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PROPORTIONAL GROWTH AND TURNPIKE THEOREMS

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Errata

p. 3, ℓ. 9, from below, before "Inada," insert: "Furuya and"

ℓ. 8, " " , read "Nikaido"

p. 9, ℓ. 6, change $(\mu)^T$ to $(\mu)^{T+1}$ in both equations.

p. 32, ℓ. 1, ℓ. 3, change Oz^* to Ox^*

p. 39, ℓ. 6, change "paths" to "path"

ℓ. 3, from below, change "M is also positive" to " $0 < M \leqq 1$ "

ℓ. 1, " " , add "where $T_\delta \geqq 0$ "

p. 40, ℓ. 3, 4, change "T_δ increases if δ is chosen" to

" , whenever $M < 1$, T_δ is bound to increase

if δ is chosen sufficiently"

PROPORTIONAL GROWTH AND TURNPIKE THEOREMS*

Tjalling C. Koopmans

1. Introduction

In 1936, John von Neumann published, in an Austrian mathematical periodical little known to economists, a paper [von Neumann, 1936] that has greatly influenced economic theory up to the present time, and of which all the ramifications have perhaps not yet become fully apparent.

One can find in von Neumann's difficult short paper starting points for three distinct and extensive subsequent developments in economic theory. Two of these are not specifically connected with capital or growth theory. The paper contains the first explicit statement, known to this author, of what has subsequently been called the activity analysis model of production. This is a model in which there is a finite number of methods of production, each of which is characterized by constant ratios of inputs to outputs, hence by constant returns to scale. The inputs and outputs involved in the various methods together make up a finite list of commodities (goods and services). The paper contains an explicit statement of the relations between commodity prices and the production coefficients (input-output ratios) that describe methods in use and methods unused but available. These relations are found to characterize both efficient use of resources (in a sense discussed below) and competitive equilibrium.

*The ideas for this paper were developed when visiting at Harvard University as Frank W. Taussig Professor of Economics in 1960-61. The paper was written after returning to Yale, as part of research undertaken by the Cowles Foundation for Research in Economics under Task NR 047-006 with the Office of Naval Research.

In fact, to exhibit a model of competitive equilibrium was the main purpose of the paper emphasized by von Neumann. In this regard, together with a preceding and a subsequent paper by Wald [1933-34, 1936], von Neumann's paper became again the starting point for a systematic development of models of competitive equilibrium that has continued up to the present time.

Finally, and most importantly for our present purpose, the paper is the first rigorous, formal, and fully explicit paper in capital theory known to this author.

Paradoxically, von Neumann's paper shows that for a piece of work to spark several new developments in economic theory, it is not necessary that it have any particular claim to realism in its portrayal of economic life. Actually, the paper is poor economics. I am not speaking merely of the assumption of an unchanging technology, a highly unrealistic postulate often justifiable by the useful start it provides. It seems equally arbitrary, and contrary to all experience about economic growth, to assume that all production and consumption activities grow in time at the same proportional rate. Worse than that, it is quixotic to ignore completely the historically given capital stock available at the beginning of the time period under consideration, and to assume instead that out of some fourth dimension one can at time $t = 0$ pull forth a capital stock of precisely that composition that enables proportional growth to take place at a maximal rate and through a continuing competitive equilibrium. Finally, it is a very poor model of consumption indeed that assumes that growth in consumption is likewise characterized by constant ratios of (consumer goods) inputs to (labor) outputs.

More than twenty years later, Dorfman, Samuelson and Solow [1958, Chapter 12] perceived an implication of von Neumann's model that removes two of the foregoing objections. Holding on to the assumption of an unchanging technology of production and consumption, they accepted a historically given capital stock. Instead of von Neumann's objective of a maximal rate of proportional growth, they adopted as objective the attainment of a maximal capital stock of some specified character at a given time point in a distant future. They then asked what growth path will best serve that objective, and suggested that if the target date is distant enough, the best growth path (in the sense indicated) will run along (or close to) the turnpike of the von Neumann path of fastest proportional growth for most of the period under consideration. By this happy conjecture the von Neumann model, thus far mainly a highly inspiring source of theoretical developments, was also given a bearing on certain real-world phenomena, to wit, the forced development of an economy in which the aim is to construct a definite productive capacity for some future date without regard for the raising of consumption levels in the meantime.

While the turnpike conjecture is basically valid, the sketches of proofs presented by its originators are not rigorous. Kuhn [1959] was the first to point to a class of overlooked exceptions. Radner [1961] defined an important class of cases for which he proved the conjectures, and Hicks [1961], Inada [1961], McKenzie [1962a, 1962b], Morishima [1961], Nikkaido [1962] and others obtained valuable additional results.

The main purpose of the present article is expository. Drawing where needed on the studies cited, we shall utilize a diagrammatic device that makes it possible to exhibit, in terms of a two-commodity world, the essentials of the von Neumann model and its maximal growth properties. Those who wish to examine an explicit mathematical discussion of the von Neumann model written without reference to the turnpike property are referred to a paper by Gale [1956] or one by

Koopmans and Bausch [1959, Topic 5]. The diagrams in the present article have been developed from those in the latter paper by the addition, in projection, of a third dimension, in order to permit separate coordinate axes to be used for inputs and for outputs.

Von Neumann's original proof of the existence of a continuing competitive equilibrium exhibiting proportional growth at a maximum rate made use of Brouwer's fixed point theorem. Gale shows that a simpler separation theorem for convex sets suffices.

2. A Model With Two Commodities.

We shall consider a technology defined as a set of feasible activities. An activity in turn is defined as a procedure whereby a pair $x = (x_1, x_2)$ of inputs (one for each commodity), available at the beginning of any period, is converted into a pair $y = (y_1, y_2)$ of outputs, available at the end of that period. The amount of each input and of each output to an activity (x, y) is positive or zero. The technology remains the same for all future periods. The activities it contains are called feasible in the sense of technical feasibility of the required production processes. Availability of inputs for each period is to be ascertained as a separate matter, hence does not enter into the definition of the technology.

We shall now list the assumptions to be made about the technology. First of all, we make an assumption of constant returns to scale (proportionality): if an activity $(x, y) = \big((x_1, x_2), (y_1, y_2)\big)$ is feasible, then all the activities $\lambda \cdot (x, y) = (\lambda x, \lambda y) = \big((\lambda x_1, \lambda x_2), (\lambda y_1, \lambda y_2)\big)$ obtained by multiplying all inputs and outputs by the same nonnegative number λ are likewise feasible. Since $\lambda = 0$ is permitted, this implies that the null activity $\big((0, 0)\ (0, 0)\big)$ with no inputs and outputs is feasible. We further assume noninterference or additivity of activities: if two activities (x, y) and (x', y') are feasible, then their sum $(x + x', y + y') = \big((x_1 + x_1', x_2 + x_2'), (y_1 + y_1', y_2 + y_2')\big)$, a new activity whose inputs and outputs are the sums of those of the two given activities, is also feasible. In a context in which different activities are pursued by different firms this assumption is often called "absence of external economies or diseconomies." Where we do not necessarily presuppose organization through firms, the term "additivity" appears more appropriate.

We express a fact of life by the assumption that with limited inputs one cannot obtain unlimited outputs. That is, corresponding to any pair of bounds ξ_1, ξ_2 on the inputs, there is a pair of bounds η_1, η_2 on the outputs, such that for all feasible activities with $x_1 \leqq \xi_1, x_2 \leqq \xi_2$, one must have*

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* Note that this implies a corresponding assumption by Gale to the effect that one cannot produce something from nothing. That is, if (x, y) is feasible, and $x_1 = x_2 = 0$, then $y_1 = y_2 = 0$. For, if we had $x_1 = x_2 = 0$ and $y_1 > 0$, say, we could by the proportionality assumption choose a λ such that $\lambda x_1 = \lambda x_2 = 0 \leqq \xi_1, \xi_2$ but $\lambda y_1 > \eta_1$.

$y_1 \leqq \eta_1$, $y_2 \leqq \eta_2$. For mathematical convenience, we also assume that the set of feasible activities (x, y) in the space with four coordinates x_1, x_2, y_1, y_2 is closed, that is, contains all points of its boundary.*

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*This assumption, devoid of empirical content because of the approximate nature of all measurement, enables us to attain logical sharpness with simpler formulations than would be required without it. It permits us to speak below of a feasible maximal growth path, rather than a possibly unfeasible maximal growth path that can be approximated arbitrarily closely by feasible growth paths.

Gale shows that closedness of the production set together with his assumption cited in the preceding footnote imply our fact-of-life assumption.

We further assume free disposal: if an activity (x, y) is feasible, and if (x', y') is an activity with no smaller inputs ($x_1' \geqq x_1$ and $x_2' \geqq x_2$) and no larger outputs ($y_1' \leqq y_1$ and $y_2' \leqq y_2$) then (x', y') is likewise feasible. The consideration thus expressed is that one can, either at the beginning or at the end of a period during which the activity (x, y) takes place, dispose without cost of any excess inputs $(x_1' - x_1, x_2' - x_2)$ or excess outputs $(y_1 - y_1', y_2 - y_2')$ that arise if (x', y') is the activity that is wanted. Finally, it is assumed (output positivity) that there is a feasible activity (x, y) of which both outputs are positive, $y_1 > 0$, $y_2 > 0$. The reason for this assumption is that we will deal only with cases where no flows of commodities enter the system from the outside -- except for what may be available at the beginning of the first period. Hence the inputs for the second period have to be found from the outputs of the first, and so on. If the output of a commodity were zero in all feasible activities, then that commodity if initially present would disappear from the system after one round of production. If, on the other hand, for either commodity there is an activity producing some of it, then the addition of these two activities yields an activity that produces some of both commodities, and our last assumption is satisfied.

All the assumptions stated above will be maintained in what follows. In the von Neumann model as originally presented one further assumption was made. This is that there is given a finite number of <u>basic feasible activities</u>, from which all other feasible activities can be derived by proportional variation, addition, disposal or combinations of these. We shall make this assumption in some of our examples below.

Since we are interested in properties of the two-commodity model that carry over into models with n commodities, there is no need to associate particular named commodities with the inputs and outputs. However, we shall in some cases below use a simple example where the two commodities are food and tools, and where the technology has a finite basis.

3. Growth Paths and Proportional Growth Paths.

Any sequence of activities $(x^t, y^t) = \left((x_1^t, x_2^t), (y_1^t, y_2^t)\right)$, carried out in successive periods labeled $t = 1, 2, \ldots, T$ will be called a <u>growth path</u> if the output pair from each period's activity equals* the input pair for that

*Since the disposal possibility has already been recognized in defining the set of feasible activities, we do not need to admit inequality here.

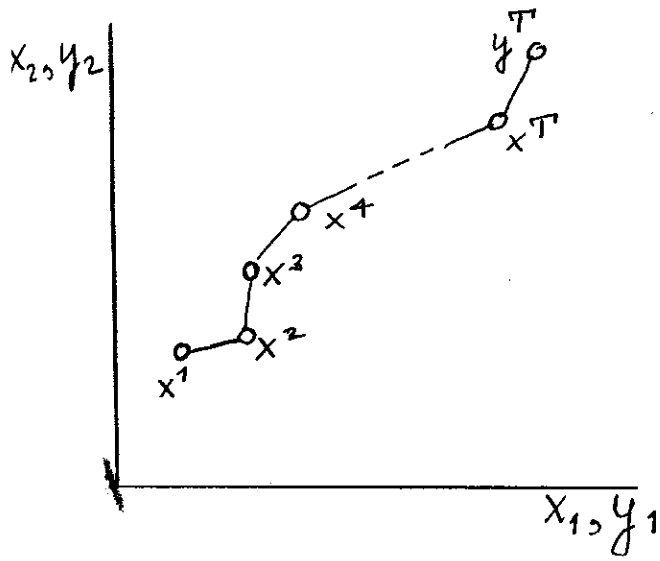

Fig. 1

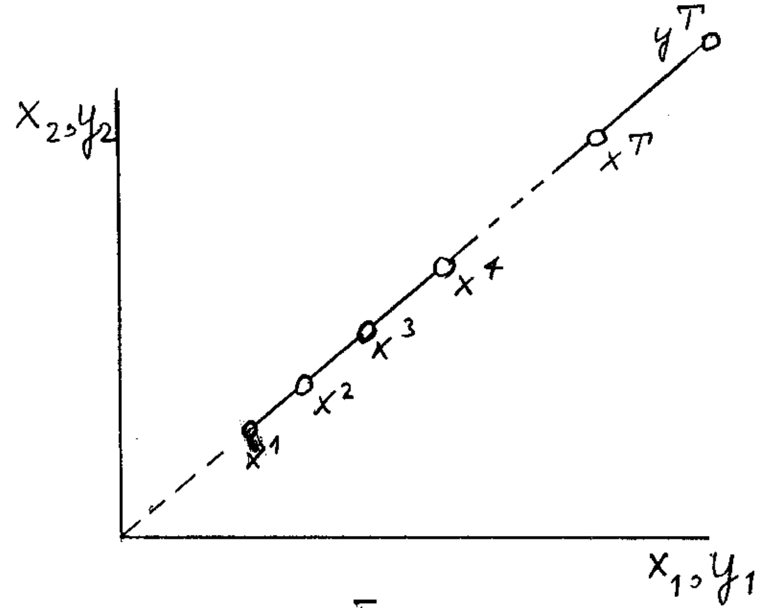

Fig. 2

of the next period,

$$y^t = (y_1^t, y_2^t) = \left(x_1^{t+1}, x_2^{t+1}\right) = x^{t+1} .$$

In the commodity space with coordinates x_1, x_2 , a growth path can be represented, as in Figure 1, by the sequence of points

$$x^1 , y^1 = x^2 , \quad y^2 = x^3 , \quad \ldots , y^{T-1} = x^T , y^T ,$$

labeled to indicate the order in which they occur. It is called a path even though actually it is a sequence of steps rather than a continuous path.

Von Neumann limited his discussion to <u>proportional growth paths</u>, in which each step is a <u>proportional growth activity</u>. The latter is defined as a non-null activity $(x, y) = \left((x_1, x_2), (y_1, y_2)\right)$ in which the outputs are multiples

$$y_1 = \mu x_1 , \quad y_2 = \mu x_2 , \qquad \mu > 0 ,$$

of the inputs, by a positive factor μ called the growth factor. There is actual growth if $\mu > 1$, stationarity if $\mu = 1$, contraction if $\mu < 1$. The terminal outputs y_1^T, y_2^T of a proportional growth path with initial inputs x_1^1, x_2^1 and a growth factor μ are, of course,

$$y_1^T = (\mu)^{T+1} x_1^1 , \quad y_2^T = (\mu)^{T+1} x_2^1 ,$$

where $(\mu)^T$ denotes " μ raised to the power T " (as distinct from the use of a time superscript to x^T, y^T). Figure 2 illustrates such a path.

Instead of the growth factor μ we will often use the growth rate $\mu - 1$.

4. Geometrical Representation of the Technology.

While the two-dimensional diagram of Figures 1 and 2 gives a clear image of a growth path, the same diagram is poorly suited to represent feasibility considerations. Actually, each activity has four coordinates, $x_1, x_2; y_1, y_2$. To represent the set of feasible activities explicitly by a geometrical figure would therefore require a four-dimensional space. However, the proportionality assumption makes it possible to cut this requirement down to three dimensions. There are many ways of doing this, out of which we shall choose a particular normalization of the inputs. If (x', y') is a non-null activity, at least one of the inputs x_1', x_2' must be positive, because if both inputs were zero the outputs would vanish also, by one of our assumptions. Hence the sum of the inputs

$x_1' + x_2'$ is positive. By applying a positive factor $\lambda = \dfrac{1}{x_1' + x_2'}$ to the given activity (x', y'), we obtain an activity

$$(x, y) = (\lambda x', \lambda y') =$$

$$\left(\frac{x'}{x_1' + x_2'} , \frac{y'}{x_1' + x_2'} \right) =$$

$$\left(\left(\frac{x_1'}{x_1' + x_2'} , \frac{x_2'}{x_1' + x_2'} \right) , \left(\frac{y_1'}{x_1' + x_2'} , \frac{y_2'}{x_1' + x_2'} \right) \right)$$

with the property that the sum of the inputs is unity:

$$x_1 + x_2 = \frac{x_1'}{x_1' + x_2'} + \frac{x_2'}{x_1' + x_2'} = 1 .$$

The fact that we are adding bushels of wheat to plows does not need to detain us here. The procedure is an arbitrary normalization device, which will yield us the same logical propositions if we choose to add pounds of wheat to dozens of plows.

Since each non-null activity is now represented by just one normalized activity, and since the coordinates of each normalized activity can again be increased or decreased proportionally to reproduce any other activity it represents, we shall have obtained a useful geometrical representation of the technology if we can exhibit the set of normalized activities. Since the two inputs add up to

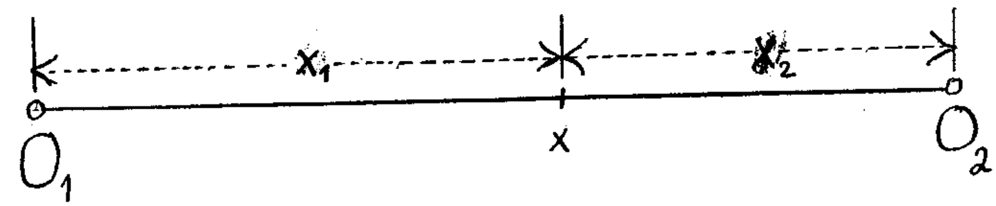

Fig. 3

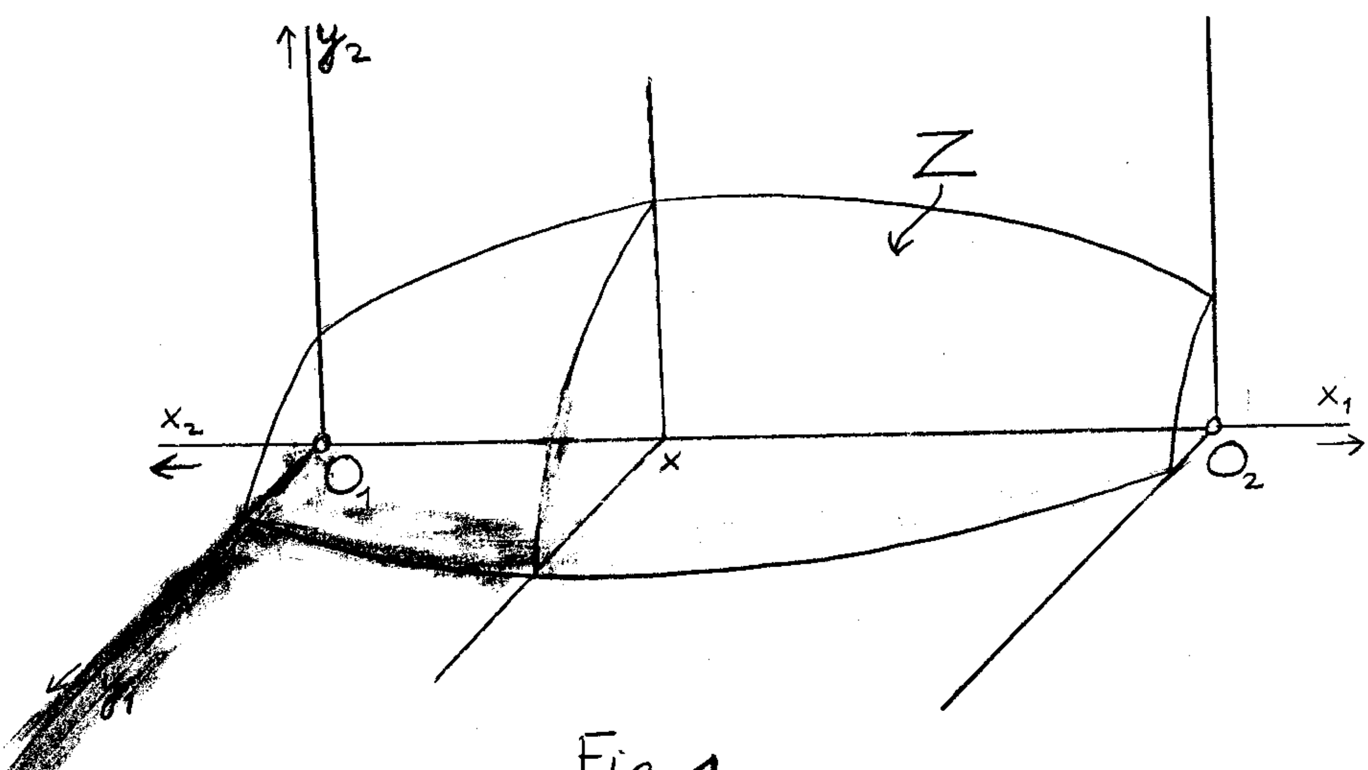

Fig. 4

unity, they can be represented together by a single point on a line segment of unit length. As indicated in Figure 3, the position of the point labeled x on that segment, relative to an origin in the left endpoint $(0, 1)$ which we shall label 0_1, fixes the first input x_1, measured toward the right. The position of that same point x relative to an origin in the right end point $(1, 0)$ labeled 0_2 fixes the second input x_2, measured toward the left.

We shall call this segment the x-segment, the line containing it the x_1-axis, with $(0, 1)$ as its origin. At right angles to the x_1-axis, we choose additional coordinate axes for y_1 and y_2, respectively, indicated in Figure 4 in stereographic projection. The technology is now represented by a set Z of points $z = (x_1, y_1, y_2)$, each representing a normalized feasible activity $\big((x_1, 1-x_1), (y_1, y_2)\big)$ in which the input $x_2 = 1 - x_1$ of the second commodity is implied in that of the first. We shall therefore hereafter freely use the expression "activity z," even though z has only three coordinates. The points of Z are all "above" or in the (horizontal) x_1-y_1-plane, "in front of" or in the (vertical) x_1-y_2-plane, and between, or in one of, two other (vertical) planes, one being the y_1-y_2-plane, through the origin $0_1 = (0, 0, 0)$, the other parallel to it through the "alternate origin" $0_2 = (1, 0, 0)$.

Since each of our normalized inputs is at most unity, our fact-of-life assumption entails that both outputs are also subject to upper bounds, $y_1 \leqq \bar{\eta}_1$, $y_2 \leqq \bar{\eta}_2$, hence that the entire set Z is <u>bounded</u> (contained in a finite box).

Furthermore, the set Z is <u>convex</u>: if two points $z = (x_1, y_1, y_2)$, $z' = (x_1', y_1', y_2')$ belong to Z, then all points of the line segment $\overline{z\,z'}$ joining z with z' belong to Z. To argue this for the midpoint z" of $\overline{zz'}$ one scales down each of the activities (x, y) and (x', y') -- where, of course $x_2 = 1 - x_1$, $x_2' = 1 - x_1'$ -- by applying a factor $\lambda = \frac{1}{2}$, and adds the resulting inputs and outputs. The proportionality and additivity assumptions assure us that the resulting activity (x", y") is again feasible. Arithmetic shows that it is again normalized,* and therefore represented in Z by the midpoint z" in

* $$x_1'' + x_2'' = \frac{x_1+x_1'}{2} + \frac{x_2+x_2'}{2} = \frac{x_1+x_2}{2} + \frac{x_1'+x_2'}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

question. Similar reasoning** applies to any other point of $\overline{zz'}$. Finally,

** Substituting λ and $1-\lambda$ for $\frac{1}{2}$ and $\frac{1}{2}$, where $0 < \lambda < 1$.

Z is again closed.***

*** As the intersection of the (closed) production set with the (closed) normalization set $x_1 + x_2 = 1$.

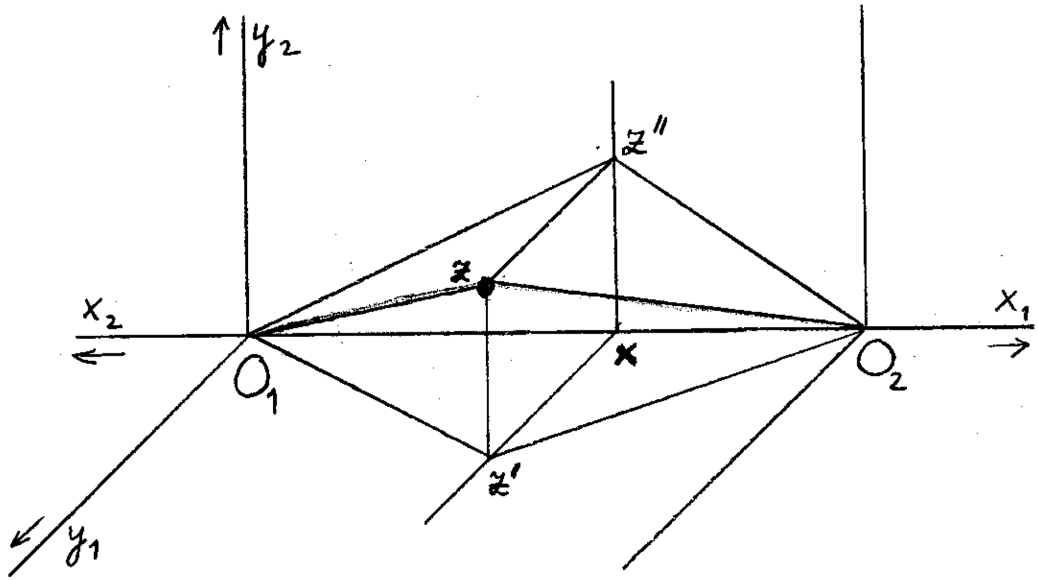

Fig. 5

Figure 4 on page 11 indicates one possible shape of Z . For any particular normalized input, such as that indicated by x in the figure, the set of feasible outputs is found by intersecting Z with a vertical plane through $x = (x_1, 0, 0)$ parallel to the y_1-y_2-plane.

Figure 5 shows the implications of free disposal for the shape of Z . If a point $z = (x_1, y_1, y_2)$ belongs to Z , then by disposal of part or all of one or both outputs one can get to any point of the rectangle $xz'zz''$ shown. Therefore all of these points belong to Z . On the other hand, the two origins O_1 , O_2 belong to <u>any</u> normalized production set, hence to Z , because they represent disposal of one unit of one or the other input followed by the null activity. By the convexity of Z , therefore, the entire polyhedron $O_1z'zz''xO_2$ must be contained in Z , any point of it being "produceable" by disposal of none, some, or all of one of the inputs, production proportional to $\big((x_1, 1-x_1), (y_1, y_2)\big)$, and disposal of none, some or all of one or both outputs. We note that, because of the assumption of output positivity, there is always a point $z = (x_1, y_1, y_2)$ in Z with both outputs y_1, y_2 positive. It follows that Z cannot be contained in a line or plane, but is always a three-dimensional body, containing interior points.

If the polyhedron $O_1z'zz''x\ O_2$ in Figure 5 is itself the technology set Z, it constitutes the simplest example of a normalized production set satisfying von Neumann's assumption of a finite basis. Apart from disposal, there is just one basic activity, represented by z. Data for a somewhat more complicated example are given in Table 1. There are two commodities, tools and food, where food as an input is identified with the labor it makes possible.

Table 1

Basic Activities at Unit Levels

	Producing Foods				Producing Tools	
	(1)	(2)	(3)	(4)	(5)	(6)
Outputs						
(1) tools		.1	.3	.5	.5	.6
(2) food	1	1.3	1.5	1.55		
Inputs						
(1) tools		.2	.5	.8		.1
(2) food=labor	1	.8	.5	.2	1	.9

The basic activities fall into two categories, those for producing food and those for producing tools. In each category, activities are labeled in an order of increasing tool- (or capital-) intensiveness. Tool-using activities yield an output of remaining serviceable tools smaller than the tool input, to allow for scrapping of worn-out tools in a ratio that decreases as capital intensity increases. The normalized production set for this example is shown in Figure 8 below. It will be clear that a technology with a finite basis always gives rise to a normalized production set Z that is polyhedral in shape.

5. Representation of Proportional Growth

In general, the highest proportional growth rate that can be achieved from given initial inputs will depend on these inputs, more precisely, on their ratio. We shall now look for the highest rate achievable with any inputs, and for the ratio of inputs that makes that rate possible. It has already been pointed out that in this formulation historically given initial inputs are not recognized. It is assumed for the time being that whatever initial inputs may be needed for a maximal proportional growth path can be procured.

To facilitate our search, let us first consider how proportional growth can be represented in our diagram without any regard to its feasibility. Since proportional growth is defined only for non-null activities, it suffices to consider only activities normalized on the inputs. Figure 6 represents all such activities showing proportional growth by a factor μ. The input pair $x^{(1)} = (0, 1)$ represented by 0_1 gives rise to an activity point $z^{(1)} = (0, 0, \mu)$ found on the positive y_2-axis at a distance μ from 0_1. The input pair $x^{(2)} = (1, 0)$ represented by 0_2 similarly leads to $z^{(2)} = (1, \mu, 0)$ found on the "alternate" positive

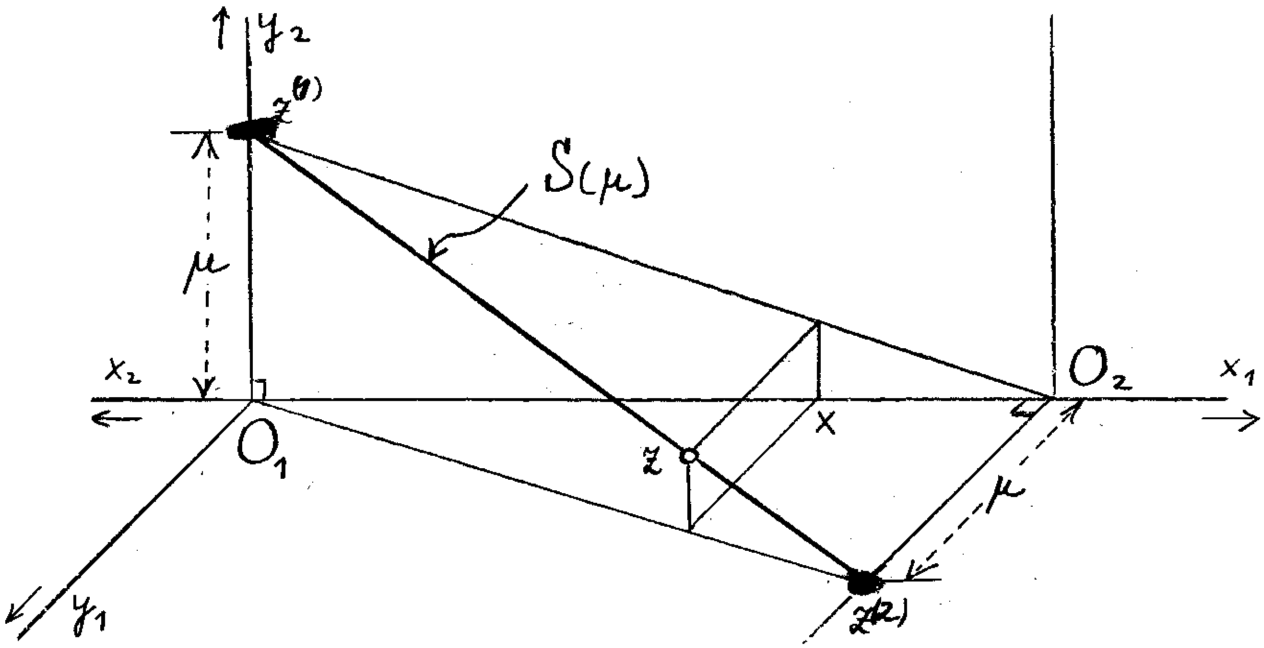

Fig. 6

y_1-axis at a distance μ from O_2 . For all other normalized input pairs $x = (x_1, x_2)$, proportional outputs $y_1 = \mu x_1$, $y_2 = \mu x_2$ are represented, separately, by points of the line segments $\overline{z^{(1)}O_2}$ and $\overline{O_1 z^{(2)}}$, respectively, and hence jointly by points z of the segment $\overline{z^{(1)}z^{(2)}}$.

A higher growth factor μ' gives rise to a similar segment with the distances from O_1 to $z^{(1)}$ and from O_2 to $z^{(2)}$ increased to μ' . To indicate its dependence on μ , we shall hereafter denote the segment $\overline{z^{(1)}z^{(2)}}$ by $S(\mu)$.

6. Determination of a Maximal Proportional Growth Path.

By putting together the diagrams of Figures 4 and 6 in Figure 7 we can now find a normalized activity that represents proportional growth at a maximal rate. We observe that feasible proportional growth by a factor μ is represented by a point of Z which is at the same time a point of $S(\mu)$. We are therefore looking for the largest value μ^* of μ for which Z and $S(\mu)$ have at least one point in common.

Now first of all $S(0) = \overline{0_1 0_2}$, the segment representing a zero growth factor (i.e., complete collapse of the economy in one period), is entirely contained in Z. Better than that, a comparison with Figure 5 shows that, for some small enough positive value $\underline{\mu}$ of μ, the segment $S(\underline{\mu})$ still has points in common with Z. At the other extreme, since we can obviously choose a $\overline{\mu}$ so large that $S(\overline{\mu})$ remains outside the box(of dimensions 1 by $\overline{\eta}_1$ by $\overline{\eta}_2$) that contains Z, it is clear that then $S(\overline{\mu})$ and Z do not intersect.

There is therefore a unique largest value μ^* of μ for which Z and $S(\mu)$ have a point in common,* and μ^* is positive but finite because

$$0 < \underline{\mu} \leqq \mu^* < \overline{\mu} \quad .$$

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* Let $d(\mu)$ be the shortest distance between a point of Z and a point of $S(\mu)$. Since both sets are closed, $d(\mu)$ exists, and $d(\mu) = 0$ if and only if Z and $S(\mu)$ intersect. It is easily seen that $d(\mu)$ is a continuous function of μ. Furthermore, if $d(\mu) = 0$ and $\mu > \mu'$, then $d(\mu') = 0$, because of free disposal. Hence the set of μ for which Z and $S(\mu)$ intersect is an interval $[0, \mu^*]$, which contains its positive end point μ^* because $d(\mu)$ is continuous.

Any common point z^* of Z and $S(\mu)$ will be called a von Neumann point, the corresponding activity a von Neumann activity. It achieves proportional growth by a maximal factor μ^*.

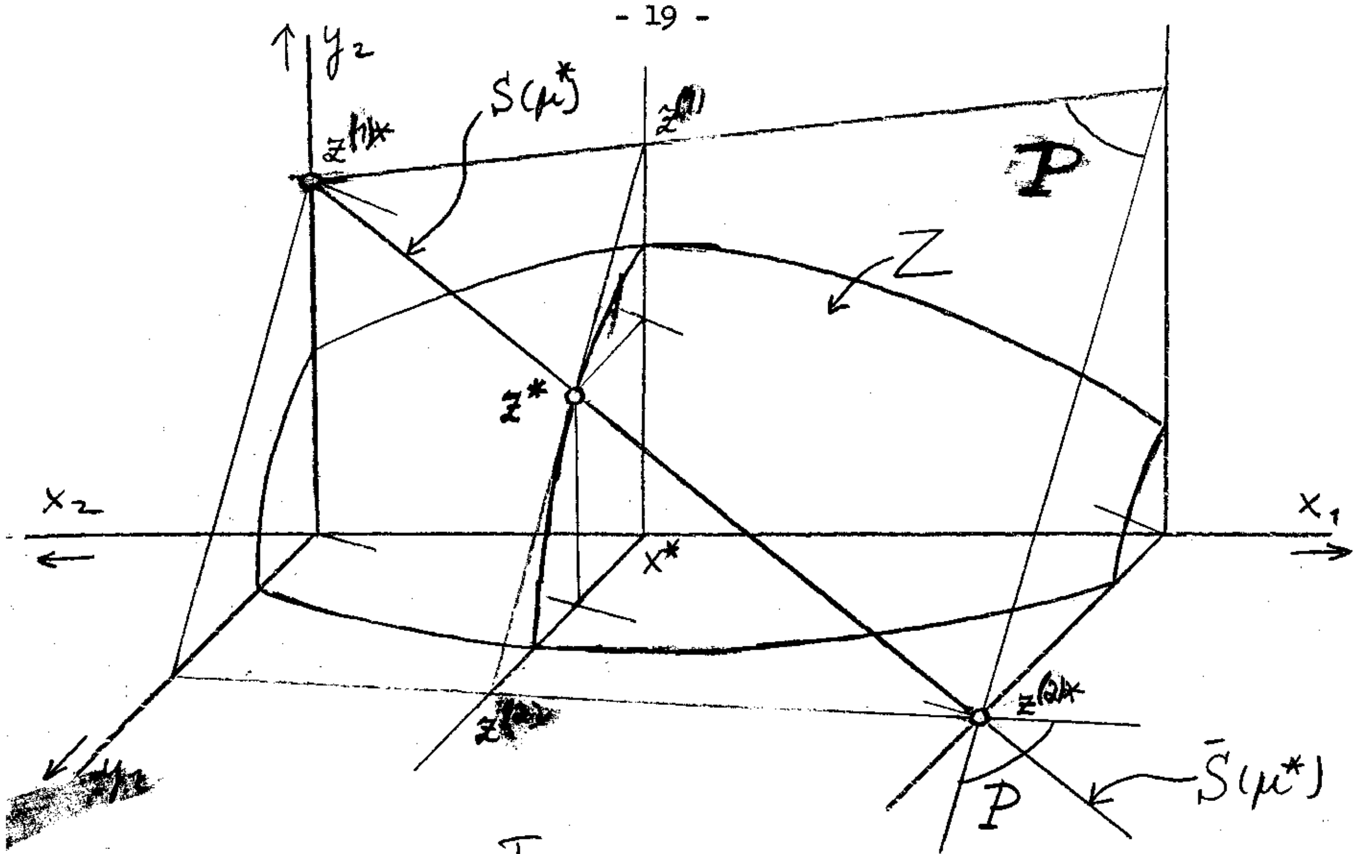

Fig. 7

Figure 7 is so drawn that there is one unique von Neumann activity $z^* = (x_1^*, y_1^*, y_2^*)$, and that this activity involves positive inputs and outputs of both commodities. (z^* different from the end points $z^{(1)*}$, $z^{(2)*}$ of $S(\mu^*)$). Other cases will be considered below.

If the outputs $y_1^* = \mu^* x_1^*$, $y_2^* = \mu^* x_2^*$ of the activity z^* are again used in the next period as inputs to a stepped up activity proportional to z^* , the second period outputs will be $(\mu^*)^2 \cdot x_1^*$, $(\mu^*)^2 \cdot x_2^*$, and if this procedure is continued, the outputs of the t-th period will be

$$(\mu^*)^t \cdot x_1^* \ , \ (\mu^*)^t \cdot x_2^* \ .$$

We shall call/such growth path a <u>von Neumann path</u>.* There is no alternative proportional growth path with a higher rate of growth.

*Regardless of whether the initial input vector x^* is or is not normalized.

7. A Price System Sustaining a von Neumann Path.

In von Neumann's discussion, the maximality of the growth rate was established as an implication of the existence of a price system that sustains a von Neumann path. Our reversal of the order in which these two properties are established is favored by the simpler mathematical tool we use. We shall use the following separation theorem for convex sets:*

*For a proof of this theorem in n dimensions, see Debreu [1959], Corollary 3, or Karlin [1959], Appendix B, Theorem B.1.2. For a discussion of the importance of this theorem in the theory of allocation of resources, see Koopmans [1957].

If A and B are convex sets in three-dimensional space which have no point in common, then there exists a plane P which separates A and B in the sense that every point of A is either in P or on one side of P , whereas every point of B is either in P or on the other side of P . The application we shall make of this theorem will at the same time serve as an illustration of its meaning and use.

If, as in Figure 7, the point z^* is not an end point of the segment $S(\mu^*)$, it will suffice to take $S(\mu^*)$ itself as the set A in the theorem. However, in order to include also cases where z^* is an end point, we shall choose for A the line $\overline{S}(\mu^*)$ obtained by extending $S(\mu^*)$ indefinitely beyond both end points.

For B we choose the interior $\overset{o}{Z}$ of the normalized production set Z , that is, all points of Z not in its boundary. We have already concluded from the free disposal and output positivity assumptions that Z does have interior points. It is intuitively clear that $\mathring{Z}$ and $\bar{S}(\mu^*)$ have no point in common.*

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* For a proof, refer to the definition of an interior point z of Z, which states that there exists a positive number δ such that all points at a distance from z no larger than δ belong to Z. Suppose $z = (x_1, y_1, y_2)$ is both in $\mathring{Z}$ and in $\bar{S}(\mu^*)$. Since z is in Z, $0 \leqq x_1 \leqq 1$, and hence z is not in the extensions of $S(\mu^*)$, for which either $x_1 < 0$ or $1 < x_1$. Hence z is in $S(\mu^*)$. But then, for any $\epsilon > 0$, $S(\mu^*(1+\epsilon))$ contains a point $z_\epsilon = (x_1, (1+\epsilon)y_1, (1+\epsilon)\, y_2)$ of which the distance from z is $\epsilon\sqrt{y_1^2 + y_2^2}$. Since by suitable choice of ϵ, this distance can be made less than δ, z_ϵ can be made to fall in Z. But then μ^* is not the maximum feasible rate of proportional growth, contrary to the definition of μ^*.

Hence there is a plane P separating $\bar{S}(\mu^*)$ and $\mathring{Z}$. Since $\mathring{Z}$ is three-dimensional, not all points of $\mathring{Z}$ can be in P, and some of them will be "behind" P. Moreover, since the boundary of Z is only a skin without thickness,** all points

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** Suppose z of Z is "in front of" P, and D is a small sphere, constructed on a center in $\mathring{Z}$ behind P, so as to be both behind P and in Z. Then, by the convexity of Z, the convex hull of D and z is contained in Z. But then E contains points of $\mathring{Z}$ in front of P, a contradiction.

---

of Z are in or behind P, hence P "separates" $\bar{S}(\mu^*)$ and Z as well. In particular, the common point z^* of $\bar{S}(\mu^*)$ and Z must lie in P. But if all points of the straight line $\overline{S}(\mu^*)$ are in or in front of P , and one of them, z^* , is in P , all points of $\overline{S}(\mu^*)$ must be in P , and a fortiori all points of $S(\mu^*)$ are in P .

Figure 7 shows the construction of the separating plane P . The diagram uses a <u>strictly convex</u>* production set Z , in which case P and Z have only the point z^* in common.

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* To be precise, we shall call Z <u>strictly convex</u> if it is the intersection of the space Ω defined by $0 \leqq x_1 \leqq 1$, $0 \leqq y_1$, $0 \leqq y_2$ with an "extended" set $\overline{Z}$ that has no straight line segment in its boundary.

However, we shall in what follows use the strict convexity assumption only where that is explicitly mentioned.

Let us write the equation of P as

$$q\, x_1 + p_1\, y_1 + p_2\, y_2 \;=\; q_2 \;.$$

Since $z^{(1)*} = (0,\ 0,\ \mu^*)$ and $z^{(2)*} = (1,\ \mu^*,\ 0)$ are in P , substitution of these coordinates for $(x_1,\ y_1,\ y_2)$ in the equation gives

$$\mu^*\, p_2 = q_2 \;, \qquad \mu^*\, p_1 = q_2 - q = q_1 \;, \qquad \text{say.}$$

Reintroducing the input $x_2 = 1 - x_1$ of the second commodity, we can therefore write the equation of P in the symmetrical form

$$\text{(P)} \qquad \pi(x,y) = p_1\, y_1 + p_2\, y_2 - \mu^*\,(p_1\, x_1 + p_2\, x_2) \;=\; 0 \;,$$

it being understood that x_2 needs to be replaced by $1 - x_1$ to make this the equation of a plane in our normalized three-dimensional space of x_1, y_1, y_2 .

We have introduced the notation $\pi(x,y)$ as a short symbol for the linear function in the middle member of (P).

The function $\pi(x,y)$ is positive on one side of P, negative on the other. Since we are still free to change the signs of both p_1 and p_2 in (P), we shall choose the signs in such a way that

$$(\overline{P}) \qquad \pi(x,y) = p_1 y_1 + p_2 y_2 - \mu^*(p_1 x_1 + p_2 x_2) \leqq 0$$

for all points in or behind P. This includes all points of Z. In particular, the $<$ sign in $(\overline{P})$ must apply to all points of Z other than z^* whenever z^* is the only point Z and P have in common.

We know that Z contains the two origins (0, 0, 0) and (1, 0, 0). Inserting of their coordinates in $(\overline{P})$ yields

$$p_1 \geqq 0 , \; p_2 \geqq 0 , \quad \text{but not} \quad p_1 = p_2 = 0 ,$$

because in the latter case (P) would not be the equation of a plane. If, as in Figure 7, both origins are "behind" P, we have*

$$p_1 > 0 , \quad p_2 > 0 .$$

---

* This will always be the case if Z is strictly convex.

We will now interpret p_1 and p_2 as <u>prices</u> of commodities "1" and "2," respectively, which we think of as constant through time. Likewise we interpret $\mu^* - 1 = r$ as an <u>interest rate</u> for one period, hence $\mu^* = 1 + r$ as an <u>interest</u>

factor. The left hand member $\pi(x,y)$ in (P) then represents the value of the outputs y_1, y_2 at the end of the period, minus the value of the inputs x_1, x_2 at the beginning of the period multiplied by an interest factor to account for the time lead of one period by which inputs must precede outputs. This is precisely the profit that arises from engaging in the activity (x,y) during one period, evaluated in the price system p_1, p_2, r for a time of reference at the end of that period. Since z^* is in P, the profit from the maximal proportional growth activity z^* is precisely zero. Moreover, by $(\bar{P})$, the profit $\pi(x,y)$ from any feasible activity (x,y) is nonpositive.* In this sense, it can be said that the

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* Strictly, Figure 7 shows this only for the normalized activities of Z. However, since any feasible activity is obtainable from a normalized activity by multiplying all coordinates with a nonnegative factor λ, the profit is likewise multiplied by λ, leaving the above statement true.

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price system p_1, p_2, r, if remaining constant through time, sustains the von Neumann growth path. It is immaterial for this statement whether the price system in question is thought of as determined in competitive markets in a growing economy, or whether it is regarded as a set of centrally determined steering prices guiding allocations in a planned economy. If the maximal growth activity is itself a basic activity that can be arranged for in a single productive process or establishment, the price system will permit continual growth of that activity if the interest charge is the only obstacle to investment, because no loss is incurred by meeting it. Neither does there exist any other feasible activity which, by yielding a positive profit, would lure resources away from the maximal growth path.

It is, of course, also possible that the (or a) maximal growth activity is a composite

$$(x^*, y^*) = \lambda_1 \cdot (x^1, y^1) + \lambda_2 \cdot (x^2, y^2) + \ldots + \lambda k \cdot (x^k, y^k)$$

of k basic activities (x^h, y^h), $h = 1, \ldots, k$ with positive weights λ_h. We shall see below that then <u>each</u> of these basic activities will also break even. Use of each of these activities is again made possible, as far as profit incentives are concerned, by the fact that there is no competing feasible activity promising a positive profit. Moreover, once the levels of these activities have the proportions $\lambda_1 = \lambda_2, \ldots, \lambda_n$ occurring in the above representation of (x^*, y^*), the outputs from each round of production will just suffice as inputs for the next round at levels stepped up by the factor μ^*.

The price ratio $p_1 \div p_2$ can be read off from Figure 7 as follows. The equations of the line $z^{(1)} z^{(2)}$ in which P intersects the plane through z^* parallel to the y_1-y_2-plane are found from (P) to be

$$p_1 y_1 + p_2 y_2 = \mu^*(p_1 x_1^* + p_2 x_2^*) = c^* \text{, say}$$
$$x_1 = x_1^*$$

Hence $- p_1/p_2$ is the slope of y_2 on y_1 as read off from that line.

In figure 7 this slope is uniquely determined by the smooth shape of Z, A limited indeterminacy in p_1/p_2 can arise, for instance, if z^* happens to be a vertex of a polyhedral Z.

It will be useful later if we also determine the slope of the line $z^{(1)*}z^{(1)}$ in which P intersects the x_1-y_2-plane. (If $p_2 = 0$ we must use the x_1-y_1-plane instead.) From (P) we have, if $y_1 = 0$,

$$p_2 y_2 = \mu^*\left(p_1 x_1 + p_2(1 - x_1)\right) = \mu^*(p_1 - p_2)x_1 + \mu^* p_2$$

Hence the slope of y_2 on x_1 is $\mu^* \cdot \frac{p_1 - p_2}{p_2}$. It depends on the price ratio p_1/p_2 and also on μ^* provided $p_1 \neq p_2$.

We have found that with any proportional growth activity with the maximal growth factor μ^* one can associate a price system p_1, p_2, $r = \mu^* - 1$ which sustains that activity. It is worth looking at the reverse question: Suppose one has obtained a proportional growth activity $(x',y') = (x',\mu'x')$ and a price system $p_1' \geqq 0$, $p_2' \geqq 0$ (not both $= 0$), $r' = \mu' - 1$ which sustains it, in the sense that $\pi'(x,y) \equiv p_1'y_1 + p_2'y_2 - \mu'(p_1'x_1 + p_2'x_2) \leqq 0$ for all feasible activities (x,y). (Inserting (x',y') in this inequality obviously gives the $=$ sign, because $y' = \mu'x'$.) Does it follow that $\mu' = \mu^*$ and hence that the given activity is a von Neumann activity?

The answer is affirmative whenever the given prices p_1', p_2' are both positive. To see this we evaluate a von Neumann activity (x^*,y^*), where $y^* = \mu^*x^*$ -- we already know there exists one -- in the given price system. Since this activity is feasible

$$\pi'(x^*,y^*) = p_1'y_1^* + p_2'y_2^* - \mu'(p_1'x_1^* + p_2'x_2^*) \leqq 0 \ .$$

On the other hand, since $y^* = \mu^* x^*$,

$$p_1' y_1^* + p_2' y_2^* - \mu^*(p_1'x_1^* + p_2'x_2^*) = 0 \ .$$

Subtracting we have, after cancellations,

$$(\mu^* - \mu')(p_1'x_1^* + p_2'x_2^*) \leqq 0 \ .$$

Now x_1^*, x_2^* are nonnegative and not both zero. Hence, if p_1', p_2' are both positive, $p_1'x_1^* + p_2'x_2^* > 0$, and therefore $\mu^* - \mu' \leqq 0$. On the other hand, since μ^* is the maximal proportional growth rate that is feasible, $\mu^* - \mu' \geqq 0$. It follows that $\mu' = \mu^*$, and hence that the given activity (x',y') is a von Neumann activity, the given price system p_1', p_2', r an associated sustaining price system.

The foregoing reverse reasoning, from a given sustaining price system to the maximality of a proportional growth activity, is important in verifying, for a numerically defined production set Z , that a proposed proportional growth activity is indeed a maximal one. Proof is rendered by exhibiting positive sustaining prices p_1', p_2' . This reasoning does not hold if, say, $p_1' = 0$, while $x_2^* = 0$ for the von Neumann activity.*

---

* An example is given by the technology with a finite basis (table turned sideways).

	Outputs (1)	Outputs (2)	Inputs (1)	Inputs (2)
Basic Activities (1)	4		1	
(2)	3	2	1	1

where activity (1) is the unique von Neumann activity with $\mu^* = 4$, but activity (2) with a growth factor $\mu' = 2$ can be sustained by the price system $p_1' = 0$, $p_2' = 1$, $r' = 1$. A diagram of the type of Figure 7 readily reveals the geometrical configuration that makes this possible.

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8. A Numerical Example

We shall illustrate the determination of the maximal proportional growth activity and the associated price system in terms of the technology with six basic activities, shown in Table 1 above. A glance at Figure 8 (before $S(\mu^*)$ is drawn in) suggests that the segment $S(\mu^*)$ will cut one, or possibly two, of the edges leading from activity z^6 to activities z^2, z^3 or z^4. Without looking for a systematic method for determining $S(\mu^*)$ and its intersection with a polyhedral Z in more complicated cases, let us guess that $S(\mu^*)$ cuts only the edge $\overline{z^3 z^6}$. Then μ^* can be determined from the condition that the point

$$(x_1^*, y_1^*, y_2^*) = z^* = \lambda z^3 + (1 - \lambda) z^6$$

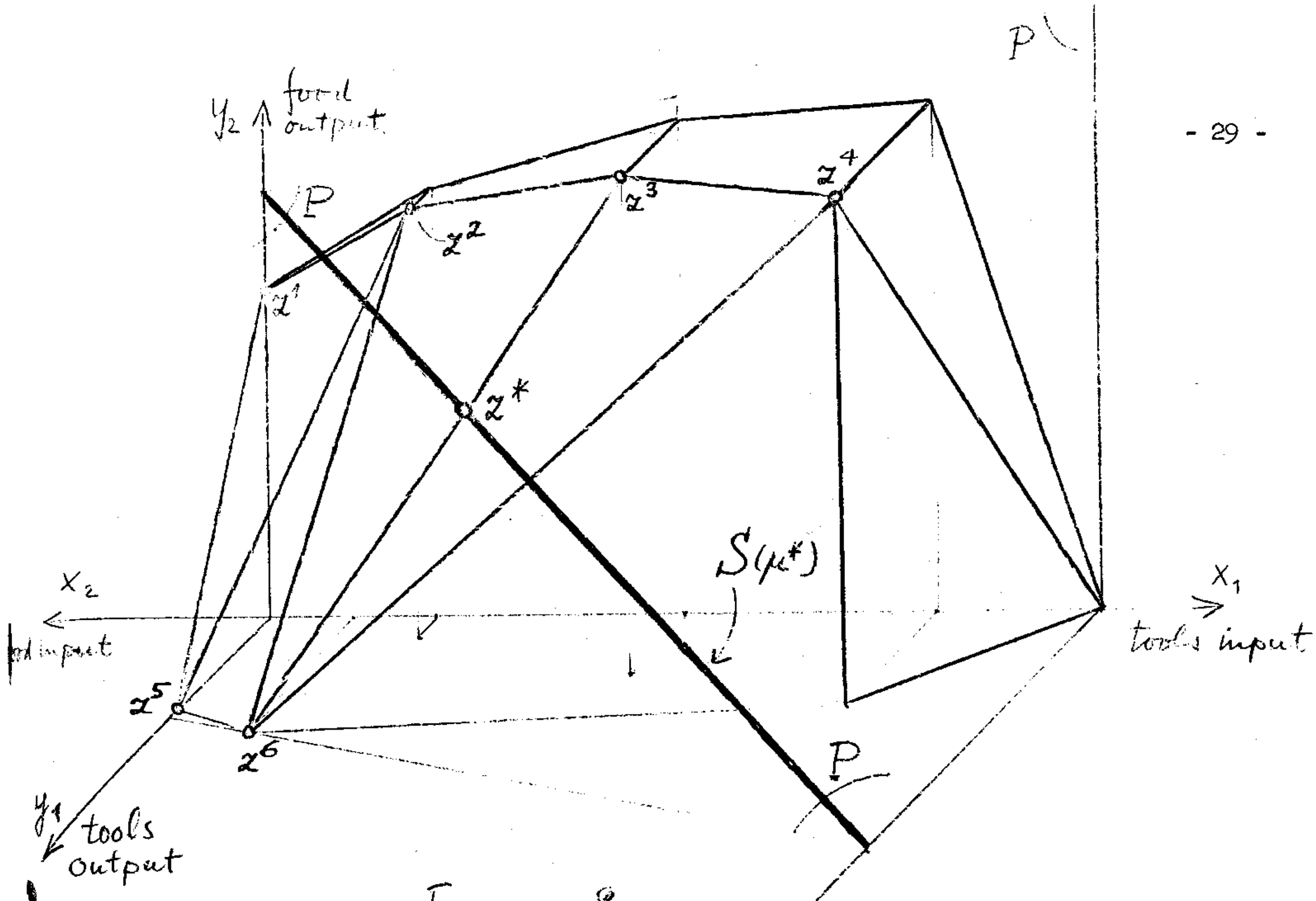

Figure 8

on the edge $\overline{z^3 z^6}$ have outputs proportional to inputs, augmented by a factor μ^*. Using the coordinates of z^3, z^6 given in Table 1, this requires

$$.3\lambda + .6(1-\lambda) = \mu^* \big(.5\lambda + .1(1-\lambda)\big)$$

$$1.5\lambda = \mu^* \big(.5\lambda + .9(1-\lambda)\big) ,$$

each of which can be solved for λ to give

$$\lambda = \frac{9\mu^*}{4\mu^* + 15} = \frac{6 - \mu^*}{4\mu^* + 3} .$$

The requirement that the two values are equal yields a quadratic equation μ^* with

$$\mu^* = 1.292 \ldots$$

as the only positive, hence meaningful, root. The corresponding von Neumann activity is found to be, within errors of rounding,

$$(x^*, y^*) = \big((.331, .669), (.427, .865)\big) ,$$

provided our initial guess was correct. To verify that point, we determine the prices p_1, p_2 from the requirement that the coordinates of z^6 (and hence also those of z^3) satisfy (P) with $\mu^* = 1.292 \ldots$. The result is

$$(p_1, p_2) = (2.48, 1) \quad ,$$

if we choose food as the numéraire. Finally we verify that the condition ($\overline{P}$) , which now runs

$$2.48\, y_1 + y_2 - 1.292\,(2.48\, x_1 + x_2) \leqq 0 ,$$

is satisfied with the $<$ sign by all activities other than z^3, z^6 (for which the $=$ sign holds). Our initial guess is thereby confirmed. The failure of activity z^4 to improve on z^3 can be interpreted to mean that with an interest rate $r = \mu^* - 1$ as high as 29.2%, the higher capital cost of activity (4) is not fully compensated for by the higher outputs and the lower labor input.

Figure 8 shows the maximal growth activity z^* , the segment $S(\mu^*)$ and the plane P for the present example.

9. The Turnpike Conjecture

There is nothing in the assumptions we have made about the technology that limits the discussion to a comparison of alternative paths of proportional growth. Dorfman, Samuelson and Solow therefore considered growth paths that are maximal in some wider sense. They accepted as given the initial inputs $x^1 = (x_1^1, x_2^1)$, and required that the terminal T-th period outputs y_1^T, y_2^T be proportional,

$$y_1^T = \kappa h_1 , \qquad y_2^T = \kappa h_2, \qquad h_1 + h_2 = 1 ,$$

to two prescribed nonnegative numbers, h_1, h_2, not both zero, which we have normalized to have a unit sum. They maintained the requirement that each period's outputs are all that is available, subject to free disposal, for use as inputs for the next period. We shall call a growth path subject to these specifications maximal (they called it optimal) if it achieves the highest value κ^* of the factor of proportionality κ attainable by any growth path meeting these specifications. They then formulated the following conjecture as to the nature of the maximal growth path:

> "...if the programming period is very long, the corresponding optimal capital program will be describable as follows: The system first invests so as to alter its capital structure toward the special von Neumann proportions. When it has come close to these proportions, it spends most of the programming period performing steady growth at the maximal rate (more precisely, something close to maximal steady growth). The

"system *expands along or close to the von Neumann ray* Ox^* until the end of the programming period approaches. Then it bends away from Ox^* and invests in such a way as to *alter the capital structure to the desired terminal proportions*, arriving at y^T as the period ends."*

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* Dorfman, Samuelson, Solow [1958] Ch. 12, p. 331, quotation changed only to correspond to present notation.

Thus the von Neumann path acts like a turnpike that attracts all discerning long-distance traffic by the shorter travel time it makes possible, even though the road mileage may be lengthened thereby. Figure 9 illustrates this idea. The path that, starting with the normalized initial input x^1 reaches the farthest point λh on the ray Oh reachable in T periods runs close to the von Neumann path Ox^* for most of its course, if T is sufficiently large.

One is reminded of the technological reference in Lenin's well-known dictum: "Soviets plus electrification equals communism." The turnpike proposition at least supports the idea that, in a given technology, a particular choice of methods of production may be most conducive to long run growth regardless of the more distant objectives of the full-grown economy.

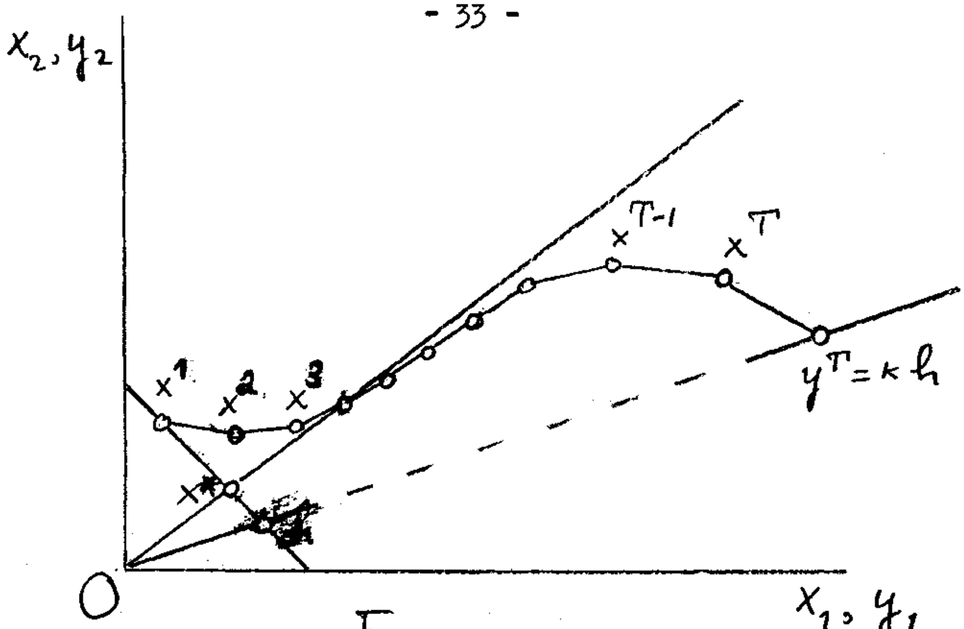

Figure 9.

We shall consider a sequence of possible cases in order to bring out why this remarkable conjecture is by and large valid, and to indicate how the exceptions noted by Kuhn and Morishima arise.

10. Access to or from the von Neumann Path

We must first exclude certain cases in which either the given initial stock $x^1 = (x_1^1, x_2^1)$ does not permit one ever to reach a point of a von Neumann path (other than the origin), or in which one cannot, from a von Neumann path that can be reached, in turn reach a point $\gamma h = (\gamma h_1, \gamma h_2)$ of the required terminal stock ratio $h_1 = h_2$ with $\gamma > 0$. In such cases the proofs of the turnpike conjecture given below are not valid, and it would seem that the conjecture itself is not valid either.

It is clear that from a given point $x = (x_1, x_2)$ one can by mere disposal achieve any desired ratio of availabilities as long as x_1 and x_2 are both positive. The inaccessibility cases can therefore arise only if either the initial stock x^1 or the von Neumann point z^* lacks one of the commodities. However, special shapes of Z are needed in addition for such circumstances to give rise to inaccessibilities that invalidate the turnpike conjecture. For instance, since in Figure 4 neither the intersection of Z with the y_1-y_2-plane, nor that with the parallel plane through 0_2, is contained in a single coordinate axis $(y_1$ or $y_2)$, absence of either commodity from the initial stock, or from a von Neumann point, does not give rise to inaccessibility.

It is not difficult to enumerate all possible cases of inaccessibility for two, or for that matter for n, commodities. We shall avoid this somewhat trivial problem by making the required accessibility a premise of the turnpike theorem. We merely state without further proof that, if from a given point (x_1, x_2) another point with a given ratio $y_1 \div y_2$ is accessible at all, it is accessible in one or two periods.*

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* More periods may be required if the model contains more than two commodities.

Using this readily verifiable fact, the premise will be stated as follows.

Assumption A (Accessibility). The points x^1, x^*, h in the commodity space representing, respectively, the initial stock, a von Neumann point, and a point defining the final stock ratio, can be supplemented by points x^2, x' such that, for suitable positive numbers ν, ξ, the activities (x^1, x^2), $(x^2, \nu x^*)$, (x^*, x'), $(x', \xi h)$ are all feasible.

11. Rates of Growth in von Neumann Value.

To associate a numerical growth rate or factor with a step or a sequence of steps in a commodity space of at least two dimensions, one has to reduce a comparison of two vectors to a scalar measure of "growth." In von Neumann's discussion this is done by considering only proportional growth, where the factor of proportionality provides the wanted scalar. We now observe that the prices $p = (p_1, p_2)$ found as a by-product of the study of maximal proportional growth can be used to extend this scalar to nonproportional growth.

We shall use the abbreviated notation

$$px = p_1 x_1 + p_2 x_2$$

for the value of a commodity pair $x = (x_1, x_2)$ for a pair of prices $p = (p_1, p_2)$. If these are von Neumann prices associated with maximal proportional growth, we shall call px the (von Neumann) value of the pair x. Since no normalization has been imposed on p, von Neumann value is determined up to a positive constant factor, except if at least two (and hence infinitely many) non-proportional von Neumann price pairs exist. In the latter case, we arbitrarily choose one price pair (with both prices positive), keeping it constant in all that follows.

Given any activity (x, y) such that the von Neumann input value px is positive, we can now use

$$\mu(x, y) = \frac{py}{px}$$

as a scalar measure of growth, to be called simply the value growth factor for the activity. Clearly, for proportional growth the value growth factor equals the factor of proportionality.

Radner's analysis is based on a study of inequalities that constrain the value growth factors for feasible activities in general, and for the activities of a maximal growth path in particular. The no-profit condition $(\overline{P})$ immediately gives that

$$\mu(x, y) \leqq \mu^*$$

for all feasible activities with a positive input value.

Now let (x^t, y^t), $t = 1, \ldots, T$ be a maximal growth path as defined in Sections 3 and 8 and illustrated in Figure 9. Then x^1 is the given initial stock, which for simplicity we normalize by $x_1^1 + x_2^1 = 1$. Furthermore, $y^t = x^{t+1}$ for all t, and $y^T = \chi^* h$ is the maximal final stock having proportions given by $h = (h_1, h_2)$. In order to be sure that the various value growth factors occurring in the analysis are defined, we must make

Assumption B (Value positivity). The normalized commodity pairs x^1, x^*, h possess positive von Neumann values px^1, px^*, ph.

This assumption is clearly satisfied if both prices are positive.

We denote the value growth factor for the t-th step of the path by

$$\mu_t = \mu(x^t, y^t) = \frac{py^t}{px^t} .$$

Then the growth factor for the entire path is*

$$\frac{py^T}{px^1} = \frac{py^1}{px^1} \cdot \frac{py^2}{px^2} \cdot \ldots \cdot \frac{py^T}{px^T} = \mu_1 \mu_2 \cdots \mu_T$$

---

* $p\,x^1 > 0$ by assumption B. If we had $px^t = 0$ for any $t \leqq T$, this would imply $py^s = 0$ for $s = t, \ldots, T$ because of $(\overline{P})$, contradicting that $py^T > 0$ by assumption B.

We shall now follow Radner in a calculation showing that only a limited number of the factors μ_t can fall substantially below the upper bound μ^*. To this end we consider a comparison path $(\bar{x}^t, \bar{y}^t)$, $t = 1, \ldots, T$ constructed as follows. The initial stock $\bar{x}^1 = x^1$ is the same as before. The first two steps are used to arrive at the highest multiple

$$\bar{y}^2 = \nu \cdot x^*$$

of the von Neumann point of Assumption A that can be attained in two steps. The next $T-4$ steps proceed along the von Neumann path at maximum growth

$$\bar{y}^t = \mu^* \bar{x}^t , \qquad t = 3, \ldots, T-2 .$$

The last two steps are used to attain the highest attainable multiple

$$\bar{y}^T = \bar{\kappa} h$$

of the prescribed bundle $h = (h_1, h_2)$, attainable from $\bar{y}^{T-2}$ in two steps. Obviously the latter multiple $\bar{\kappa}$ cannot exceed the highest multiple κ^* attainable from x^1 in T steps, which is attained along the maximal path. Therefore

$$py^T = \kappa^* \cdot ph \geqq \bar{\kappa} \cdot ph = p\bar{y}^T$$

If we now factorize the value growth factor for the entire comparison path in a similar manner, we find that*

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* Again all denominators and numerators are positive by Assumption B.

$$\frac{p\bar{y}^T}{px^1} = \frac{p\bar{y}^T}{p\bar{x}^1} = \bar{\mu}_1 \bar{\mu}_2 \cdot (\mu^*)^{T-4} \cdot \bar{\mu}_{T-1} \bar{\mu}_T ,$$

where the $\bar{\mu}_t = \mu(\bar{x}^t, \bar{y}^t)$ are value growth factors for the steps of the comparison path. Using all these results together, we find that

$$\mu_1 \mu_2 \cdots \mu_T \geqq \bar{\mu}_1 \bar{\mu}_2 (\mu^*)^{T-4} \bar{\mu}_{T-1} \bar{\mu}_T$$

Now let us choose any small positive number δ , and, remembering that $\mu_t \leqq \mu^*$ for all t , let us denote by T' the number of value growth factors μ_t in the maximal growth paths that fall short of μ^* by more than δ ,

$$\mu_t < \mu^* - \delta \quad \text{in } T' \text{ out of } T \text{ cases.}$$

Then clearly

$$(\mu^*)^{T-T'} (\mu^*-\delta)^{T'} \geqq \mu_1 \mu_2 \cdots \mu_T \geqq \bar{\mu}_1 \bar{\mu}_2 \bar{\mu}_{T-1} \bar{\mu}_T (\mu^*)^{T-4}$$

or, dividing through by $(\mu^*)^T$,

$$\left(\frac{\mu^* - \delta}{\mu^*}\right)^{T'} \geqq \frac{\bar{\mu}_1 \bar{\mu}_2 \bar{\mu}_{T-1} \bar{\mu}_T}{(\mu^*)^4} = M \text{ , say} \quad .$$

Since δ and μ^* are positive,

$$0 < \frac{\mu^* - \delta}{\mu^*} = 1 - \frac{\delta}{\mu^*} < 1 \quad \text{as long as } \delta < \mu^* \quad .$$

It follows that $(1 - \delta/\mu^*)^n$ decreases as the integer n increases, and approaches zero as n increases beyond bound. Since M is also positive, there must therefore be a largest value of n for which $(1 - \delta/\mu^*)^n \geqq M$. Denoting this value by T_δ we must then have

$$T' \leqq T_\delta \quad .$$

Thus the calculation has shown that there is an upper bound T_δ on the number of steps in a maximal growth path with $\mu_t < \mu^* - \delta$. This bound is determined from a condition involving δ , μ^* and M in such a way that T_δ increases if δ is chosen smaller. T_δ also depends, through M , on the initial and final stock ratios given by x^1, h, and, of course, on the shape of the production set Z . The important point is, however, that T_δ does <u>not</u> depend on the length T of the path, because $\bar{\mu}_{T-1} \cdot \bar{\mu}_T$ depends only on the normalized points x^* , h rather than on their respective multiples $\bar{y}^{T-2}$, $\bar{y}^T$.

The simple reason for this beautiful result is clear. As the path length T increases, the value growth factor for the entire comparison path keeps piling up factors μ^* . Since value growth factors in excess of μ^* are impossible in any path, too many factors less than $\mu^* - \delta$ in the maximal growth path would cause its terminal value py^T to fall below that of the comparison path. More precisely, for any given δ the upper bound T_δ to the number of such factors is determined by the relative factors $\bar{\mu}_1/\mu^*$, $\bar{\mu}_2/\mu^*$, $\bar{\mu}_{T-1}/\mu^*$, $\bar{\mu}_T/\mu^*$, associated with the "weak links" in the comparison path. If more nearly maximal paths of comparison can be found, they will lead to sharper bounds.

12. "Profit" Effects of an Interest Rate Reduction.

There is a natural economic interpretation for those $T - T'$ steps in the maximal growth path whose value growth factors μ_t stay within δ from the maximum μ^*,

$$\mu_t \geqq \mu^* - \delta \ .$$

Consider a price system having the same commodity prices, but an interest rate r' reduced by δ,

$$r' = r - \delta \ .$$

In this modified price system, the profit function is

$$\pi'(x, y) = py - (\mu^* - \delta) \cdot px$$

By dividing both sides by px we see that the steps in question are precisely those that yield a positive or zero profit in the modified price system. Hence our result can also be formulated thus: For any reduction in the interest rate, no matter how small, the number of steps in a maximal growth path with given initial and terminal stock composition that remains unprofitable is limited regardless of the length of the path.

13. A Variant of Radner's Turnpike Theorem.

The implications of this result for the course of a maximal growth path depend on the precise shape of the production set Z . The simplest case is defined by

Assumption C . *The separating plane P can be so chosen that P and the production set Z have only the point z^* in common.**

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* It follows directly that z^* is the only (normalized) von Neumann activity, since every such activity is in $S(\mu^*)$ and $S(\mu^*)$ is in P .

This assumption implies that both prices p_1 , p_2 are positive, since otherwise one or the other of the two origins would be both in Z and in P , and z^* would not be the only such point. Hence Assumption C implies Assumption B . The converse is not true.

Assumption C is necessarily satisfied if Z is strictly convex. It can also be valid, for instance, if z^* happens to be a vertex of a polyhedral Z .

By applying Assumption C to the results of Section 7, we find that the profit $\pi(x, y)$ in the original price system (p_1, p_2, r) is negative in any point z of Z other than z^* . On the other hand, the condition

$$(\underline{P}') \qquad \pi'(x, y) = py - (\mu^* - \delta) \cdot px \geqq 0$$

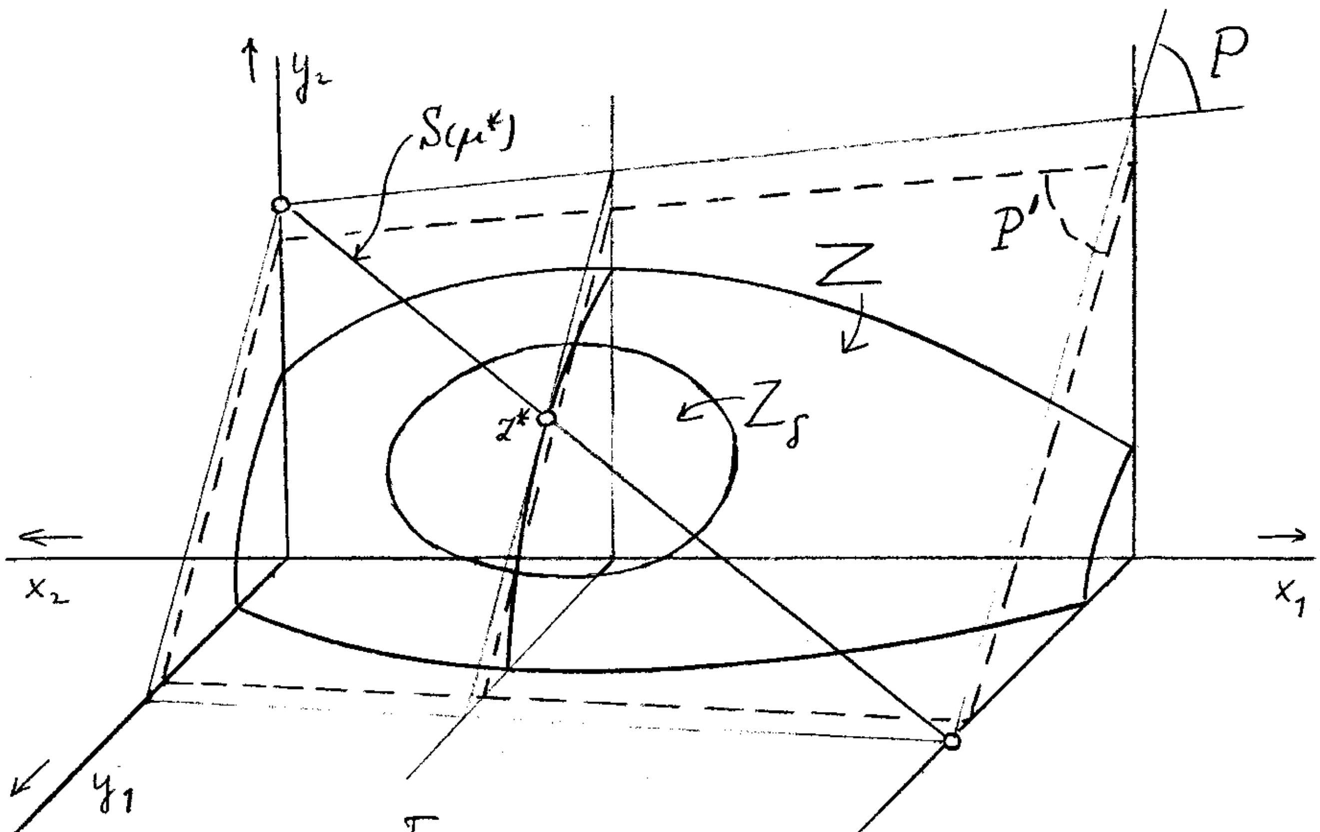

Figure 10

of <u>nonnegative</u> profit in the modified price system (p_1, p_2, r') cuts a slice, Z_δ say, off the production set. In Figure 10 this slice, and the plane

$$(P') \qquad \pi'(x,y) = py - (\mu^*-\delta) \cdot px = 0$$

that cuts it off, are shown for a production set similar to that of Figure 7.

Clearly, in the limiting case $\delta = 0$, the slice Z_δ consists of the single point z^*. Moreover, z^* is contained in Z_δ for every positive δ. The significance of assumption C is that it makes z^* the only point with the latter property. For any other point z of Z, since the original profit is negative in z, there exists an interest rate reduction δ small enough to leave the modified profit negative in z . It follows that the slices Z_δ , which are nested one inside the other as δ becomes smaller, shrink down to the point z^* as δ tends to zero.

This mathematical fact is responsible for the turnpike theorem in its present, simplest, version. It also suggests that it will be convenient to use δ as a measure of the distance, in the (three-dimensional, normalized) activity space, between z^* and any point z of Z , instead of Radner's measure of angular distance between points of the (two-dimensional) commodity space. Specifically, given a feasible point z , we will define $\delta(z)$ as the smallest value of δ for which z is still in Z_δ . In economic terms, $\delta(z)$ is the smallest interest rate concession that prevents z from yielding a loss. From continuity considerations, that concession will then just make z break even.

We can now summarize the results of our reasoning in

PROPOSITION 1. *If Assumptions A and C are satisfied, then there exists for each (small) positive number δ an integer T_δ such that, in a maximal path of any length, $\delta(z^t) \geqq \delta$ for at most T_δ steps of the path. Here $\delta(z)$ is a measure of the proximity of a normalized activity z to the von Neumann activity z^* of Assumption A. This measure of proximity is that reduction in the interest rate that will make the activity z break even. It reflects the shape of the production set in a neighborhood of z^* . The upper bound T_δ to the number of steps which upon normalization are more than δ "away from" z^* increases as δ decreases, and depends also on the initial and final stock proportions x^1, h, but not on the length of the path.*

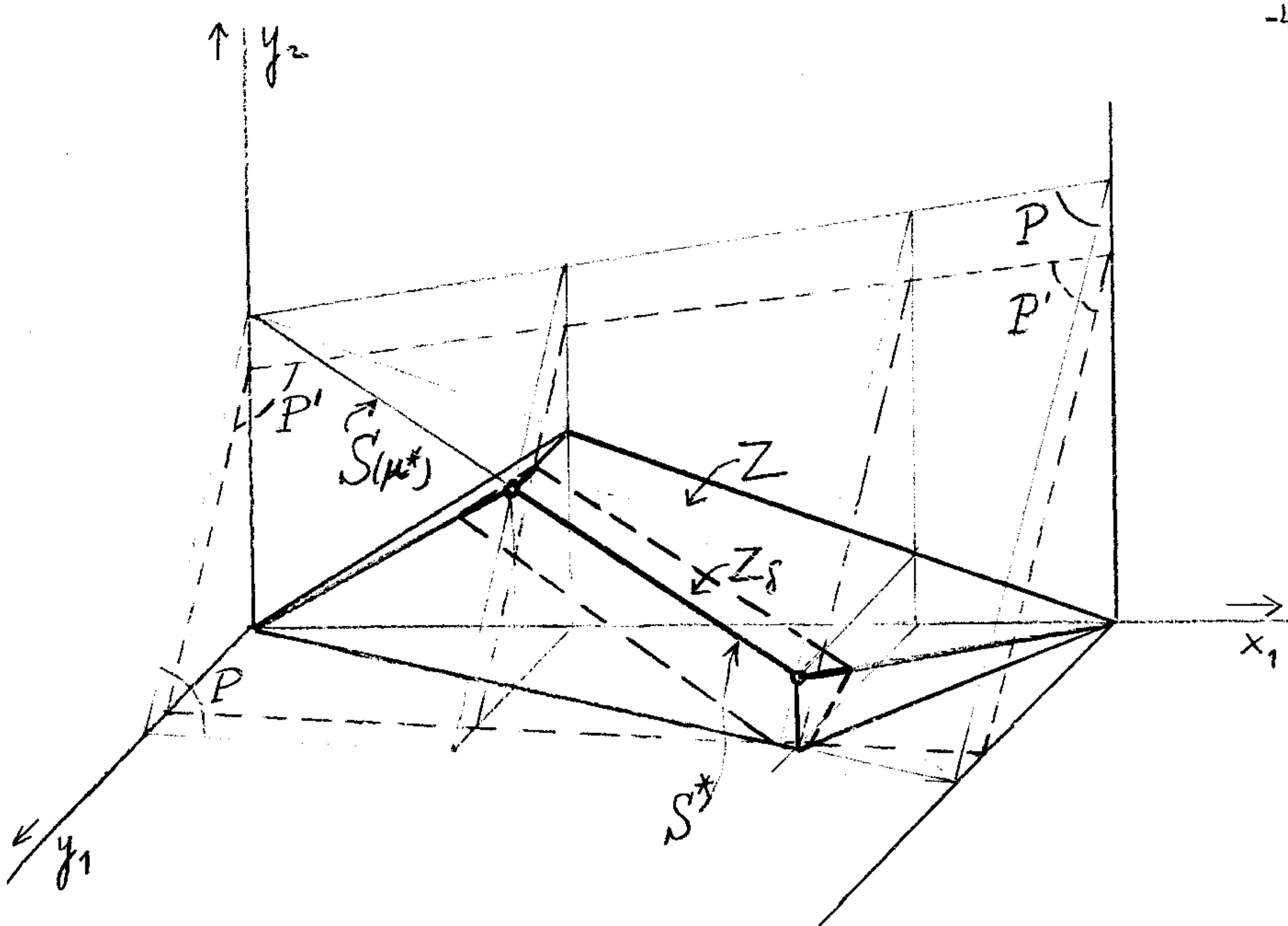

Figure 11

Note that this proposition is silent on the question where in the maximal growth path the exceptional activities more than δ removed from z^* may be found. We return to this question in Section 16 below.

14. Other Cases.

Matters remain relatively simple if the common points of P and Z make up a line segment S^* <u>contained in</u> $S(\mu^*)$. This case is illustrated by a polyhedral example in Figure 11. All points of S^* are now points of maximal proportional growth, and define as many von Neumann paths, all capable of being sustained by the same price system (p_1, p_2, r) . The slices Z_δ all contain

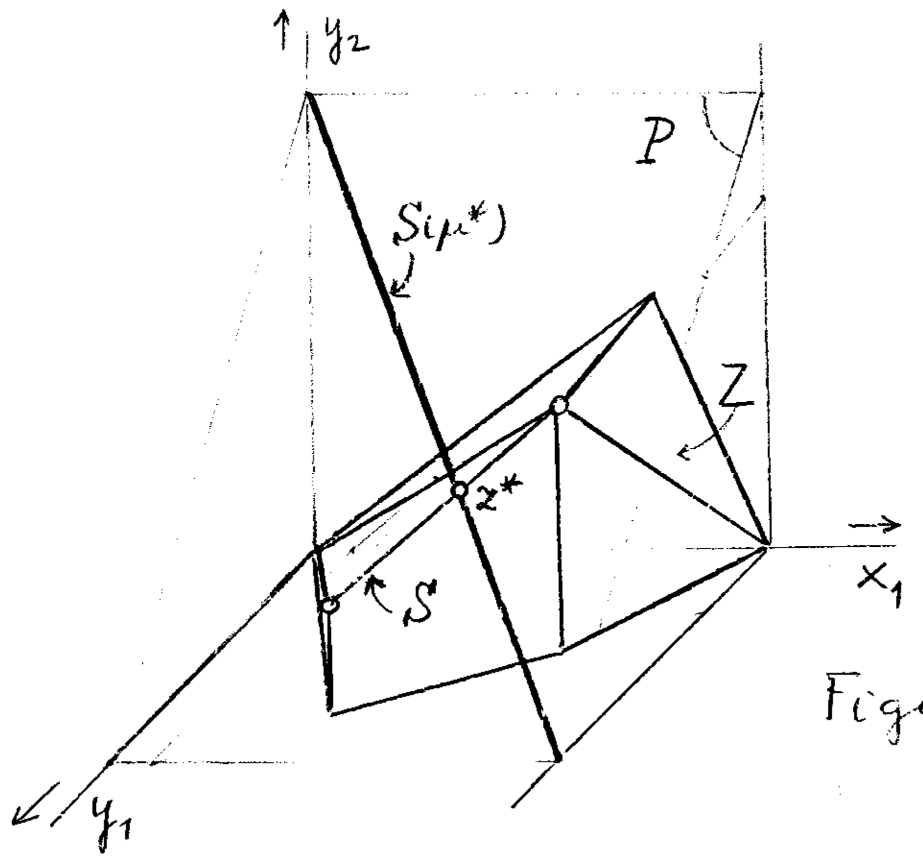

Figure 12

S^* , and shrink down to S^* as δ tends to zero. Hence, in any maximal growth path the number of activities outside a Z_δ which closely hugs S^* is limited. This is all that can be said in this case.

Further complications arise if P and Z intersect in a line segment S <u>not</u> contained in $S(\mu^*)$. In that case, S and $S(\mu^*)$ have only the unique von Neumann activity z^* in common. To avoid still further complications, let us assume that positive prices p_1, p_2 can be associated with z^* . The situation is then as illustrated in Figure 12. It remains true that the slices Z_δ shrink down to S , and that the number of activities in a maximal growth path outside a given slice is limited. It no longer follows without further analysis that therefore a maximal growth path has to be close to the von Neumann path except for a limited number of steps. The new complication is that S may contain points "far" removed from z^* . The turnpike conjecture would therefore be true in the present case only if we can show that, in order to be close to points of S most of the time, a growth path actually has to be close to z^* most of the time. This possibility arises because we have not yet made use of the equality $y^t = x^{t+1}$ between a period's outputs and the next period's inputs. (So far we have only needed to use the value equality $py^t = px^{t+1}$.)

In order to examine this question briefly, we first observe that it can be clarified by the study of growth paths that consist of zero-profit activities only, hence are entirely constructed from activities in S . By the preceding analysis, any such paths are themselves maximal growth paths, and are centers of attraction for all other maximal growth paths, in the sense indicated.

Next it will help to change the units of the two commodities in such a way that their prices become equal,

$$p_1 = p_2 = 1 , \qquad \text{so} \quad px = x_1 + x_2 ,$$

making a normalized input identical with an input of unit value. Thereafter we make the units of both commodities dependent on the time period t by multiplying each unit by $(\mu^*)^t$. This makes the new maximal "growth" factor equal to unity, the new "interest rate" equal to zero. The service rendered by this somewhat artificial redefinition of the units is that a zero-profit activity (x, y) now satisfies

$$px - py = 0 ,$$

and hence converts a normalized input x into a normalized output y . Figure 12 has already been drawn on this basis.

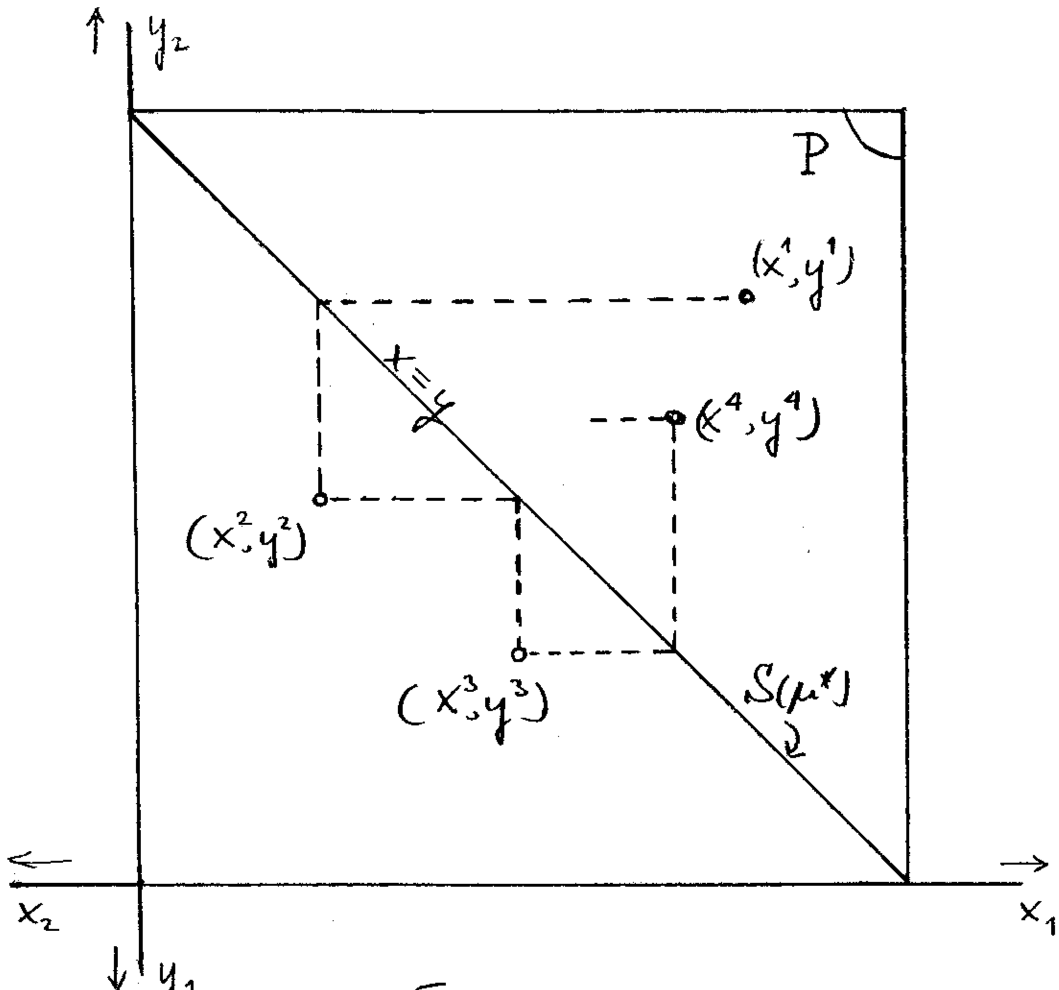

Figure 13

The analysis can now be completed within the zero-profit plane P , which we can project, out of its position in Figure 12, onto the vertical plane, identified with the plane of the paper in Figure 13. In this diagram the measurement of y_1 and y_2 along the vertical axis has become completely symmetrical to that of x_1 and x_2 along the horizontal axis, described in Section 4.

Disregarding feasibility, any growth path consisting of zero-profit activities only is now representable by a sequence of points (x^t, y^t), $t = 1, 2, \ldots$, connected by the identity $y^t = x^{t+1}$ in the manner indicated in Figure 13. Feasibility considerations are introduced by insisting in addition that all points (x^t, y^t) belong to the segment S .

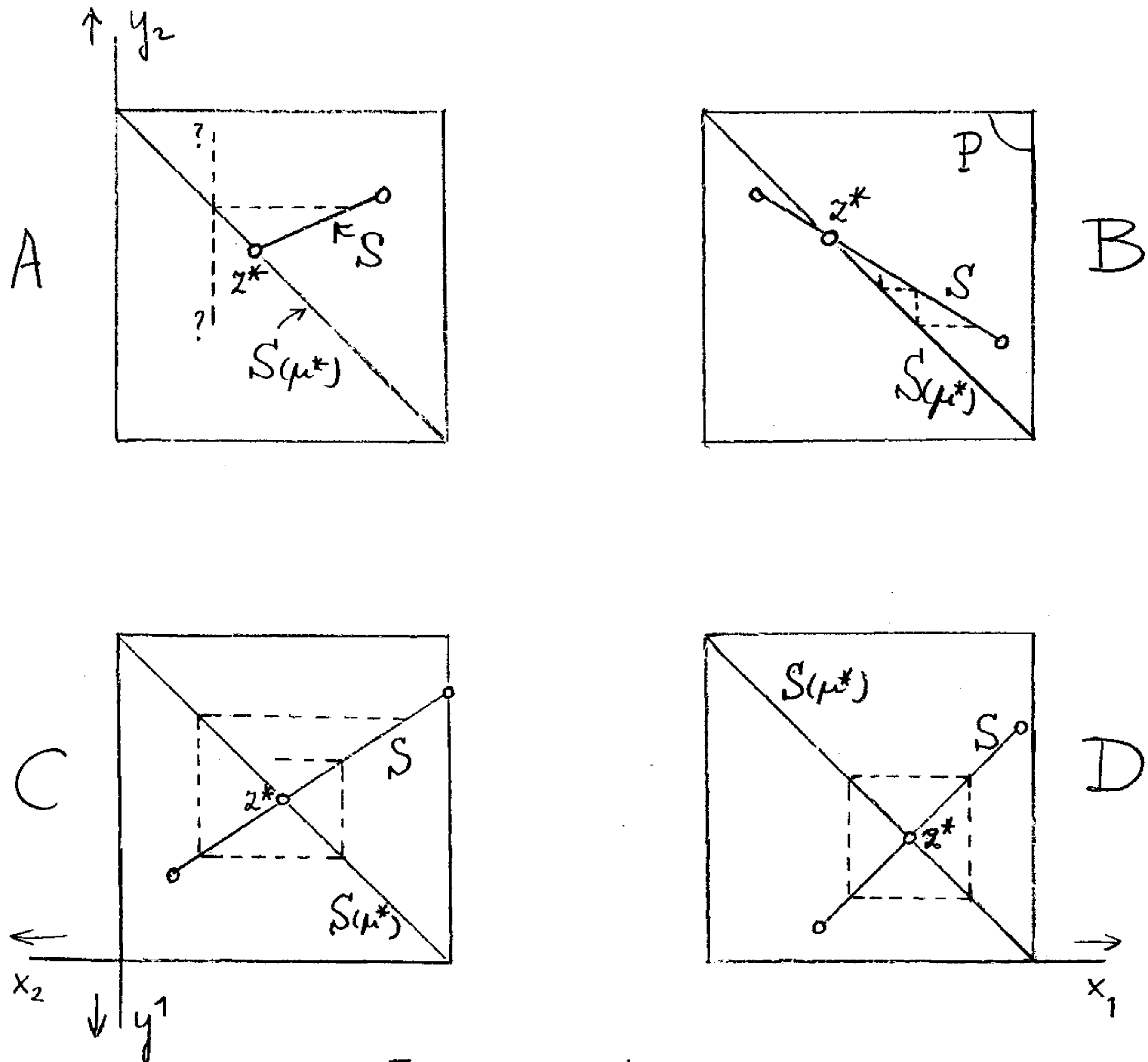

Figure 14

If, as in Figure 14A, the von Neumann point z^* is an end point of S and if at the same time the slope of S (in the present diagram) is opposite in sign to that of $S(\mu^*)$, then the only feasible sequence of zero-profit activities consists of a repetition of the von Neumann activity z^*. Hence the turnpike conjecture is valid in this case as well. If, as in Figure 14B the slope of S is of the same sign as that of $S(\mu^*)$, then there exist infinitely many feasible sequences of zero-profit activities, that either converge toward, or move away from, z^*, depending on whether the slope of S is absolutely smaller (as in Figure 14B) or larger than that of $S(\mu^*)$, and this holds regardless of whether z^* is an end point of S or not. Finally, if the slope of S is opposite in sign but not equal in absolute value to that of $S(\mu^*)$ there are again infinitely many permitted sequences, oscillating on S toward or away from z^*, depending on whether the slope of S is absolutely smaller (as in Figure 14C) or larger than that of $S(\mu^*)$. Since in all these cases the permitted sequences converge to z^* if time is taken forward or backward as may be needed, the turnpike conjecture is still valid. However, for a given δ, a larger bound T_δ on the number of steps more than δ away from z^* than would otherwise apply must be allowed in the present case, because the feasible sequence (x^t, y^t) of zero-profit activities to which some maximal growth path is "close" may itself take a long time getting close to, or moving away from, z^*. Thus the turnpike assertion for these cases is the more academic, the closer one comes to the following last case.

In this case, the counterexample discovered by Kuhn (Figure 14D), the slopes of S and $S(\mu^*)$ are opposite in sign and absolutely equal. There now exist infinitely many oscillating maximal growth paths consisting of zero-profit activities, that never come near the von Neumann point z^*, as well as many other maximal growth paths close to each of these, for which the same is true. Hence the turnpike conjecture is false in this case.

Further counterexamples arise in the case where P and Z have a two-dimensional convex set in common. We shall not examine these possibilities further.

15. A Remark about the Counterexamples.

The counterexamples are important from a logical point of view, as part of the intellectual process whereby a conjecture becomes a theorem. From a realistic point of view, however, it is hard to take the counterexamples seriously. They all involve an oscillation, or a more complicated continual change, in the methods of production used. The assumptions of our model ignore a consideration that weakens the claim of the counterexamples to a long-run growth capacity equalling that of maximal proportional growth. It is a well-documented fact of experience that the mere repetition of a production process facilitates its gradual improvement, through learning of the operations rather than through the introduction of new technological principles. Much of this advantage is lost in a path in which a substantial part of the labor force oscillates between different methods of production. While clearly this consideration lies outside the model here being studied, it may influence the degree of detail in which its study is pursued.

16. Concluding Remarks.

We have already observed that the turnpike proposition formulated in Section 13 does not touch on the question where in a maximal growth path the exceptional activities more than δ removed from a von Neumann activity may be found. Further light is shed on this in a paper by Nikaido [1962] through an ingenious argument in which Radner's reasoning is applied twice in succession. Since his paper has not yet been published, I shall not go into the details of his reasoning. His result is that, if in addition to Assumptions A and C it is given that the von Neumann point x^* has only positive components $(x_1^* > 0 , x_2^* > 0)$, then for any positive δ there exists an integer T'_δ such that, in a maximal path of any length T with initial and final constraints given by x^1 , h , the only steps which, upon normalization, are more than δ removed from z^* are found among the first T'_δ and the last T'_δ of the T steps of the path. This important finding has already been expressed in Figure 9. It confirms what was no doubt in the minds of the originators of the turnpike conjecture.

It was stated in Section 1 that the von Neumann model, in its original version quite remote from any real world problem, was given some bearing on the problem of forced economic growth by the discovery of the turnpike propositions. It should now be admitted that the problem of growth at a maximal rate is still a somewhat narrow and perhaps unnatural one. One would want to go on to the further study of optimal growth where the criterion of optimality expresses a concern with the desire for consumption levels that, if possible, are at all times above the minimum needed for self-reproduction of the labor force at the growth rate envisaged. One may want further to leave scope for an uncertain degree and kind of technological progress, and for a desire for flexibility in future consumer's preference.* Granted

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* See Koopmans [1962].

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the need for these further steps of generalization, however, the study of economic growth at a maximal rate in a constant technology seems nevertheless to have some justification beyond its immediate results. The solutions of problems obtained by generalizing or complicating another simpler problem often continue to bear some of the traits of the solution of the simple problem. Thus we may find as time goes on that the study of growth at a maximal rate is yielding returns that go beyond the confines of the original formulations.

Note: Upon completing this discussion paper I came upon a mimeographed paper by Inada [undated] in which very similar techniques of reasoning are used.

REFERENCES

Dorfman, R., P. A. Samuelson, and R. M. Solow, Linear Programming and Economic Analysis, McGraw Hill, 1958.

Gale, David, "The Closed Linear Model of Production," Linear Inequalities and Related Systems, H. W. Kuhn and A. W. Tucker, Eds., Princeton University Press, 1956, pp. 285-303.

Hicks, T. R., "Prices and the Turnpike, I, The Story of a Mare's Nest," Review of Economic Studies, Vol. XXVIII No. 2, February, 1961, pp. 77-88.

Furuya, H., and Inada, K., "Balanced Growth and Intertemporal Efficiency in Capital Accumulation," International Economic Review, Vol. 3, No. 1, January 1962, pp. 94-101.

Inada, Ken-ichi, "Some Structural Characteristics of Turnpike Theorems," mimeographed, 23 pp., undated.

Koopmans, T. C., and A. F. Bausch, "Selected Topics in Economics Involving Mathematical Reasoning," The SIAM Review, Vol. 1, No. 2, July 1959, pp. 79-148.

Koopmans, Tjalling C., "On Flexibility of Future Preference," Cowles Foundation Discussion Paper, No. 150, December 1962, 16 pp.

Kuhn, H. W., "Comments on the Turnpike Theorems," mimeographed, December 1959.

McKenzie, Lionel W., "The Dorfman-Samuelson-Solow Turnpike Theorem," March 1962, revised August 1962, mimeographed, 24 pp.

________, "Three Turnpike Theorems for a Generalized Leontief Model," January 1962, mimeographed, 37 pp.

________, "The Turnpike Theorem of Morishima," undated, 16 pp.

Morishima, M., "Prices and the Turnpike, II, Proof of a Turnpike Theorem: The 'No Joint Production Case,'" Review of Economic Studies, Vol. XXVIII No. 2, February 1961, pp. 89-97.

Nikaido, Mukukane, "Persistence of Continual Growth near the Von Neumann Ray: A Strong Version of the Radner Turnpike Theorem." Discussion Paper 27, Institute of Social and Economic Research, Osaka University, June, 1962, 20 pp.

Radner, R., "Prices and the Turnpike, III, Paths of Economic Growth That are Optimal with Regard only to Final States: A Turnpike Theorem," Review of Economic Studies, Vol. XXVIII No. 2, February 1961, pp. 98-104.