rates depending upon the good. Thus a constant percentage of the quantity of any specific capital good existing at t is lost through depreciation, but the remaining quantity at $t+1$ has the same physical properties as any other unit of this good available at $t+1$.

The capital services yielded by the capital goods within the economic horizon in each market are considered separately from the corresponding capital goods. Namely, if one unit of D_h is available at t , it is considered as one unit of C_h available at t , $(1-\xi_h)$ units ^{7/} of C_h available at $t+1$, and as $(1-\xi_h)^2$ units of D_h available at $t+2$. Finally, consumption units demand capital goods available at $t+2$ ^{8/} only for the sequence of capital services generated by them. They are not interested in the actual type of the capital good but only in the sequence of net rentals which the various types are capable of sustaining.

2.8. The paper money existing at t is assumed to perform two distinct functions in the market convention at t . First, paper money is the actual medium with which payments for the transactions, which follow the

^{7/} ξ_h is the depreciation coefficient for the capital good D_h .

^{8/} Any demand for a capital good available at t is thought of as demand for its capital services at t , and at $t+1$, and as demand for the remaining quantity of the capital good at $t+2$.

closure of the convention, are made. Second, paper money can perform the same tasks in the succeeding market conventions. If an appropriate mechanism for money loans among the participants in the convention at t is provided for, then it is not necessary for a consumption or production unit to own paper money in the beginning of the convention in order to be able to execute their plans. Only a sufficient quantity of services of paper money available at t is needed.

Consequently, we have to distinguish completely between the demand by production and consumption units for the services of paper money available at t and the demand by consumption units for paper money available at $t+1$. When sufficient conditions for the existence of the first demand are given, the existence of a demand for paper money available at $t+1$ is easily accounted for. Paper money, like any other capital good, yields services. Thus the motives behind this latter demand should be ~~similar~~ to those behind the demand for capital goods in general.

The following assumption about the timing of the payments after a market convention is made:

Payments Assumption: Payments for all purchases are made immediately after the closure of the market convention at t , while the receipts for all sales are received only at the end of the time period t .

This lack of synchronization between payments and receipts creates a need for each participant to hold adequate money balances for the convention at t , i.e., to have available at t a quantity of services of paper money. In our model all consumption or production units are able to acquire in the convention at t the quantity of services of paper money, which they need for

their transactions at t , by demanding an appropriate quantity of services of paper money at t .^{9/}

Under the above payments assumption, which can be traced back to the quantity theory of money, the consumption as well as the production units operate not only subject to their wealth restraint but also subject to a new restraint which we may call the financial restraint.

The above payments assumption appears very strong and unrealistic. It is true that the choice of the timing patterns of the payments for the transactions concluded during each convention is dictated -- and in fact limited -- by the assumption that the market operates through a \hat{t} âtonnement process. Once the \hat{t} âtonnement assumption is strictly adhered to, paper money can be entirely dispensed with. The economy can operate on the basis of an accounting system, i.e., under a pure barter regime. However, in the absence of a complete and rigorous non- \hat{t} âtonnement scheme of operation of the economy as a whole, the combination of the above \hat{t} âtonnement process and payments assumption may be considered as a not too inaccurate idealization of the workings of an actual economy.^{10/}

^{9/} Since these services are needed for the whole period between two consecutive conventions, the delivery of this particular good is made immediately after the closure of the market convention (in accordance with the rule of § 2.2). The payment of the accounting value of the services of paper money purchased at t (i.e., the interest charge for the money loan) is not made immediately after the closure of the convention, along with all other payments. Since the lack of synchronization between payments and receipts does not appear in the market for services of paper money, we assume that interest on money loans at t is paid at the end of t along with the return of the money loan.

^{10/} Our payments assumption is in fact the limiting case of the well-known assumption introduced by Patinkin [14, pp. 86-95], (namely, that payments are made in a random manner during the time interval between the closure of the convention and the opening of the next one), if all participants are keenly afraid of the risks of default. Then they are actually constrained in demanding only such quantities of goods that their money value is equal to the quantity of services of paper money in their possession at t .

3. PRICE EXPECTATIONS AND THE SYSTEM OF INTEREST RATES

3.1. When the market convenes at t , trial prices for all traded goods are announced. ^{11/} At the same time the market forms collectively estimates about the prices of all the goods to be traded at $t+1$, which are expected to be established in the next market convention. ^{12/13/}

We assume that these market expectations refer only to estimates of the average

^{11/} These announced prices include the prices of the commodities available at t and $t+1$, p_t^t and p_{t+1}^t ; the price of the services of paper money available at t , p_{mt}^t ; that of paper money available at $t+1$, $p_{m t+1}^t$; and the price of Q_3 available at $t+1$, a fictitious good to be introduced in § 3.3. With respect to the notation, the superscript refers to the time of the convention, whereas the subscript refers to the time of the availability of the particular commodity. p_t^t and p_{t+1}^t are n -dim vectors. Also $p_t^t = (p_{1t}^t, p_{2t}^t, p_{3t}^t)$, $t = t, t+1$, with the subscripts 1, 2, and 3, used to denote capital services (as well as capital goods), primary factors, and final commodities, respectively.

^{12/} These prices, which are expected at t to be established at $t+1$, include the prices of the commodities available at $t+1$, \hat{p}_{t+1}^{t+1} ; the prices of the capital services available at $t+2$, $\hat{p}_{1 t+2}^{t+1}$; the price of the services of paper money available at $t+1$, $\hat{p}_{m t+1}^{t+1}$; that of paper money available at $t+2$, $\hat{p}_{m t+2}^{t+1}$; and the price of Q_3 available at $t+2$.

^{13/} The full implications of this assumption should be made very clear. The market forms price expectations only for the convention at $t+1$. Hence it is really assumed as expecting that the prices which will be established in the remaining future conventions are equal to the corresponding prices expected to be established at $t+1$. In general, a sequence of price systems which are expected at t to be established at $t+1, \dots$, should be formed. However, the certainty with which these price expectations could be held should diminish as we consider expected prices for periods further into the future, since more periods have to be included for the data of which nothing is known at t . Thus it is reasonable that there is a future period such that the price expectations from that period and on are static, i.e., such that the market at t holds the same price expectations for all time periods after that particular period. The real restriction imposed by our assumptions is that this future period is the $t+1$ period.

prices of all the goods to be traded at $t+1$.^{14/} These expected prices reflect, as much as they can, the different degrees of certainty, with which they are held; see below § 3.6.

For our purposes we do not need to introduce any particular assumptions about the generation of the price expectations; in general, they are based on (among other things) the equilibrium price systems of previous conventions and on the currently announced prices.

3.2. The consumers may regularly devote a part of their wealth to commodities available at $t+1$ with the purpose of selling them back in the next market convention. The only motive behind this speculative demand is the expectations of changes in the equilibrium price system at $t+1$ relatively to that at t . Provided that all commodity-price expectations are held with the same degree of certainty, there is no preference towards any particular commodity, since no gain through use is sought after. Therefore, the consumers will maximize the expected value of these "holdings" per accounting unit devoted to them. This clearly implies the assignment of all speculative demand to the purchase of the commodity (-ies) with the highest ratio of expected to actual price at time t .

^{14/} Namely, we follow the traditional route of Hicks [7, pp. 124-127], in trying to subsume all the information provided by the frequency distribution associated with each expected price in the estimate of the average expected price.

We are thus led to the consideration of the first of the market interest rates, which will be distinguished in our economy:

Definition 1: The short-term interest factor on commodities from t to $t+1$ is defined by ^{15/}

$$(1) \quad \beta_t = \max_{g \in N} \hat{p}_{g, t+1}^t / p_{g, t+1}^t, \quad \text{where } N = \{1, \dots, n\} .$$

The short-term interest rate on commodities from t to $t+1$ is given by

$$(2) \quad i_t = \beta_t - 1 .$$

Consequently, we may subsume the speculative demand of the consumers by introducing a fictitious good, Q_1 , available at $t+1$ and measured in terms of the accounting unit. The demand for Q_1 is embodied in those commodities for which the maximum in (1) is realized. Generally, if 1 accounting unit is devoted to demand of Q_1 , the number of units of Q_1 bought is equal to the number of accounting units expected to be earned at $t+1$, on the basis of the interest rate i_t , by selling the commodities bought for this purpose. Hence the price at t of Q_1 available at $t+1$ is

$$(3) \quad \Theta_1^t = 1/\beta_t .$$

3.3. Capital goods are held by the consumers for the sequence of services generated by them. Provided that the price expectations about capital goods available at $t+2$ are held with the same degree of certainty, all capital goods

^{15/} Namely, β_t is the maximum of the ratio of expected future spot prices to current future prices of all commodities. Its interpretation is simple: If a consumer gives one accounting unit at t , he can buy

$1/p_{g, t+1}^t$ units of C_g available at $t+1$. He then expects to get $\hat{p}_{g, t+1}^t / p_{g, t+1}^t = \beta_t$ accounting units at $t+1$ by selling the commodity in the market then, provided of course that C_g achieves the maximum in (1).

are identical from this standpoint irrespective of their physical properties.^{16/}
The demand for capital goods available at $t+2$ is thus a demand for the sequence of perpetual net rentals derived from them, which will begin to be paid at $t+1$. Hence, it is equivalent to the demand of a fictitious good Q_3 available at $t+1$ and measured in accounting units. The ownership of one unit of Q_3 entitles its owner to one accounting unit at $t+1$, ..., ad infinitum.^{17/}

It is clear that since a demand for D_h can only be a demand for Q_3 available at $t+1$, this demand will be directed only to those capital goods available at $t+2$ which are expected to yield the highest return. Thus at equilibrium these returns must be equal for all capital goods available at $t+2$. It is convenient not to consider the relevant phase of the \hat{t} atonnement process in each convention but to assume it conducted in such a way that these returns are always equal. Thus whenever we want to consider the demand for capital goods in general we can equivalently consider the demand for Q_3 available at $t+1$.

^{16/} See Walras [16, pp. 268-276], and Schumpeter [15, pp. 1016].

^{17/} Namely, Q_3 is like an annuity or perpetual bond. We may suppose that there exists a fictitious institution which owns the capital goods in behalf of the consumption units. These have in their possession at t a quantity of perpetuities equal in number to the quantity of Q_3 available at $t+1$ which they own at t . The institution offers in each convention the capital services which are then marketable and it uses the proceeds in order to pay the interest on the perpetuities which are outstanding. At the same time the institution offers to the market new perpetuities and uses the proceeds to demand capital goods.

If h_{t+1}^t is the price of one unit of Q_3 available at $t+1$, then

Definition 2: The long-term interest rate on capital goods at t , (for one period), is defined by ^{18/}

$$(4) \quad r_t = \frac{1}{h_{t+1}^t} .$$

However, the returns (in accounting units) which are secured by this demand for Q_3 depend on the prices which will be established at $t+1$, $t+2$,

Hence in our model -- where static expectations from $t+1$ and on are assumed -- these returns depend on the price system expected to be established at $t+1$.

Let then \hat{r}_{t+1} be the expected at t long-term interest rate on capital goods

for the $t+1$ period, and let $\hat{h}_{t+2}^{t+1} = \frac{1}{\hat{r}_{t+1}}$. The returns expected to be

derived from the ownership, (for one period), of one accounting unit's worth

of Q_3 are equal to $\frac{1 + h_{t+2}^{t+1}}{h_{t+1}^t}$ accounting units.

Definition 3 ^{19/}: The effective short-term interest rate on capital goods from t to $t+1$ is given by

$$(5) \quad \rho_t = \frac{1 + h_{t+2}^{t+1}}{h_{t+1}^t} - 1 .$$

^{18/} This is easily seen if we consider a consumer who devotes at t one accounting unit to the demand of Q_2 . He buys $1/h_{t+1}^t = r_t$ units of Q_3 available at $t+1$ and is thus entitled to r_t accounting units at $t+1$, ..., ad infinitum.

^{19/} See Hicks [7, p. 149].

It is ρ_t which is really important for the behavior of each consumer.

Since a corresponding quantity of Q_3 available at $t+2$ can always be purchased at $t+1$, each consumer is mainly interested in the effective short-term rate. Equivalently, a consumer is influenced in taking decisions not only by r_t but by \hat{r}_{t+1} also. Consequently, with respect to consumer decisions the units in which Q_3 is defined must be changed so that if one accounting unit is devoted to the demand of Q_3 , the number of units of Q_3 bought (as far as the consumers are concerned) is equal to the number,

$\frac{1 + \hat{h}_{t+1}}{h_{t+1}}$, of accounting units which are expected to be earned on the basis

of the effective interest rate ρ_t . Thus if $\Theta_{3t+1}^t = 1/(1 + \rho_t)$, Θ_{3t+1}^t is the price of one unit of Q_3 defined in these units which are appropriate for consumption decisions.

3.4. Paper money available at $t+1$ is demanded by the consumers because of the services that it can yield in the succeeding market conventions.

A consumer devoting at t one accounting unit buys $\frac{1}{P_{m t+1}^t}$ money units available at $t+1$. He then expects ^{20/} to receive at $t+1$

$\frac{\hat{p}_{m t+1}^{t+1} + \hat{p}_{m t+2}^{t+1}}{P_{m t+1}^t}$ accounting units.

^{20/} We assume again that the market forms collectively price expectations for the services of paper money at $t+1$ and for paper money at $t+2$, and moreover that these expectations are held with the same degree of certainty.

Again, we have:

Definition 4: The long-term money interest rate at t (for one period) is defined by

$$(6) \quad r_{m,t} = \frac{\hat{p}_{m,t+1}}{P_{m,t+1}^t},$$

the expected at t long-term money interest rate at $t+1$ is defined by

$$(7) \quad \hat{r}_{m,t+1} = \frac{\hat{p}_{m,t+1}}{\hat{p}_{m,t+2}^{t+1}},$$

and the effective short-term money interest rate from t to $t+1$ is defined by

$$(8) \quad \rho_{m,t} = r_{m,t} + \frac{r_{m,t}}{\hat{r}_{m,t+1}} - 1.$$

It is apparent that, similarly to the consumers' demand for Q_1 and Q_3 , it is the proceeds obtainable at $t+1$ which constitutes the object of the demand for paper money available at $t+1$. Therefore the units in which paper money available at $t+1$ is measured must be redefined in order to make it commensurate with consumer preferences. It is not a quantity of paper money available at $t+1$ which is the object of the demand by a consumer but a certain quantity of proceeds which is expected at t to be realized at $t+1$ by the ownership of this quantity of paper money. Thus we introduce another fictitious good, Q_2 available at $t+1$. Q_2 is measured in accounting units and is defined analogously with Q_1 and Q_3 . If one accounting unit is devoted to the demand at t of Q_2 , the number of units of Q_2 bought is equal to the number of accounting units which are expected to be earned at $t+1$ on the basis of $\rho_{m,t}$. Thus if $\theta_{2t+1}^t = \frac{1}{1 + \rho_{m,t}}$, θ_{2t+1}^t is the price of one unit of Q_2 .

3.5. Consumers' demand for Q_1 , Q_2 , and Q_3 available at $t+1$
versus their demands for all other commodities available at t :

The incorporation into the general theory of choice of those goods which are not demanded for their direct use in meeting consumption needs is now complete. We see that, whenever a good is demanded by the consumption units not for the satisfaction of consumption needs by its direct use but for other purposes, we have to find out what is the real object of this demand and then consider for each such distinct demand a corresponding fictitious good defined in appropriate units. Following this line of thought we see that the demand for paper money available at $t+1$ is but a part of the general category of a "demand for assets," along with the speculative demand and the demand for capital goods. We must carefully distinguish these demands from the demand for services of paper money during each convention. The rationale of this last demand is provided in the present model by our Payments Assumption.

3.6. In our presentation of the consumers' demands which are not directly made for consumption purposes we assumed (for expository purposes) that homogeneous markets for Q_1 and Q_3 exist. A basic objection to this assumption is that in reality different risks are associated with each commodity or capital good if they are held as means of furnishing purchasing power at $t+1$. Thus different interest rates should be distinguished. However, the question about how many such interest rates (or equivalently how many such fictitious goods) will be distinguished in an actual economy cannot of course be settled a priori; it depends on the actual circumstances.

Our treatment follows a middle course. Namely, we allow for the possibility that the degree of certainty with which our short-term interest rates, i_t , ρ_t , and $\rho_{m,t}$, are held by the market cannot be made the same.^{21/} This is recognized by the consumers, which thus prefer to diversify their holdings of purchasing power available at $t+1$, within certain limits imposed by the current differences among i_t , ρ_t , and ρ_{mt} .

3.7. We may also take the opportunity and comment on a subject to which much attention has been paid in the writings of Malinvaud [10] and Koopmans [8]. The problem is the place of the rate(s) of interest in an intertemporal general equilibrium model. If we consider an intertemporal planning model, namely an economy in which only one market convention takes place, then the concept of an interest rate is not actually needed for the analysis of a competitive allocation of the goods over the entire horizon. The reason is simply that the whole course of the price system throughout the -- finite or infinite -- horizon is simultaneously determined.^{22/}

^{21/} This is reasonable because essentially different factors will influence the establishment in the next market convention of the prices of the goods in these three categories. The determining influence on the prices of the commodities available at $t+1$, will be exercised by the demands of the production units, whereas that on the prices of the capital goods by the demands of the consumption units. Finally, although paper money is a capital good, it is a capital good performing a very special function, that of a medium of transactions. Money's general acceptability in exchange is the basic distinguishing characteristic of paper money from any other good. Thus the factors which are going to influence in the next convention the demand for services of paper money can very well behave differently from those influencing the demand for the commodities in which Q_1 is embodied, as well as from those influencing the demand for capital goods and services.

^{22/} It is of course possible to define a system of interest rates in such a model, but this only amounts to introducing a superfluous "superstructure of monetary concepts." For excellent discussions on this point see Debreu [2, pp. 32-35], Koopmans [8, pp. 113-115], and Malinvaud [10].

However, in a recursive model the situation is not the same. Here, the various interest rates play an important and natural role in the allocation of the traded goods between successive market conventions. Thus, each interest rate is uniquely related to a particular market in which a well-defined good is traded. Moreover, the system of interest rates which emerges in a recursive model is much more informationally efficient when compared with the systems of current future prices and expected future spot prices, from which it is derived.

4. THE EXISTENCE OF A COMPETITIVE EQUILIBRIUM IN EVERY MARKET CONVENTION

4.1 The general price system at t : The price vector

$$(9) \quad p = (P_t^t, P_{t+1}^t, P_{mt}^t, P_{mt+1}^t, h_{t+1}^t)$$

constitutes the announced price system at t . p consists of the prices of all traded goods, which are announced in each round of the tatonnement process during the convention at t .^{23/} Furthermore, the price vector

$$(10) \quad \hat{p} = (\hat{P}_{t+1}^{t+1}, \hat{P}_{1t+2}^{t+1}, \hat{P}_{mt+1}^{t+1}, \hat{P}_{mt+2}^{t+1}, \hat{h}_{t+2}^{t+1})$$

constitutes the price system expected at t to be established in the convention at t+1 . We assume that

(E) \hat{p} is a continuous, and positive homogeneous of degree one, function of p . Moreover, $\hat{p}(p) > \Theta$ for all $p \geq \Theta$.

Finally, the general price system at t,²⁰ includes all the announced and expected prices, as well as all other prices which are derived from them (as e.g. the prices Θ_{it+1}^t of Q_i , $i = 1, 2, 3$; or the prices of the various capital goods D_h available at t+2 , P_{ht+2}^t , which will be considered in § 4.8).^{24/}

4.2. The consumption units: The sth consumer's preferences describe his tastes with respect to his demands and offers at t . The consumer demands at

^{23/} $p \in R^G$, where R^G , the goods space at t , is an $(2n + 3)$ -dim Euclidean space.

^{24/} Since \hat{p} is a function of p , we will often speak about the price system at t referring to p .

t (a) bundles of commodities available at t, x_t^{st} , for consumption purposes; (b) services of paper money available at t, x_{mt}^{st} , which he is constrained to use in financing his action plan; and (c) quantities of Q_i available at t+1, q_{it+1}^{st} , $i = 1, 2, 3$, which are held for one period mainly for protection against an unknown future beyond the horizon.

A feasible action of the s^{th} consumer,

$$(11) \quad x^s = (x_t^{st}, x_{mt}^{st}, q_{it+1}^{st})$$

is called an action plan at t. The set X^s of all feasible actions of the s^{th} consumer is called his action set at t. X^s is contained in the positive orthant Ω ^{25/} of the action space R^a , an (n+4)-dim Euclidean space. We assume that,

(C₁) X^s is closed and convex.

The introduction of the demands under (b) and (c) creates a situation which would not exist if only demands for purely consumption purposes were considered. In the latter case we can assume as a first approximation the independence of each consumer's preferences from any other economic variable also determined in the convention at t. But when demands of goods not destined to be used directly for consumption purposes are introduced, there is a difference. Thus the adequacy of the quantity of services of paper money demanded at t depends on the prices of all goods in terms of that for paper money available at t+1. Similarly the extent to which services of paper money in the succeeding conventions have been secured (by the acquisition of a corresponding quantity of paper money available at t+1) depends on the prices

^{25/} Ω will denote the positive orthant of any particular Euclidean space under consideration.

which are expected to be established in the future. In our model, therefore, it depends on the expected prices for $t+1$. The same of course holds true with respect to the quantities of Q_1 and Q_3 demanded at t . Thus each consumer's preferences must refer to the "real" content of the various quantities of Q_i , $i = 1, 2, 3$, available at $t+1$ / ^{which are} demanded at t . Since this "real" content is determined by the price system p , each consumer's preferences depend on the price system in each market convention.

Furthermore, with the payments assumption introduced in § 2 each consumer operates under a financial restraint, in addition to his wealth restraint. He is also not supposed to have any preferences ^{26/} for services of paper money at t in excess of his strict transaction requirements.

These two special properties of each consumer's preferences are expressed in (C_6) and (C_7) below.

We thus assume that

(C_2) X^s is a completely ordered subset of R^a by a given preference ordering \succ_s^p , depending on the price system at t ^{27/};

^{26/} In other words, assuming that consumer preferences are described in terms of a "utility function", the services of paper money at t do not enter into his utility function.

^{27/} Since the preference ordering depends on p , we examine a particular case of that where consumer preferences are considered to be dependent on the action plans of the other consumption units as well as on prices. The general case is treated by McKenzie [11].

- (C₃) $\overset{p}{\succsim}_s$ is closed, i.e., for given $p^k \rightarrow p$, $x^k \rightarrow x$, $x'^k \rightarrow x'$, such that $x^k \overset{p^k}{\succsim}_s x'^k$ holds for all k , then $x \overset{p}{\succsim}_s x'$ 28/;
- (C₄) $\overset{p}{\succsim}_s$ is convex, i.e., given p , if for $x, x' \in X^s$, $x' \overset{p}{\succsim}_s x$, then $\lambda x + (1-\lambda) x' \overset{p}{\succsim}_s x$, for $0 < \lambda < 1$;
- (C₅) insatiability of $\overset{p}{\succsim}_s$, i.e., for any $x \in X^s$, there exists $x' \in X^s$ with $q_{it+1}^t = q_{it+1}^t$, $i = 1, 2, 3$, such that $x' \overset{p}{\succsim}_s x$;
- (C₆) absence of price illusion in $\overset{p}{\succsim}_s$, i.e., the ordering between any two action plans of the s^{th} consumer is not affected by a uniform change of the prices of all goods, with the prices of Q_i , $i = 1, 2, 3$, (and hence the interest rates) constant. 29/ 30/
- (C₇) absence of preference for services of paper money at t , i.e., for all $x \in X^s$, $x \overset{p}{\succsim}_s x'$, $x' \in X^s$, if all components of x' , except that referring to the services of paper money at t , are equal to those of x .

4.3. When the market convenes at t each consumer has just acquired a commodity bundle, \bar{x}_t^{st} , representing the part of the economy's capital services, primary factors, and final commodities, which are made available at t , and which the consumer bought at $t-1$. Moreover, each consumer knows exactly his endowment with primary factors at $t+1$, \bar{x}_{2t+1}^{st} .

28/ (C₃) extends for the case where $\overset{p}{\succsim}_s$ depends on p , the familiar postulate, which insures the continuity of consumer's preferences.

29/ When only uniform changes of prices are involved, (C₆) implies that consumer preferences refer only to the "real" demands for Q_i , $i = 1, 2, 3$.

30/ For a detailed discussion on the concept of "money illusion" which is involved here see Patinkin [14, pp. 23-24].

Each consumer is also the owner of capital goods, \bar{c}_{t+1}^{st} , available for yielding capital services from $t+1$ and on. In accordance with § 2.7, he is thought of as the owner of \bar{x}_{1t+1}^{st} ($=\bar{c}_{t+1}^{st}$) capital services at $t+1$, and of $\bar{c}_{t+2}^{st}=(1-\xi)\bar{c}_{t+1}^{st}$ capital goods available at $t+2$. Thus $\bar{x}_{t+1}^{st} = (\bar{x}_{1t+1}^{st}, \bar{x}_{2t+1}^{st}, \Theta)$ is the consumer's endowment with commodities available at $t+1$.

Similarly, each consumer carries to t a quantity of paper money available at t , \bar{c}_{mt}^{st} . He therefore has \bar{x}_{mt}^{st} ($=\bar{c}_{mt}^{st}$) units of services of paper money available at t , and \bar{c}_{mt+1}^{st} ($=\bar{c}_{mt}^{st}$) units of paper money available at $t+1$. Finally, all profits of the production units are distributed to the consumers. Let γ^{rs} be the proportion of the r^{th} producer's profit at t which is distributed to the s^{th} consumer; $\gamma^{rs} \geq 0$, and $\sum_s \gamma^{rs} = 1$ for all r .

For any price system p , the s^{th} consumer's wealth at t is given by 31/

$$(12) \quad w^s(p) = p_t^t \bar{x}_t^{st} + p_{t+1}^t \bar{x}_{t+1}^{st} + p_{mt}^t \bar{x}_{mt}^{st} + p_{mt+1}^t \bar{c}_{mt+1}^{st} + p_{t+2}^t \bar{c}_{t+2}^{st} \\ + \sum_r (\gamma^{rs} \times \text{profits of the } r^{th} \text{ producer}).$$

31/ We denote by $\bar{q}_{1t+1}^{st} = \frac{1}{\Theta_{1t+1}^t} (p_{t+1}^t \cdot \bar{x}_{t+1}^{st})$ the quantity of Q_1 , which the s^{th}

consumer owns in the beginning of the market convention at t , under the price system p . If the demand for Q_1 can be embodied in all commodities available at

$t+1$, then we immediately see from (1) that $\bar{q}_{1t+1}^{st} = \hat{p}_{t+1}^{t+1} \cdot \bar{x}_{t+1}^{st}$. We may also denote

by $\bar{q}_{2t+1}^{st} = \frac{1}{\Theta_{2t+1}^t} (p_{mt+1}^t \bar{c}_{mt+1}^{st})$, and $\bar{q}_{3t+1}^{st} = \frac{1}{\Theta_{3t+1}^t} (p_{t+2}^t \cdot \bar{c}_{t+2}^{st})$, the quantities

of Q_2 and Q_3 , which the s^{th} consumer has in effect (i.e., under the price

system p) carried from $t-1$. We will see below in §4.8 that $\bar{q}_{2t+1}^{st} = \hat{p}_{mt+1}^{t+1} \bar{c}_{mt+1}^{st}$, and $\bar{q}_{3t+1}^{st} = \hat{p}_{t+2}^{t+1} \bar{c}_{t+2}^{st}$, as it should be.

Any feasible action of the s^{th} consumer is attainable if, first, his wealth restraint, given by

$$(13) \quad p_t^t x_t^{st} + p_{mt}^t x_{mt}^{st} + \sum_{i=1}^3 \theta_{it+1}^t q_{it+1}^{st} \leq w^s(p),$$

is satisfied.^{32/} In addition, his financial restraint, given by^{33/}

$$(14) \quad x_{mt}^{st} \geq \frac{1}{P_{mt+1}^t} \left\{ p_t^t (x_t^{st} - \bar{x}_t^{st})' + \theta_{1t+1}^t (q_{1t+1}^{st} - \bar{q}_{1t+1}^{st})' + \theta_{3t+1}^t (q_{3t+1}^{st} - \bar{q}_{3t+1}^{st})' \right\}$$

must also be satisfied.

4.4. We can now introduce the first condition for a competitive equilibrium at t . Let $H^s(p) = \{x^s \in X^s \mid (13) \text{ and } (14) \text{ are satisfied}\}$ be the s^{th} consumer's attainable action set at p , and $C^s(p) = \{x^s \in X^s \mid x^s \succeq_s^p x^s \text{ for all } x^s \in H^s(p)\}$

^{32/} Let x_{t+1}^{st} , c_{mt+1}^{st} , and c_{t+2}^{st} , denote the commodities available at $t+1$, the paper money available at $t+1$, and the capital goods available at $t+2$, respectively into which the demands for Q_i , $i = 1, 2, 3$, are embodied. Then the wealth restraint is equivalently given by

$$(13') \quad p_t^t x_t^{st} + p_{mt}^t x_{mt}^{st} + p_{t+1}^t x_{t+1}^{st} + p_{mt+1}^t c_{mt+1}^t + p_{t+2}^t c_{t+2}^{st} \leq w^s(p)$$

^{33/} In (14) the dashes indicate that only excess demands are considered. Demand for services of paper money at t is created only when a consumer demands a good in quantities larger than his own endowment with it at t . Also, the second and third

terms in the RHS of (14) can be given equivalently by $p_{t+1}^t (x_{t+1}^{st} - \bar{x}_{t+1}^{st})'$ + $p_{t+2}^t (c_{t+2}^{st} - \bar{c}_{t+2}^{st})'$. Finally, it is clear that the financial restraint cannot include a term referring to paper money available at $t+1$. The demand for paper money available at $t+1$ is the only demand which by its very nature does not require any financing.

be his preferred set at p . Then,

(I) The s^{th} consumer, faced with p , selects an $x^s \in X^s$ such that $x^s \in H^s(p) \cap C^s(p)$.

Such an x^s is called an optimum action plan at t of the s^{th} consumer. Let $\zeta^s(p)$ consist of all optimum action plans of the s^{th} consumer at prices p ^{34/}. The correspondence ζ^s from R^g to X^s is the demand correspondence of the s^{th} consumer at t .

We now state certain properties of $H^s(p)$, $C^s(p)$, and $\zeta^s(p)$: ^{35/ 36/}

- (a) If X^s is bounded and if p is such that $\min p_a \cdot X^s < w^s(p)$, then H^s is continuous at p ^{37/};
- (b) If X^s is bounded and if p is such that $\min p_a \cdot X^s < w^s(p)$, then $H^s(p) \cap C^s(p) \neq \emptyset$ ^{38/};
- (c) If $x^s \in C^s(p)$, then $p_a \cdot x^s \geq w^s(p)$ ^{38/};
- (d) $C^s(p)$ is convex and closed, and $C(p) = \sum_s C^s(p)$ is closed; ^{38/}
- (e) If X^s is bounded and H^s is continuous at p , then ζ^s is upper semi-continuous at p ; ^{39/} finally it can be shown that
- (f) If ζ^s is upper semicontinuous at p , then x_{t+1}^{st} is upper semicontinuous at p .

^{34/} We must note that it is possible that, for some prices, there may not be any attainable actions plans of the s^{th} consumer. It is also possible that, even if attainable plans exist, optimum action plans do not exist. Conditions for the existence of attainable, and of optimum action plans are given below.

^{35/} In the following p_a denotes the price vector $(p_t^t, p_{mt}^t, \theta_{it+1}^t)$.

^{36/} We note that the condition " p_a is a continuous function of p " is indispensable for most of these properties. This condition is easily seen to result from (E); see § 4.7 below.

^{37/} See e.g. Debreu [2; pp. 67-68].

^{38/} See e.g. McKenzie [12; p. 68].

^{39/} See e.g. Debreu [2; p. 72].

4.5. The production units: The technological possibilities of production of the k capital goods and of the m final commodities by the r^{th} producer are described by his production set at t , T_t^r . T_t^r is the set of pairs, $(-u, v)$, of n -dim commodity vectors $u = (u_1, u_2, u_3)$, and $k+m$ -dim capital goods and final commodities vectors $v = (v_1, v_3)$, such that the production of the output vector $v \geq \Theta$ at $t+1$ from the input vector $u \geq \Theta$ at t is technically possible if and only if $(-u, v) \in T_t^r$. Each $(-u, v) \in T_t^r$ is called a production process at t of the r^{th} producer.

We assume the absence of external (dis-)economies between the various producers; thus $T_t = \sum_r T_t^r$ is the economy's production set at t . We also assume,

- (T_1) possibility of inaction, i.e., $\Theta \in T_t^r$;
- (T_2) T_t is closed and convex;
- (T_3) free disposal, i.e., if $(-u, v) \in T_t$ and $u' \geq u$, $v \geq v' \geq \Theta$, then $(-u', v') \in T_t$;
- (T_4) impossibility of free production, i.e., $(-\Theta, v) \in T_t$ implies that $v = \Theta$. ^{40/}

Finally, with any output vector v_{t+1} we may associate a commodities produced vector y_{t+1} (such that $y_{1 t+1} = v_{1 t+1}$, $y_{2 t+1} = \Theta$, $y_{3 t+1} = v_{3 t+1}$), and a capital goods produced vector $b_{t+2} (= (1-\xi) v_{1 t+1})$.

4.6. If p is the price system at t , and if the r^{th} producer selects

^{40/} We note that T_t^r exhibits by definition (time) irreversibility of production, since inputs are nonpositive and outputs are nonnegative.

$(-u, v) \in T_t^r$, then under our payments assumption he has to buy at t a quantity of services of paper money equal to the money value of the inputs, $\frac{1}{P_{mt+1}^t} (p_t^t \cdot u_t)$.

The accounting value of his demand for such services is thus $\frac{p_{mt}^t}{P_{mt+1}^t} (p_t^t \cdot u_t)$.

Then, $\left(\frac{p_{mt}^t}{P_{mt+1}^t} + 1 \right) (p_t^t \cdot u_t)$ is the total cost associated with the selected production process, and

$$(15) \quad p_{t+1}^t y_{t+1} + p_{t+2}^t b_{t+2} - \left(\frac{p_{mt}^t}{P_{mt+1}^t} + 1 \right) (p_t^t \cdot u_t)$$

is the profit of the r^{th} producer at t .

The second condition for a competitive equilibrium at t is

(II) The r^{th} producer, faced with p , selects a production process such that his profit is maximum over T_t^r .

This choice is called an optimum production plan at t of the r^{th} producer. Let $\pi^r(p)$ be his maximum profit at p , and let $\eta^r(p)$ consist of all optimum production plans of the r^{th} producer at prices p . The correspondence η^r from R^G to T_t^r is the supply correspondence of the r^{th} producer at t .

The following results are easily established:

(a) $(-u^r, v^r) \in T_t^r$ implies that $(-u^r, v^r) \in \eta^r(p)$, for all r , if and only if $(-u, v) = \sum_r (-u^r, v^r) \in T$ is an element of $\eta(p) = \sum_r \eta^r(p)$; and

(b) If T_t^r is compact, then for any $p \in R^G$, $\pi^r(p)$ are defined. Moreover, η^r is an upper semicontinuous correspondence and π^r is a continuous function. 41/

41/ See Debreu [2, p. 48].

4.7. Admissible prices and interest rates: It is advisable to restrict our attention only to nonnegative prices. If e.g. we assume that all goods can be immediately disposed without any cost, then all their prices are nonnegative. Moreover, (E) implies that then all expected prices are nonnegative. On the other hand (C₅) excludes the possibility that $P_n = (p_t^t, p_{t+1}^t, p_{mt}^t, p_{mt+1}^t) = \Theta$. Hence, we may normalize the announced price system $p = (p_n, h_{t+1}^t)$, so that

$$(16) \quad p_n \in P_n = \left\{ p_n \in \Omega \mid \sum_g p_{gt}^t + \sum_g p_{gt+1}^t + p_{mt}^t + p_{mt+1}^t = 1 \right\}.$$

Moreover, it is easily shown that the short-term interest factor on commodities, β_t , is a continuous function of p , with range $0 \leq \beta_t \leq +\infty$ ^{42/}.

Consequently, the price of Q_1 , θ_{1t+1}^t is a continuous function of p , and $0 \leq \theta_{1t+1}^t \leq +\infty$.

It is also seen that p_{mt} and θ_{2t+1}^t are continuous functions of p , with $-1 \leq p_{mt} \leq +\infty$, and $0 \leq \theta_{2t+1}^t \leq +\infty$.

Certain difficulties appear when we consider capital goods. In order to insure finite prices of the capital goods ^{43/} we introduce the following assumption:

^{42/} Thus the range of values of the rate i_t is $[-1, +\infty]$. We have no a priori reasons for excluding a negative short-term interest rate i_t . It may very well happen that consumers devote a part of their wealth to Q_1 , even though the expected value of their holdings is less than their present value.

^{43/} See § 4.8 below. The need for the following assumption (C₆) corresponds to the first reason given by Malinvaud [10, § 17] for the interest rate to be nonnegative.

(C₈) For each consumer there exists a $\rho^s > 0$ such that if the price system p implies an effective short-term interest rate on capital goods ρ_t less than or equal to ρ^s , then the consumer wishes to decumulate capital goods irrespective of the remaining prices ruling in the market. 45/ 46/

It is easily seen from the market equilibrium conditions in §4.11 below that (C₈) implies that we can confine our attention only to values of ρ_t greater than or equal to $\rho = \min_s \rho^s$, $\rho > 0$. Thus Θ_{3t+1}^t is such that $1 \leq \Theta_{3t+1}^t \leq +\infty$. Furthermore, (C₈) implies that $r_t \geq r > 0$.

Consequently, $h_{t+1}^t = \frac{1}{r_t}$ is contained in a compact interval $H = [0, h]$, where $h = \frac{1}{r}$. 47/

Naturally, one would expect $Q_1, Q_2,$ and Q_3 , to be close substitutes in the consumers' preferences, and thus that large deviations between $\Theta_{1t+1}^t, \Theta_{2t+1}^t$, and Θ_{3t+1}^t cannot subsist.

4.8. Capital goods prices: By our assumptions in § 2.7 a homogeneous capital goods market has been created. We have also assumed that the following relation holds throughout the tatonnement process,

$$(17) \quad \frac{1 + \hat{h}_{t+2}^{t+1}}{\hat{h}_{t+1}^t} = \frac{\hat{p}_{ht+2}^{t+1} + (1 - \hat{\xi}_h) \hat{p}_{ht+3}^{t+1}}{\hat{p}_{ht+2}^t}, \quad h = 1, \dots, k,$$

45/ Namely, if $\rho_t \leq \rho^s$, then $\Theta_{3t+1}^t (q_{3t+1}^{st} - \bar{q}_{3t+1}^{st}) \leq 0$ with the strict inequality if $\bar{q}_{3t+1}^{st} > 0$.

46/ Assumption (C₈) corresponds to assumption (iii) in Morishima [13]. In fact, it can be traced back to Walras [16, pp. 275, 531], for whom the function of aggregate savings in capital goods exhibited the analogous property.

47/ We may note that the compactness of H is indispensable for the proof of the existence theorem.

namely, that all capital goods available at $t+2$ are expected to yield equal returns in the next convention. By means of these assumptions both P_{t+2}^t and \hat{P}_{t+3}^{t+1} can be derived. Thus it can be shown that

$$\hat{P}_{ht+3}^{t+1} = \frac{\hat{h}_{t+2}^{t+1}}{1 + \xi_h \hat{h}_{t+2}^{t+1}} \hat{P}_{ht+2}^{t+1}, \quad h = 1, \dots, k, \quad \text{holds.}^{48/}$$

Then by (17) we see that

$$(18) \quad P_{ht+2}^t = \frac{h_{t+1}^t}{1 + \xi_h h_{t+2}^{t+1}} \hat{P}_{ht+2}^{t+1} \quad \text{holds.}$$

Finally, if we wish to consider a capital good, D_h , available at t or $t+1$, its price is derived from P_{ht+2}^t , and the prices of the capital services, P_{ht}^t , and P_{ht+1}^t , by means of ^{49/}

$$(19) \quad P_{ht+1}^t = P_{ht+1}^t + (1 - \xi_h) P_{ht+2}^t, \quad P_{ht}^t = P_{ht}^t + (1 - \xi_h) P_{ht+1}^t, \quad h = 1, \dots, k.$$

Consequently, all capital goods' prices are uniquely determined by the price system at t , p . Obviously, they also are continuous functions of p . ^{50/}

^{48/} The infinite series of discounted returns which is involved here is convergent provided that $\hat{r}_{t+1} > -\xi_h$, $h = 1, \dots, k$. This inequality is satisfied, since $r_{t+1} > 0$ by (C₈).

^{49/} Again (19) holds throughout the tatonnement process.

^{50/} Similar to (19) conditions are satisfied between the expected prices of the capital goods available at $t+1$ and $t+2$, and those for their corresponding services. Namely,

$$\hat{P}_{ht+2}^{t+1} = \hat{P}_{ht+2}^{t+1} + (1 - \xi_h) \hat{P}_{ht+3}^{t+1}, \quad \text{and} \quad \hat{P}_{ht+1}^{t+1} = \hat{P}_{ht+1}^{t+1} + (1 - \xi_h) \hat{P}_{ht+2}^{t+1}.$$

Then, one immediately sees that

$$\hat{P}_{ht+2}^{t+1} = \frac{1 + \hat{h}_{t+2}^{t+1}}{1 + \xi_h \hat{h}_{t+2}^{t+1}} \hat{P}_{ht+2}^{t+1} = \frac{1}{\Theta_{3t+1}^t} P_{ht+2}^t \quad \text{holds true.}$$

4.9. Market Equilibrium: In order to analyse the behaviour of consumers and producers we considered all goods which enter into their action or production sets defined in appropriate units so as to be made commensurate with consumer preferences or production decisions. In the market however all goods are considered in their physical units. Thus e.g. the market cannot recognize any demand for Q_1 , like q_{1t+1}^{st} , but only the demands for the goods in which q_{1t+1}^{st} is embodied.^{51/}

Therefore to each action plan x^s there corresponds a derived demand $d^s = (x_t^{st}, x_{t+1}^{st}, x_{mt}^{st}, c_{mt+1}^{st}, q_{3t+1}^{st})$. The set of all d^s 's is the derived demand set D^s of the s^{th} consumer. D^s is contained in the positive orthant of R^6 and is closed and convex.

Let $\delta^s(p) = \{ d^s \in D^s \mid d^s \in d^s(\zeta^s(p)) \}$. $\delta^s(p)$ consists of all derived demands corresponding to optimal action plans of the s^{th} consumer. The correspondence δ^s

^{51/} Similarly, to a demand for Q_2 , q_{2t+1}^{st} , there corresponds a demand for paper money available at $t+1$, $c_{mt+1}^{st} = \frac{1}{\hat{p}_{mt+1}^{t+1} + \hat{p}_{mt+2}^{t+1}} q_{2t+1}^{st}$.

Obviously, $\theta_{2t+1}^t q_{2t+1}^{st} = p_{mt+1}^t c_{mt+1}^{st}$.

Finally, Q_3 is measured in its original units (each one of which entitles its owner to one accounting unit at $t+1, t+2, \dots$), and thus to q_{3t+1}^{st} there corresponds a

demand for $q_{3t+1}^{st} = \frac{1}{1 + h_{t+2}^{t+1}} q_{3t+1}^{st}$ "physical" units of Q_3 . Again

$\theta_{3t+1}^t q_{3t+1}^{st} = h_{t+1}^t q_{3t+1}^{st}$ holds true, and thus the definitions are consistent."

from R^G to $D^S \subset R^G$ is the derived demand correspondence of the s^{th} consumer.

It is easily seen that δ^S is upper semicontinuous at p , if ζ^S is such.

Similarly, to each $(-u, v) \in T_t^r$ there corresponds a derived supply s^r defined by

$$s^r = (-u_t, y_{t+1}, -\frac{1}{p_{mt+1}} (p_t^t \cdot u_t), 0, r_t(p_{t+2}^t \cdot b_{t+2})) .$$

s^r represents the net offer of commodities available at t and $t+1$, of services of paper money available at t , of paper money available at $t+1$, and of Q_3 available at $t+1$, which is in effect made by the r^{th} producer by operating the process $(-u, v)$.

Let S^r be the derived supply set of the r^{th} producer, and $\sigma^r(p) = \{s^r \in S^r \mid s^r \in s^r(\eta^r(p))\}$. σ^r from R^G to $S^r \subset R^G$ is the derived demand correspondence of the r^{th} producer. S^r and $S = \sum_r S^r$ have the following properties, which follow from (T_1) , (T_2) , and (T_4) :

$$(S_1) \quad \emptyset \in S^r ;$$

$$(S_2) \quad S \text{ is a closed, convex subset of } R^G ;$$

$$(S_4) \quad S \cap \Omega = \{\emptyset\} .$$

We also have, in accordance with the remarks in § 4. 7,

$$(S_3) \quad S \supset (-\Omega) : \text{ (Free disposal) } .$$

4.10. Every price system p satisfying (II) must be contained in the polar

cone of the asymptotic cone of S ^{52/}, $(AS)^*$.

Since $S \supset AS \supset -\Omega$ holds, $\Omega \supset (AS)^*$. Also, since $S \cap \Omega = \{\emptyset\}$, $(AS)^* \cap (\text{Int } \Omega) \neq \emptyset$ ^{53/}.

Letting $P = (AS)^* \cap (P_n XH)$ we can easily show that P is a non-empty, compact, and convex subset of the positive orthant of R^G . Thus in the search for an equilibrium at t , we can restrict the announced price system so that

$$(III) \quad p \in P = (AS)^* \cap (P_n XH).$$

Moreover, the continuity properties of the demand correspondence stated above in § 4.4 depend on the adequacy of each consumer's endowment at t , i.e. on having $\min p_a \cdot X^S < w^S(p)$ satisfied. Several assumptions insuring this condition have been considered in the literature. Here we assume

$$(C_9) \quad \text{For every consumer, } D^S \cap (\text{Int } AS + \{w^S\}) \neq \emptyset,$$

where $w^S = (\bar{x}_t^{st}, \bar{x}_{t+1}^{st}, \bar{x}_{mt}^{st}, \bar{c}_{mt+1}^{st}, \bar{q}_{\bar{z}t+1}^{st})$ denotes the resources of the s^{th} consumer at t ^{54/}. Namely, each consumer can subsist for one period (without spending

^{52/} For the concept of the asymptotic cone of a set see Debreu [2, p. 22]; given a set X in a vector space V , the asymptotic cone of X is defined by $AX = \bigcap_{k > 0} \Gamma(X^k)$, where $X^k = \{x \in X \mid \|x\| \geq k\}$ and $\Gamma(X^k)$ is the least closed cone with vertex zero containing X^k . The polar cone of a closed and convex cone C in V , is defined by $C^* = \{v \in V \mid v \cdot c \leq 0 \text{ for every } c \in C\}$; see e.g. Fenchel [6, pp. 9-10].

^{53/} See e.g. Koopmans and Bausch [9, p. 97].

^{54/} The assumption made here is very restrictive since it requires each consumer to possess a positive quantity of at least each primary factor available at t , $t+1$, and each capital service and final commodity available at t . Various weaker assumptions have been considered in Arrow and Debreu [1], McKenzie [11], [12], Debreu [4]. The existence proof can be carried through on the basis of either of them, but it becomes more complicated.

all his wealth) even in isolation from other consumers on the basis of his endowment and of the technological possibilities of the economy. This is easily seen if we consider a derived demand $d^s \in D^s \cap (\text{Int } AS + \{w^s\})$. Then, $p \cdot d^s = p \cdot s + p \cdot w^s$ where, $s \in \text{Int}(AS)$, and $p \in P$. Since $p \cdot s < 0$, and $p \cdot w^s = w^s(p)$, then $p \cdot d^s = p \cdot s + p \cdot w^s < w^s(p)$ for some $x^s \in X^s$.

Let $X = \sum_s X^s$, $D = \sum_s D^s$, $T_t = \sum_r T_t^r$, and $w = \sum_s w^s$. For every $x \in X$, and $(-u, v) \in T_t$, the corresponding point $d - s - w$ of R^E , $d \in D$, $s \in S$, is denoted by e , the excess demand at t . Given $p \in P$, for every $x \in \zeta(p)$, $(-u, v) \in \eta(p)$, the excess demand e is a point of the set $\epsilon(p) = \delta(p) - \sigma(p) - \{w\}$, $\epsilon(p) \subset E = D - S - \{w\}$. The correspondence ϵ from P to E is the excess demand correspondence at t .

4.11. Market equilibrium conditions: The market for any good is considered to be in equilibrium when the excess demand for that good is equal to zero. Thus:

$$(IV) \quad e = \Theta, \quad \text{or} \quad d - s = w.$$

Actually, since free disposal of all goods is permitted, the excess demand for any good can be negative in equilibrium if its price is zero. Hence,

$$(20) \quad e \leq \Theta, \quad \text{and} \quad p \cdot e = 0,$$

holds in equilibrium. These market equilibrium conditions are given in more detail by ^{55/}

$$x_t^t + u_t - \bar{x}_t^t \leq \Theta, \quad (\text{commodities at } t),$$

$$x_{t+1}^t - y_{t+1}^t - \bar{x}_{t+1}^t \leq \Theta, \quad (\text{commodities at } t+1),$$

^{55/} x_{mt}^t is the aggregate of all consumer demands for services of paper money at t . Thus x_{mt}^t is greater than or equal to the aggregate of the financial restraints given by (14).

$$(21) \quad x_{mt}^t + \frac{1}{P_{mt+1}^t} (p_t^t \cdot u_t) - \bar{x}_{mt}^t \leq 0, \quad (\text{services of money at } t),$$

$$c_{mt+1}^t - \bar{c}_{mt+1}^t \leq 0, \quad (\text{paper money at } t+1),$$

$$(q_{jt+1}^t - \bar{q}_{jt+1}^t) - r_t (P_{t+2}^t \cdot b_{t+2}) \leq 0, \quad (\text{"capital goods" at } t+1).$$

On the other hand, for any $p \in P$ and any $x^s \in \zeta^s(p)$, and $(-u, v) \in \eta^r(p)$,

$$(22) \quad p \cdot d^s \leq p \cdot w^s + \sum_r \gamma^{rs} p \cdot s^r,$$

holds. Consequently, $p \cdot d \leq p \cdot w + p \cdot s$, or $p \cdot e \leq 0$, is also satisfied, with the equality holding whenever no consumer is satiated. This equality,

$$(23) \quad p \cdot e = 0,$$

has been named Walras' Law. (23) holds throughout the $\hat{\text{tatonnement}}$ process at t . Therefore, as far as the "market" is concerned, Walras' Law is an identity.

4.12. Existence of a competitive equilibrium:

Def.: A set of vectors $(x^1, \dots, x^s, (-u^1, v^1), \dots, (-u^r, v^r), p)$ is said to be a competitive equilibrium at t if conditions (I) - (IV) are satisfied.

The existence of a competitive equilibrium for the present model can be established analogously to the existence theorems in McKenzie [9], or Debreu [4].

The theorem is based on the following

Lemma: The set $D \cap (S + \{w\})$ is compact.

Proof: It follows that in McKenzie [12; pp. 61-62].

Thus all equilibrium action plans, (i.e., action plans which are optimal with respect to each consumer and such that all equilibrium conditions can be satisfied), lie

in a bounded region, and we have:

Theorem: The economy has a competitive equilibrium at t if $(C_1) - (C_9)$, $(S_1) - (S_4)$, and (E) hold.

This theorem is a special case of the theorem in Debreu [4] and [3]. However, due to our relatively stronger assumptions, its proof is much simpler than that in Debreu.

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