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ON THE CONSUMER'S LIFETIME ALLOCATION PROCESS

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by

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1. Introduction

Recent investigations of the allocation of a consumer's resources over time tend, in one way or another, to neglect the consumer's bequest motive. By the bequest motive we have in mind the hypothesis that consumer preferences take into account not only the rates of consumption of goods and services but also the stock of unconsumed resources at the end of the planning period. When the planning period is taken to be an arbitrary time interval, the term "accumulation motive" may be more suitable than the term "bequest motive"; it signifies the consumer's concern with the future which lies beyond the arbitrarily chosen horizon. Since Irving Fisher's analysis of consumer allocation over time [3] it has become customary to impose on the consumer a restraint requiring that net accumulation over the planning period be zero. (In cases where an infinite horizon is assumed, the limit of net indebtedness is required to exist and be equal to zero.) Fisher himself was not unaware of the bequest motive, as indicated by various comments scattered in his book, but his main emphasis was on saving as a means to achieving a desired consumption profile over time. Marshall, on the other hand, refers to the

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bequest motive as "the chief motive of saving" [7, p. 228]. Some of the more recent studies (e.g., Tobin [10]) take full account of the bequest motive, but the studies which have received the most attention seem, as has already been remarked, to neglect it. Duesenberry [1, p. 34] refers to the rational motives of saving as being the provision for retirement and the provision for contingencies. Any accumulation motive, which is referred to as "having assets merely for the sake of having assets" is considered irrational. Friedman [4 , p. 9], in the theoretical sketch which precedes the main part of his study, says: "If we suppose that the two years stand for the whole future for which plans are being made, there is nothing the [consumer] unit can gain by not spending all it receives." Finally, Modigliani and Brumberg [8], while recognizing the importance of the bequest motive (the "estate motive" in their terminology) then proceed to disregard it by assumption. Assumption I of their study states that net lifetime accumulation by the consumer must be equal to zero.

In the present essay an attempt is made to study consumer allocation over time in a framework which allows explicit recognition of the bequest motive. As will be shown, the traditional Fisher-type case, i.e., the case where net lifetime accumulation enters as a budget constraint rather than explicitly in the consumer's preferences, can in some sense be thought of as a special case of the bequest-motive case. Hence, many of the results which will be presented are valid under both hypotheses, and may be of some interest regardless of the bequest motive issue.

The present study is entirely under the assumption of perfect certainty. The effects of introducing in the analysis various elements of uncertainty will be examined at a later time. A preliminary version of this supplementary discussion can be found in [11].

2. Notation and Assumptions

Let T be the consumer unit's horizon. T is an arbitrary non-negative real number. The closed interval $[0, T]$ will be referred to as the consumer's lifetime and alternatively as the consumers planning-period.

The unit's consumption plan is a real valued function c on $[0, T]$. $c(t)$ is required to be non-negative for all t in $[0, T]$.

The consumer unit expects with certainty that a rate of interest of $j(t)$ per unit time will prevail at time t. j is assumed to be a bounded, continuous real function on $[0, T]$.

The unit's income stream (i.e., the stream of earnings other than interest) is a real valued function m on $[0, T]$. We require of the function m that it be Riemann-integrable, and we shall assume that it is, for the most part, positive. For convenience we assume that $m(t)$ and $c(t)$ are measured in the same unit.

Assuming that the only assets which can be held are notes bearing a rate of interest $j(t)$ at time t, we have that the consumer's assets at time t, $S(t)$, are given by

$$(1) \qquad S(t) = S(0) + \int_0^t e^{\int_\tau^t j(u)du} \left\{m(\tau) - c(\tau)\right\} d\tau ,$$

for t in $[0, T]$. For typographical convenience, we shall make two assumptions: (i) Initial assets are zero; $S(0) = 0$. (ii) The rate of interest is constant on $[0, T]$; $j(t) = j$, $0 \leqq t \leqq T$. The definition of $S(t)$ now reduces to

$$(2) \qquad S(t) = \int_0^t e^{j(t-\tau)} \left\{ m(\tau) - c(\tau) \right\} d\tau , \quad 0 \leqq t \leqq T .$$

The assumption concerning the constancy of the rate of interest will be reconsidered when the time comes to examine the effects of changes in the rate on optimal behavior.

The consumer's <u>bequests</u> are given by the quantity $S(T)$. It will be convenient to express $S(T)$ in terms of the consumer's <u>lifetime wealth</u>, to be denoted M:

$$(3) \qquad M = \int_0^T e^{j(T-t)} m(t) dt$$

$$(4) \qquad S(T) = M - \int_0^T e^{j(T-t)} c(t) dt .$$

Turning now to the consumer's preferences, our first assumption is that they can be represented by a utility function. Clearly, this assumption alone does not carry us very far. In models based on discrete time-parameter a common assumption is that the utility function is homogeneous of arbitrary degree (e.g. Friedman [4] and Modigliani-Brumberg [8].) In models of continuous time-parameter the common assumption is that preferences are independent over time, i.e., that consumption in one period does not affect the preference-ranking of consumption alternatives in other periods. This assumption allows one to write the utility function in the form of a simple integral, and since integrals are the most studied functionals on function spaces, this procedure is very convenient. Let V denote the utility functional. Under the independence assumption, V can be written as follows:

$$V(c) = \int_0^T \psi[t, c(t)]dt \,, \tag{5}$$

where the function ψ can be given the interpretation of a utility function for each moment of time. As a matter of convenience rather than necessity we shall assume further that ψ can be written as a product of two functions:

$$\psi(x, y) = \alpha(x)\, g\,(y) \,, \tag{6}$$

so that

$$V(c) = \int_0^T \alpha(t)\, g[c(t)]\, dt \,. \tag{7}$$

The function α has the interpretation of a <u>subjective</u> <u>discount</u> function, and we assume that it is non-negative and once differentiable. We may also normalize α in such a way that $\alpha(0) = 1$.

The function g is interpreted as a utility associated with the current rate of consumption at every moment of time. g is defined on the half-line $[0,\infty)$ and we assume that it is twice differentiable and strictly concave.

Strict rather than weak concavity is by no means an indispensable assumption, but it has the advantage of guaranteeing that the optimal consumption plan (if it exists) will be unique and continuous.

The functional V, as it now stands, clearly does not reflect any bequest motive in the consumer's preferences. When V is the relevant functional, the consumer's decision problem is customarily stated to be the maximization of V subject to a Fisher-constraint of the type $S(T) \geqq 0$. Accordingly, we shall refer to V as the *Fisher-constraint utility functional*.

To incorporate bequests in the utility functional, we add to V one more term and thus obtain another functional, to be denoted U:

$$(8) \qquad U(c) = \int_0^T \alpha(t)\, g[c(t)]\, dt + \varphi[S(T)] ,$$

where φ is a real function defined for all real values and assumed to be twice differentiable and concave. φ is to be interpreted as the utility of bequests. To distinguish the functional U from the Fisher constraint functional V, we shall refer to U as the *bequest motive utility functional*.

A word needs to be said about the monotonicity properties of the functions g and φ. In Fisher-constraint problems it is common to make a non-saturation assumption in order to convert the inequality constraint, $S(T) \geqq 0$, into an equality constraint, $S(T) = 0$. Accordingly, in all Fisher-constraint cases we shall be assuming that the function g is monotone increasing. However, in the bequest-motive cases a weaker assumption will suffice. The assumption which we shall make is that at least one of the two functions g and φ is monotone increasing. In fact, it is often more reasonable to assume that φ is monotone than that g is.

3. Optimal Behavior

In this section we shall examine both the Fisher-constraint case and the bequest-motive case with a view towards achieving a characterization of optimal behavior. The Fisher-constraint, i.e., the assertion that the consumer's lifetime accumulation is equal to zero, is sometimes adopted as a "simplifying assumption" (as, for instance, in Modigliani-Brumberg [8].) It will become clear as we proceed that, at least in the present framework, the Fisher-constraint offers no such gain in simplicity. If anything, the case of the bequest-motive is the simpler of the two. This observation, however, is merely a by-product of the main effort of this section which is to draw the details of the optimal consumption plan. Most of the characteristics of the optimal plan will be common to both the Fisher-constraint and the bequest-motive cases.

3.1 The Fisher-Constraint Case:

Consider the problem

$$\text{(9)} \qquad \max V(c)$$

$$\text{subject to:} \quad c(t) \geqq 0 \qquad 0 \leqq t \leqq T ,$$

$$S(T) \geqq 0 .$$

With the definitions of the various symbols, and under the assumption that g is monotone, (9) reduces to:

$$(10) \qquad \max \int_0^T \alpha(t)\, g[c(t)]dt$$

$$\text{subject to:} \quad c(t) \geqq 0 \qquad 0 \leqq t \leqq T ,$$

$$\int_0^T e^{j(T-t)}\, c(t)dt = M .$$

In a problem of this sort, one's first concern is usually the question of the existence of a solution. More precisely, one has to specify exactly the class of functions c which will be admitted into competition in the maximization, and then attempt to state conditions under which a maximum is attained in this class. Such an admissibility class is not, in general, a finite-parameter family and for this reason the investigation of attainment of a maximum is a rather difficult task. This task will be undertaken separately, and for present purposes we shall limit the discussion to stating conditions which a solution must satisfy if it exists.

It can be shown that essentially without loss of generality we can pick as our admissibility class the class of all functions which are non-negative on $[0, T]$ and right-continuous on $[0, T)$. To find a first order condition for a maximum one proceeds as follows: Denote the solution, i.e., the optimal consumption plan, if it exists, by c^*. Let R be the set on which c^* is strictly positive. By right-continuity, R must be the union of a set of half-open intervals from $[0, T]$, or else it coincides with $[0, T]$. Now let x be a function on $[0, T]$ which satisfies the following conditions

$$(11) \qquad a. \quad x(t) = 0 \quad \text{outside } R$$

b. $\int_0^T e^{j(T-t)} x(t)dt = 0$.

Aside from these two restrictions, x is an arbitrarily chosen right-continuous function. Consider now the consumption plan $\bar{c} = c^* + \epsilon x$, where ϵ is a real number. This consumption plan meets the Fisher-constraint $S(T) = 0$, and for small ϵ it is certain to be non-negative. In short, $\bar{c}$ is admissible for small ϵ . The utility of the plan $\bar{c}$, $V(\bar{c})$, is given by

(12) $$V(\bar{c}) = \int_0^T \alpha(t)\ g[c^*(t) + \epsilon x(t)]dt \ .$$

For small ϵ , $V(\bar{c})$ may be approximated as follows:

(13) $$V(\bar{c}) = V(c^*) + \epsilon \left.\frac{dV(\bar{c})}{d\epsilon}\right|_{\epsilon=0}$$

$$= V(c^*) + \epsilon \int_0^T \alpha(t)\ g'[c^*(t)]x(t)dt \ .$$

A necessary condition for a maximum is that $\left.\frac{dV(\bar{c})}{d\epsilon}\right|_{\epsilon=0}$ vanish for all admissible choices of the function x :

(14) $$\int_0^T \alpha(t)\ g'[c^*(t)]x(t)dt = 0$$

or

(15) $$\int_0^T \left(e^{j(t-T)}\ \alpha(t)\ g'[c^*(t)]\right)\left(e^{j(T-t)}\ x(t)\right) dt = 0$$

for all choices of x satisfying (11). In view of condition b in (11), the only way in which (15) can hold for all possible x is for the following condition to be satisfied:

$$(16) \qquad e^{j(t-T)}\,\alpha(t)\,g'[c^*(t)] = k \qquad \text{for all } t \text{ in } R\,,$$

where k is a constant. Condition (16) says that the discounted marginal utility of consumption, at every point where consumption is not zero, should be made equal to a constant. This constant is usually given the interpretation of the marginal utility of wealth.

Concerning the segments in which the consumption levels are zero, i.e., the segments outside the set R, one can state the following:

$$(17) \qquad e^{j(t-T)}\,\alpha(t)\,g'[c^*(t)] \leqq k \qquad \text{for all } t \text{ outside } R\,.$$

To prove this assertion assume the contrary:

$$(18) \qquad e^{j(t_o-T)}\,\alpha(t_o)\,g'[c^*(t_o)] > k \qquad \text{for some } t_o \text{ outside } R\,.$$

Clearly, there exists an interval I_o about t_o such that (18) hold everywhere in I_o. However, another interval I_1, whose length is equal to that of I_o, can be found (in the set R) such that

$$(19) \qquad e^{j(t-T)}\,\alpha(t)\,g'[c^*(t)] = k \qquad \text{for all } t \text{ in } I_1\,.$$

Now construct the function x as follows:

$$(20) \qquad \begin{aligned} x(t) &= e^{j(t-T)} && \text{for } t \text{ in } I_o \\ &= -\,e^{j(t-T)} && \text{for } t \text{ in } I_1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Consider the consumption plan $\bar{c} = c^* + \epsilon x$. Once again, this plan is admissible for small and positive ϵ , and it satisfies the Fisher constraint. Hence, for small $\epsilon > 0$ we can write the approximation

$$(21)\quad V(\bar{c}) = V(c^*) + \epsilon \left[\int_{I_o} \alpha(t)\ g'[c^*(t)]e^{j(t-T)}dt - \int_{I_1} \alpha(t)\ g'[c^*(t)]e^{j(t-T)}dt \right],$$

but by construction the expression in brackets is positive, contradictory to the optimality of c^* . Thus condition (17) is established. Conditions (16) and (17) may now be merged to obtain:

$$(22)\quad \begin{aligned} & e^{j(t-T)}\ \alpha(t)\ g'[c^*(t)] \leqq k \\ & c^*(t) = 0 \quad \text{whenever} \quad < \ \text{holds} . \end{aligned}$$

Next, it is important to investigate the continuity of the optimal plan c^* (if it exists). Under the assumption that the function g is strictly concave, we can assert that <u>the optimal consumption plan</u>, c^* , <u>is continuous</u>.

<u>Proof</u>: Since c^* is right-continuous, a discontinuity at some point t would mean:

$$(23)\qquad c^*(t) \neq c^*(t-0) .$$

If g has a strictly decreasing first derivative, (23) implies that

$$(24)\qquad g'[c^*(t)] \neq g'[c^*(t-0)] .$$

Since c^* satisfies (22) and both $\alpha(t)$ and $e^{j(t-T)}$ are values of continuous functions of t , there are only two ways in which (24) could hold.

Either

$$e^{j(t-T)} \alpha(t) \; g'[c^*(t)] = k \tag{25}$$

while $$e^{j(t-T)} \alpha(t) \; g'[c^*(t-0)] < k$$

or

$$e^{j(t-T)} \alpha(t) \; g'[c^*(t)] < k \tag{26}$$

while $$e^{j(t-T)} \alpha(t) \; g'[c^*(t-0)] = k \; .$$

The case where both $e^{j(t-T)}\alpha(t) \; g'[c^*(t)]$ and $e^{j(t-T)}\alpha(t) \; g'[c^*(t-0)]$ are less than k is ruled out because then both $c^*(t)$ and $c^*(t-0)$ are zero and there is no discontinuity.

Consider (25). It implies that

$$g'[c^*(t-0)] < g'[c^*(t)] \; , \tag{27}$$

but on the other hand, in view of (22), it also implies that

$$c^*(t) > 0 \quad \text{while} \quad c^*(t-0) = 0 \tag{28}$$

and (27) together with (28) contradict the strict concavity of g. The argument is entirely analogous in the case of (26). This completes the proof that the optimal plan must be continuous.

The next item in the investigation of the optimal plan is its monotonicity properties. We shall find it easiest to determine these properties by first deriving a basic differential equation in c^* . In order to do so, we must differentiate equation (16) with respect to t , so our first question concerns the legitimacy of such differentiation. Under the assumption that g has a second derivative everywhere, the only problem is the differentiability of the optimal plan c^* itself. In this respect we assert that <u>if the function g has a second derivative which is everywhere negative then the optimal plan c^* is differentiable on R</u> .

<u>Proof</u>: Since $e^{j(t-T)}$, $\alpha(t)$ and k are all differentiable with respect to t , equation (16) implies that $\frac{d}{dt}g'[c^*(t)]$ exists for t in R . By the mean value theorem, we can write

$$(29) \quad \frac{g'[c^*(t+h)] - g'[c^*(t)]}{h} = \frac{c^*(t+h) - c^*(t)}{h} g''[c^*(t) + \omega]$$

where $\omega \to 0$ as $h \to 0$. Since the limit as $h \to 0$ of the left-hand-side of (29) exists, the limit of the right-hand-side must also exist. Hence, if $g'' < 0$ the derivative of c^* at t exists.

Upon differentiation of (16), and letting a dot above a symbol denote differentiation with respect to time, one obtains

$$(30) \quad je^{j(t-T)}\alpha(t)\, g'[c^*(t)] + e^{j(t-T)}\dot{\alpha}(t)\, g'[c^*(g)] + e^{j(t-T)}\alpha(t)\, g''[c^*(t)]\dot{c}^*(t) = 0$$

which reduces to

$$(31) \quad \dot{c}^*(t) = -\left[\dot{j} + \frac{\dot{\alpha}(t)}{\alpha(t)}\right]\frac{g'[c^*(t)]}{g''[c^*(t)]}, \quad t \text{ in } R .$$

Equation (31) is essentially the Euler differential equation for the problem at hand. Computing the actual optimal plan would, in most specific cases, involve the solution of (31), followed by a reconciliation of the constant of integration with both the parameter M and the requirement that c^* be non-negative.

Equation (31) provides information about the monotonicity of the optimal plan. In view of the monotonicity and the strict concavity of g, (31) implies that whenever $c^*(t)$ is not zero,

$$\text{(32)} \qquad \operatorname{sgn} \dot{c}^*(t) = \operatorname{sgn} \left[\dot{j} + \frac{\dot{\alpha}(t)}{\alpha(t)} \right].$$

$-\dot{\alpha}(t)/\alpha(t)$ is the rate of subjective discount at time t, so what (32) says is that the optimal consumption plan is increasing, constant or decreasing according to whether the rate of interest is greater than, equal to or less than the rate of subjective discount. In short, the optimal consumption plan has the same monotonicity as does $\alpha(t)e^{jt}$. This fact can be shown to hold also under weaker assumptions than twice differentiability and strict concavity of g. In other words, this property holds also when the optimal plan is not necessarily continuous.

Before making further comments, we turn to the case where the consumer's preferences contain a bequest-motive.

3.2 The Bequest-Motive Case:

Here the problem is to maximize $U(c)$, as defined in (8), subject only to the requirement that c be non-negative.

$$(33) \qquad \max \left\{ \int_0^T \alpha(t) \quad g[c(t)]dt + \varphi[S(T)] \right\}$$

$$\text{subject to} \quad c(t) \geqq 0 \,, \quad 0 \leqq t \leqq T \,.$$

Once again, we take as admissible any non-negative plan which is right-continuous on $[0, T)$. Suppose that an optimal plan c^* exists in this admissibility class. Let the region on which this optimal plan is positive be denoted R. Further, let x be an admissible function which vanishes outside R and is arbitrary otherwise, and form the consumption plan $\bar{c}$, where $\bar{c}(t) = c^*(t) + \epsilon x(t)$ for all t in $[0, T]$. For small ϵ, $\bar{c}$ is an admissible plan. The utility of $\bar{c}$ is given by

$$(34) \qquad U(\bar{c}) = \int_0^T \alpha(t) \quad g[c^*(t) + \epsilon x(t)]dt + \varphi\left[M - \int_0^T e^{j(T-t)} \left\{c^*(t) + \epsilon x(t)\right\} dt\right]$$

and a first-order condition for a maximum is obtained by requiring that

$\left.\frac{dU(\bar{c})}{d\epsilon}\right|_{\epsilon=0}$ vanish for all choices of the function x.

$$(35) \qquad \left.\frac{dU(\bar{c})}{d\epsilon}\right|_{\epsilon=0} = \int_0^T \alpha(t) \quad g'[c^*(t)]dt - \varphi'[S^*(T)] \int_0^T e^{j(T-t)} \, x(t)dt \,,$$

where $S^*(T)$ is the bequest level resulting from the adoption of the consumption plan c^*.

The desired necessary condition for a maximum is now obtained directly from (35), and one gets

$$(36) \qquad \alpha(t) \;\; g'[c^*(t)] = \varphi'[S^*(T)]e^{j(T-t)} \;, \quad t \;\text{ in }\; R \;.$$

This last condition is the same as (16), the analogue in the Fisher-constraint case, with $\varphi'[S^*(T)]$ taking the place of the constant k . From here on, theorems and proofs follow exactly the lines of their analogues in the Fisher-constraint case.

The complete first-order condition for a maximum is given by

$$(37) \qquad e^{j(t-T)} \, \alpha(t) \;\; g'[c^*(t)] \leqq \varphi'[S^*(T)]$$

$$c^*(t) = 0 \quad \text{whenever} \quad < \quad \text{holds} \;,$$

for all t in $[0, T]$. In the event that g is strictly concave, the optimal plan c^* is continuous everywhere and differentiable wherever it is positive. The differential equation (31) holds without modification, and the optimal plan once again has the same monotonicity as $\alpha(t)e^{jt}$.

3.3 A comparison of the Two Approaches:

The bequest-motive approach is more general than the Fisher-constraint approach simply because it does not assume a Fisher constraint. This fact is important in itself if we are to explain, say, the existence of positive personal savings in an economy where such a phenomenon cannot be attributable to a shift in population structure or to a shift in tastes. Also from the theoretical point of view, the Fisher constraint has some rather restrictive implications (in the area of the wealth-elasticity of consumption) which do not go over into the bequest-motive case.

From the point of view of simplicity, both approaches are not very intricate, and much of the analysis in both is identical. The only difference is that in the bequest-motive case we have an unconstrained maximization (except for the non-negativity requirement) and unconstrained maximizations are usually considered simpler than constrained maximizations.

Finally, it may be of some interest to mention how the consumption plan of the Fisher-constraint case can actually be derived as a limiting case of the plan in the bequest-motive case. Let the utility-of-bequests function φ be given by

$$(38) \qquad \varphi(x) = \nu \cdot \psi(x) \; , \qquad x \geqq 0$$

where ν is a positive real number and ψ is such that

$$(39) \qquad \lim_{x \to 0} \psi'(x) = +\infty \; .$$

The optimal plan c^* under this definition of φ is a function of the parameter ν . Let us denote this optimal plan c^*_ν . If we now let ν approach 0 , the optimal plan c^*_ν will approach some plan c^*_o uniformly, and c^*_o will be the optimal plan for the Fisher-constraint problem. At the present moment I can only state this as a conjecture, because the existence of regions of zero consumption makes this proposition somewhat difficult to prove. In any case, note that as ν varies, the optimal plan c^*_ν constantly satisfies the differential equation (31), and the changes in ν affect only the constant of integration. Under conditions which are necessary to ensure the existence and uniqueness of the optimal plan it can be shown that the plan is continuous, in a properly defined sense, in any parameter which affects only the constant of integration.

As ν approaches 0, $S^*(T)$ approaches 0 also (if g is monotone) and the product $\nu\psi'[S^*(T)]$ approaches the value of the Lagrangian constant k. This shows once again that the bequest-motive approach is more general than the Fisher-constraint approach.

To sum up we have reached a fairly detailed description of the optimal plan under both the Fisher-constraint and the bequest motive cases. For example, if $\alpha(t)e^{jt}$ is monotone increasing in $[0, T]$, then the optimal plan is also monotone increasing and may have at most one interval of zero consumption which is necessarily connected to the origin $t = 0$; if $\alpha(t)e^{jt}$ is monotone decreasing in $[0, T]$, then the optimal plan is likewise decreasing and may have at most one interval of zero consumption, necessarily connected to the extreme $t = T$. In this manner we are able to describe the general characteristics of the optimal plan for any time-shape of $\alpha(t)e^{jt}$ even without being given exact specifications of the functions g and φ.

Finally, intervals of zero consumption-rate are not only analytically bothersome, but also economically rather untenable. For this reason one may choose in many examples to assume that the function g is such that $g'(x) \to +\infty$ as $x \to 0$. This would prevent the optimal consumption level from dropping to zero so long as the subjective discount function α is positive.

4. Wealth Effects and Income Effects

It is common to investigate the effects of changes in the parameters of a system by differentiating the optimal decision rule with respect to the parameter. In the present case, however, the optimal decision rule is an element in a function space, and it is not clear what is meant by differentiation of it with respect to, say, M -- the wealth parameter. To define the derivative of the optimal plan $c^*(t)$ with respect to M we proceed as follows: Let $c^*(t, h)$ be the optimal plan corresponding to a wealth-level of $M+h$. Thus, $c^*(t, 0)$ is what we have been referring to as $c^*(t)$. A function $c_M(t)$ will be called the derivative of c^* with respect to M if for every $\epsilon > 0$ there is an h such that

$$\rho \left[\frac{c^*(t,h) - c^*(t,0)}{h} , \quad c_M(t) \right] < \epsilon , \tag{40}$$

where the distance function ρ is defined in such a way that if x and y are functions on $[0, T]$, then

$$\rho(x, y) = \sup_{0 \leq t \leq T} | x(t) - y(t) | . \tag{41}$$

The existence and the properties of limits of this type is a matter whose investigation would take us far afield. Suffice it to say that under conditions where an optimal plan $c^*(t)$ exists and is unique, continuous and differentiable (wherever positive) a derivative $c_M(t)$ exists and is obtainable as an ordinary derivative of $c^*(t)$, for fixed t, with respect to M, taken for each t separately.

A second conceptual problem arises in connection with the definition of "income effects" in a world where what we have called the income stream does not in itself play any role whatever. Under perfect certainty, income affects consumption only through its effects on wealth. However, in this kind of a world income is also <u>defined</u> in terms of wealth. The function $m(t)$, which Friedman calls "measured income" is, theoretically speaking, not income. As a definition of income consider first <u>Hicksian income</u> [6, Chapter XIV] to be denoted $h(t)$:

(42) $$h(t) = je^{j(t-T)}M \, , \qquad 0 \leqq t \leqq T \, .$$

Hicksian income is defined as the rate at which the consumer could consume while keeping his wealth intact. Secondly, consider <u>normal income</u> (following Farrell [2]) to be denoted n:

(43) $$n = \frac{jM}{e^{jT} - 1} \, .$$

Normal income is that constant stream which, if accumulated and compounded at the market rate of interest j, would accumulate to become equal to M. In some sense, Hicksian income is "net" while normal income is "gross."

These definitions of income, while by no means the only ones possible, serve to show that income would most likely be defined as some fraction of M, whether a constant fraction or one which varies with time. The analysis of how changes in M affect the optimal plan, i.e., the analysis of wealth effects, is therefore an analysis of income effects as well.

We shall analyze the wealth and income effects in the framework of the bequest-motive case. The analogous results for the Fisher-constraint case are, in most instances, immediately obtainable. To start off, let f denote total attainable utility, which in the present section is taken as a function of wealth. Thus, $f(M)$ is given by

$$f(M) = \int_0^T \alpha(t) \; g[c^*(t)]dt + \varphi[S^*(T)] \; . \tag{44}$$

Let $c_M(t)$ be the derivative (in the above sense) of c^* with respect to M (taken at the point M). We note that $c_M(t) = 0$ if $c^*(t)$ is a corner, $c^*(t) = 0$, at t. Let S_M be the derivative of $S^*(T)$ with respect to M:

$$S_M = 1 - \int_0^T e^{j(T-t)} \; c_M(t)dt \; . \tag{45}$$

Differentiating f with respect to M yields

$$f'(M) = \int_0^T \alpha(t) \; g'[c^*(t)]c_M(t)dt + S_M \, \varphi'[S^*(T)] \; . \tag{46}$$

Using (22) and (45), we obtain:

$$(47) \qquad f'(M) = \varphi'[S^*(T)]$$

$$= e^{j(t-T)}\ \alpha(t)\ \ g'[c^*(t)]\ , \quad \text{for all } t \text{ in } R\ .$$

Hence, $f'(M)$ is positive.

By differentiating (46) again with respect to M and simplifying, we obtain an expression for $f''(M)$:

$$(48) \qquad f''(M) = \int_0^T \alpha(t)\ \ g''[c^*(t)]c_M^2(t)dt + S_M^2\ \varphi''[S^*(T)]$$

which is negative. Maximum attainable utility, as expected, is an increasing and strictly concave function of wealth.

To get the analogues of the above relationships for the Fisher-constraint case, simply substitute 0 for S_M and k for $\varphi'[S^*(T)]$.

If we evaluate $f''(M)$ from (47) rather than from (48) and use the knowledge that $f''(M) < 0$, we get

$$(49) \qquad c_M(t) \geqq 0 \quad \text{for all } t \text{ in } [0, T]$$

$$> 0 \quad \text{for } t \text{ in } R\ ,$$

and also

$$(50) \qquad S_M > 0\ ,$$

with $S_M = 0$ for the Fisher-constraint case. Equations (45) and (50) together imply

$$(51) \qquad 0 < \int_0^T e^{j(T-t)} c_M(t)\,dt < 1$$

where the right-hand inequality becomes an equality in the Fisher-constraint case.

The quantity S_M is in a sense a measure of what we ordinarily refer to as the marginal propensity to save. More precisely, $1 - S_M$ is some average of the consumer's marginal propensities to consume over his lifetime. To show this, let income be defined as <u>normal income</u> n of equation (43). Let $c_n(t)$ be the derivative of c^* with respect to n . By definition,

$$(52) \qquad c_M(t) = \frac{c_n(t)}{\int_0^T e^{j(T-t)}dt} .$$

Hence, by (45)

$$(53) \qquad 1 - S_M = \frac{\int_0^T e^{j(T-t)}\, c_n(t)dt}{\int_0^T e^{j(T-t)}\, dt} ,$$

which expresses $1 - S_M$ as a weighted arithmetic average of $c_n(t)$. For small j , the following approximation holds:

$$(54) \qquad 1 - S_M = \frac{1}{T} \int_0^T c_n(t)dt .$$

If we could estimate the quantity $1 - S_M$ from actual samples, we would have an estimate of the marginal propensity to consume which would be free of bias due to transitory effects on income. Unfortunately, the data needed for making such an estimate are not readily available. Efforts in this direction are currently being undertaken.

It is of course quite logical to require that $1 - S_M$ be, in some sense, an average of the marginal propensities to consume, since $1 - S_M$ is the rate of change of accumulated lifetime spending with accumulated lifetime earnings. Let us therefore consider the marginal propensities to consume when income is defined to be Hicksian income of (42) and then ask what happens if we require that $1 - S_M$ be an average of these. Let $c_h(t)$ be the derivative at time t of the optimal consumption level with respect to Hicksian income. $c_h(t)$ is related to $c_M(t)$ by

$$(55) \qquad c_h(t) = \frac{c_M(t)}{je^{j(t-T)}} .$$

Hence, by (45)

$$(56) \qquad 1 - S_M = j \int_0^T c_h(t)dt .$$

If we now require $1 - S_M$ to be a simple arithmetic mean of $c_h(t)$, we get the awkward requirement

$$(57) \qquad j = \frac{1}{T} .$$

In my opinion, this shows the inadequacy of Hicksian income. This, however, is far from being an accepted opinion. Friedman, for instance, defines income both in [4] and in [5] as Hicksian income, and indeed, he requires in [5] that the horizon and the rate of interest be reciprocals of each other. This causes at least one of these two to become rather artificial. In this case, since Friedman has evidence to support the hypothesis that $T = 3$, he has to construct a personal rate of interest (not to be confused with a subjective discount rate) which is defined as the own-rate of the consumer's human and non-human wealth, which for some mysterious reason is approximately equal to 1/3. Friedman arrives at this reciprocal relationship in a manner which is quite different from ours. In fact, he defines the "personal rate of interest" as $1/T$. However, the argument that this rate represents the intrinsic rate of interest of the consumer's wealth derives, in my opinion, from thinking in terms of Hicksian income.

Let us turn now to a brief investigation of the elasticities of c^* and $S^*(T)$ with respect to the parameter M. These elasticities, denoted $\eta_{cM}(t)$ and η_{SM}, are defined as follows:

$$\eta_{cM}(t) = \frac{Mc_M(t)}{c^*(t)} , \qquad t \text{ in } R \tag{58}$$

$$\eta_{SM} = \frac{MS_M}{S^*(T)} .$$

Note that $\eta_{cM}(t)$ is at the same time the wealth-elasticity of consumption and the income-elasticity of consumption, provided income is defined as some constant or variable fraction of M.

Consider first the Fisher-constraint case. What the Fisher constraint requires is that

(59) $$\int_R e^{j(T-t)} c^*(t)dt = M .$$

Differentiating with respect to M and expressing the result in terms of elasticities, we get

(60) $$\int_R e^{j(T-t)} c^*(t) \eta_{cM}(t)dt = M .$$

Comparing (59) and (60) we see immediately that either $\eta_{cM}(t)$ is equal to one for all t or else $\eta_{cM}(t)$ is greater than one for some t and less than one for some t. This is a somewhat disturbing result if one believes that there should be a possibility for <u>all</u> $\eta_{cM}(t)$ to be less than unity. The Fisher constraint precludes this possibility.

A somewhat similar result is obtained by Herbert Scarf in [9], where he proves that wealth-elasticity of consumption cannot be less than one for all rates of interest. However, Scarf's definition of wealth is different from ours, and it corresponds more to a dynamic programming view of the allocation process than to our variational view of it. In contrast with our definition of wealth at time t, which is simply $e^{j(t-T)}M$, Scarf defines wealth at time t, $W(t)$, as the solution of the following differential equation

(61) $$\dot{W}(t) = jW(t) - c(t)$$

$$W(0) = e^{-jT}M .$$

This is the consumer's wealth, in the ex-post sense, at time t . It is net of consumption outlays which will have taken place up to time t . When taking elasticities of the consumption levels with respect to W , Scarf finds that they cannot always be less than unity under the Fisher constraint. There is no immediate relationship between his result and ours, since the elasticities are defined with respect to different parameters.

In the bequest-motive case, however, there is no reason to reject a-priori the hypothesis that $\eta_{cM}(t)$ is less than unity for all t . The analogue of (59) in this case is

$$\int_R e^{j(T-t)} c^*(t)dt = M - S^*(T) , \tag{62}$$

and differentiation with respect to M followed by conversion to elasticities yields

$$\int_R e^{j(T-t)} c^*(t) \ \eta_{cM}(t)dt = M - S^*(T)\eta_{SM} . \tag{63}$$

It is now possible for η_{SM} to be greater than unity and for $\eta_{cM}(t)$ to be less than unity for all t .

Finally, we can obtain one more result by differentiating the marginal utility condition (36) with respect to M and then convert to elasticities:

$$\alpha(t)c^*(t) \ [\text{illegible}]''[c^*(t)]\eta_{cM}(t) = e^{j(T-t)} \ S^*(T)\varphi''[S^*(T)]\eta_{SM} , \quad t \text{ in } R . \tag{64}$$

Dividing each side of (64) by the corresponding side of (36), one obtains

$$(65) \qquad \frac{c^*(t)g''[c^*(t)]}{g'[c^*(t)]}\,\eta_{cM}(t) = \frac{S^*(T)\varphi''[S^*(T)]}{\varphi'[S^*(T)]}\,\eta_{SM} \quad \text{for all } t \text{ in } R .$$

Now consider a Friedmanian consumption function:

$$(66) \qquad c^*(t) = \gamma(t) \,.\, M \qquad t \text{ in } [0, T] .$$

If this relationship holds, $\eta_{cM}(t) = 1$ for all t, and by (63) we have that $\eta_{SM} = 1$ as well. This means that

$$(67) \qquad S^*(T) = \lambda M$$

for some constant λ. Equation (65) now reduces to

$$(68) \qquad \frac{M\gamma(t)g''[M\gamma(t)]}{g'[M\gamma(t)]} = \frac{\lambda M\varphi''[\lambda M]}{\varphi'[\lambda M]} \quad \text{for all } t \text{ in } [0, T] .$$

We know, however, that $\gamma(t)$ is not, in general, constant since it depends among other things on α and on j. Hence it must be that $xg''(x)/g'(x)$ is constant for all x in the range of c^*. This means that $g(x)$ must be some linear transformation of either $\log x$ or x^{ℓ} for some $0 < \ell < 1$. Given that g is a member of this family, we may now vary M and obtain that $x\varphi''(x)/\varphi'(x)$ must also be constant. Hence φ must also be of the family of $\log x$ or x^{ℓ}. Thus, taking the consumption function to be of the type in (66) constitutes a severe restriction on the utility function.

5 Rate of Interest Effects

Until now, the assumption that the rate of interest is constant in $[0, T]$ has not caused any loss of generality. All the foregoing results remain virtually unchanged if we introduce a variable rate of interest $j(t)$. The only changes which have to be made to accommodate this variable rate of interest are substitutions of $e^{\int_t^T j(x)dx}$ and $j(t)$ for $e^{j(T-t)}$ and j respectively, wherever the latter two appear. However, when changes in the rate of interest are to be considered, having a single rate j which varies uniformly for all t in $[0, T]$ does indeed reduce the generality of the analysis. For instance, this assumption rules out cases in which the rate of interest increases for some t and simultaneously decreases for other t. The constancy assumption in effect ignores all considerations which have to do with the timing of changes in the rate of interest. However, in the present model, information about the timing of changes in the rate of interest turns out to be of no use whatever unless we know the time-profile of the earnings function $m(t)$. In the absence of such information, all we can get in the variable interest case is exactly what we can also get in the constant rate case. For this reason, we shall retain the assumption that the rate of interest is a constant, j, for all t.

The discussion of rate-of-interest effects will be entirely in terms of the bequest-motive case.

Let $c_j(t)$ denote the derivative of c^*, at time t, with respect to j. Once again, $c_j(t)$, taken for all t in $[0, T]$, can be shown to be a meaningful function for which the various operations which are performed below (e.g., integration) are well defined. Similarly, let S_j be the derivative of $S^*(T)$ with respect to j.

Differentiating the marginal utility condition (36) with respect to j, one obtains

$$(69)\quad \alpha(t)\ g''[c^*(t)]c_j(t) = (T-t)\alpha(t)\ g'[c^*(t)] + e^{j(T-t)}\varphi''[S^*(T)]S_j \quad \text{for } t \text{ in } R ,$$

$$(70)\qquad c_j(t) = (T-t)\ \frac{g'[c^*(t)]}{g''[c^*(t)]} + \frac{e^{j(T-t)}}{\alpha(t)}\ \frac{\varphi''[S^*(T)]}{g''[c^*(t)]}\ S_j \qquad t \text{ in } R .$$

For t outside R, $c_j(t)$ vanishes. We shall assume that at the point T the optimal consumption level is interior, i.e., that T is in R. Evaluating (70) at $t = T$:

$$(71)\qquad c_j(T) = \frac{\varphi''[S^*(T)]}{\alpha(T)g''[c^*(T)]}\ S_j ,$$

and hence --

$$(72)\quad c_j(t) = (T-t)\ \frac{g'[c^*(t)]}{g''[c^*(t)]} + e^{j(T-t)}\ \frac{\alpha(T)}{\alpha(t)}\ \frac{g''[c^*(T)]}{g''[c^*(t)]}\ c_j(T), \text{ for } t \text{ in } R .$$

From (71) and (72) we may derive two preliminary conclusions:

(73) $$\operatorname{sgn} S_j = \operatorname{sgn} c_j(T) \ ,$$

(74) If $c_j(T) \leqq 0$

Then $c_j(t) < 0$ for all t in R .

Consider next the change in $S^*(T)$ due to a change in j in and of itself, before revision of the optimal plan. We shall refer to the rate of this change as the partial derivative of $S^*(T)$ with respect to j and denote it by S^j:

(75) $$S^j = \int_o^T (T-t)\ e^{j(T-t)} \left\{ m(t) - c^*(t) \right\} dt \ .$$

The sign of S^j depends upon the time-profile of the saving stream $m(t) - c^*(t)$ which in turn depends on the time-profile of the earnings stream $m(t)$. For any optimal consumption plan there exist earnings streams which will make S^j either positive or negative. The relationship between S_j and S^j is given by

(76) $$S_j = S^j - \int_o^T e^{j(T-t)} c_j(t) dt \ .$$

Consider first the case $S^j \geqq 0$. We wish to show that under this assumption it is true that

$$(77) \qquad c_j(t) \geqq 0 \qquad \text{for some } t \,.$$

Assume the contrary:

$$(78) \qquad c_j(t) < 0 \qquad \text{for all } t \text{ in } [0, T] \,.$$

By (76) and by the fact that $S^j \geqq 0$ we have --

$$(79) \qquad s_j > 0 \,,$$

and hence, by (73) --

$$(80) \qquad c_j(T) > 0$$

contradicting (78). Thus, (77) is established which, by (73) and (74) also establish (79) and (80). In other words, in the case where $S^j \geqq 0$ we have that

$$(81) \qquad c_j(T) > 0$$

$$\text{and} \qquad s_j > 0 \,.$$

Under the assumption that $S^j \geqq 0$, a rise in the rate of interest stimulates both saving (bequests) and consumption, at least in periods near the horizon T .

The statement that a rise in the rate of interest "depresses consumption" is therefore not quite true. Such a rise stimulates lifetime saving but at the same time it stimulates consumption, possibly in all periods.

Now consider the case where $S^j < 0$. In a manner similar to the foregoing argument it can be shown that

$$c_j(t) < 0 \quad \text{for some } t \text{ in } [0, T]. \tag{82}$$

Hence, if $S^j < 0$ a rise in the rate of interest does depress consumption in some periods, but there is nothing to say that it does not also depress saving (i.e., that $S_j < 0$) or, for that matter, that it does not stimulate consumption in other periods.

Should the consumer hope for a rise or for a decline of the rate of interest? To answer this question we have to write total attainable utility f as a function of the rate of interest:

$$f(j) = \int_0^T \alpha(t)\, g[c^*(t)]dt + \varphi[S^*(T)] . \tag{83}$$

Differentiate with respect to j :

$$f'(j) = \int_0^T \alpha(t)\, g'[c^*(t)]c_j(t)dt + \varphi'[S^*(T)]\, S_j . \tag{84}$$

By (36), this equation reduces to

$$f'(j) = S^j \varphi'[S^*(T)] . \tag{85}$$

Hence --

(86) $$\operatorname{sgn} f'(j) = \operatorname{sgn} S^j .$$

If a rise of the rate of interest, in and of itself, causes the consumer's lifetime savings to become greater (which depends on the time-profile of his earnings stream) then he should hope for such a rise. If, on the other hand, such a rise causes lifetime savings to decline, then he should hope for a decline of the rate.

Finally, a point for which I am indebted to Arthur M. Okun: In considering the effects of changes in the rate of interest one should take into account the fact that at time T it may be the value of the consumer's stock of savings, rather than its earning power, which is of importance. A rise of the rate of interest allows the consumer to earn higher interest-income but on the other hand it lowers the value of his assets if he is to sell them at time T. Certainly, if this rise of the rate of interest comes at a time close enough to T, the increase in income will not be sufficient to outweigh the decline in asset value. Unfortunately, the framework of the foregoing discussion is one of instantaneous recontracting, an idealization which is hardly capable of capturing capital gains and capital losses, since the consumer's assets are always evaluated at current prices.

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