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ON AN ASYMPTOTIC NON-SUBSTITUTION THEOREM IN THE TWO-SECTOR CLOSED PRODUCTION MODEL

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ERRATA

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p. 19, line 6	$p^1 = \Pi^1(p^o)$, x_1^1	should read:	$p^1 = \pi^1(p^o)$, x_1^1
last line	$p^T = \Pi^T(p^o)$.	" "	$p^T = \pi^T(p^o)$.
p. 20, line 5	${}_1^o(p^o)$,	" "	$\chi_1^o(p^o)$,
p. 29, line 1	${}_1^o(\underline{p}^{o1}) = y^o$	" "	$\chi_1^o(\underline{p}^{o1}) = y^o$
line 2	${}_2^o(\bar{p}^{o1}) = y^o$	" "	$\chi_2^o(\bar{p}^{o1}) = y^o$
line 8	$\psi^2(p^o)$,	" "	$\equiv \psi^2(p^o)$,
p. 30, line 4	$[\underline{p}^{o1} , \bar{p}^{o1}|$ holds	" "	$[\underline{p}^{o1}, \bar{p}^{o1}]$ holds.
last line	$[\pi^T(\underline{p}^{oT}) , \pi^T(\bar{p}^{oT})$,	" "	$[\pi^T(\underline{p}^{oT}) , \pi^T(\bar{p}^{oT})]$,
p. 31, last line	if $\Pi^{t+1}(p^t)$	" "	if $\pi^{t+1}(p^t)$
p. 33, line 2	$= \pi^{t+1}(p^t)$	" "	$= \Pi^{t+1}(p^t)$
line 9	$[\log \underline{p}^{oT} , \log \bar{p}^{oT})$,	" "	$[\log \underline{p}^{oT}, \log \bar{p}^{oT}]$,
last line	I_T^o .	" "	$\subset I_T^o$.
p. 36, line 3	After word "large" change semi-colon to comma.		
line -2	$= \Pi^{t+1}(p^t)$	should read:	$= \pi^{t+1}(p^t)$
line -2	P_t^*	" "	p_t^*

p. 42, line 8	$p^{t+1} = 2(p^t)^{-1/3} p^t$.	should read	$p^{t+1} = 2(p^t)^{-1/3}$.
p. 45, line 7	Thus $p^{*t} =$	" "	Thus $p_t^* =$
p. 48, line 8	uncouraging	" "	encouraging
p. 54, line 2	$x_i^o = \ {}_i^o(p^{o''})$,	" "	$x_i^o = \chi_i^o(p^{o''})$,
line 3	$y^1 = \dfrac{f_{10}(x_1^o)}{f_{20}(x_2^o)} = \dfrac{\ell_1^o}{\ell_2^o}$,	" "	$y^1 = \dfrac{f_{10}(x_1^o)}{f_{20}(x_2^o)} \dfrac{\ell_1^o}{\ell_2^o}$,

ON AN ASYMPTOTIC NON-SUBSTITUTION THEOREM IN THE TWO-SECTOR CLOSED PRODUCTION MODEL **

Emmanuel M. Drandakis*

1. INTRODUCTION

1.1. In this paper we consider a two-sector closed production model with a time structure. In each time period the available quantities of the two goods are used for the production of the same goods available in the next period. Production conditions may change from period to period, but they are known throughout the horizon under consideration. Each good is produced separately under constant-returns-to-scale and diminishing rates of input substitution, and there are no external (dis-)economies.

We focus our attention to <u>T-period accumulation programs</u>. E.g., we may assume that the proportions in which the two goods will be available

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at time T are exogenously prescribed and aim at maximizing the quantities of both goods at T in these proportions, given the initial endowment at time 0. This will be called a forward T-period program. On the other hand, we may inquire about the minimal quantities of the two goods at time 0, which are required for the availability at T of given quantities of the two goods. This will be called a backward T-period program.

1.2. Our main objective is the examination of the shape of the t-period, $t = 1, 2, \ldots,$ efficient attainable production set (given the initial endowment at time 0), i.e., the shape of the envelope of all t-period production-possibility loci. Any point on a t-period envelope is producible from the initial endowment at 0, and any movement to another point on it necessitates a reduction in the quantity produced of one of the goods. These envelopes are described by concave curves (towards the origin) in the two-dimensional commodity space.

We will establish that the t-period envelope tends to become less concave as t increases, and moreover that it converges to a straight line as $t \to + \infty$. Equivalently, we will show that the range of the t-period price-ratios, at which an "interior" efficient accumulation program for

t periods can be carried out, tends to decrease as t increases.[1/]

1/ In the special case of a constant technology one can show that as $t \to +\infty$ this range of t-period price-ratios converges to a unique price-ratio, which is the von Neumann price-ratio.

If all price-ratios throughout the program horizon are set equal to the von Neumann price-ratio, then a balanced efficient accumulation program is feasible (provided of course that the initial endowment is also appropriate).

1.3. Let us briefly indicate the main reasons for the rôle of the length of the program horizon on the concavity of the envelope of the last period's production-possibility loci. The 1-period envelope coincides with the production-possibility locus determined by the initial endowment. This locus is described by a strictly concave curve, if the production conditions in the two sectors are essentially different.

Considering any 1-period program, a specific change in the output proportions will correspond to a certain change in the output price-ratio, depending on the initial endowment as well as on the difference in the production conditions in the two sectors.

If on the other hand, we consider a T-period program, with T fairly large, the same change in the final output proportions will correspond to a much smaller change in the final price-ratio. For, in an efficient accumulation program, any change in the final price-ratio is associated with appropriate changes in the price-ratios and thus in the output proportions in all previous periods. Thus the limitations imposed by the initial endowment at time 0 , and by differences in the production conditions in the two sectors in the possibilities of output substitution at the end of the program horizon become less severe as the horizon becomes longer.2/ The above argument also shows that

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2/ It appears that this result is a manifestation of the Le Chatelier principle of Thermodynamics; see Samuelson [8 , pp. 36-39]. As the program horizon increases the constraints imposed by the initial endowment and differences in the production conitions in the two sectors are weakened and thus the change in the final output proportions resulting from any change in the final price-ratio becomes larger.

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constancy of the technology in successive periods is not needed for establishing this result. What is needed is the existence of some production possibilities in every period (and not merely the availability of free storage for both goods).

1.4. The title of the paper obviously refers to the well known non-substitution theorem of Samuelson [9].[3] It is easily seen why the

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[3] In [9] an n-sector model is considered, in which the n goods are produced separately by the use of the same goods and of a primary factor, labor, under constant-returns-to-scale. Samuelson showed that the prices of all the goods in terms of the wage rate are determined by technological conditions alone. Since the input proportions in each sector are determined by these prices, they are constant and cannot be altered by any change in the final demand. The state of final demand determines only the relative importance (i.e., the level of operation) of each sector.

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present theorem can be classified as an asymptotic non-substitution theorem[4]

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[4] The observation that the asymptotic flattening of the t-period envelope as $t \to +\infty$ corresponds to an asymptotic non-substitution theorem is due to Professor T. Koopmans. See however Section 1.5.

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analogous to the static non-substitution theorem of Samuelson. If the program horizon is fairly long, the price-ratios in the time periods near the end of the horizon are almost constant (for each period) and they are determined by technology alone. Thus the input proportions in both sectors, which are determined by the price-ratio of the same period, are almost constant (for each period) in these last periods. Consequently, no change in the final output proportions can result in an appreciable change in prices and input proportions in the last periods. Changes in the final output proportions can only affect the relative importance of the two sectors in the last periods.

1.5. The above non-substitution theorem is not of course true only in a two-sector model. It also holds in an n-sector generalized Leontief model with neoclassical production functions. The proof of the theorem depends on a basic property of the sequence of successive inter-temporally efficient prices which is manifested in a model with no joint production. E.g., in the case of constant technology this sequence converges to the von Neumann price vector. This result has been already proved by e.g. Morishima [7], McKenzie [6], and Uzawa [10]. However, in [7], [6], [10], attention is focused on the turnpike property of efficient accumulation paths, rather than on a direct examination of the shape of the loci of all such efficient paths in successive time periods.

On the other hand, reference must also be made to a paper by Hicks [3]. In [3] a remarkable combination of solid economic intuition and of simple mathematical argument is exhibited in proving the turnpike theorem in the two sector model with Cobb-Douglas production functions. Hicks also describes in detail the tendency of the t-period envelopes to become straightlines. I am only sorry that I was not fully acquainted with [3] when I was working on this paper. As it now stands the present paper is a modest extension of [3] for more general production functions and for changing production conditions from period to period.

2. THE TWO-SECTOR CLOSED PRODUCTION MODEL

2.1. We consider a two-sector production model in which two goods are produced separately, with a uniform lag of one time period, under constant-returns-to-scale, positive and diminishing rates of input substitution, and no external (dis-)economies. The conditions of production may be different in successive periods, but they are known throughout the horizon. The model is closed because any consumption of the two goods which is not purely exogenous is assumed away. Exogenous consumption can be handled without any difficulty. For simplicity however, we assume that there is no even exogenous consumption.

2.2. The production possibilities in each time period are described by two production functions

$$Y_i^{t+1} = F^{it}(X_{1i}^t, X_{2i}^t) \equiv F^{it}(X_i^t) \qquad i = 1, 2, \quad t = 0, 1, 2, \dots , \tag{1}$$

where Y_i^{t+1} is the quantity of the i^{th} good at $t+1$ produced at t by the use of X_{1i}^t and X_{2i}^t units of the first and second good, respectively, available at t .

The functions 5/ F^i, defined for all nonnegative inputs $X_i = (X_{1i}, X_{2i})$,

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5/ The time superscript is suppressed whenever no explicit reference to a specific time period is made.

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have the following properties:

(2)

(a) $F^i(X_i) \left\{ \begin{matrix} = \\ \geqq \\ > \end{matrix} \right\} 0$ for $X_i \left\{ \begin{matrix} = \\ \geq \\ > \end{matrix} \right\} \Theta$;

(b) F^i is twice continuously differentiable;

(c) $F^i(\lambda X_i) = \lambda F^i(X_i)$, for all $\lambda \geqq 0$ and $X_i \geqq \Theta$;

(d) $F^i_j(X_i) > 0$, $j = 1, 2$, for all $X_i > \Theta$; and

(e) $F^i(\mu X_i + (1-\mu) X_i') > \mu F^i(X_i) + (1-\mu) F^i(X_i')$, for all $1 > \mu > 0$, and all nonproportional $X_i, X_i' \geq \Theta$. 6/

6/ Namely, F^i is a strictly concave function for nonproportional inputs.

Then, $F^{it}(X_{1i}^t, X_{2i}^t) = X_{2i}^t F^{it}(X_{1i}^t/X_{2i}^t, 1) \equiv X_{2i}^t f_{it}(x_i^t)$, for $X_{2i}^t > 0$,

where $x_i^t = \dfrac{X_{1i}^t}{X_{2t}^t}$ denotes the input proportions in the i^{th} sector, and

f_{it} is defined as above.[7]

7/ We see from (2) that f_i, defined for all nonnegative x_i, is twice continuously differentiable, that $f_i(x_i) \begin{pmatrix} \geqq \\ > \end{pmatrix} 0$ for $x_i \begin{pmatrix} = \\ > \end{pmatrix} 0$, and that $f_i'(x_i) > 0$, $f_i(x_i) - x_i f_i'(x_i) > 0$, $f_i''(x_i) < 0$, for all $x_i > 0$.

2.3. Considering the <u>forward T-period programming problem</u>, let $Y^o = (Y_1^o, Y_2^o) > \Theta$ denote the quantities of the two goods available to the economy at time 0 , and let $y_i^T \geqq 0$, $i = 1, 2,$ $y_1^T + y_2^T = 1$, be the prescribed configuration of the two goods at time T . The quantities of the two goods, which are available to the economy in each time period, namely, Y_i^t , $t = 0, 1, \ldots , T - 1$, are allocated to the two sectors for further production purposes. We assume that both goods are freely transferable as inputs from one sector to the other.

Thus, in every feasible accumulation program for T periods, the following inequalities hold:

$$X_{i1}^{t} + X_{i2}^{t} \leqq Y_{i}^{t} ,$$

$$Y_{i}^{t} \geqq 0 , \quad X_{ij}^{t} \geqq 0 , \quad i, j = 1, 2, \quad t = 0, 1, \ldots, T - 1$$

2.4. An efficient accumulation program starting from Y_i^o, and having outputs at T in the prescribed proportions, y_i^T, $i = 1, 2$, can be found as a solution to the following programming problem (I):

Maximize μ

subject to

$$
\text{(I)} \qquad
\begin{aligned}
&X_{11}^{o} + X_{12}^{o} \leqq Y_{1}^{o} \\
&X_{21}^{o} + X_{22}^{o} \leqq Y_{2}^{o} \\
&X_{11}^{1} + X_{12}^{1} \leqq Y_{1}^{1} = F^{1o}(X_{11}^{o}, X^{o}_{21}) \\
&X_{21}^{1} + X_{22}^{1} \leqq Y_{2}^{1} = F^{2o}(X_{12}^{o}, X_{22}^{o}) \\
&- - - - - - - - - - - - - - - - - \\
&\mu\, y_{1}^{T} \leqq Y_{1}^{T} = F^{1T-1}(X_{11}^{T-1} , X_{21}^{T-1}) \\
&\mu\, y_{2}^{T} \leqq Y_{2}^{T} = F^{2T-1}(X_{12}^{T-1} , X_{22}^{T-1})
\end{aligned}
$$

and $\qquad X_{ij}^{t} \geqq 0 , \quad i, j = 1, 2, \quad t = 0, 1, \ldots , T - 1 .$

The above programming problem is a problem of concave programming in linear spaces. The existence of a solution to (I) is insured by our assumptions.[8] Furthermore, the Kuhn and Tucker results [9] provide us with

8/ Namely, that $Y_1^o, Y_2^o < +\infty$, $T < +\infty$, $0 \leqq y_1^T, y_2^T$ and $y_1^T + y_2^T = 1$, and the continuity of the functions F^{it}.

9/ See Kuhn and Tucker [5], and Uzawa [12].

necessary and sufficient conditions for a solution to (I).

The Lagrangean function associated with (I) is defined by:

(3)
$$L(X_{ij}^t, p_i^t) = \mu + p_1^o(Y_1^o - X_{11}^o - X_{12}^o) + p_2^o(Y_2^o - X_{21}^o - X_{22}^o)$$
$$+ \dots + p_1^T\,[F^{1T-1}(X_{11}^{T-1}, X_{21}^{T-1}) - \mu\, y_1^T] + p_2^T\,[F^{2T-1}(X_{12}^{T-1}, X_{22}^{T-1})$$
$$- \mu\, y_2^T] \,.$$

Since all F^{it} are concave functions, a particular (X_{ij}^t), $X_{ij}^t \geqq 0$, $i, j = 1, 2$, $t = 0, \dots, T-1$, achieves the maximum if and only if there exists (p_i^t), $p_i^t \geqq 0$, $i = 1, 2,$ $t = 0, \ldots, T,$ such that the following intertemporal efficiency conditions are satisfied: 10/

(4) $$p_j^t \geqq p_i^{t+1} F_j^{it}(X_{1i}^t, X_{2i}^t) ,$$

with equality if $X_{ji}^t > 0$, $i, j = 1, 2,$ $t = 0, \ldots, T - 1$,

and

(5) $$X_{11}^o + X_{12}^o \leqq Y_1^o$$

$$- - - - - - - - - -$$

$$\mu\, y_2^T \leqq F^{2T-1}(X_{12}^{T-1}, X_{22}^{T-1})$$

with equality if $p_i^t > 0$. $t = 0, \ldots, T\text{-}1$.

10/ See Kuhn and Tucker [5 , Theorem 3], and Uzawa [12, Theorem 2]. We note that, in accordance with our assumptions, there exists a feasible (X_{ij}^t) such that the restraints are satisfied as strict inequalities. Thus the Slater condition is satisfied and consequently we may apply Theorem 2 of Uzawa and then Lemma 1 and 2 of Kuhn and Tucker.

3. EXISTENCE OF A UNIQUE AND POSITIVE EFFICIENT ACCUMULATION PROGRAM FOR T PERIODS

3.1. We observe initially that, because of the strict concavity of the functions F^{it} for nonproportional inputs, the solution to (5) is unique and such that all conditions in (5) are satisfied as equalities. We will see in Section 4 that the conditions in (4) are also satisfied as equalities.

Let now $y^t = \dfrac{Y_1^t}{Y_2^t}$, $p^t = \dfrac{p_2^t}{p_1^t}$, $(t = 0, \ldots, T)$, and

$x_i^t = \dfrac{X_{1i}^t}{X_{2i}^t}$, $\ell_i^t = \dfrac{X_{2i}^t}{Y_2^t}$, $(t = 0, \ldots, T - 1)$. 11/ The intertemporal

11/ If any of the denominators in these ratios is zero, that ratio is defined to be equal to $+\infty$.

efficiency conditions (4) and (5) become:

$$p^o = \frac{f_{io}(x_i^o)}{f'_{io}(x_i^o)} - x_i^o , \quad i = 1, 2,$$

(6) $$p^1 = \frac{f_{i1}(x_i^1)}{f'_{i1}(x_i^1)} - x_i^1 = \frac{f'_{1o}(x_1^o)}{f'_{2o}(x_2^o)} , \quad i = 1, 2,$$

- -

$$p^T = \frac{f'_{1T-1}(x_1^{T-1})}{f'_{2T-1}(x_2^{T-1})} ,$$

and

$$x_1^o \ \ell_1^o + x_2^o \ \ell_2^o = y^o ,$$

(7) $$\ell_1^o + \ell_2^o \quad = 1 ,$$

$$- - - - - - - - - - -$$

$$x_1^t \ \ell_1^t + \ell_2^t \ x_2^t = y^t = \frac{f_{1t-1}(x_1^{t-1})}{f_{2t-1}(x_2^{t-1})} \ \frac{\ell_1^{t-1}}{\ell_2^{t-1}} ,$$

$$\ell_1^t + \ell_2^t = 1 ,$$

$$- - - - - - - - - - - - - - -$$

$$y^T = \frac{f_{1T-1}(x_1^{T-1})}{f_{2T-1}(x_2^{T-1})} \ \frac{\ell_1^{T-1}}{\ell_2^{T-1}} ,$$

where y^o and y^T are exogenously given and $y^o > 0$. The conditions (6) (as well as (7)) are exactly the intertemporal efficiency conditions which were first considered in the celebrated Chapter 12 of Dosso [1 , pp. 310-316]. Namely, the condition $p^t = \dfrac{f'_{1t-1}(x_1^{t-1})}{f'_{2t-1}(x_2^{t-1})}$ refers to the rate-of-output-transformation between the two goods considered as outputs emerging at t. On the other hand, the conditions $p^t = \frac{f_{it}(x_i^t)}{f'_{it}(x_i^t)} - x_i^t$, $i = 1, 2$, refer to the rate-of-input-substitution between the two goods considered as inputs to the subsequent production at t in the two sectors. We thus see that for intertemporally efficient production the ROT between the two goods at t must be equal to their RIS at t in both sectors. This is illustrated in Figures 1 and 2. In Figure 1 the curve E^t shows the envelope of all t-period production-possibility loci, whereas E^{t+1} shows the t+1-period envelope. The curve L_A shows a particular t+1-period production-possibility locus, that one producible from the point A. On the other hand, the curve I_B describes the aggregate input isoquant for the production of B. The derivation of the isoquant I_B from the individual isoquants I_{B_1} and I_{B_2} is illustrated in Figure 2. We see in Figure 1 that the intertemporal efficiency conditions for the t^{th} period are satisfied at the point A, where the t-period envelope E_t is tangential to the aggregate input isoquant for the production of B at $t+1$. $\frac{f'_{1t}(x_1^t)}{f'_{2t}(x_2^t)}$ is equal to the slope of E_t, whereas $\frac{f_{it}(x_i)}{f'_{it}(x_i)} - x_i$, $i = 1, 2$, is equal to the slope

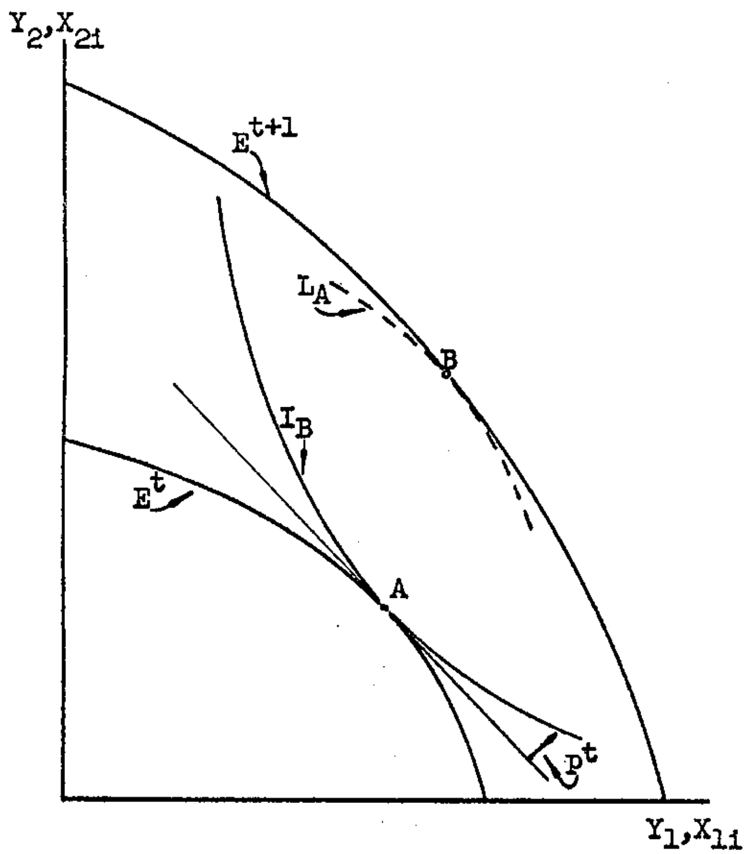
Y_2, X_{2i}
E^{t+1}
L_A
B
I_B
E^t
A
p^t
Y_1, X_{1i}

FIG. 1

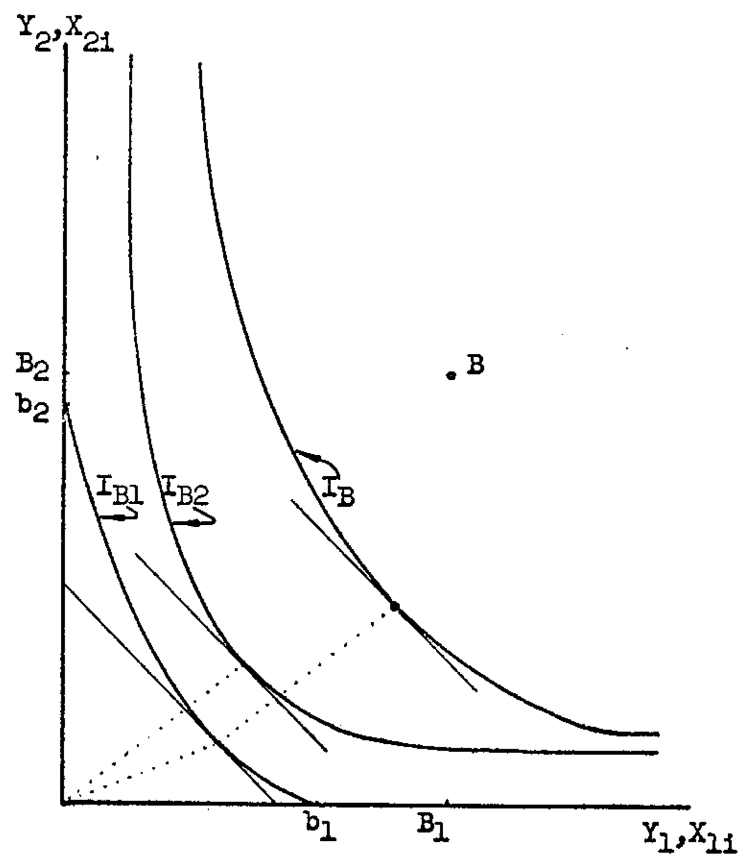
Y_2, X_{2i}
B_2
b_2
B
I_{B1}
I_{B2}
I_B
b_1
B_1
Y_1, X_{1i}

FIG. 2

of I_B (as well as of I_{B_1} and I_{B_2}). In the following sectors we will closely examine some important properties of both E^t and I_B.

3.2. Let

$$\alpha_{it}(x_i) = \frac{f_{it}(x_i)}{f'_{it}(x_i)} - x_i \,, \quad (t = 0, \ldots, T-1) \,. \tag{8}$$

We know that $\alpha_{it}(x_i) > 0$ for $x_i > 0$ holds, and we can easily see that $\alpha'_{it}(x_i) > 0$ for $x_i > 0$. We will assume that the production functions in every period are such that 12/

$$\lim_{x_i \to 0} \alpha_{it}(x_i) = 0 \,, \quad \lim_{x_i \to +\infty} \alpha_{it}(x_i) = +\infty \,. \tag{A}$$

12/ As it is shown in Appendix 1, (A) is satisfied by a wide class of production functions, as e.g., the constant-elasticity-of-substitution production functions. Figures 1 and 2 have been drawn on the basis of (A). E.g., we see that the production conditions at t are such that good 1 can be produced even if the input of the good 1 or 2 is zero. However the input-isoquant I_{B_1} is tangential to the axes at both b_1 and b_2, in conformity to the assumption (A). Similarly, I_{B_2} is asympototic to parallels to the axes, indicating that both goods are needed as inputs for the production (at t) of the good 2.

Hence for each $p > 0$ there exists a unique $x_i > 0$ for which $p = \alpha_{it}(x_i)$.

Therefore, for every positive price-ratio p^t , $(t = 0, 1, \ldots, T-1)$, the corresponding equation in (6) can be solved for a unique and positive input-ratio $x_i^t = \chi_i^t(p^t)$ in each sector.

We thus see from (6) that, on the basis of (A), we can determine for any $p^o > 0$ unique and positive 13/ $x_i^o = \chi_i^o(p^o)$, $p^1 = \mathscr{P}^1(p^o)$, $x_i^1 = \chi_i^1(p^o)$, ... ,

13/ We must emphasize the following point about the notation used in this paper. In writing any function, as e.g., $\chi_i^t(p^o)$, the first time superscript refers to the time period corresponding to the function value, namely x_i^t . However, the time superscript on the argument of the function is also needed for its complete specification. Thus $\chi_i^t(p^{t'})$ is an entirely different function from $\chi_i^t(p^{t''})$, if $t' \neq t''$, where t' , $t'' \leqq t$. For this reason symbolisms like $\chi_i^t(p^o)$ will denote both the relevant functions and the particular function values. The short-hand symbolisms like x_i^t will be used only when the time dimension of the argument of the function has been clearly indicated.

$x_i^{T-1} = \chi_i^{T-1}(p^o)$, $p^T = \mathscr{P}^T(p^o)$.

3.3. The question which immediately arises is whether the remaining conditions, (7), can be satisfied for any $p^o > 0$. It is apparent that this depends on, first of all, the value of y^o also. We may illustrate the situation by means of Figure 3. In Figure 3 the functions $\chi^o_i(p^o)$, determined by (8), are plotted along with the value of y^o.

For simplicity in the text of this paper the following assumption will be made.

Input-Intensity-Assumption:

For each period, $\alpha_{1t}(x_1) = \alpha_{2t}(x_2)$, with $x_1 > 0$, implies that either $x_1 > x_2$ or $x_2 > x_1$.

We will show in Appendix 2 that IIA is not critical for any of results in this paper. Now considering the first three conditions in (7), we immediately see from Figure 3 that the initial-endowment-ratio y^o determines a compact interval $[\underline{p}^{o1}, \bar{p}^{o1}]$ of positive price-ratios p^o for which y^1 is nonnegative. As p^o increases from $\underline{p}^{o1}$ to $\bar{p}^{o1}$ the relative importance of the two sectors (i.e., the value of ℓ^o_1 versus that of ℓ^o_2) is changing, with that of sector 1 constantly declining (for the case illustrated in Figure 3(a)). Thus the value of y^1 is constantly

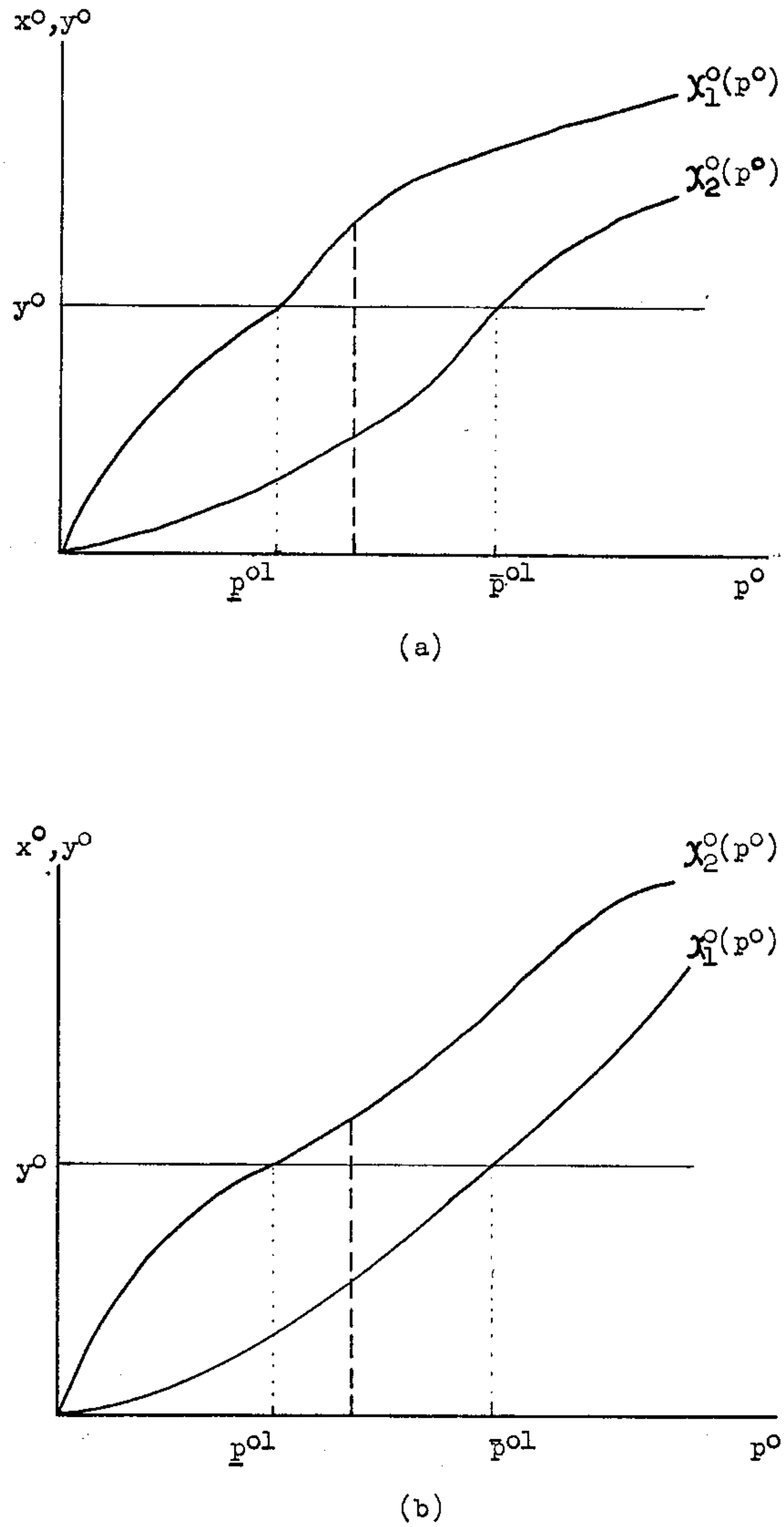
x^o, y^o
$\chi_1^o(p^o)$
$\chi_2^o(p^o)$
y^o
$\underline{p}^{o1}$
$\bar{p}^{o1}$
p^o
(a)
x^o, y^o
$\chi_2^o(p^o)$
$\chi_1^o(p^o)$
y^o
$\underline{p}^{o1}$
$\bar{p}^{o1}$
p^o
(b)

FIG. 3

declining from $+\infty$ at $\underline{p}^{o1}$ to 0 at $\bar{p}^{o1}$, and we can safely exclude from any consideration all p^o which are outside the interval $[\underline{p}^{o1}, \bar{p}^{o1}]$. Actually, since the aggregate input-ratio y^t must assume a value between the equilibrium input-ratios in the two sectors, $x_i^t = \chi_i^t(p^o)$, depending on IIA, it will be seen in the next section that a much smaller interval $[\underline{p}^{oT}, \bar{p}^{oT}] \subset [\underline{p}^{o1}, \bar{p}^{o1}]$ contains all (0-period) price-ratios p^o, which are feasible for our T-period programming problem.

For all $p^o \in (\underline{p}^{oT}, \bar{p}^{oT})$ unique and positive values of ℓ_i^t $t = 0, \ldots, T-1$, and y^t, $t = 1, \ldots, T$, are determined by

$$\ell_1^o = \frac{y^o - \chi_2^o(p^o)}{\chi_1^o(p^o) - \chi_2^o(p^o)} \equiv \lambda_1^o(p^o), \quad \ell_2^o = \frac{\chi_1^o(p^o) - y^o}{\chi_1^o(p^o) - \chi_2^o(p^o)} \equiv \lambda_2^o(p^o),$$

$$(9)\quad y^1 = \frac{f_{1o}[\chi_1^o(p^o)]}{f_{2o}[\chi_2^o(p^o)]} \quad \frac{y^o - \chi_2^o(p^o)}{\chi_1^o(p^o) - y^o} \equiv \psi^1(p^o),$$

- -

$$y^T = \frac{f_{1T-1}[\chi_1^{T-1}(p^o)]}{f_{2T-1}[\chi_2^{T-1}(p^o)]} \quad \frac{\psi^{T-1}(p^o) - \chi_2^{T-1}(p^o)}{\chi_1^{T-1}(p^o) - \psi^{T-1}(p^o)} \equiv \psi^T(p^o).$$

3.4. Since in our programming problem (I) y^T is exogenously prescribed, we must see if a unique solution for $p^o \in (\underline{p}^{oT}, \bar{p}^{oT})$, through (9) is possible.

From

$$\psi^t(p^o) = \frac{f_{1t-1}(x_1^{t-1})}{f_{2t-1}(x_2^{t-1})} \frac{y^{t-1} - x_2^{t-1}}{x_1^{t-1} - y^{t-1}}, \quad t = 1, \ldots, T, \tag{10}$$

we get

$$\frac{1}{\psi^t(p^o)} \frac{d\,\psi^t(p^o)}{d\,p^o} = - \frac{p^{t-1} + y^{t-1}}{(x_1^{t-1} - y^{t-1})\,(p^{t-1} + x_1^{t-1})} \frac{d\,\chi_1^{t-1}(p^o)}{d\,p^o} \tag{11}$$

$$- \frac{p^{t-1} + y^{t-1}}{(y^{t-1} - x_2^{t-1})\,(p^{t-1} + x_2^{t-1})} \frac{d\,\chi_2^{t-1}(p^o)}{d\,p^o} + \frac{x_1^{t-1} - x_2^{t-1}}{(x_1^{t-1} - y^{t-1})\,(y^{t-1} - x_2^{t-1})} \frac{d\,\psi^{t-1}(p^o)}{d\,p^o}$$

The sign of $\frac{d\,\psi^t(p^o)}{d\,p^o}$ for all successive values of t from 1 to T

depends on the signs of $\frac{d\,\chi_i^{t-1}(p^o)}{d\,p^o}$, $i = 1, 2$, $t = 1, \ldots, T$, which in

their turn depend on the sign of $\frac{d\,\pi^t(p^o)}{d\,p^o}$, $t = 1, \ldots, T$.

We thus have to examine the function

$$\pi^t(p^{t-1}) = \frac{f'_{1t-1}(x_1^{t-1})}{f'_{2t-1}(x_2^{t-1})} \tag{12}$$

where $x_i^{t-1} = \chi_i^{t-1}(p^{t-1})$ are determined from $p^{t-1} = \alpha_{it-1}(x_i^{t-1})$, $t = 1, \ldots, T$.

We have:

$$\frac{p^{t-1}}{\pi^t(p^{t-1})} \; \frac{d\pi^t(p^{t-1})}{dp^{t-1}} = \frac{p^{t-1}}{p^{t-1} + x_2^{t-1}} - \frac{p^{t-1}}{p^{t-1} + x_1^{t-1}} \tag{13}$$

It is thus seen that $\frac{d\pi^t(p^{t-1})}{d\,p^{t-1}} \neq 0$ wherever IIA holds. Its sign depends on the particular form that IIA takes at $t = 0, \ldots, T-1$: 14/

If $\chi_1^{t-1}(p^{t-1}) \begin{Bmatrix} > \\ < \end{Bmatrix} \chi_2^{t-1}(p^{t-1})$ then $\frac{d\pi^t(p^{t-1})}{d\,p^{t-1}} \begin{Bmatrix} > \\ < \end{Bmatrix} 0$.

14/ We note that the range of $\pi^t(p^{t-1})$, for $p^{t-1} > 0$, may possibly be a proper subset of the positive real numbers $R^>$.
For Cobb-Douglas production functions, however,

$$\lim_{p^{t-1} \to 0} \pi^t(p^{t-1}) = 0 \ , \quad \lim_{p^{t-1} \to \infty} \pi^t(p^{t-1}) = +\infty \ .$$

Therefore, if IIA holds for all $t = 0, \ldots, T-1$, all $\frac{d\pi^1(p^o)}{d\,p^o}$,

$$\frac{d\chi_i^1(p^o)}{d\,p^o}, \quad \frac{d\psi^1(p^o)}{d\,p^o}, \quad \frac{d\pi^2(p^o)}{d\,p^o}, \quad \frac{d\chi_i^2(p^o)}{d\,p^o}, \quad \frac{d\psi^2(p^o)}{d\,p^o}, \; ------ \; ,$$

$\frac{d\psi^T(p^o)}{d\,p^o}$, are non-zero and, consequently, <u>for any T-period output-ratio</u> y^T, <u>a unique and positive price-ratio</u> p^o <u>is determined by (10).</u>

Having determined a unique and positive p^o from (10), unique and positive solutions for $p^t = \pi^t(p^o)$ $x_i^t = \chi_i^t(p^o)$, $\ell_i^t = \lambda_i^t(p^o)$, $y^t = \psi^t(p^o)$, are thereby obtained as it was shown in Section 3.3.[15] We note that the

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15/ We may note that if the range of values of any of the $\pi^t(p^o)$, $t = 1, \ldots, T-1$, is a proper subset of $R^>$, then the ranges of all succeeding $\chi_i^{t'}(p^o)$ and $\pi^{t'}(p^o)$ are also proper subsets of $R^>$ which are moreover diminishing as t increases.

---

price-ratio p^o determined by (14) is such that

$$(14) \qquad \chi_1^t(p^o) > \psi^t(p^o) > \chi_2^t(p^o)$$

holds for $t = 0, \ldots, T-1$.

3.5. We have thus proved the following:

<u>Theorem 1</u>: Under the input-intensity-assumption, for any $y^{0} > 0$, $y^{T} \geqq 0$, the intertemporal efficiency conditions (4) and (5) of the programming problem (I) are satisfied by unique and positive (X_{ij}^{t}) and (p_{i}^{t}) .

4. THE ENVELOPE OF ALL t-PERIOD PRODUCTION-POSSIBILITY LOCI

4.1. The function

$$(14) \qquad y^t = \frac{f_{1t-1}[\chi_1^{t-1}(p^o)]}{f_{2t-1}[\chi_2^{t-1}(p^o)]} \; \frac{\psi^{t-1}(p^o) - \chi_2^{t-1}(p^o)}{\chi_1^{t-1}(p^o) - \psi^{t-1}(p^o)} \equiv \psi^t(p^o) \;, \quad \text{for} \quad t = 1, \ldots, T$$

and for p^o in a compact interval in $R^>$ to be specified below, describes the envelope of t-period production-possibility loci given the initial endowment y^o. (The fact that under IIA y^t is uniquely given the initial endowment y^o.) The fact that under IIA y^t is uniquely determined by p^o implies that the t-period envelope is described by a strictly concave curve in the 2-dimensional commodity space.

4.2. Let us consider each one of these envelopes more carefully. We have,

$$y^1 = \frac{f_{1o}[\chi_1^o(p^o)]}{f_{2o}[\chi_2^o(p^o)]} \; \frac{y^o - \chi_2^o(p^o)}{\chi_1^o(p^o) - y^o} \equiv \psi^1(p^o) \;,$$

with $y^o > 0$ given; see Figure 4. As it was explained in Section 3.3, under the form of IIA at time 0 considered, there exists a unique $\underline{p}^{o1} > 0$ for

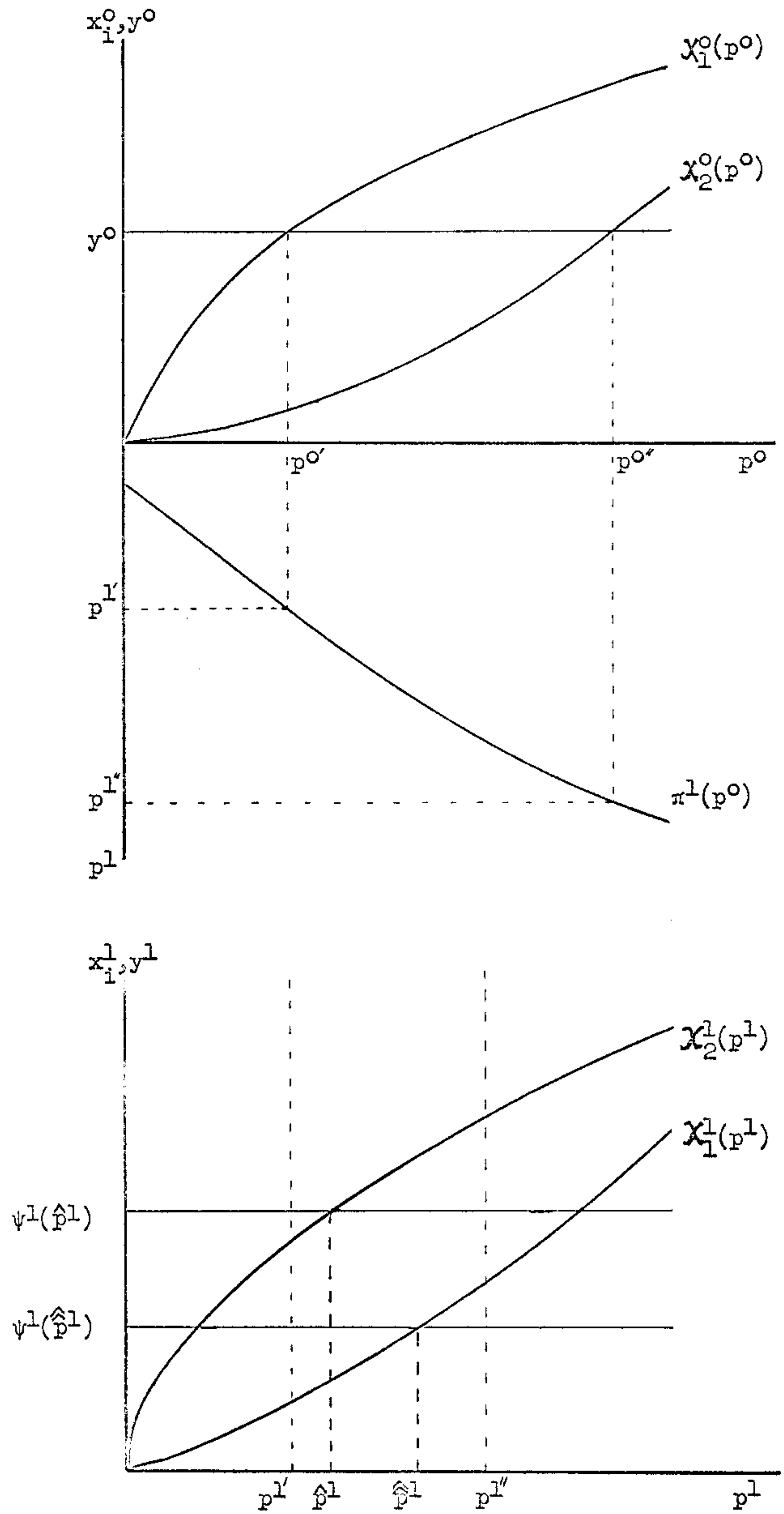
x_i^o, y^o
$\chi_1^o(p^o)$
$\chi_2^o(p^o)$
y^o
$p^{o\prime}$
$p^{o\prime\prime}$
p^o
$p^{1\prime}$
$p^{1\prime\prime}$
$\pi^1(p^o)$
p^1
x_i^1, y^1
$\chi_2^1(p^1)$
$\chi_1^1(p^1)$
$\psi^1(\hat{p}^1)$
$\psi^1(\hat{\hat{p}}^1)$
$p^{1\prime}$
$\hat{p}^1$
$\hat{\hat{p}}^1$
$p^{1\prime\prime}$
p^1

FIG. 4

which $\chi_1^o(\underline{p}^{o1}) = y^o$ and thus $\psi^1(\underline{p}^{o1}) = +\infty$, and a unique $\bar{p}^{o1}$ for which $\chi_2^o(\bar{p}^{o1}) = y^o$ and thus $\psi^1(\bar{p}^{o1}) = 0$.

$[\underline{p}^{o1}, \bar{p}^{o1}]$ is compact, and for any $p^o \in [\underline{p}^{o1}, \bar{p}^{o1}]$, the corresponding output price-ratio p^1 is given by $p^1 = \pi^1(p^o) \in [\pi^1(\underline{p}^{o1}), \pi^1(\bar{p}^{o1})]$. $\pi^1(p^o)$ is here an increasing function of $p^o \in [\underline{p}^{o1}, \bar{p}^{o1}]$.

4.3. Let us now consider a 2-period problem and thus the 2-period envelope described by

$$y^2 = \frac{f_1[\chi_1^1(p^o)]}{f_2[\chi_2^1(p^o)]} \; \frac{\psi^1(p^o) - \chi_2^1(p^o)}{\chi_1^1(p^o) - \psi^1(p^o)} \equiv \psi^2(p^o) ,$$

under the conditions contemplated in Figure 4. $\chi_i^1(p^o)$ is strictly increasing function of $p^o \in [\underline{p}^{o1}, \bar{p}^{o1}]$ onto a proper subset of $R^>$. On the other hand $\psi^1(p^o)$ is a strictly decreasing function of $p^o \in [\underline{p}^{o1}, \bar{p}^{o1}]$ onto $R^{\geq}$. Thus there exists a unique $\underline{p}^{o2} \in (\underline{p}^{o1}, \bar{p}^{o1})$ for which $\chi_2^1(\underline{p}^{o2}) = \psi^1(\underline{p}^{o2}) > 0$, and thus $\psi^2(\underline{p}^{o2}) = 0$. Similarly, there exists a unique

$\bar{p}^{o2} \in (\underline{p}^{o1}, \bar{p}^{o1})$ for which $\chi_1^1(\bar{p}^{o2}) = \psi^1(\bar{p}^{o2}) > 0$, and thus $\psi^2(\bar{p}^{o2}) = +\infty$.

Since both $\psi^1(\underline{p}^{o2})$ and $\psi^1(\bar{p}^{o2})$ are positive and finite, we see that

$$\underline{p}^{o1} < \underline{p}^{o2} < \bar{p}^{o2} < \bar{p}^{o1} ,$$

namely, $[\underline{p}^{o2}, \bar{p}^{o2}] \subset [\underline{p}^{o1}, \bar{p}^{o1}]$ holds.

We also have

$$[\pi^1(\underline{p}^{o2}), \pi^1(\bar{p}^{o2})] \subset [\pi^1(\underline{p}^{o1}), \pi^1(\bar{p}^{o1})] .$$

4.4. We can proceed in the same manner and examine the 3-, ... , T-period envelope. We thus establish that the following relations hold:

$$[\underline{p}^{o1}, \bar{p}^{o1}] \supset [\underline{p}^{o2}, \bar{p}^{o2}] \supset \dots \supset [\underline{p}^{oT}, \bar{p}^{oT}] ,$$

$$[\pi^t(\underline{p}^{ot}), \pi^t(\bar{p}^{ot})] \supset [\pi^t(\underline{p}^{o\ t+1}), \pi^t(\bar{p}^{o\ t+1})] \supset \dots \supset [\pi^t(\underline{p}^{oT}), \pi^t(\bar{p}^{oT})] ,$$

for $t = 1, 2, \dots, T$.

Considering our T-period programming problem, the intervals

$i_T^o = [\underline{p}^{oT}, \bar{p}^{oT}]$, $i_T^1 = [\pi^1(\underline{p}^{oT}), \pi^1(\bar{p}^{oT})]$, ..., $i_T^T = [\pi^T(\underline{p}^{oT}), \pi^T(\bar{p}^{oT})]$, are, respectively, <u>the sets of all T-optimal 0-period, 1-period, ..., T-period, price-ratios</u>. The main purpose of the paper is to establish certain relations between the intervals i_T^o, i_T^1, ..., i_T^T.

4.5. We have seen that the function $p^{t+1} = \pi^{t+1}(p^t)$ is a strictly increasing (decreasing) function of $p^t > 0$ if $\chi_1^t(p^t) > \chi_2^t(p^t)$ ($\chi_2^t(p^t) > \chi_1^t(p^t)$). Consequently,[16] $P^{t+1} = \log p^{t+1}$ is a function of $P^t = \log p^t$, given by

$$P^{t+1} = \Pi^{t+1}(P^t), \quad P^t \in R,$$

such that

$$(15) \qquad 0 < \left| \frac{d\,\Pi^{t+1}(P^t)}{d\,P^t} \right| = \left| \frac{p^t}{p^t + \chi_2^t(p^t)} - \frac{p^t}{p^t - \chi_1^t(p^t)} \right| < 1$$

16/ This transformation to the logarithms of the original variables is the basic tool used by Uzawa [11] for the proof of the turnpike theorem in the 2-sector model.

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holds for all $P^t \in R$. $\Pi^{t+1}(P^t)$ is an increasing (decreasing) function of P^t if $\pi^{t+1}(p^t)$ is an increasing (decreasing) function of p^t.

We may secure that

$$(16) \qquad 0 < \left| \frac{d\Pi^{t+1}(P^t)}{d\, P^t} \right| \leqq \delta < 1$$

for all $t = 1, \ldots, T - 1$, and all $P^t \in R$, by strengthening IIA to

<u>IIA':</u> Then exists a finite positive scalar M such that for every $t = 0, \ldots, T - 1$,

$\alpha_{1t}(x_1) = \alpha_{2t}(x_2)$, with $x_i > 0$, implies that

$$M > \frac{x_i}{x_j} > 1 \, , \quad i \neq j \, , \quad i, j = 1, 2 \, .$$

With IIA' the condition (16) is satisfied.[17]

17/ Since $\left|\frac{d\,\Pi\,(P)}{d\,P}\right| < 1$ holds for all $P \in R$, $\left| \frac{d\,\Pi\,(P)}{d\,P} \right| \leqq \delta < 1$ may fail to hold for all $P \in R$ only if

$$\lim_{P \to (\pm)\infty} \left| \frac{d\,\Pi\,(P)}{d\,P} \right| = \lim_{p \to (+_o\infty)} \left| \frac{p}{p + \chi_2(p)} - \frac{p}{p + \chi_1(p)} \right| = 1.$$

It is easily seen that this can happen if and only if

$$\lim_{p \to (+\infty_o)} \frac{d\chi_i(p)}{d\,p} = +\infty \text{ and } \lim_{p \to (\pm\infty_o)} \frac{d\chi_j(p)}{d\,p} = 0 \, , \quad i \neq j \, .$$

This possibility however is excluded by IIA'.

Considering any $P^t, P'^t \in R$, with $P^t > P'^t$, we get

$$P^{t+1} - P'^{t+1} = \Pi^{t+1}(P^t) - \Pi^{t+1}(P'^t) = \frac{d\,\Pi^{t+1}(\hat{P}^t)}{d\,P} (P^t - P'^t), \text{ for some}$$

$\hat{P} \in (P^t, P'^t)$. Since $0 < \left| \dfrac{d\,\Pi^{t+1}(\hat{P}^t)}{d\,P^t} \right| \leqq \delta < 1$ holds, $\Pi^{t+1}(P^t)$ is a real-valued function on R such that there exists a number $\delta < 1$ and

$$(17) \qquad |P^{t+1} - P'^{t+1}| \leqq \delta \, | P^t - P'^t|$$

for any $P^t, P'^t \in R$.

4.6 Let us now return to our T-period programming problem and examine the intervals

$$I_T^o = [\log \underline{p}^{oT}, \log \bar{p}^{oT}], \ldots, I_T^T = [\log \pi^T(\underline{p}^{oT}), \log \pi^T(\bar{p}^{oT})].$$

They are all compact intervals.

For any $P^o, P'^o \in I_T^o$, $P^o > P'^o$,

$$P^1 - P'^1 = \Pi^1(P^o) - \Pi^1(P'^o) = \frac{d\,\Pi^1(\hat{P}^o)}{d\,P^o} (P^o - P'^o),$$

for some $\hat{P}^o \in (P^o - P'^o) \subset I_T^o$. By the argument in Section 4.5,

(18) $$0 < \left| \frac{d\,\Pi^{t+1}(P^t)}{d\,P^t} \right| \leqq \delta < 1$$

holds for all $P^t \in I_T^t$, and all t and T . Hence,

$$|P^1 - P'^1| \leqq \delta \; | P^o - P'^o| \; .$$

Similarly,

$$|P^2 - P'^2| = |\Pi^2(P^1) - \Pi^2(P'^1)| = \left| \frac{d\,\Pi^2(\hat{P}^1)}{d\,P^1} \right| |P^1 - P'^1|$$ holds, and thus

$$|P^2 - P'^2| \leqq \delta |P^1 - P'^1| \leqq \delta^2 |P^o - P'^o| \; .$$

Proceeding in the same manner we get

(19) $$|P^t - P'^t| \leqq \delta^t |P^o - P'^o| \; , \quad t = 1, 2, \ldots, T \; .$$

(19) immediately shows that <u>the lengths of the intervals</u> I_T^t <u>are strictly decreasing as</u> t <u>increases, and also that the length of</u> I_T^t <u>converges to zero as</u> t , $T \to +\infty$.

Since all I_T^t are compact intervals, (19) also shows that the lengths of the intervals i_T^t are similarly decreasing as t increases,[18] and that they converge to zero as t , $T \to +\infty$.

18/ The lengths of i_T^t are not necessarily monotonically decreasing as t increases. However, (19) shows that for each $t > 0$, there exists a finite $t' > 0$, for which $|i_T^t| > |i_T^{t+t'}|$, $T \geqq t + t'$. To indicate this property we will say that the length of i_T^t is quasi-decreasing as t increases.

4.7. For every T , the T-period envelope corresponds to the set of T-optimal T-period price-ratios, i_T^T , since the final output proportions, y^T , can be given as a function of the final price-ratio $p^T \epsilon\, i_T^T$. Since the length of i_T^T is quasi-decreasing as T increases and it converges to zero as $T \to +\infty$, it becomes evident that the concavity of the T-period envelope tends to diminish 19/ as T increases and that the T-period envelope converges to a straight line as $T \to +\infty$.

19/ Since the intervals $[\pi^t(\underline{p}^{ot}) , \pi^t(\bar{p}^{ot})]$, $t = 1, 2, \ldots, T$, properly contain the T-optimal t-period intervals, $i_T^t = [\pi^t(\underline{p}^{oT}) , \pi^t(\bar{p}^{oT})]$, the tendency for each period's envelope to become less concave is even stronger than it is indicated by the result that $|I_T^t| > |I_T^{t+1}|$, $t = 1, 2, \ldots, T$.

We have thus proved the following theorem for the production conditions specified in Section 2.2-3.2:

<u>Theorem 2</u>: Given the initial endowment with goods at time zero, the envelope of all t-period production-possibility loci tends to be less concave as t increases and it converges to a straight line as $t \to +\infty$.

4.8. We must be careful in our interpretation of the above result. It shows that the lengths of the intervals I_T^t are strictly decreasing as t increases. However, the interval I_T^t itself does not converge to a unique point as $t, T \to +\infty$, as long as the sequence of functions $(\Pi^{t+1}(P^t))$ does not converge to a function $\Pi(P)$. Consequently, although the T-period envelope approaches a straight-line, for T large, the normal to this line is not the same from period to period.

In Section 4.6 we have actually shown that each one of the functions $\Pi^{t+1}(P^t)$ is a contraction mapping defined on R into itself.[20] By the main

[20] A mapping A on a metric space X into itself is a contraction mapping if there exists a number $\alpha < 1$ such that for any $x, y \in X$

$$\rho(Ax, Ay) \leqq \alpha\rho(x, y),$$

where ρ is a metric on X.
For an analysis of the principle of contraction mappings and its applications see e.g., Kolmogorov and Fomin [4, pp. 43-51].

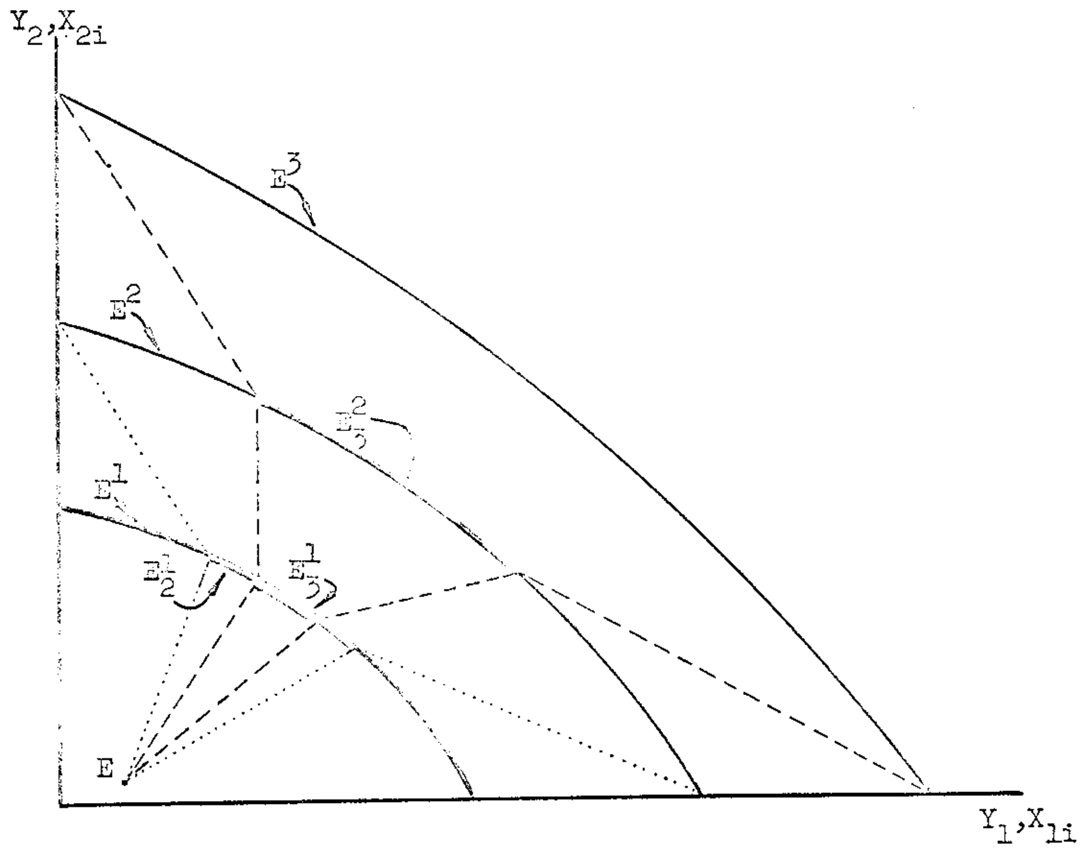
theorem on contraction mappings each $\Pi^{t+1}(P^t)$ has a unique fixed point, i.e., the equation $P^{t+1} = \Pi^{t+1}(P^t)$ has a unique solution in which $P^{t+1} = P^t \equiv P_t^*$. Consequently, the price-ratio p_t^*, with $P_t^* = \log p_t^*$, is the unique solution of $p^{t+1} = \pi^{t+1}(p^t)$ for which $p^{t+1} = p^t$. p_t^* may be called the von Neumann price-ratio under the production conditions at time t.

It is very interesting to find out if there is any relationship between the intervals i_T^t and the corresponding von Neumann price-ratios p_t^* , at least in some simple cases of continuous (neutral or biased) technological progress.

5. FINAL REMARKS AND SOME EXAMPLES

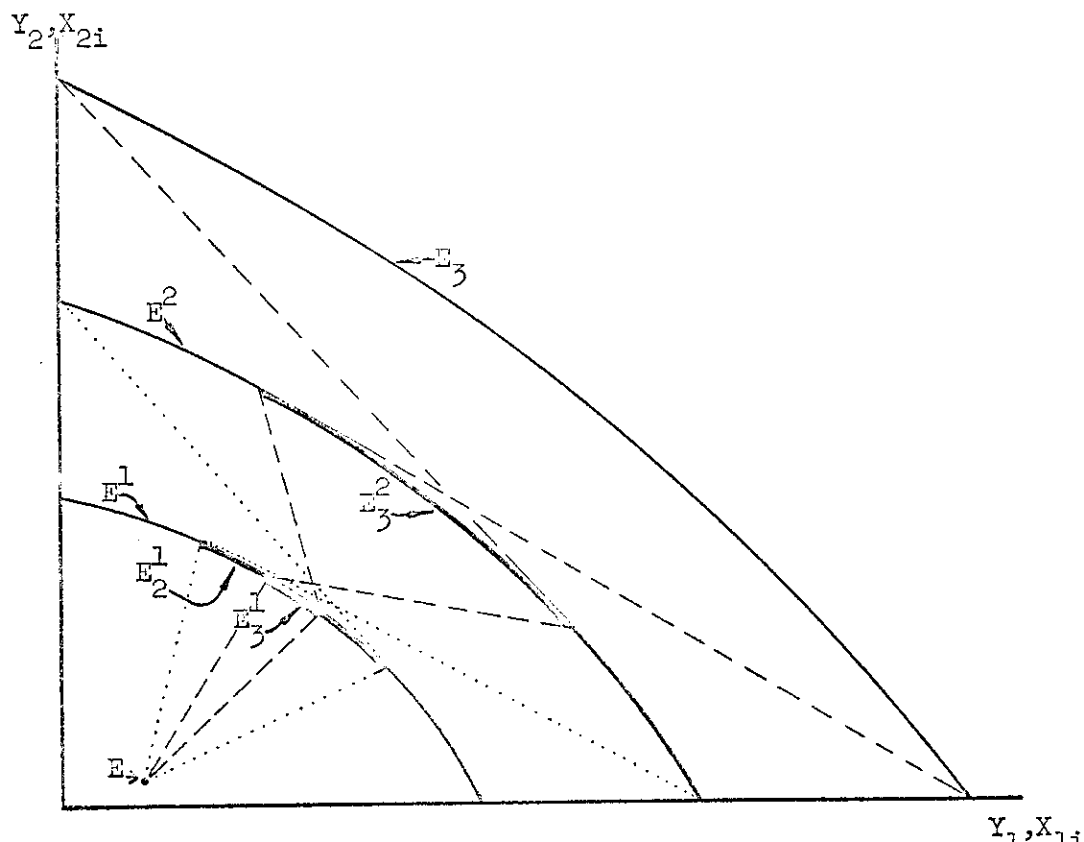
5.1. Theorem 2 established that the lengths of the intervals, i_T^t, of all T-optimal price-ratios at time t are quasi-decreasing as t increases, and that they converge to zero as $t, T \to +\infty$. To these sets of price-ratios i_T^t there correspond subsets of the 1-period, ..., T - 1 period, envelopes, which may be called the T-optimal subsets of the 1-period, ..., T - 1 period, envelopes, $E_T^1, \ldots, E_T^{T-1}$, respectively.

Let us consider $E_T^1, \ldots, E_T^{T-1}$, as T increases from 1 to $+\infty$. A careful examination of the argument of Section 3.1 and 3.2 shows that E_{T+1}^t is a proper subset of E_T^t for any t, T. This follows immediately from the observation that not all points on the t-period envelope, E^t, can lead us to a point on the t+1-period envelope, E^{t+1}; see e.g., Figure 5. This last property is, in general, a consequence of the strict concavity of the t-period envelope, E^t, and the strict convexity of the aggregate input isoquant at t for the production of any specified output combination at $t+1$; because, as we saw in Section 3.1, all points on E^t which can lead us to a point on E^{t+1} are points at which E^t is tangential to some aggregate input isoquant for the production of the corresponding point on E^{t+1}. In particular,



Y_2, X_{2i}
E^3
E^2
E_3^2
E^1
E_2^1
E_3^1
E
Y_1, X_{1i}

(a)



Y_2, X_{2i}
E_3
E^2
E_3^2
E^1
E_2^1
E_3^1
E
Y_1, X_{1i}

(b)

FIG. 5

a relatively strong sufficient condition for this property is our assumption (A) of Section 3.2, that the slope of the input isoquants becomes parallel to the axes if one wishes to produce a specified output with continuously increasing quantitities of only one of the inputs. Therefore, although the production of both goods may be possible by the use of only one of the inputs, such production is never intertemporally efficient if (A) holds true. Thus the t+1-optimal subset of the t-period envelope, E^t_{t+1}, is a proper subset of E^t. Similarly, E^{t+1}_{t+2} is a proper subset of E^{t+1}, and thus E^t_{t+2} is a proper subset of E^t_{t+1} and in general E^t_{t+1} is a proper subset of $E^t_{t'}$, $t' = t, \ldots, T$. We thus see that, under the production conditions specified in Section 2.2 and 3.2, as the length of the horizon increases all intertemporally efficient accumulation paths move very closely together in the beginning periods. This result may be formulated as follows:

Theorem 3: Given the initial endowment with goods at time zero, the T- optimal subset E^t_T of the t-period envelope E^t is a proper subset of it, for any t, T, with $t < T$. The sequence of E^t_T, as T increases, is a strictly decreasing sequence converging to one point as $T \to +\infty$.

The above theorem proves for our very special model a general conjecture that in any multisector production model, even under changing production conditions, all intertemporally efficient accumulation paths keep closely together for most of their duration if the program horizon is long enough.[21]

21/ This property, which generalizes the turnpike property of efficient accumulation paths under constant technology, was recently conjectured by G. Debreu and is currently explored by T. Koopmans and others at the Cowles Foundation.

5.2. Nothing so far has been said about the <u>speed of convergence</u> of the envelopes of all t-period production-possibility-loci to straight lines as t increases. A common regrettable feature of all papers dealing with properties of efficient accumulation paths in closed models is the absence of any quantitative information about the length of the horizon required for the convergence of the path to a neighboring cone of the von Neumann ray for a specified part of the horizon. In other words, in all these papers asymptotic properties of efficient paths are established without any information on the speed of convergence of these paths.

Casual examination of simple examples for our two-sector model appeared to support the conjecture that the convergence of the envelopes established by Theorem 2 is very rapid. In order to get accurate information on this question, some examples with Cobb-Douglas and constant-elasticity-of-substitution production functions have been worked out, by means of a simple computer algorithm. The algorithm consists of an iterative procedure based on (6) and (7) of Section 3.1.

5.3. In our first example we consider a constant technology model described by the following Cobb-Douglas production functions,

$$Y_1^{t+1} = A\, X_{11}^{t^{\alpha}}\, X_{21}^{t^{1-\alpha}}, \quad \text{and} \quad Y_2^{t+1} = B\, X_{12}^{t^{\beta}}\, X_{22}^{t^{1-\beta}}, \quad \text{with } A = 4, \; B = 2 \quad \alpha = 1/3, \beta = 2/3$$

The von Neumann price-ratio is $p^* = 1.68$, while the von Neumann output-ratio is $y^* = 1.68$.[22] Also $x_1^t = 1/2\, p^t$, $x_2^t = 2\, p^t$, and $p^{t+1} = 2(p^t)^{-1/3}$. With

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22/ The von Neumann price-ratio, p_t^*, and output-ratio, y_t^*, under the production conditions prevailing at t, are the unique solutions of

$$p^{t+1} = \pi^{t+1}(p^t) \equiv \frac{f'_{1t}[\chi_1^t(p^t)]}{f'_{2t}[\chi_2^t(p^t)]}, \quad \text{and} \quad y^{t+1} = \frac{f_{1t}[\chi_1^t(p^t)]}{f_{2t}[\chi_2^t(p^t)]} \quad \frac{y^t - \chi_2^t(p^t)}{\chi_1^t(p^t) - y^t},$$

for which $p^{t+1} = p^t$ and $y^{t+1} = y^t$, respectively. The existence of unique and positive p_t^* and y_t^* can easily be demonstrated by the use e.g. of the principle of contraction mappings; see above Section 4.

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initial endowment $Y_1^o = 1$, $Y_2^o = 1$, the envelopes for the first three periods appear in Figure 6. The data for this example are summarized in Table 1 in Appendix 3.

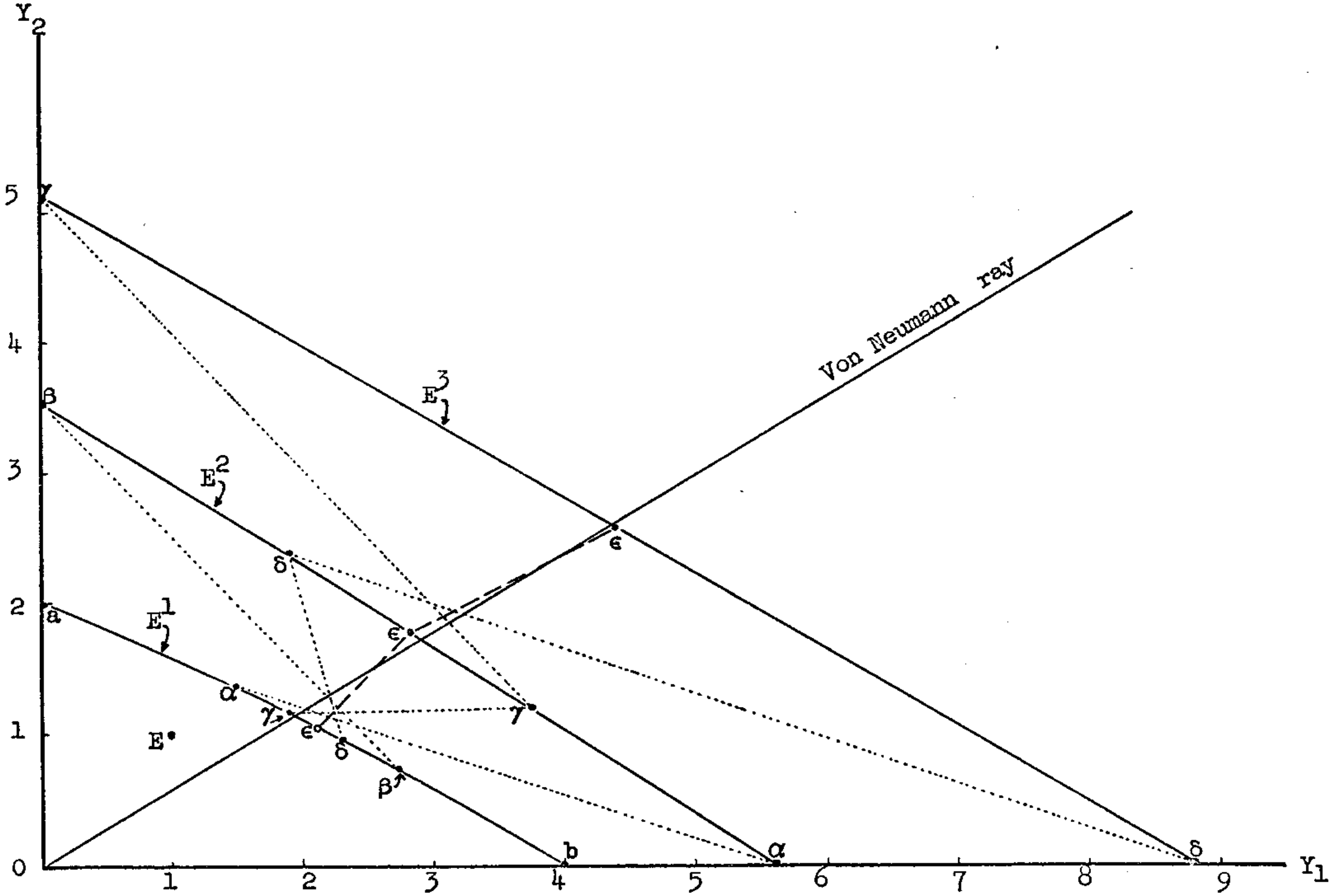


FIG. 6

We witness the rapid flattening of the successive envelopes. As we see from the various intertemporally efficient paths which are drawn in Figure 6, the "turnpike property" of intertemporally efficient paths manifests itself only for the path $\epsilon\epsilon$ [23/] The turnpike property is a property for a really long program horizon, while the above exhibited property

23/ The path $\epsilon\epsilon$ actually consists of all intertemporally efficient paths for 20 time periods and over, assuming that the technology remains the same over the whole horizon.

of the successive envelopes to become less concave is not.

Finally, we see that in Example 1 all efficient paths move in a wave-like fashion. The necessity for this kind of motion is due solely to the fact that in this case $\chi_2^t(p^t) > \chi_1^t(p^t)$ holds, i.e., that sector 1 uses relatively more of input 2 than sector 2 does.[24/]

24/ This wave-like motion of efficient paths should not be confused with the well-known "cyclic" exceptions to the Turnpike theorem. As a matter of fact, in our model the input matrix (at the von Neumann prices) is strictly positive. Thus its positive eigenvalue is strictly greater in absolute value than the second eigenvalue, and the "cyclic" exception to the Turnpike theorem cannot occur.

We may note however that, whenever $x_1^t > x_2^t$ holds both eigenvalues are positive, while if $x_2^t > x_1^t$ the second eigenvalue is negative.

5.4. In Example 2 we consider a changing technology model with a program horizon of 20 periods. We have:

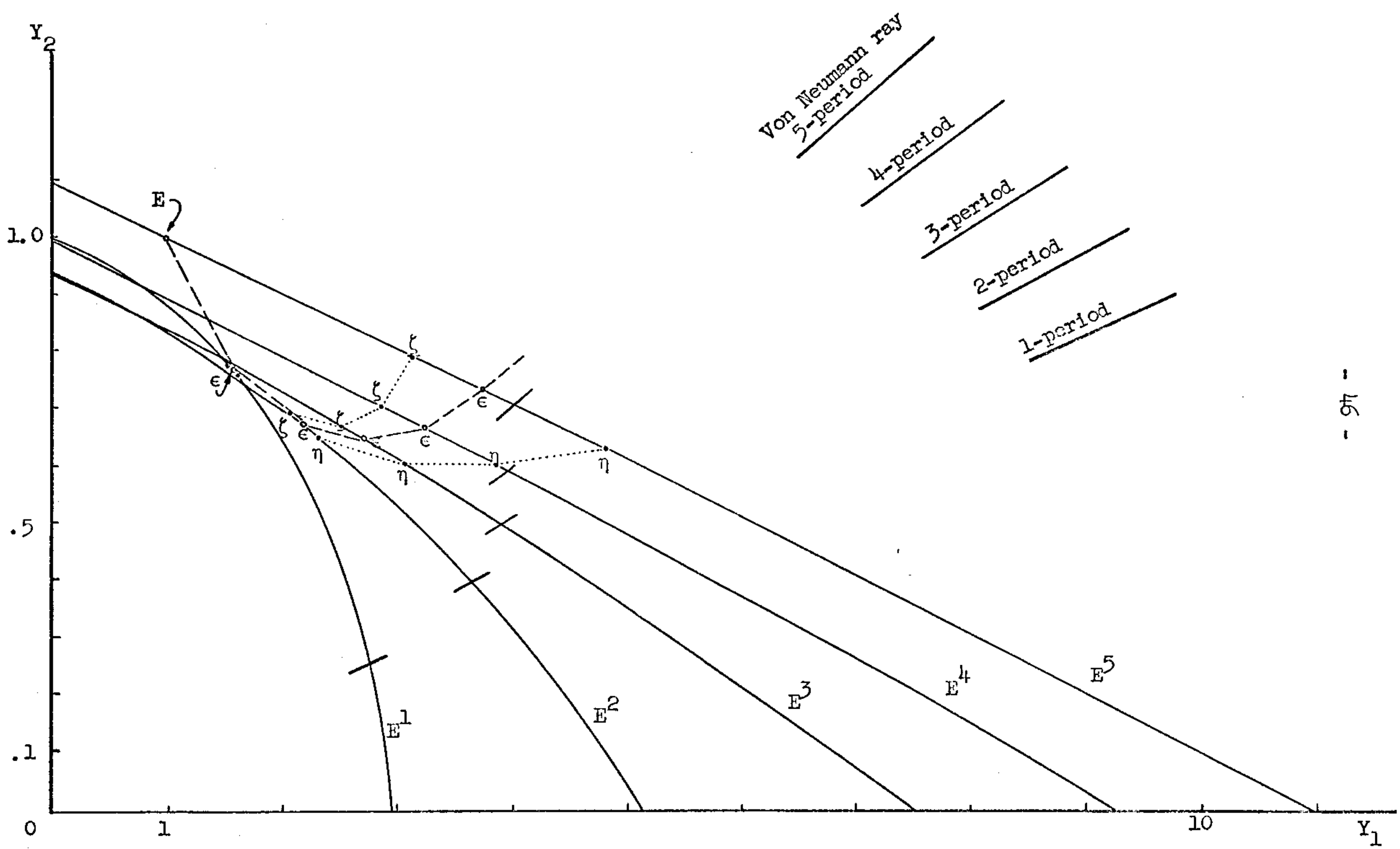
$$Y_1^{t+1} = A(t)\ X_{11}^{t^{\alpha(t)}}\ X_{21}^{t^{1-\alpha(t)}}\ , \quad \text{and} \quad Y_2^{t+1} = B(t)\ X_{12}^{t^{\beta(t)}}\ X_{22}^{t^{1-\beta(t)}}\ , \quad \text{with}$$

$A(t) = 3$, $B(t) = 1.05^t$, $\alpha(t) = .8$, $\beta(t) = .1$, $t = 0, \ldots, 20,$ and $Y_1^o = 1$, $Y_2^o = 1$. Namely, we examine the simple case of neutral technological progress, which is unequal in the two sectors. In this case we have:

$x_1^t = 4\ p^t$, $x_2^t = \frac{1}{9}\ p^t$, and $p^{t+1} = \frac{2.518}{1.05^t}\ (p^t)^{.7}$. Thus $p_t^* = \frac{21.71}{1.177^t}$ and $y_t^* \doteq \frac{10.85}{1.177^t}$

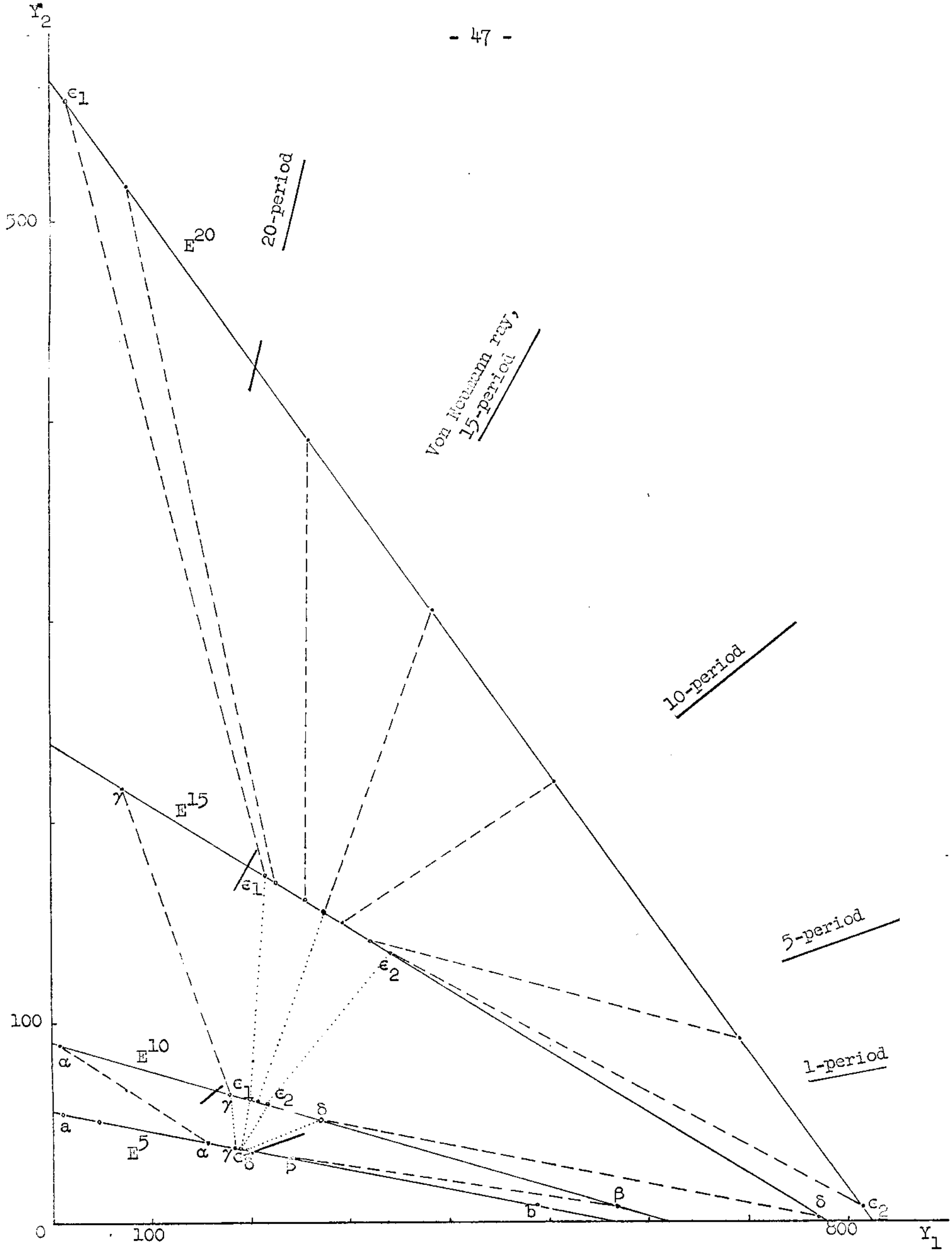
The envelopes for the first five periods appear in Figure 7, while those for the 5-, 10-, 15-, and 20-period are drawn in Figure 8.[25] The data for Example 2 are summarized in Tables 2 and 3 in Appendix 3.

25/ The scale in the axes of Figure 8 refers to the 20-period envelope. In order to get the actual scale of the 15-period, 10-period, and 5-period, envelopes, this scale must be divided by 8, 25, and 50, respectively. Of course, absolutely no change in the essential features of Figure 8 can result from such a normalization.



Y_2
E
1.0
.5
.1
0
1
10
Y_1
E^1
E^2
E^3
E^4
E^5
ε
ζ
η
Von Neumann ray
5-period
4-period
3-period
2-period
1-period

FIG. 7



Y_2
500
100
0
100
800
Y_1
E^{20}
E^{15}
E^{10}
E^5
20-period
Von Neumann ray,
15-period
10-period
5-period
1-period

FIG. 8

First we observe again the flattening of the successive envelopes. The envelopes from the 10^{th} period and on are virtually straight-lines. However, their corresponding price-ratios are continuously declining, because of the neutral technological progress in sector 2.

Second, we see in Figure 7 and 8 (and also in Tables 2 and 3) that the 20-optimal subsets of the 1-period, ..., 5-period, and even of) the 10-period envelopes are extremely small. This strong verification of Theorem 3 above is naturally very encouraging. It clearly indicates the importance of long-run planning. Our economy is severely restrained in its selection of a multiperiod accumulation path, if this path is to be intertemporally efficient. Long-run planning appears to be indispensable. It would be very interesting to test the importance of long-run planning by a systematic examination of such models (with Cobb-Douglas, or more general C-E-S production functions), and to derive an index (as well as some quantitative estimates) of the loss of efficiency which can result if only short-run (e.g., 5-period) planning is practiced.

5.5 It is of course true that the production conditions which are likely to prevail in the future are never known with certainty today. Moreover, the uncertainty surrounding them increases as we consider longer planning periods. A satisfactory answer to this problem is not possible within the confines of our model. However, the following example is offered for illustrating purposes. We consider our economy as described by Example 2 for $t = 0, \ldots, 9$, but we now suppose that from the 11^{th} period and on neutral technological progress occurs faster in sector 1 than in sector 2. Namely we have:

$A(t) = 3\,(1.10)^{t-10}$, and $B(t) = B(10)$, for $t = 10, \ldots, 20$. Under the changed conditions, our economy starting again with $Y_1^0 = 1$ and $Y_2^0 = 1$ at time 0, has naturally very different production possibilities in the last 10 periods. These possibilities are summarized in Table 4 of Appendix 3. However, as the data in this table for the 1-period, 5-period, and 10-period envelope show, the 20-optimal subsets of these envelopes are hardly changed as a result of the new production conditions in the latter half of the planning horizon. This is an interesting, and at the same time a very intriguing, finding. If such a property of intertemporally efficient paths is true under fairly general conditions, then the existence of uncertainties with respect to the conditions of production in the more distant future does not render useless all long-run planning conducted on the basis of known production conditions.

APPENDIX I

1. The functions $x_i^t = \mathcal{X}_i^t(p^t)$, $i = 1, 2,$ under the assumption (A) of Section 3.1, are (strictly increasing for all $p^t > 0$ and) such that $\lim\limits_{p^t \to 0} \mathcal{X}_i^t(p^t) = 0$, $\lim\limits_{p^t \to +\infty} \mathcal{X}_i^t(p^t) = +\infty$. Assumption (A) is satisfied if the production functions are of the constant-elasticity-of-substitution type, which includes both the Cobb-Douglas and the Leontief (fixed coefficients) types. For example, let F^i be given by

$$\text{(i)} \qquad Y = A_i \left[\alpha_i X_{1i}^{-\rho_i} + (1-\alpha_i) X_{2i}^{-\rho_i} \right]^{-\frac{1}{\rho_i}} ,$$

with $1 > \alpha_i > 0$ and $\sigma_i = \frac{1}{1+\rho_i}$, $\sigma_i > 0$, as the (constant) elasticity of substitution. Rewriting conditions (6) in the text on the basis of (i) we easily see that $\mathcal{X}_i(p)$ is given by

$$\text{(ii)} \qquad x_i = \left(\frac{\alpha_i}{1-\alpha_i} \right)^{\sigma_i} p^{\sigma_i} .$$

Therefore, the limits indicated above hold true for C-E-S production functions.

2. We may note that C-E-S production functions have the following properties (see [2]):

(a) If $\sigma_i > 1$, then a positive output is possible even with one of the inputs. In fact, $Y_i = A_i(1-\alpha_i)^{-\frac{1}{\rho_i}} X_{2i}$ if $X_{1i} = 0$, and $Y_i = A_i \alpha_i^{-\frac{1}{\rho_i}} X_{1i}$ if $X_{2i} = 0$. However, any input isoquant is tangential to the axes in the points of contact.

(b) If $\sigma_i < 1$, then any input isoquant is asymptotic to parallels to the axes, i.e., some minimal positive quantity of input 1 or 2 is always needed for the production of any specified $Y_i > 0$, $i = 1, 2$.

(c) If $\sigma_i = 1$, (i.e., the Cobb-Douglas type), then any input isoquant is asymptotic to the axes.

APPENDIX II

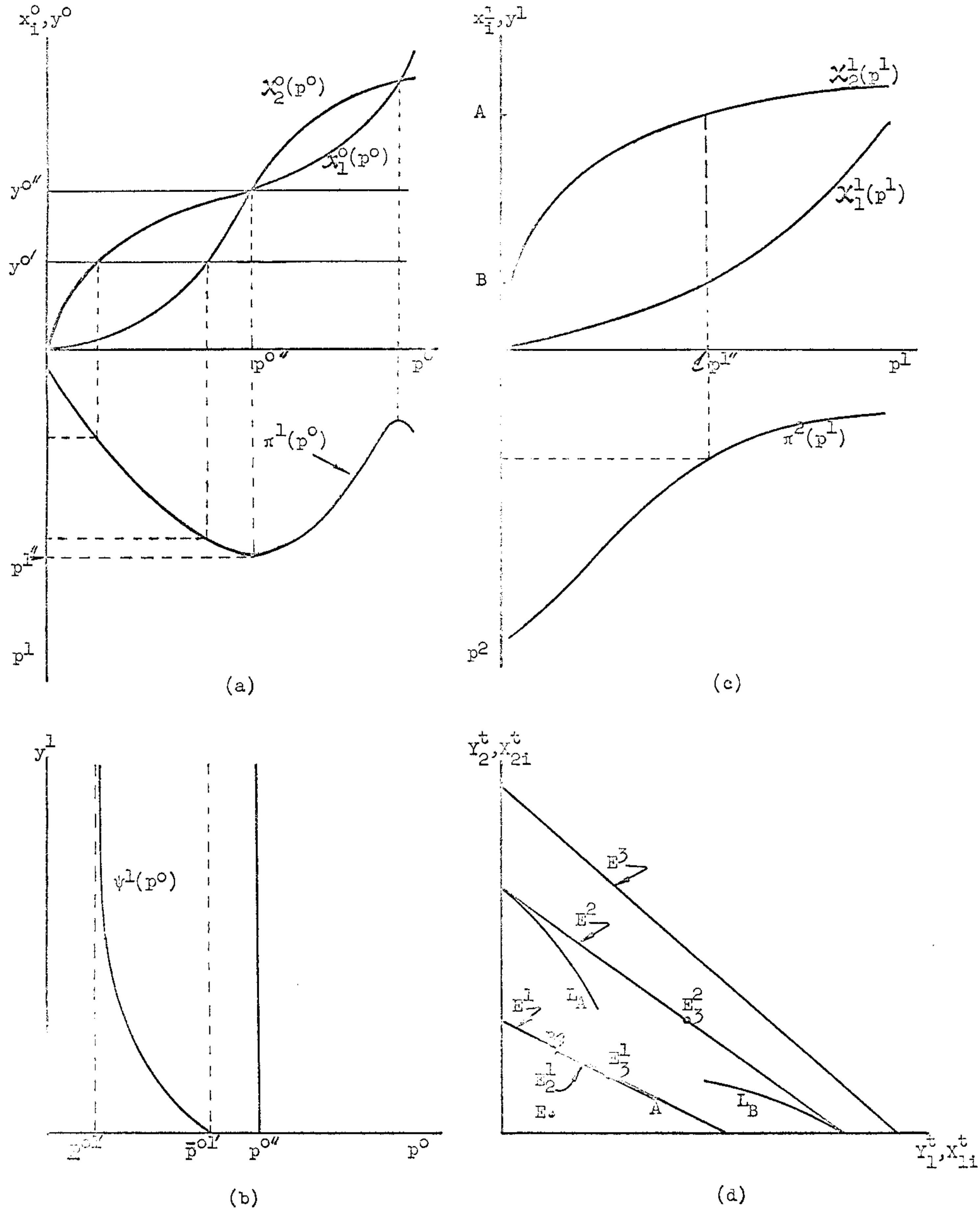
1. In the text we carried out our analysis on the basis of (IIA). We have now to indicate why (IIA) is not critical for the proof of the existence of a unique and positive solution to the programming problem (I).

For simplicity, let us first consider a one-period problem and suppose that the functions $\mathcal{X}_i^o(p^o)$ are as in Figure 9(a).

The price-ratio at 1 , p^1 , as a function of the price-ratio at 0 is also described in Figure 9(a). We note that the inverse function of π^1 for all nonnegative p^o does not exist in this case. However, this is of no importance for our problem because we will immediately see that the initial-endowment-ratio y^o always determines an interval of feasible p^o's over which the inverse function of π^1 exists.

If e.g. the initial-endowment-ratio is equal to $y^{o'}$, then in accordance with the argument of Section 3, a compact interval $i_1^{o'} = [\underline{p}^{o1'}, \overline{p}^{o1'}]$ of feasible 0-period price-ratios is determined. π^1 is strictly increasing over $i_1^{o'}$ and thus the inverse function of π^1 over $i_1^{o'}$ exists. Also, in this case $y^1 = \psi^1(p^o)$ is a decreasing function of $p^o \in i_1^{o'}$, with $\psi^1(\underline{p}^{o1'}) = +\infty$, and $\psi^1(\overline{p}^{o1'}) = 0$. Thus $\psi^1(p^o)$ over $i_1^{o'}$ may be illustrated as in Figure 9(b).

If $y^o = y^{o''}$, we see that the intersection of $\mathcal{X}_i^o(p^o)$ and of $y^o = y^{o''}$ occurs at a unique point $p^{o''}$, since both $\mathcal{X}_i^o(p^o)$ are strictly increasing functions of $p^o \geqq 0$. Thus again the inverse of π^1 at $p^{o''}$ exists.



x_i^o, y^o
$\mathcal{X}_2^o(p^o)$
$\mathcal{X}_1^o(p^o)$
$y^{o''}$
$y^{o'}$
$p^{o''}$
p^o
$\pi^1(p^o)$
$p^{1''}$
p^1
(a)
x_i^1, y^1
$\mathcal{X}_2^1(p^1)$
A
$\mathcal{X}_1^1(p^1)$
B
$p^{1''}$
p^1
$\pi^2(p^1)$
p^2
(c)
y^1
$\psi^1(p^o)$
$\bar{p}^{o1'}$
$p^{o''}$
p^o
(b)
Y_2^t, X_{2i}^t
E^3
E^2
L_A
E^1
E_3^2
E_3^1
E_2^1
E
A
L_B
Y_1^t, X_{1i}^t
(d)

FIG. 9

Since in this programming problem y^1 is exogenously prescribed, and since $x_i^o = \chi_i^o(p^{o''})$, we can determine unique and positive values of ℓ_i^o from

$$y^1 = \frac{f_{10}(x_1^o)}{f_{20}(x_2^o)} \frac{\ell_1^o}{\ell_2^o}, \quad \text{and} \quad \ell_1^o + \ell_2^o = 1 .$$

By changing y^1 , we only move both inputs from one sector to the other without changing the sector-input-ratios, x_i^o , or the prices. The relation between y^1 and p^o in this case is also described in Figure 9(b). It is clear that the production-possibility-locus is a straight-line with normal equal to $\pi^1(p^{o''})$.

2. The situation in multi-period problems is similar. If e.g. we consider a T-period problem, in which the 0- and 1-period conditions are as described in Figure 9(a) and (c), and $y^o = y^{o''}$, we see that from $p^o = p^{o''}$ all subsequent price-ratios, p^t , and sector-input-ratios, x_i^t , are determined by (6). Since y^T is exogenously given, unique and positive values of ℓ_i^{T-1} are determined from $y^T = \frac{f_{1\,T-1}(x_1^{T-1})}{f_{2\,T-1}(x_2^{T-1})} \frac{\ell_1^{T-1}}{p_2^{T-1}}$, and

$\ell_1^{T-1} + \ell_2^{T-1} = 1$, those of y^{T-1} are determined from $x_1^{T-1}\,\ell_1^{T-1} + x_2^{T-1}\,\ell_2^{T-1} = y^{T-1}$,

etc., till unique and positive values of y^1 and ℓ_i^o are determined from

$$x_1^1\,\ell_1^1 + x_2^1\,\ell_2^1 = y^1 , \quad y^1 = \frac{f_{10}(x_1^o)}{f_{20}(x_2^o)} \quad \frac{\ell_1^o}{\ell_2^o} , \quad \text{and} \quad \ell_1^o + \ell_2^o = 1 .$$

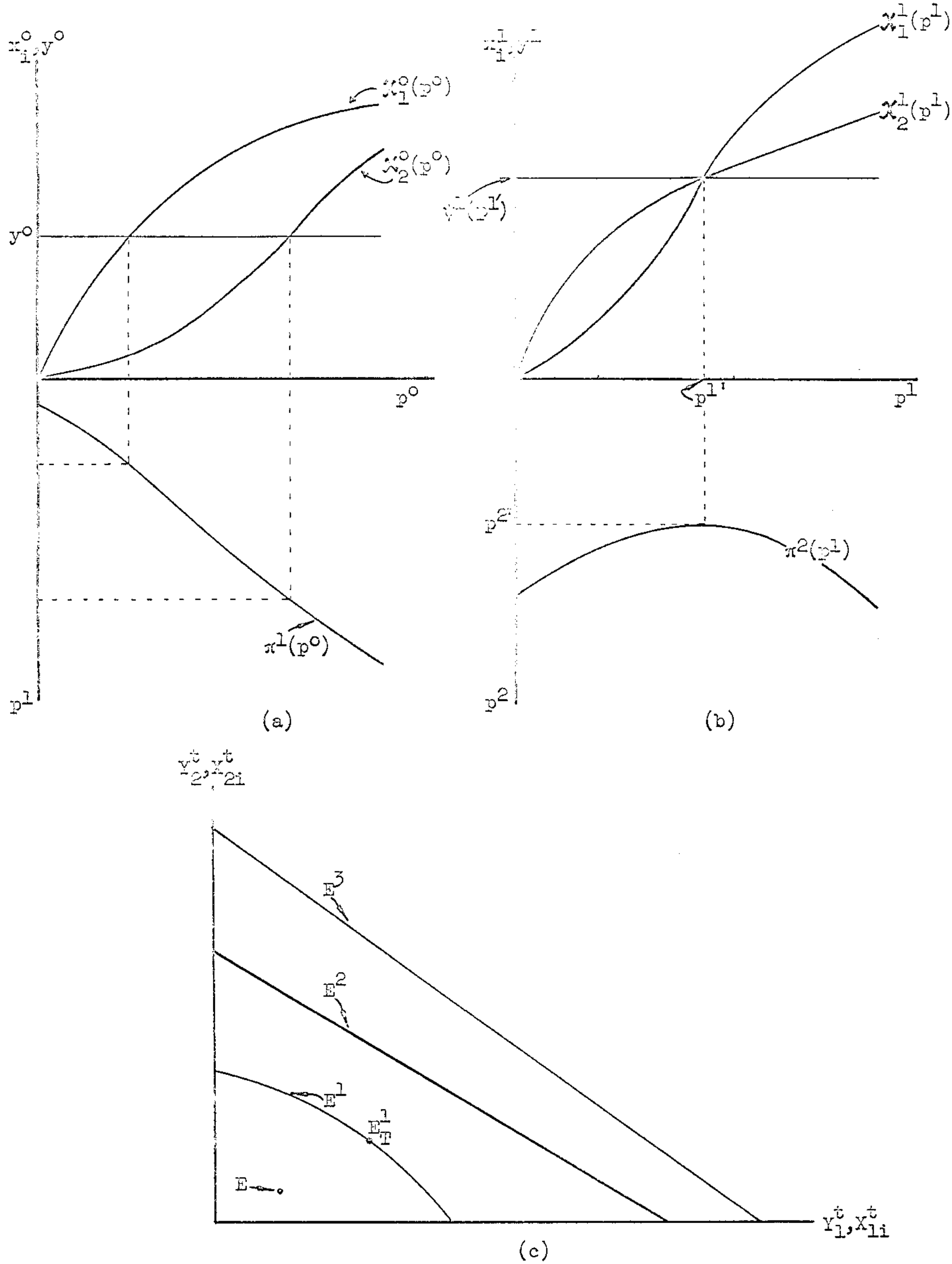
In this case not only the production possibility-locus at 1, but all subsequent envelopes are straight-lines (with normals equal to $\pi^t(p^{o''})$). If for some t, $1 \leqq t < T$, $\mathcal{X}_1^t(p^{o''}) \neq \mathcal{X}_2^t(p^{o''})$, (as e.g. in Figure 9(d) for $t = 1$), then as y^T moves from zero to $+\infty$, y^t moves from $\mathcal{X}_2^t(p^{o''})$ to $\mathcal{X}_1^t(p^{o''})$. Clearly if $\mathcal{X}_1^t(p^{o''}) = \mathcal{X}_2^t(p^{o''})$, then $y^t = \mathcal{X}_i^t(p^{o''})$ for any y^T.

3. Finally, let us suppose that in a multi-period problem the 0- and 1-period conditions and the initial-endowment-ratio y^o are as in Figure 10. Further, suppose that the 1-period conditions are such that for $p^o = p^{o'}$ $\mathcal{X}_1^1(p^{o'}) = \psi^1(p^{o'}) = \mathcal{X}_2^1(p^{o'})$. Then $p^{o'}$ is the unique 0-period price-ratio which is feasible for our T-period programming problem. With $p^t = \pi^t(p^{o'})$ and $x_i^t = \mathcal{X}_i^t(p^{o'})$ we can determine for any given y^T unique and positive ℓ_i^{T-1}, y^{T-1}, ..., y^2, ℓ_1^1, by

$$y^T = \frac{f_{1\ T-1}(x_1^{T-1})}{f_{2\ T-1}(x_2^{T-1})} \frac{\ell_1^{T-1}}{\ell_2^{T-1}}, \quad \ell_1^{T-1} + \ell_2^{T-1} = 1, \ \ldots, \quad x_1^2\, \ell_1^2 + x_i^2\, \ell_2^2 = y^2,$$

$$y^2 = \frac{f_{11}(x_1^1)}{f_{21}(x_2^1)} \cdot \frac{\ell_1^1}{\ell_2^1}, \quad \ell_1^1 + \ell_2^1 = 0,$$ respectively. $y^1 = \psi^1(p^{o'})$ is equal to

$\mathcal{X}_i^o(p^{o'})$ and $\ell_i^o = \lambda_i^o(p^{o'})$. Consequently, we obtain again a unique and positive solution to (6) and (7). Here, although the production-possibility-locus at 1 is concave, all the succeeding envelopes are straight-lines; see e.g. Figure 10(c).



x_i^o, y^o
$x_1^o(p^o)$
$x_2^o(p^o)$
y^o
p^o
$\pi^1(p^o)$
p^1
(a)
x_i^1, y^1
$x_1^1(p^1)$
$x_2^1(p^1)$
$y^1(p^{1'})$
$p^{1'}$
p^1
$p^{2'}$
$\pi^2(p^1)$
p^2
(b)
Y_2^t, X_{2i}^t
E^3
E^2
E^1
E_T^1
E
Y_1^t, X_{1i}^t
(c)

FIG. 10

APPENDIX 3

TABLE 1 (For Figure 6)

	Point	Y_1^t		Y_2^t	p^t	
1-period Envelope	a	0		2.00	2.52	
	α	1.48		1.37	2.16	
	γ	1.90		1.16	2.05	
	ϵ	2.12	E_3^1	1.06	2.00	i_3^1
	δ	2.32		.96	1.95	
	β	2.74		.74	1.86	
	b	4.00		0	1.59	
2-period Envelope	α	5.62		0	1.55	
	γ	3.75		1.19	1.57	
	ϵ	2.82	E_3^2	1.78	1.59	i_3^2
	δ	1.90		2.38	1.60	
	β	0		3.54	1.63	
3-period Envelope	γ	0		5.12	1.72	
	ϵ	4.39		2.56	1.71	i_3^3
	δ	8.80		0	1.71	

	Length of the interval of the t-period price-ratios	Length of i_3^t
1	.93	.10
2	.08	.03
3	.01	.01

TABLE 2 26/ (for Figure 7)

	Point	Y_1^t		Y_2^t	p^t	
1-period Envelope	ζ	1.55		.78	4.209	
	ϵ	1.5825	E_{20}^1	.775	4.0930	i_{20}^1
		1.5852	E_{20}^1	.774	4.0831	i_{20}^1
	η	1.64		.76	3.882	
2-period Envelope	ζ	2.08		.69	6.557	
	ϵ	2.1704	E_{20}^2	.677	6.4302	i_{20}^2
		2.1782	E_{20}^2	.675	6.4192	i_{20}^2
	η	2.33		.65	6.196	
3-period Envelope	ζ	2.53		.67	8.517	
	ϵ	2.7191	E_{20}^3	.649	8.4015	i_{20}^3
		2.7353	E_{20}^3	.647	8.3915	i_{20}^3
	η	3.07		.61	8.186	
4-period Envelope	ζ	2.89		.71	9.741	
	ϵ	3.2281	E_{20}^4	.672	9.6486	i_{20}^4
		3.2575	E_{20}^4	.669	9.6405	i_{20}^4
	η	3.87		.60	9.475	
5-period Envelope	ζ	3.15		.80	10.192	
	ϵ	3.726	E_{20}^5	.741	10.1239	i_{20}^5
		3.7763	E_{20}^5	.736	10.1180	i_{20}^5
	η	4.82		.63	9.996	

26/ All paths leading to the 20-period envelope are given by the path $\epsilon\epsilon$. Those leading to the 10-period envelope are included between the paths $\zeta\zeta$ and $\eta\eta$.

TABLE 3 (for Figure 8)

	Point	Y_1^t		Y_2^t	p^t	
5-period Envelope	a	.17		1.08	10.561	
	α	3.15		.78	10.192	
	γ	3.67		.75	10.130	
		3.73	E_{20}^{5}	.74	10.124	i_{20}^{5}
	ϵ	3.78		.74	10.118	
	δ	3.95		.72	10.098	
	β	4.82		.63	9.996	
	b	9.72		.13	9.468	
10-period Envelope	α	.33		3.58	6.7660	
	γ	7.20		2.57	6.7591	
	ϵ_2	7.97	E_{20}^{10}	2.45	6.7584	i_{20}^{10}
	ϵ_2	8.64		2.35	6.7577	
	δ	10.93		2.02	6.7554	
	β	22.73		.27	6.7439	
15-period Envelope	γ	9.09		27.23	3.21034	
	ϵ_1	27.11	E_{20}^{15}	21.62	3.21028	i_{20}^{15}
	ϵ_2	42.80		16.73	3.21023	
	δ	96.46		.01	3.21005	
20-period Envelope	ϵ_1	17.48	E_{20}^{20}	559.58	1.440145	i_{20}^{20}
	ϵ_2	814.13		6.40	1.440141	

	Length of the interval of the t-period price-ratios	Length of i_{20}^{t}
5	1.093	.006
10	.0221	.0007
15	.00029	.00005
20	.000004	.000004

TABLE 4 27/

	Point	Y_1^t		Y_2^t	p^t	
5-period Envelope	γ	3.67		.75	10.130	
	ϵ_1	3.73	E_{20}^{5}	.74	10.124	i_{20}^{5}
	ϵ_2	3.78		.74	10.118	
	δ	3.95		.72	10.098	
10-period Envelope	γ	7.25		2.56	6.7591	
	ϵ_1	7.96	E_{20}^{10}	2.46	6.7584	i_{20}^{10}
	ϵ_2	8.63		2.36	6.7577	
	δ	10.91		2.02	6.7554	
15-period Envelope	γ	26.95		16.28	13.9522	
	ϵ_1	70.91	E_{20}^{15}	13.13	13.9520	i_{20}^{15}
	ϵ_2	112.29		1.16	13.9517	
	δ	252.74		.10	13.9510	
20-period Envelope	ϵ_1	97.45	E_{20}^{20}	157.36	59.08440	i_{20}^{20}
	ϵ_2	9200.83		3.28	59.08427	

27/ The paths recorded in Table 4 have the same relative position in the 15-, and 20-period envelopes for this example as the corresponding paths in Table 3.

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