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Price Guides in Decentralized Organization\*

Andrew Whinston

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E R R A T A

(Because of the technicalities of reproduction it is necessary to include corrections separately.)

Page 1, line 7, 2nd para.	Substitute "he" for "we"
21, line 1, last para.	Subs. "separable" for "responsible"
22, line 3	Subs. " $\sum_i g_{ij}(x_i) \equiv g_{ij}(x_i) + \dots +$ " for " $\sum_i g_{ij}(x_i) \equiv g_{ig}(x_i) + \dots +$ "
22, line 3, 2nd para.	Subs. " $f(x_1, \dots, x_n) \equiv \sum f_i(x_i)$ " for " $f(x_1, \dots, x_n) \equiv \sum f_i(x_i)$ "
22, line 4, 2nd para.	Subs. " $\text{Max} \sum_{i=1}^n f_i(x_i)$ " for " $\text{Max} \sum_{i=1}^n f_i(x_i)$ "
23, line 3	Subs. "convex" for "concave"
30, line 1, 2nd para.	Subs. "criterion" for "interior"
30, line 1, 2nd para.	Subs. " $f_i(x_i)$ " for " $j_i(x_i)$ "
30, line 6, 2nd para.	Subs. "convex" for "concave"
32, footnote, line 2	Subs. " $\lambda_j$ " for " $\mu_j$ "
33, line 7	Subs. "improving" for "improvising"
33, footnote, line 3	Insert " $\lambda$ " before " $j(t)$ "
41, line 9	Subs. "context" for "contest"
44, line 10	Subs. "divisions" for "diversions"
57, line 12	Subs. " $\overline{H(E)} \not\subseteq X$ " for " $\overline{H(E)} \cap X$ "
57, line 13	Subs. " $X^* \notin \overline{H(E)}$ " for " $X^* \cap \overline{H(E)}$ "

## 1. Introduction

This paper deals, primarily, with the use of prices as a means for achieving efficient allocation of resources within a decentralized system such as a business firm. We shall be mainly concerned with two general classes of problems: efficiency of resource allocation and coordination of the results of operations. These problems are examined with special reference to cases when external effects are present.

By a decentralized decision making system we refer to the following: Given  $m$  decision or actions to be made and  $n$  decision makers ( $1 < n \leq m$ ) each decision maker is assigned a subset of the  $m$  decisions. For the overall system there is given a criterion function and a space of possible choices involving the  $m$  decisions. Each decision maker is assigned a space of possible choices and a criterion function involving at least the decision variables which we can partially or totally control. In this paper we shall be studying decentralized decision systems which use prices as a guide for the decisions to be made in choosing between alternate courses of action. Here, however, we view prices as primarily in terms of their information content and explore their uses and limitations from this point of view along with other possible devices -- e.g., administrative-organization arrangements -- that can be used for reenforcement and extensions of the indicated price guides.

We can perhaps distinguish this information theoretic approach to prices by distinguishing them as, say, the "transfer prices" of industrial

practice as these have emerged from the cost accounting transfers that double entry bookkeeping had previously developed around the cost collection centers for decentralized decisions. Thus, in the context of controlling transfer prices these cost collection centers become profit centers with some degree of freedom allowed for managerial negotiations and price setting between the centers. The objective is to secure a best or efficient set of operations (a) in each profit center and (b) in the total company. By an efficient decentralized system we refer to a comparison with the achievement of the overall goal by the actions of each decision maker and the highest possible level of the criterion function achievable in terms of the allowable decision space.

By an external effect - - or more briefly, an "externality" - - we refer to events that flow from the decisions of particular managers and affect the criteria or space of possible decisions that are used by others to guide their actions. In particular, externalities are said to be present when the relevant decision variables are not entirely under the control of the manager whose decision is being considered. We need to know, then, how such a person might, or should behave in such circumstances. In particular, where prices are used to guide such decisions we need to know the consequences of sole reliance upon, say, a mechanism that might be used to form such prices and also when such mechanisms might yield prices that do not achieve full efficiency (or coordination) in the decisions that affect the allocation of

resources. In cases where the mechanisms fall short then it is natural to inquire in what ways they might be altered or supplemented to provide an attainable bettering of these decisions.

## 2. Some Observations on Practices and Problems in Company Decentralization

### 2-1. Introduction

The first part of this paper will report on certain findings that were obtained from field interviews and discussions with officials of certain companies that utilize mechanisms of decentralization and coordination which, in part, depend on transfer prices and like devices. Such discussions were carried out in depth with only a limited number of officials from two particular companies. Hence, though this is supplemented by some further discussions with others, the results can hardly be regarded as providing a representative picture of the current state of such corporate practices or problems. These discussions, however, did prove fruitful, at least to the author, in providing insights into the nature of specific kinds of problems that are likely to be encountered when such devices of decentralization are used. This aspect of the interpretations gained from these field inquiries will be emphasized in the following.

Further such descriptive material will provide the reader with an introduction and background to problems of a more theoretical and normative nature as will be discussed in the second part of this paper.

An official statement from one of the visited companies reads:

"The direct responsibility for managing company lines operations rests with the general manager of the division. Under our form of organization, each division represents a separate profit center for the purpose of management control, and the general

manager is accountable for earning a satisfactory rate of return on the assets employed in his operation."

The intent of this statement is clear, of course, but it is also susceptible to various interpretations. Partly this is due to a natural ambiguity which arises from the fact that there is, in general, no precisely defined collection of words which meets the needs of managerial usage. Indeed, even the terms "decentralization" and "centralization" are usually characterized only as a matter of contrast or emphasis - - i.e., degree of centralization or decentralization - - from one organization situation to another and by reference to the particular aspects of the organization which are under immediate consideration.<sup>1/</sup>

For purposes of theoretical economics we may assume, of course, that decentralization occurs at the entrepreneurial (or customer) level when each individual is subject only to the limits imposed by market mechanisms<sup>2/</sup> and the resources that are available for his disposition. That is, the decentralization which is implicit in a freely functioning price mechanism is relatively complete in the sense that each entrepreneur, say, has whatever autonomy he desires in making decisions to commit such resources as he may have available. In the absence of externalities the decisions are his own in the sense that operational mechanisms are available, at least in principle, so that each entrepreneur can analyze the variations he encounters and isolate the parts

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<sup>1/</sup> See, e.g., H. Simon et.al., Centralization vs. Decentralization in Organizing the Controller's Department, Controllership Foundation, Inc.

<sup>2/</sup> Other social institutions like the law and mores are, of course, also operative.

due to his own decisions from the parts that are due to actions by others. Moreover, when decisions are successful then still further resources are placed at his command, and so on.

The manager of a company division encounters a somewhat different situation. "Externalities" are likely to be present so that either (a) some further direction is necessary in order to provide him with supplementary information as required for him to guide even his own decisions or (b) further coordinating mechanisms may be imposed to insure the range of benefit-penalty relations arising from externalities is taken into account. The imputation mechanism is further clouded by hierarchical layers along with tiers of line-staff relations so that it may not be possible to determine precisely his own role in effectuating any particular decision. It is true, of course, that promotion mechanisms and other devices may be used to reward "successful" managers or to punish "unsuccessful" ones in ways that are analogous to the penalty-reward systems that are provided in theoretical economics for transferring the responsibility for administering larger or smaller amounts of the social assets between "better" and "worse" entrepreneurs. But the possibility of different situations must also be considered for internal management. Thus, one division manager may have his profit center "earnings" transferred to another part of the company, even when he is "successful" and, in any event, the resulting profit accumulations are rarely left to his own autonomous discretion for further disposition.

Similar remarks apply, of course, to possible access to further sources of funds from "outside" the company and an individual division

manager is rarely, if ever, given complete autonomy to acquire or dispose of significant amounts of capital facilities. This suggests that, perhaps, a treatment of the topic of autonomy for a completely decentralized system might best be focused on the decisions involving current operations such as acquiring and disposing of the materials needed in production. To some extent this is true, but even here some qualification is needed where, for instance, the possibility of procurement or sale via "outside" sources is encountered in the various divisions of a firm.

#### 2.-2. Decentralization on Current Account in Two Companies

We shall return to this topic for further general treatment after first considering the issues in more concrete form. For this purpose we now summarize the general policies of certain companies where outside purchases are permitted as part of the autonomy granted to managers under certain conditions. In this case, the division managers must, as a general rule,<sup>1/</sup> buy from inside sources in all cases where the company has facilities available for "efficient" use in producing the indicated items. The following statements, which are actually patterned after a policy memorandum of one company but which accurately describes the procedures of both, provide a guide in such cases:

1. The company should produce internally items which are important because of "design, performance, or other characteristics or because of the company's investment in unique facilities."
2. If the company has a large investment in facilities,

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<sup>1/</sup> I.e., proposed deviations must first be appealed and receive justification for authorization before the completed action is taken.



and they can be operated profitably and efficiently, then the company's facilities should be used.

On the other hand, outside sources may be used when the following is true:

1. If investment in facilities are small and qualified suppliers exist and they charge a smaller price than the present internal supplier.

The following points were also made:

1. Outside "distress" prices should not be used in sourcing decisions. (This instruction evidently raises some question as to what is a valid outside price.)
2. An attempt should be made to "split" certain items where valuable technical data may be obtained. A split item is one which is supplied partly internally and partly from an outside company. (This characterization of split term procurements indicates that, in fact, divisional managers may have very little say about whether purchases are to be made within the company or outside it.)

We shall next describe the procedure used as to the "sourcing" <sup>1/</sup> of a part within the company. Here certain questions naturally arise such as (1) the discretion that will be permitted a division manager in regard to which of the various divisions it would be best to order a part, especially if the part had not previously been fabricated in the company or <sup>2/</sup> (2) the discretion that should be given to a potential supplying division for deciding whether it would be worthwhile to invest capital in order to supply the part. Company officials indicated that the following procedures were utilized to resolve such questions: Ordering divisions are to take the initiative and have primary responsibility for sourcing all items, but old sourcing patterns are to continue.

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<sup>1/</sup> This term is borrowed from the company's usage where it has the following meaning: The determination at the origin of production of some specified item.

<sup>2/</sup> Plus, possibly, imputed variances from other departments.

When a question arises as to how a new part is to be sourced, it is up to the manufacturing staff (a central staff unit) to indicate what division should be the supplying division. All questions of sourcing which involve capital expenditure or new investment are subject to central staff review and approval.

Even after the sourcing problem is decided, however, there remains a further question as to the price at which such goods are transferred. In a "centralized" firm the price would be either a standard cost<sup>1/</sup> - - if a standard cost system<sup>2/</sup> obtained - - or some version of an actual accounting cost. In a "decentralized" firm, a possibly different price may be used to effect such transfers in order to allow for some profit-or-loss margin, in addition to the accounting costs, inclusive or exclusive of overhead.

Since a transfer price may differ sometimes, considerably, from the related standard or actual cost figures, a question naturally arises as to how such prices should be set. Some have suggested the use of competitive outside prices, and in a rough general way this practice is often followed.<sup>3/</sup> There are, however, certain qualifications and refinements which need attention. For instance, in one of the firms that was visited, three different classes of goods with associated criteria were distinguished as follows:

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<sup>1/</sup> Plus, possibly, imputed variances from other departments.

<sup>2/</sup> Cf. E. L. Kohler, A Dictionary for Accountants

<sup>3/</sup> See, e.g., Joel Dean, "Decentralization and Intracompany Pricing," Harvard Business Review 1955 pp. 65-74.

Group I.

Items for which prices of competitive producers are not available and cannot be reliably approximated by comparative analysis.

Group II.

Items purchased from both outside suppliers and company sources. (Split items)

Group III.

Items, other than split items, for which prices of competitive producers are available, or can be reliably approximated by comparative analysis.

We shall discuss the pricing of items in Groups II and III first. These present the least difficulty to the organization. The policy for Group II (split items) is for the transfer price of a split item to be the same as the price paid to the outside supplier with various possible adjustments for differences in specification, volume, engineering, services, royalties, inbound freight, etc. Thus group II prices may be estimated in a fairly unambiguous manner. However, at the firms interviewed, split items tended to be a minor consideration in practice. To be sure, various qualifications to the use of outside prices may need to be made in order to allow for other considerations. This may lead to some difficulty, but, at least in the firms that were interviewed, little use was made of any possible deviation from the outside price in Group II cases.

Group III items present the same problem as Group II items and, with suitable qualifications the usual point of reference is an outside market price. A supplying division may, of course, refer to one or more qualifications in order to justify a higher-than-market price but such appeals

are usually ignored by the central staff unit concerned with arbitrating disputes. Below, further elaborations will be made and certain tentative hypothesis suggested in order to provide a possible rationale for this presumably "biased" procedure.

As far as the companies are concerned, Group I prices cause considerable difficulty because their determination, to a certain extent, seems to them arbitrary. In terms of dollar value they are heavily predominant over Groups II and III. A company statement reads:

"Group I prices are to be established on the basis of the estimated costs of an efficient producer plus a markup on the assets utilized."

Material costs, direct labor, and overhead are computed to reflect what an efficient producer would charge. Current assets, such as working capital and accounts receivables, are estimated and a return is thereby determined and applied to total costs in order to reflect the utilization of such assets by an independent producer in the current conduct of their business. Longer range factors are also considered. Fixed assets are included at their undepreciated book value. A further profit markup is then computed from this source and added to the costs to obtain the transfer price.

Such practices are, of course, open to dispute and disagreement because they lack an objective verified basis in fact of the kind, say, used to provide voucher and invoice support for ordinary accounting cost entries. Therefore certain further machinery is supplied in accordance with the following statement, which is an excerpt from a policy memorandum, of the procedures to be followed:

"If the buying or selling division believes that the profit markup being applied to a particular part [in Group I] is inappropriate, the issue should be referred to the Vice-President Finance for discussion."

The above case does not exhaust the possible variations for computing transfer prices, of course, and for further perspective we now refer to the specific practices of one of the companies that was visited. These are, somewhat similar to, although not identical with, those practices already discussed. Here listed in the order of preference accorded to them by company policy are the criteria to be used in determining transfer prices in this case:

- "a. Lowest current bona fide price or quotation of an outside vendor, consistent with sound purchasing practice
- b. Estimated cost of the design differences between the part to be priced and a similar part for which a current bona fide price or quotation is available
- c. Estimated cost differences due to design, volume, engineering, service, royalties and other relevant factors between the part to be priced and a similar part for which an acceptable price exists.
- d. Estimated cost of an efficient producer plus the amount of profit that an efficient producer can expect to receive
- e. Estimated cost of producing the part or assembly with efficient utilization of major equipment and facilities which exist at the source plant plus a normal profit for that type of operation."

Further administrative arrangements are also utilized which allow for possible challenges and other adjustments in accordance with the following procedures: (a) Supplying division submits to an ordering division its price quotation, which includes cost of operation, engineering specifications and prints as determined from estimates based on its

accounting records. On this basis the ordering division will quote a price. (b) The ordering division has a responsibility to check on the cost quotation furnished by the supplying division. (c) Each division has a staff of cost analysts who check the data for possible challenges. A challenge is generally made on the basis of information about competitors' costs, but challenges can also be based on the cost figures or the rules under which they have been synthesized.<sup>1/</sup>

A distinction formulated along the line of the preceding distinction between Group I and other group categories may also be used here for clarification. A Group I product price, as already observed, lacks the kind of objective justification that is normally available for items in Groups II and III. A Group I price can also give rise to trouble because this kind of product (for which no competitive suppliers are available) also suggests the lack of any real alternative for the ordering department. On the other hand, careful analysis by reference to the company's costing conventions does permit a challenge and adjustment possibility that may prove extremely profitable for the ordering division.

Presumably under a centralized management system such challenge and increased profit possibilities would not exist. Under the indicated decentralization and transfer price system, haggling could continue indefinitely, possibly even as a game strategy, with resulting injury, perhaps, to other company operations. To foreclose this possibility, however, the following rules are utilized: Price settlements are to be

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<sup>1/</sup> E.g., by referring to company manuals or other authoritative documents on accepted accounting practices.

completed within 30 days.<sup>1/</sup> Furthermore, once a price is established it holds for an entire production run of the product. (Hence gains or diseconomies from small "nuisance" orders thereby tend to be eliminated.)

Exceptions to the latter rules of course, are allowed, but these must be justified by reference to alterations of the following kind: (a) design of the product, (b) general level of wages or material costs, (c) competitive price components used in establishing the costs that underlie the derived price. On the other hand, cost variations resulting from efficiency variations in the supplying department are not generally regarded as justifying a further adjustment in an already established price for a given production run.

A distinction must be made, of course, between managerial and other sources of efficiency variations. For instance, technological changes may cause a shift in the relevant cost functions, or the dynamics of a learning-curve situation also may have to be allowed for in new item production. Dynamic factors like these are subject to negotiation and adjustment. But, except in unusual circumstances, once the dynamic path of the cost curve is agreed upon it cannot be altered on the basis of actual results--which are thus seen to be implicitly imputed to managerial efficiency only.

In one company the Product Analysis Department handled price disputes. A price dispute over a Group II or III item has usually

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<sup>1/</sup> In the event that a price is not settled upon within the time allocated for negotiations the matter is referred to the staff of the Vice-President for Finance, who is empowered to make a final determination, subject only to the qualifications noted above for further post facto adjustments.

revolved around a price quotation for a similar part obtainable from an outside vendor. In this case detailed data on the vendor is supplied including such items as reputation, tool capacity, and details of any deviation in the vendor's product. A challenge involving a Group I product generally involves a listing of specific disagreements on certain cost items. In such cases, the Product Analysis Department generally reviews the data and issues a recommended price. This may be appealed to an Arbitration Board which makes a final binding decision.

2.-3. Composition of Disputes over Transfer Prices:

The use of one kind of administrative mechanism for arbitrating or otherwise resolving disputes over transfer prices has just been noted. But the disputes are also of interest per se for supplying insight into the use of such decentralization devices and some discussion of them is warranted. Consider, therefore, the following case which was examined and discussed with relevant company officials: A supplying division claimed that it rightly should be able to charge a price higher than the market price on one of its Group III items. The market price was based on the selling price of a vendor who possessed a modern, efficient plant. The manager of the supplying division, however, was required to operate with an old outdated plant. Since the division manager had no authority to make the capital expenditures needed to modernize this plant, he claimed that he was penalized for something beyond his control. He claimed that such an organizational situation created bad "managerial psychology."



In this case, the office of the Vice President- Finance<sup>1/</sup> made the following decision:

"Since the primary objective of the intra company pricing system is to provide a means of measuring performance against known competitive levels, competitive practice with regard to costs and piece price must be followed. In the case of this part, there is an adequate knowledge concerning competitive practice to make this possible."

This case was discussed with company officials who had assisted the Vice President in the analyses leading up to this conclusion. They did not deny that the supplying manager's plant was outdated. Their justification was that, once he was forced to accept a transfer price which would cause him to lose money under his current operating procedures, the division would be able to find sufficient ways to save on expenses in order to show again a profit. (Note, therefore, that they were not assuming, as in economic theory, that the supplying division's manager was in fact utilizing an optimum production function. Rather they were using the transfer price mechanism as one device for bringing about a closer approach to the technologically feasible optimum. On the other hand, they did not admit of a still further change in this function that might be brought about from additional plant investment in this case.)

A similar dispute arose in another company. One division supplied a certain part to several other divisions within the company. Other companies in the industry purchased this same part from one independent producer. Thus the independent supplier had a much higher volume than

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<sup>1/</sup> Cf. Footnote 2, p. 11.

the internal supplier. The internal supplier claimed that his division was entitled to a higher transfer price than the outside price, because his smaller volume arose from company wide considerations that were outside his division's control. The Product Analysis Department decided against this division's claim for much the same reason as in the previous dispute. Established competitive prices were said to provide a desirable norm and the fact that the supplying division could not increase its volume (e.g. by outside sale) to obtain the supposed economies of scale was ignored.

Before proceeding to further cases we may pause to make certain remarks here. In the traditional approach to budgetary accounting a commonly accepted opinion is that managers should only be held responsible for costs which they at least partially control. The above disputes suggest that this is not always true. Volume considerations, the condition of the plant that must be used, etc., may well lie entirely beyond an individual manager's control but the resulting costs are nevertheless charged against his budget. On the other hand, the question of optimum practice even of a non-optimal plant or volume must also be considered, as it apparently was in the preceding cases. Thus the transfer price issues discussed here have also helped to highlight certain aspects of budgetary practice. From a budgetary point of view the transfer pricing mechanism may be used, as in the

above cases, as a way of setting an upper bound on an "acceptable" cost level. If a division manager, when faced with a lowered acceptable cost level (as in the above disputes) is able, in fact, to lower costs, then this procedure may be considered justifiable.<sup>1/</sup>

We now return to further cases. Another basis for dispute in one company arose over the method of scheduling production. When transfer prices were negotiated, a tentative schedule of deliveries was agreed upon. However, the ordering division, because of varying demand for its product, sometimes found it necessary to alter the delivery schedule. This alteration, which affected the supplying division's costs, was not permitted to affect the price at which the goods were transferred. Supplying divisions continually point out that the fluctuations are not their responsibility and therefore they should not be penalized. On the other hand, the indicated fluctuations were also beyond the control of the ordering division. Evidently, then, no beneficial incentive effect could be expected from merely raising the transfer price and, in any event, the supplying division's appeal was denied. No change in the transfer price was permitted.

Other disputes can arise because of interactions in the productive processes. A particularly striking case of this occurred in one of the companies examined where two different divisions alternatively shared the same production line. One division claimed that the other was not providing proper maintenance and, by saving on its own maintenance, was causing excess maintenance costs to be shouldered by the

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<sup>1/</sup> A similar argument has been made by A. Stedry on the general problem of what are good budgets. See his Budget Control and Cost Behavior, Prentice Hall, 1960.

other division. That is, the complainant division argued that its maintenance costs could be reduced if only the alternate user of this production line were somehow made to conduct its own maintenance "efficiently." Because of such frequent occurrences of these cases, this company established a policy under which only units with separate assets could be formed into decentralized divisions. In this particular case, other solutions are conceivable. For example, each unit could be directed to budget for a minimum level of maintenance, or a special maintenance unit could be formed to service the production process. But note that these proposals would be organizational or "mixed system" rather than pure price solutions. In the next section, we shall study in a more formal manner some problems raised here. We shall be concerned, in particular, with the question of how a large decentralized organization can effectively generate the transfer prices both when externalities are absent and present.

### 3. Price Guides Within a Decentralized Organization—a Normative Approach

In contrast to the last section of this paper where we were mainly interested in a descriptive approach of decentralized behavior, we will focus here on how a decentralized organization might best be coordinated from a price (information-theoretic) standpoint. For the purposes of this section, we shall introduce several models representing organizational decision making. The aim of these models will be to allow us to focus on the role of price guides, on question of the implications of externalities and other features, and to do so in terms of easily manipulated mathematical expressions where complexes of interacting variables and constraining conditions are involved.

### 3.1. Firm as a Decentralized Decision Maker

We begin with the consideration of the case for a degree of decentralization in the firm. Two main points are to be made here. The first has to do with problems of motivation and the second with the consequence of human cognitive limitations for the managerial span of control.

While we emphasize the formulation of a desirable overall plan of behavior for the organization via decentralization, e.g., the planning problem, in fact discussions of decentralization in the business literature very often emphasize the control or motivational aspects. Writers have claimed that with the introduction of profit centers top management may be better informed as to weakness and strengths of the organization in specific areas.<sup>1/</sup> Writers such as Argyris and Likert,<sup>2/</sup> for example, have emphasized factors that motivate middle and lower management by means of certain characteristics of a decentralized organization. Setting management goals and rewards, both salary and promotional, as some functions of the profit of the decentralized unit may be effective forms of budgeting.<sup>3/</sup>

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1/ The control aspect is emphasized by J. Dean. See "Decentralization and Intercompany Pricing," Harvard Business Review, July-August, 1955.

2/ See C. Argyris, "Personality and Organization," Harper's New York, and R. Likert, "Measuring Organizational Performance," Harvard Business Review, 1958.

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3/ For a discussion of experimental approaches to motivation and budgeting, see A. Stedry, Budget Control and Cost Behavior, Prentice-Hall, 1960.

On the other hand, many of these writers have failed to relate these aspects of subunit behavior and motivation to the problem of overall optimization in either its coordination or efficiency senses. In particular, they have tended to ignore any consideration of information theoretic devices--e.g., prices--which can be used to guide each subunit into a coordinated scheme of overall efficiency by reference only to the motivational devices that they suggest for consideration.

A fundamental argument for decentralization which underlies our interest in the subject, but is not reflected in our formal discussion is the concept of "bounded rationality" as presented by Hayek and Von Mises.<sup>1/</sup>

We may do no better than quote Hayek:

"As decentralization has become necessary because nobody can consciously balance all the considerations bearing on the decisions of so many individuals, the coordination can clearly not be effected by 'conscious control,' but only by arrangements which convey to each agent the information he must possess in order effectively to adjust his decisions to those of others. And because all the details of the changes constantly affecting the conditions of demand and supply of the different commodities can never be fully known, or quickly enough be collected and disseminated, by any one centre, what is required is some apparatus of registration which automatically records all the relevant effects of individual actions, and whose indications are at the same time the resultant of, and the guide for, all the individual decisions.

"This is precisely what the price system does under competition, and which no other system even promises to accomplish. It enables entrepreneurs, by watching the movement of comparatively few prices, as an engineer watches the hands of a few dials, to adjust their activities to those of their fellows--The more complicated the whole, the more dependent we become on that division of knowledge between individuals whose separate efforts are co-ordinated

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<sup>1/</sup> Hayek, F., The Road to Serfdom, London, 1946. Von Mises, L., Bureaucracy, New Haven, 1944. See also J. March and H. Simon, Organizations, John Wiley, 1958, p. 203.

by the impersonal mechanism for transmitting the relevant information known by us as the price system."

Then we may note, in particular, that Hayek assumes that it is always possible to arrange a price mechanism so that each individual can, in fact, effect the necessary cost-price calculations needed to effectuate his own best interests. Of course, when this is true, then under certain very general circumstances an efficient overall decision will emerge when each unit pursues its own goals efficiently. There are certain questions which require further attention, however, before this kind of assumption and this kind of generalization is extended from a free market-oriented economy into the actual operation of decentralized (but managed) business firms.

### 3.2. Models of Resource Allocation Within the Firm:

The general purpose of this section is to study decentralized resource allocation within the firm. For this we need a model of resource allocation and decision making.

Let  $x_1, \dots, x_n$  be certain decision variables such as level of output, level of a service, etc. In general,  $x_i$ , ( $x_i \geq 0$ ,  $i = 1, \dots, n$ ), represents a quantifiable operation within the firm. Of course, certain organizational and physical constraints are imposed on these  $x_i$ . Let  $g_{ij}(x_i)$  be the amount (in appropriate units) of the  $j^{\text{th}}$  constraint utilized by the  $i^{\text{th}}$  decision variable when some value of  $x_i$  is specified. Then we can write  $\sum_i g_{ij}(x_i) \leq K_j$  for each of the  $j = 1, \dots, m$  constraints that we shall consider.

We define a function  $\Phi(x_1, \dots, x_n)$  to be separable if it can be written in the form  $\sum_i \Phi_i(x_i)$  where each  $\Phi_i$  is a function of only one variable. Note that the functions involved in the model are all separable.

In a subsequent section of this paper we discuss the implications of relaxing this condition but, for the present, we assume that the  $g_{ij}(x_i)$  are separable and write  $\sum_i g_{ij}(x_i) = g_{ig}(x_i) + \dots + g_{nj}(x_n) \leq K_j$  for each of the  $j = 1, \dots, m$  constraints.

We should next observe that these constraints need not be restricted to problems in economic resource limitations only. An example of such a constraint originating from a different quarter might be given as follows: Assume some organization, while wishing to maximize profit, is also concerned that it does not "overly penetrate" the market. (This constraint may be derived from a fear that an overall domination of several markets may result in anti-trust suits.) Let  $K_j$  be measured in percentage of market penetration so that  $K_j$  itself measures the maximum amount of total increase in market penetration that this organization will allow. Evidently then each  $g_{ij}(x_i)$  in  $\sum_i g_{ij}(x_i)$  measures, as a function of  $x_i$ , the increase in market penetration which results from any specified  $x_i$ .

We shall assume that within the specified structural constraints the firm is attempting to maximize an overall profit function <sup>1/</sup> of the form  $f(x_1, \dots, x_n) = \sum f_i(x_i)$ . Then our immediate relevant model is

$$I. \quad (1) \quad \text{Max} \quad \sum_{i=1}^n f_i(x_i)$$

subject to

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<sup>1/</sup> As measured by the difference between revenue and variable plus fixed costs.



$$(2) \quad \sum_{i=1}^n g_{ij}(x_i) \leq K_j \quad j = 1, \dots, m$$

$$(3) \quad x_i \geq 0$$

We further assume that every  $g_{ij}$  is concave, which in classical economics terminology, means that we are reaching an assumption of decreasing marginal productivity. We also assume that each  $f_i$  is concave, which implies decreasing marginal profitability in each  $f_i$ .

With these assumptions we proceed to obtain the following sufficient conditions <sup>1/</sup> for a choice of  $x$  that will maximize  $\sum f_i$ :

$$(1) \quad \frac{\partial f}{\partial x_i} - \sum_j \lambda_j \frac{\partial g_{ij}}{\partial x_i} \leq 0 \quad i = 1, \dots, n$$

II.

$$(2) \quad x_i^0 \left( \frac{\partial f_i}{\partial x_i} - \sum_j \lambda_j \frac{\partial g_{ij}}{\partial x_i} \right) = 0 \quad i = 1, \dots, n$$

$$(3) \quad \sum g_{ij}(x_i^0) \leq K_j$$

$$(4) \quad \lambda^0 \left( \sum_{i=1}^n g_{ij}(x_i^0) - K_j \right) = 0 \quad j = 1, \dots, m$$

$$(5) \quad x_i^0, \lambda_i^0 \geq 0 \quad (\text{all } i, j)$$

Next we provide certain interpretations. Thus the  $\lambda_j$  are identified as prices, in fact, each  $\lambda_j$  is a so-called efficiency price associated

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<sup>1/</sup> These are known as the Kuhn Tucker conditions. See H. Kuhn and A. Tucker "Non Linear Programming" Proceedings of the Second Berkeley Symposium in Mathematical Statistics and Probability (University of California Press 1951)

with the  $j^{\text{th}}$  constraint. Thus if the constraint refers to the transfer of goods between units then  $\lambda_j$  would be referred to as a transfer price -- ie. the price per unit at which a purchasing or selling unit trades the good -- and at an optimum, as above, the  $\lambda_j$  would be associated with efficiency in achieving the overall objective. For the case where the constraint refers to a physical limitation the price is a unit charge on the use of the facility so that, all things considered, this facility will be used efficiently as the price,  $\lambda_j$ , serves to ration the desires of all managers to utilize the services that it supplies.

With this interpretation of the  $\lambda_j$  in mind we may now describe an iterative procedure<sup>1/</sup> which reflects the kinds of mechanisms that might be used to generate a set of efficient prices. There is, of course, a problem in that we will not, in general, be able to supply (always) an effective solution procedure. But our immediate purposes will be served if we define an iterative procedure to solve problems like the above if only they may ultimately be interpreted in terms of prices for decentralization. To obtain our immediate objective we proceed "organizationally" as follows: We refer to the variable  $x_i$  as an activity of the organization. To each activity we assign a manager who is responsible for choosing an optimal level of the activity. The manager is instructed to make his decision on the

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<sup>1/</sup> The procedure has been discussed extensively by K. Arrow and L. Hurwicz. See their paper "Decentralization and Computation in Resource Allocation" Essays in Economics and Econometrics ed. R. Pfouts (Chapel Hill: University of North Carolina Press 1961)

basis of the various efficiency prices of the "inputs" of fixed resources needed, the direct revenues and costs of the unit and the technology relevant to the unit -- e.g., the input requirements of fixed resources needed for different levels at the activity. Each manager is, in fact, instructed to choose on the basis of the information just enumerated, an activity level which will maximize the profits of his unit and, when efficiency prices prevail, this should (at least under certain circumstances) produce an overall efficient state for the entire firm as well.

The iterative procedure can then be accorded the following interpretative development. For any given  $\lambda_j(t)$  each manager chooses  $x_i^*(t)$  so that it is the maximizer of

$$f_i(x) - \sum_{j=1}^n \lambda_j(t) g_{ij}(x_i)$$

This defines, then, a function  $\Phi_i(\lambda_1(t), \dots, \lambda_n(t))$  for each manager. Prices are adjusted by the following rule

$$(5) \quad \frac{d\lambda_j}{dt} = \begin{cases} 0 & \text{if } \lambda_j = 0 \text{ and } K_j - \sum_{i=1}^n g_{ij} [x_i(t)] > 0 \\ \ell (\sum_{i=1}^n g_{ij} [x_i(t)] - K_j) & \text{otherwise} \end{cases}$$

$$j = 1, \dots, m$$

$\ell$  = dimensionality factor

where  $x_i(t) = \Phi_i [\lambda_1(t), \dots, \lambda_n(t)]$  .

Systems of this sort<sup>1/</sup> were first discussed by Koopmans in his original paper on decentralized resource allocation<sup>2/</sup> in which he introduced the terminology of helmsmen, custodians and managers. His development was, of course, only static and dealt with the maintenance only of already achieved efficient production possibilities.

We may, in fact, follow Koopmans, at least in part, and view each "custodian" as an official who is in charge of allocating a specific fixed resource and who proceeds according to (5) or its difference equation equivalent. Roughly the custodian raises the price if the resource is in excess demand and lowers the price if it is in excess supply. "Helmsmen," on the other hand, are officials who do not know the fixed resources of how they are being utilized. Their job is to supply suitable criteria via the  $f_i(x_i)$  to "managers." The latter (i.e., managers) are also assigned a role in the resource allocation and this role is played in accordance with the following rules:<sup>1/</sup>

"Do not engage in activities that have negative profitability.  
Maintain activities of zero profitability at a constant level.

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<sup>1/</sup> As we have formulated the model of a decentralized solution to the organizational decision problem there may be in principle an infinite number of calculations to perform by each manager. Thus in any practical application of the ideas suggested, modification would be required. One writer in discussing this aspect of the problem has suggested that a truncation of the process be made after a given interval of time. See T. Marschak, "Centralization and Decentralization in Economic Organizations," Econometrica, July 1959.

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<sup>2/</sup> T. C. Koopmans, Ed., Activity Analysis of Production and Allocation, New York: John Wiley and Sons, Inc., 1951.

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<sup>3/</sup> Loc. cit. pp. 94 ff. See also A. Charnes and W. W. Cooper, Management Models and Industrial Applications of Linear Programming, Chapter IX.

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<sup>4/</sup> Ed., Koopmans, *ibid.* p. 94

Expand activities of positive profitability by increasing orders for the necessary inputs with, and offers of the outputs in question to, the custodian of those commodities."

Convergence and stability properties of systems like this can be verified by reference to Uzawa<sup>1/</sup> and so we shall not find it necessary to supply a detailed mathematical development for this.

Assuming that the required convergence and stability properties are at hand we can provide an interpretation as follows: The manager for each activity, given internal prices, reaches an optimal decision on the basis of his own technology. Custodians or a central staff group proceeds on the basis of information supplied by managers about their decision and allocate correctly the jointly used resources of the company by varying the internal prices. To operate effectively each manager need concern himself only with his internal operations and the information received from a central staff group.

We have now provisionally achieved one of our objectives -- on the indicated assumptions -- in that the equilibrium solution to the above iterative decentralized scheme is (a) an optimal solution for the overall system that is compatible with (b) efficiency in each subunit. By optimal we mean that no better solution exists in terms of the given organizational goal.<sup>2/</sup>

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<sup>1/</sup> H. Uzawa, "On the Stability of Dynamic Processes," Technical Report No. 61, Stanford University, November 1958.

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<sup>2/</sup> We refer to a profit goal only for convenience and definiteness. Any quantifiable goal is admitted.

### 3.3 Cost Accounting.

In the previous section we considered several problems associated with the management of decentralized organizations. However, our discussion proceeded at a theoretical level and we made no reference to problems associated with the specification of the kinds of data which are required for the decentralized decision making system to proceed effectively.

In this section we first interpret the ideas already introduced in terms of certain basic concepts in cost accounting. After noting similarities and divergencies we shall stress the important uses certain cost concepts currently not used in accounting contain for problems of planning and managerial decision making. Mainly the discussion will be limited to aspects of cost accounting developed to deal with managerial planning problems as distinct from financial reporting.

A fundamental distinction is made in cost accounting between period costs and direct costs. This distinction has been stated in the following terms:<sup>1/</sup>

Period costs are those costs incurred for keeping manufacturing and marketing capacity in readiness regardless of the extent to which such capacity is utilized. Included are not only costs associated with plant and equipment, but also costs of maintaining a basic organization and expenditures for advertising and research which management has committed itself to make. In the absence of change in capacity or other existing commitments, the amount of period cost incurred runs with time but is independent of variations in activity within the range for which provision has been made. Direct costs are the additional costs incurred only if specific goods are produced and sold. In total, direct costs tend to vary directly with volume of production or other cost

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<sup>1/</sup> National Association of Accountants Research Report No. 37.

incurring activity. Within limited ranges of volume, this variation also tends to be proportional to volume and hence unit cost tends to be constant in amount.

Using this distinction between the two types of costs are two divergent cost accounting methods. In one type of cost accounting system only the direct product or activity costs are recorded in the accounts of the particular product or activity. In a "fully allocated cost system" period costs are assigned to various products in some arbitrary fashion. Thus product costs at various stages of fabrication would be different for the two methods of recording costs.

Advocates of direct costing have stressed its use as a tool in managerial decision making:<sup>1/</sup>

"By knowing the rate at which profit varies with volume (i.e., the marginal income ratio), management can determine the expected addition to profit from a proposed increment in volume when the capacity provided by period costs is not fully utilized. The same data also provides a guide to selecting the most profitable products, customers, or other segments when the available volume of business exceeds capacity of existing facilities."

Note that the use of a fully allocated cost system does not necessarily imply that information finally used for decision making would diverge between the two cost types of systems. The major emphasis is on the desirability of having direct cost figures immediately available from the account books instead of having

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<sup>1/</sup> National Association of Accountants Research Report No. 37, p. 31.

to adjust the cost figures to eliminate the period cost elements. In a large organization with several levels of fabrication the processing of cost figures may become a large task and in practice the time and costs involved in translating the cost figure may actually result in decisions based on fully allocated costs. The following comments by one company as reported in the National Association of Accountant's Research Report No. 37 are indicative of this:<sup>1/</sup>

"Under whole (fully allocated costs, we would not know the profit contribution of this product except by recalculating our costs and breaking out the fixed expenses. Frequently, there is not time to do this before we quote. We would decide under whole (fully allocated) costs that we do not want this business, not knowing it would contribute \$14,000."

In the interior function we have used  $j_i(x_i)$  as the profitability of the  $i^{\text{th}}$  unit. This can be evidently represented in terms of  $r_i(x_i) - c_i(x_i)$  where  $r_i(x_i)$  is the revenue factor and  $c_i(x_i)$  represents the direct costs e.g., labor, material, cost for various outputs  $x_i$ . We further assume that each  $c_i(x_i)$  is a continuous, concave, differentiable function of the variable  $x_i$ . A more realistic situation would be represented by a piecewise continuous function indication that in different ranges of production, different ranges of production, different

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<sup>1/</sup> National Association of Accountants Research Report No. 37.



costs are operative.<sup>1/</sup> Costs which caused the discontinuities would typically be referred to in the literature of cost accounting as other semi-variable or semi-direct costs. Fixed costs were not explicitly introduced in the model although they would be represented in the criterion function as any costs independent of the decision variables.

The correctness of using direct costs as the basis for short run decisions on price and output when the organization is operating below capacity is generally acknowledged at least in theoretical discussions of cost accounting. However, for the case where the firm operates near capacity there is no systematic way for synthesizing cost data that will enable the managers to decide on the most efficient use of the scarce fixed facilities. Thus consider the following statement:<sup>2/</sup>

"When the volume of goods that can be sold exceeds the production capacity of available machines, the highest net profit results from using machine capacity to produce those products which return the largest marginal income per machine hour."

For the case where only one facility is used to capacity, and costs, revenues and utilization of the particular facility is

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<sup>1/</sup> The problem of maximizing with discontinuities could in principle be handled by integer programming techniques. See Charnes and Cooper, Vol. II, for example.

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<sup>2/</sup> "Current Applications of Direct Costing," N.A.A. Research Report No. 37, p. 32.

constant per unit over the entire range of output of the various goods, is this a correct procedure. With variability in costs, etc., the procedure can give only rough approximation depending on the level of outputs for which the measurements are made. For the case when several fixed resources are used to capacity there would seem to be no guide available to decide on the correct output combination.

In principal a correct solution to the problem of determining the optimal output combination for an organization could be solved by formulating and solving a large programming problem. This would entail a large problem of gathering and processing the data on the technology of the firm and also pose a serious computational problem. For the moment, however, we wish to consider decentralized schemes for generating correct decisions and the required data.

We have seen that the imputed prices play a fundamental role in allocating these fixed resources. For correctly chosen  $\lambda_j$  <sup>1/</sup> the fixed resource will be properly allocated. The per unit opportunity cost  $\lambda_j$  for the resource is considered, from the point of view of the divisional unit, on the same basis as direct money cost. However,

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<sup>1/</sup> For a discussion of computational schemes for determining values of  $\mu_j$  see the last section of this paper.

it does not represent a money cost to the overall firm.<sup>1/</sup> Thus a cost accounting system for its managerial decision making uses should include, with the money direct costs, the opportunity costs if the information which the cost prices designate are to impound in themselves the benefits and penalties that will be incurred over all of the alternatives that are available to each of the Decentralized departments. We may thus regard this result as one mode of improvising the costing and transfer pricing devices that are now being used by industrial firms.

#### 3.4 Difficulties in Guiding Decentralized Decision Making Due to the Externality Problem.

We have discussed one of a variety of models that might be used to represent decisions for efficiency within a decentralized firm. Under the assumptions made decentralized resource allocation supplemented with price guides was seen to be useful. However, it was indicated that further study and experimentation was needed

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<sup>1/</sup> The charging of divisions for the use of fixed resources should not be confused with a fully allocated cost system since, as we are concerned to emphasize,  $c_i(t)$  is an opportunity cost determined by reference to outside as well as inside opportunities. The usual fixed charge (fully allocated cost) approach is more difficult to rationalize, however, except possibly as a rule of thumb long run average approach or else as a device for cash conservation for eventual repurchase of new equipment, uncertainty allowance, etc., of course, the assumptions underlying such approaches may not be valid at all--e.g., for optimization--and besides other approaches now available could be used instead. For instance, actual constraint could be formulated and charges made for the use of cash on a strict opportunity cost basis. Similar remarks could be made with respect to "user cost" in view of the utilization of historical depreciation expenses, etc.

before definite conclusions could be drawn concerning the practicality of such an approach.

In this section we consider some further difficulties with the use of price guides when we weaken the assumptions concerning a decomposable technology--i.e., separability of the functions. These difficulties raise serious problems of how to organize a meaningful decentralized price system. We shall also see that traditional problems discussed in the organization theory literature such as conflict, communication, misbehavior among subsidiary units, etc., can also be interpreted for such further light as can be cast on them by our more general model.

We shall proceed by successively generalizing the static model of an organization that was introduced earlier. Suppose we have

$$(1) \quad \text{Max } f_1(x_1, x_2) + f_2(x_2) + \dots + f_n(x_2)$$

subject to

$$(2) \quad \sum_{i=1}^n g_{ij}(x_i) \leq K_j \quad j = 1 \dots m$$

$$(3) \quad x_i = 0$$

This model is identical with the earlier one except that the profit for unit one depends on the activity level of unit two. This could arise in the case where associated with management of  $x_2$  affected, say, the demand curve for the output of activity  $x_1$ .

For the case when increases in  $x_2$  shifts the demand curve-- e.g., for  $x_1$ , to the right thereby increasing the demand for  $x_1$  while causing a loss--or reduced profit--on  $x_2$  then the latter is sometimes referred to as a "loss leader."

The original stipulation to manager two was to choose  $x_2^*$  for given prices as the maximizer of

$$f_2(x_2) - \sum_{j=1}^m \lambda_j g_{2j}(x_2)$$

$$x_2 \geq 0$$

In the present case, from the point of view of overall welfare of the firm, manager two should choose rather  $\hat{x}_2$  as the maximizer of

$$f_1(x_1, x_2) + f_2(x_2) - \sum_{j=1}^m \lambda_j g_{2j}(x_2)$$

$$x_2 \geq 0$$

In general  $x_2^*$  will differ from  $\hat{x}_2$ .

Note that the immediate result of this is that manager two is required to act in a way which is inconsistent with the presumed goal of maximal profit for his own particular unit. A further difficulty for manager two even if (possibly altruistically) he decided to act solely in the best interest of the company, he still would not have the information--from prices only--that would enable him to do this to everyone's best advantage. This is true because

for given prices his optimal choice depends, via  $f_1(x_1, x_2)$ , on the choice of  $x_1$  by the manager of activity one. The difficulty is even further compounded in that the latter chooses  $x_1^*$  as the maximizer of

$$f_1(x_1, x_2) - \sum_{j=1}^m \lambda_j g_{1j}(x_1)$$

$$x_1 \geq 0$$

for given prices. The optimal choice for manager one will, assuming  $f_1(x_1, x_2)$  cannot be written as  $f_{11}(x_1) + f_{12}(x_2)$  (i.e., separable in the variables), depend on the choice of manager 2. Even if manager two were correctly motivated and the form of the function  $f_1(x_1, x_2)$  were communicated to him there still remains the difficulty for each manager of choosing the optimal value because of the simultaneity of the decision.

On this kind of point Professor H. Simon has written the following:

"...each individual, in order to determine uniquely the consequences of his actions, must know what will be the actions of the others. This is a factor of fundamental importance for the whole process of administrative decision-making.

...Instability may result even if activity is cooperative, provided the participants are insufficiently informed."

These examples indicate the class of problems which develop when separability of the functions is no longer assumed. When this

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1/ H. Simon, Administrative Behavior, Second ed., New York: The Macmillan Company, 1959.

assumption is relaxed we have the problem of "externalities" whose presence--at least in certain forms--means that the price guides no longer give sufficient information to guide the individual decision makers in making correct decisions in terms of the overall organizational goals and constraints.

There are two problems which arise: The first is that a manager is not motivated to take into account the consequences of his actions or the welfare of other divisions since these consequences do not affect his own reward structure. The second arises from the fact that separability is closely associated with the uncertainty that occurs from lack of separability. This uncertainty results in a loss of efficiency for the decentralized system unless some way can be found to reduce or eliminate the ambiguity which ensues because one managers decision variables are dependent, at lease in part, on decisions by others in such a way that he cannot ascertain from prices only as to how his own conduct should be guided.

The above case provides a simple illustration which can be handled in a variety of ways, but in more complex cases a reorganization of the entire heirarchical structure of a firm may be needed. For instance, such a reorganization might be effected by combining two divisions into one single decision making unit in order to eliminate the externality problem. This may be desirable--or even necessary--under certain circumstances, but some caution is needed in that the purpose of decentralization

thereby tend to be frustrated and certain advantages, stressed in the paper, may thereby be lost.

Consolidation of units affected by externalities is, of course, only one approach. Another involves the use of organizational constraints. This may be illustrated as follows. Assume that two divisions affected by each others externalities sell in a competitive market. We can then represent, for given outside and internal price guides the profits of each unit in the following form.

$$p_i x_i - C_i(x_1, x_2) \text{ for } i = 1, 2.$$

Now suppose that the manager of one of these units--call this "unit one"--is faced with the problem of choosing  $x_1$  without being informed concerning the choice,  $x_2$ , of division two. Also unit one fails to take into consideration his effect on unit two. As an extreme case we assume that the manager of division one has no information concerning the intentions of division two.

A reasonable course of action for the manager of unit one to follow in this circumstance is to select an output level which provides a certain security level<sup>1/</sup>--i.e.,

$$\text{Max}_{x_1} \text{Min}_{x_2} \{ p_1 q_1 - C_1(x_1, x_2) \}$$

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<sup>1/</sup> The question of a reasonable course of action in these circumstances is a debated point. See chapter 13 of Luce and Raiffa, Games and Decisions, (New York: John Wiley & Sons, Inc., 1957), on this point. However, Charnes and Cooper in an interesting reformulation of the problem have demonstrated that criteria suggested other than the max min rule have implicit in their formulation certain information about the opponent's behavior. For the moment, however, we wish to impose a strict assumption of no information. See Charnes and Cooper, op. cit., chapter XI.



We obtain as necessary conditions for the achievement of these objectives:

$$(1) \quad P_1 - \frac{\partial C}{\partial q_1} \leq 0$$

$$(2) \quad - \frac{\partial C}{\partial q_2} \geq 0$$

From (1) and (2) a solution  $(\hat{x}_1, \hat{x}_2)$  is obtained. By a similar argument a max min solution for division two is obtained  $(\tilde{x}_1, \tilde{x}_2)$ . In general  $(\hat{x}_1, \hat{x}_2) \neq (\tilde{x}_1, \tilde{x}_2)$  which means, in turn, that each unit will act on the basis of information which, ex-post, may turn out to be incorrect.

In order to illustrate the role of constraints assume that division one causes large diseconomies to division two at low outputs--e.g.,  $C_2(x_1, x_2)$  is a monotonically decreasing function of  $x_1$  for each  $x_2$  --but that the price for division one's output rarely falls so low that it will produce at such low output levels. Consider the effect of a central unit announcing that unit one is now subject to a constraint of the form

$$x_1 \geq K$$

This constraint is arranged so that it will be redundant for division one and thus does not alter its max min level. However, from the viewpoint of division two this is an important piece of information.

Division two selects  $x_2$  in accordance with

$$\text{Max}_{x_2} \text{Min}_{x_1} \{ p_2 x_2 - C_2(x_1, x_2) \}$$

subject to

$$x_1 \geq K > 0$$

Now the necessary conditions become

$$(3) \quad p_2 - \frac{\partial C}{\partial q_2} \leq 0$$

$$(4) \quad -\frac{\partial C}{\partial q_1} - \lambda \geq 0 \quad \text{or} \quad \frac{\partial C}{\partial q_1} \leq -\lambda$$

$$(5) \quad \lambda (q_1 - K) = 0$$

By our assumptions  $\tilde{x}_1 = K$  and  $\tilde{x}_2$  is the solution of (3) when  $x_1 = K$ . Note

$$(6) \quad C_2(K, x_2) < C_2(0, x_2)$$

for every  $x_2$ , by the monotonicity assumption.

We now wish to show that the security level has increased. Assuming that the original max min solution for decision two without constraints is  $(0, x_2^*)$  where zero is chosen because of the monotonicity condition the solution  $(K, x_2^*)$  is still attainable. But by (6) this is already a definite improvement and thus the security level, is improved.<sup>1/</sup>

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<sup>1/</sup> This suggests that a possible area for research is the design of optimal constraints. Note that while a constraint may increase the security levels of the individual units this is not necessarily an actual improvement in overall return to the organization. This question must also be further investigated.

### 3.5 Problems of Price Generation:

In this paper we have focused on the uses of prices as guide for decentralized resource allocation within the firm. In earlier sections we indicated the particular form in which price guides would enter into the delegated preference ordering of a decentralized manager. We also discussed difficulties which may arise in pure reliance on individual managers in such a system. In this section we now turn to the question of how price guides may be generated within the firm and we do this in a general context that considers externalities as being either present or absent.

Assume that the technology of the organization is such that it lends itself to some degree of decentralization, or that suitable administrative devices can be imposed so as to make feasible decentralized resource allocation at a particular level. There still remains the question of how suitable price guides may be generated. In dealing with the economic system as a whole, conceived of as a collection of independent economic units, it is reasonable to think of price guides as generated automatically through the market mechanism. However, most discussions of business organizations treat the firm not as a collection of independent economic units, but as a structure characterized by a high degree of hierarchical interdependence. Therefore, in order to complete our treatment of the uses of price guides in business organizations it will be necessary for us to show how these prices also may be generated in an interdependent hierarchical system.

Specifically we will develop an iterative model with the following characteristics. At each stage of the iterative process, divisional managers make decisions with respect to price guides provided by the central staff. The central staff, which need not have knowledge of the divisional technology receives these decisions from the divisions and determines price guides based on these decisions, the overall organizational constraints and the criterion function of the overall organization.

Earlier in the paper a gradient adjustment mechanism was presented as a type of decentralized scheme for determining price guides. This was presented as analogous to the device presumed to be used in the clearing of economic markets. A drawback in this method as proposed arises in connection with the problem of how to deal with a truncated form of the process. Since in general, the process only converges to an optimum with infinite time only a truncated version has any practical use. The major difficulty is that any time before the achievement of the optimum the constraints are not satisfied. Thus, with a truncation, the question arises as to how the demands and supplies of the various units should be finally determined. It is also acknowledged that the gradient adjustment method is generally inefficient for solving nonlinear programming problems, which is the form we have presented the organizational decision problem. For these two reasons we present an alternative scheme.

3.5.1 Price Generation in the Absence of Externalities.

We begin by developing a model for a hierarchically structured organization, with each division considered initially as a single decision unit. Let  $X$  denote the set of all possible activities in the organization. We partition the set  $X$  into  $m$  blocks or collections of activities, each block representing the activities carried out by a particular division. Thus let

$$x_i^j = i^{\text{th}} \text{ activity in } j^{\text{th}} \text{ division.}$$

Corresponding to this partition we have the following model of the organization:

$$\text{Max } \sum_j \sum_i c_i^j x_i^j$$

subject to

$$\begin{array}{ll} \text{I} & \sum_j \sum_i a_{zi}^j x_i^j \leq b_z \quad z = 1 \dots \ell \\ \text{A} & \\ \text{II} & g_q^j(x_1^j \dots x_{i_j}^j) \leq R_q^j \quad q = 1 \dots m_j \\ & j = 1 \dots m \\ & x_i^j \geq 0 \end{array}$$

Type I constraints are interdivisional while type II are intra-divisional--i.e., they are constraints on variables in a particular division. We assume that  $g_q^j$  are convex differentiable functions and  $g_q^j(0) = 0$ .  $c_i^j$  is the net profit per unit of  $i^{\text{th}}$  activities in  $j^{\text{th}}$  division.

The administrative process may be described in the following manner: Each division supplies tentative proposed activity levels  $X^{j0} = (x_1^{j0} \dots x_{i_j}^{j0})$ <sup>1/</sup> to a central staff group assigned to coordinate the interdivisional constraints. The initial proposals are made with knowledge of the part of the criterion function affecting its activities and the relevant type II constraints. Verbally the task of the central staff group is to scale down or re-evaluate the proposals of the various divisions in some optimal fashion. Based on the scaled down proposals, the central staff issues tentative price guides for use by divisions in making their next proposals. The process continues for a fixed number of iterations after which the central staff based on the proposals of the divisions issues a set of activity levels that each division should produce. The final set of activities satisfies all constraints since alterations generally will have been made in the divisional proposals.

In order to amplify the above remarks, let  $X^{j0} = (x_1^{j0} \dots x_{i_j}^{j0})$  for  $j = 1 \dots m$  be the initial proposals of the various divisions.

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<sup>1/</sup> The second superscript refers to the number of iteration.

Then the central staff problem is the following:<sup>1/</sup>

$$\text{Max } \sum_{j=1}^m \sum_{i=1}^l \lambda^{j1} c_i^{j0} x_i^{j0}$$

subject to

$$\sum_{j=1}^m \sum_{i=1}^l \lambda^{j1} a_{zi}^j x_i^{j0} \leq b_z \quad z = 1 \dots l$$

$$B \quad 0 \leq \lambda^{j1} \leq 1 \quad j = 1 \dots m$$

This is a linear programming problem for the unknown variables  $\lambda^{j1}$ .<sup>2/</sup>

<sup>1/</sup> The reader will note that we are drawing on a computational scheme suggested for linear programming problems made by Dantzig and Wolfe in "Decomposition Principles for Linear Programs," Operations Research, January-February, 1960. They give a brief interpretation of their method for decentralized decision making within the firm. See page 102, paragraph 1.

<sup>2/</sup> By assuming linearity we achieve the simplification of using the simplex method to determine the tentative price guides. Since the constraints will very often refer to transfer of goods a linearity assumption is not unreasonable. However, the basic ideas developed do not depend on linearity of the organizational constraints or the criterion function. Thus let the criterion function be of the form:

$$f(x_1^1 \dots x_{i_1}^1, x_1^2 \dots x_{i_2}^2 \dots x_1^m \dots x_{i_m}^m)$$

concaved for each  $x_i^j$  and the constraints of the form

$$g_z(x_1^1 \dots x_{i_1}^1 \dots x_1^m \dots x_{i_m}^m) \leq b_z \quad z = 1 \dots l$$

convex for each  $x_i^j$ . We proceed by substituting for the first iteration

$\lambda^{j1} x_i^{j0}$  for the variable  $x_i^j$  for each  $i$  and  $j$  and solve the non-linear programming model for  $\lambda^{j1} \quad j = 1 \dots m$

$$0 \leq \lambda^{j1} \leq 1$$

Methods for solving nonlinear programming problems are presented in G. Zoutendijk, Methods of Feasible Direction, New York: Elsevier Publishing Company, 1960.

Solving this problem by the simplex method<sup>1/</sup> we obtain a set of dual variables for the constraints which are interpretable as imputed prices. These can be interpreted as the tentative price guides.

Using the tentative price guides each division "j" solves the following nonlinear programming problem:

$$\text{Max } \sum_i c_i^j x_i^j - \sum_z \sum_i \pi_z^1 a_{zi}^j x_i^j$$

subject to

$$C \quad g_q^j (x_1^j \dots x_{i_j}^j) \leq R_q^j \quad q = 1 \dots m_j$$

$$x_i^j \geq 0$$

where  $(\pi_1^1 \dots \pi_\ell^1)$  are given prices.

For the moment we simply assume that each division can solve the divisional problem to obtain a solution  $(x_1^{j1} \dots x_{i_j}^{j1})$ .

Each division communicates the new tentative proposals to the central staff group. The procedure already described could be repeated, solving for  $\lambda^{j2}$  in model B (With values  $\{x^{j1}\}$  instead of  $\{x^{j0}\}$ ). However, we proceed in a different fashion by finding the optimal convex combination of the initial proposal with the present proposal in each division. This procedure guarantees

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<sup>1/</sup> Some adaptation of it to take into account the upper bound on  $\lambda^{j1}$ . See, e.g., H. Wagner, "The Dual Simplex Algorithm for Bounded Variables," Naval Research Logistics Quarterly, 1958.



monotonicity for the value of the criterion function.<sup>1/</sup> Thus we have the following central staff problem

$$\text{Max} \quad \sum_{j=1}^m \sum_i^{i_j} \lambda_0^{j1} c_i^j x_i^{j0} + \sum_{j=1}^m \sum_i^{i_j} \lambda_1^{j1} c_i^j x_i^{j1}$$

subject to

$$\sum_j \sum_i^{i_j} \lambda_0^{j1} a_{zi}^j x_i^{j0} + \sum_j \sum_i^{i_j} \lambda_1^{j1} a_{zi}^j x_i^{j1} \leq b_z \quad z = 1 \dots l$$

D

$$\lambda_0^{j1} + \lambda_1^{j1} = 1$$

for each j

$$\lambda_0^{j1} \lambda_1^{j1} \geq 0$$

New price guides are obtained and in turn new divisional proposals are generated.

Each division then is instructed to produce

$$E \quad (\tilde{X}^{jT}) = \lambda_0^{jT} (X^{j0}) + \lambda_1^{jT} (X^{j1}) + \dots + \lambda_T^{jT} (X^{jT})$$

where, of course,  $\sum_{\tau=1}^T \lambda_{\tau}^{jT} = 1$

for each j.

We wish to note several characteristics of our truncated solution. First the output vector  $X^{jT}$  for each j satisfies all the constraints of the model. It certainly satisfies the type I

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<sup>1/</sup> This is easily seen since the space of possible solutions for the second iteration contains the space of possible solution to the first iteration. Dantzig and Wolfe use this approach in their paper op. cit.

interdivisional constraints by construction. To show that type II intra-divisional constraints are satisfied we prove the following:

Lemma I: The set  $X = (x_1 \dots x_n)$  with the property that  $g(x_1 \dots x_n) \leq 0$  is convex where  $g$  is a convex function. Let  $X^1$  and  $X^2$  two points  $R^n$  which satisfy the constraint. Then

$$g(\lambda X^1 + (1 - \lambda) X^2) \leq \lambda g(X^1) + (1 - \lambda) g(X^2) \leq 0$$

Lemma II: We now wish to show that if  $X^1 \dots X^m$  are  $m$  points  $R^n$  which satisfy the constraint, then

$$X^z = \sum_{i=1}^m \lambda_i X_i$$

$$\sum \lambda_i = 1$$

also satisfies the constraint. We may alternatively prove this by noting that by Lemma I  $X_1 \dots X_m$  are points in a convex set and thus if we show that  $X^z$  also must lie in this convex set we have our result. We prove this last point, by induction. For  $m = 2$  the result follows from the definition of convexity. Assume it is true for  $m = n$  and prove it for the case

$$X^z = \lambda_1 X_1 + \dots + \lambda_n X_n + \lambda_{n+1} X_{n+1}$$

Assuming  $\lambda_{n+1} < 1$  (if  $\lambda_{n+1} = 1$  then the lemma is proven)

we may write

$$X^z = (\lambda_1 \dots + \lambda_n) \left( \frac{\lambda_1}{\lambda_1 + \dots + \lambda_n} X_1 + \dots + \frac{\lambda_n}{\lambda_1 + \dots + \lambda_n} X_n \right) + \lambda_{n+1} X_{n+1}$$

By the inductive hypothesis

$$\frac{\lambda_1}{\lambda_1 + \dots + \lambda_n} X_1 + \dots + \frac{\lambda_n}{\lambda_1 + \dots + \lambda_n} X_n \text{ is a point in the}$$

convex set. Then  $X^z$  being a convex combination of this point and  $X_{n+1}$  it must be in the convex set.

We wish to characterize the optimum solution. We do this with the following:

Theorem I If for some  $T$  we have for all  $j$

$\tilde{X}^{jT} = X^{jT+1}$ , where vector equality means equality in each component and  $\tilde{X}^{jT}$  is defined by  $E$ , then we have achieved an overall optimum for the problem.

Proof:

We proceed by showing that  $x_i^{j,T+1} = \tilde{x}_i^{jT}$  satisfies the Kuhn-Tucker conditions for each  $j$  and  $i$  noting that with our assumptions of convexity these conditions are sufficient. The solution  $x_i^{j,T+1}$  must satisfy for each  $i$  and  $j$ , the following conditions:

$$(1) \quad c_i^j - \sum_{z=1}^{\ell} \pi_z a_{zi}^j - \sum_{q=1}^{m_i} \mu_q^j \frac{\partial g_q^j}{\partial x_i^j} \leq 0$$

$$(2) \quad x_i^{j,T+1} \left( c_i^j - \sum_{z=1}^{\ell} \pi_z a_{zi}^j - \sum_{q=1}^{m_i} \mu_q^j \frac{\partial g_q^j}{\partial x_i^j} \right) = 0$$

$$(3) \quad x_i^{j,T+1} \geq 0$$

for all  $i, j$ .

$$(4) \quad \sum_j \sum_i a_{zi}^j x_i^{jT+1} \leq b_j$$

$$(5) \quad \Pi_z \left( \sum_j \sum_i a_{zi}^j x_i^{j,T+1} - b_j \right) = 0$$

$$(6) \quad g_q^j (x_1^{jT+1} \dots x_{1_u}^{jT+1}) \leq R_q^j$$

$$(7) \quad \mu_q^j \left( g_q^j (x_1^{j,T+1} \dots x_{1_j}^{j,T+1}) - R_q^j \right) = 0$$

$$(8) \quad \Pi_z \mu_q^j \geq 0$$

for all  $\Pi_z \mu_q^j$ .

Note that for a particular set of values  $\hat{\Pi}_1 \dots \hat{\Pi}_\ell x_i^{j,T+1}$  satisfy conditions (1), (2), (3), (6), (7) since they are solutions to the divisional problems. Condition (4) and (5) are

$$\left( \sum_j \sum_i a_{zi}^j \hat{x}_i^{jT} \leq b_j \right)$$

$$\hat{\Pi}_z \left( \sum_j \sum_i a_{zi}^j \hat{x}_i^{jT} - b_j \right) = 0$$

since  $\hat{\Pi}_z$  was determined as a solution to the central staff problem and  $\hat{x}_i^{jT} = x_i^{jT+1}$  by assumption.

Hence it is possible for central management within a decentralized organization to utilize prices (price guides) for coordination. Furthermore in this method of solution this is done without central management having particular knowledge of the technologies of various divisions.

### 3.5.2 Generalization to Case Where Externalities are Present

The above discussion assumed the absence of externalities. The question naturally arises as to whether pricing schemes can be utilized within a divisionalized organization when externalities are present. In this section we shall consider an extension of the scheme just presented to account for such externalities. In order to maintain the continuity of the argument we view the decision problem on the divisional level. Thus externalities exist when the technology and criterion function of at least one division is affected by the activities contained in another division.

Suppose that division  $\eta$  affects the technology of a subset of the rest of the divisions which we refer to as  $Y$ . For simplicity of the description each division in  $Y$  is affected by the same activity in division  $\eta$ . Call this activity  $X_{i*}^{\eta}$ . We proceed by asking each division in  $Y$  to propose in addition to its own tentative activity levels a tentative level of  $X_{i*}^{\eta}$  which would be optimal for that division. In the game theory analogy each player specifies not only his own strategies but the strategy he would like his opponents to play. In general, each player's specification of the strategies will differ. The central staff group, with only the knowledge of the overall criterion function and the tentative proposals, solves a programming problem (linear if the criterion function is linear) in order to determine a preliminary set of "consistent" strategies. It uses these to determine preliminary price guides. Each player in this  $N$  person non-zero sum game now

faces a new payoff matrix, and again specifies the strategies for the players involved. This process continues for a predetermined number of iterations after which the central staff group issues a set of consistent strategies for each division to follow. As in the previous discussion the central staff need not have any knowledge of the technology.

In order to formalize and clarify this verbal discussion let us consider the following model. This assumed organization is of the same form as presented in model A except that externalities are introduced as indicated below. Thus our model is of the form:

$$\text{Max } \sum_j \sum_i c_i^j x_i^j$$

subject to

$$\text{I. } \sum_j \sum_i a_{zi}^j x_i^j \leq b_z \quad z = 1 \dots t$$

$$\text{II. } g_q^j (x_1^j \dots x_{i_j}^j) \leq R_q^j \quad j \in Y$$

$$g_1^j (x_1^j \dots x_{i_j}^j, x_{i^*}^j) \leq R_q^j \quad j \in Y$$

$$q = 1 \dots m_j$$

$$x_i^j \geq 0 \quad j = 1 \dots m$$

For each division  $j \in Y$  we define a new variable  $x_{i_{j+1}}^j$  to replace  $x_{i^*}^j$ . Thus we have for  $j \in Y$

$$g_q^j (x_1^j \dots x_{i_j}^j x_{i_{j+1}}^j) \leq R_q^j$$

and

$$x_{i_{j+1}}^j = x_{i^*}^\eta$$

for each  $j \in Y$ . The new model obtained by this transformation of variables is obviously equivalent to the original problem in the sense that the optimal solutions to both models are the same. We shall proceed using the modified model. We write the model in the form:

$$\text{Max } \sum_j \sum_i c_i^j x_i^j$$

subject to

$$\text{I. } \sum_j \sum_i a_{zi}^j x_i^j \leq b_z \quad z = 1 \dots l$$

$$x_{i_{j+1}}^j - x_{i^*}^\eta = 0 \quad j \in Y$$

$$\text{II. } g_q^j (x_1^j \dots x_{i_j}^j) \leq R_q^j \quad j \notin Y$$

$$q_q^j (x_1^j \dots x_{i_{j+1}}^j) \leq R_q^j \quad j \in Y$$

$$x_i^j \geq 0$$

Note that the consistency condition on the selection of strategies is now part of the type I constraints. We now proceed in the exact same manner as in the previous case of no externalities.

We have illustrated an approach to generating price guides in

a decentralized or quasi-decentralized firm. The partitioning of the information is still achieved since the central staff only need be informed of the projected plans of the various units. In contrast to earlier schemes the central staff makes the final decision on plans.

4. Conclusion:

The examples presented in this paper are illustrative rather than exhaustive even for the two companies that were examined in any detail. Nevertheless they have served to help us point up certain issues in a concrete fashion. Decentralization as understood and practiced in these companies bears, at best, only a partial relation to the economist's conception of a Smithian "invisible hand" which leads each unit to act to the best interests of an overall society by striving only to promote its own best interests by reference to a suitably arranged system of prices.<sup>1/</sup>

Additional points have also emerged from our discussion of these company practices. From the point of view of planning, i.e., deciding on output, type of product etc., little authority was decentralized. However, from the point of view of cost control, we found that much responsibility was delegated to divisional managers. Each division was highly motivated through a profit incentive (which

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<sup>1/</sup> For a recent discussion of decentralization in the firm from the point of view of classical economics see K. J. Arrow, "Optimization, Decentralization and Internal Pricing in Business Firms," Contributions to Scientific Research in Management, The University of California Press, pp. 9-19.



affected the promotion possibilities of the executives of the division) to improve its own cost structure and to challenge and thus improve the cost basis of the other divisions from which it purchased products.

Note, now, how this differs from the traditional preoccupation of economic theorists with decentralization in planning decisions only. Partly this arises because of the different kinds of institutional structures that are assumed. (For instance, there is no "higher authority" to which an entrepreneur can appeal for a higher price merely because he has been assigned a less than adequate plant or volume.) It also arises because of other simplifying assumptions that cannot be made in practical company operations. For instance, the assumption of equality of access to available technology, at a price, cannot be made in the latter cases. Neither can the primitive organization assumptions of economic analysis be carried over into this context. Control of cost incurrence by delegated subordinates does not exist as a problem in economic theory and therefore the organization and information forms needed to secure such cost control are also non-existent.<sup>1/</sup>

In the second part of the paper we discussed in a more formal manner questions of decentralization decision making. We were concerned with the organization of a decentralized system and the specific role of price guides once a formal model is presented.

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<sup>1/</sup> Even in the activity analysis models of T. C. Koopmans--see T. C. Koopmans, Activity Analysis of Production and Allocation, John Wiley and Sons, Inc. New York, 1951--the technological matrix is assumed to be given and not subject to further variation.

We saw that price guides were used to coordinate independent decision makers when certain resource or other types of constraint limited the possible choices open to each decision maker.

By adding certain complications into the model in the form of nonseparable functions we found that difficulties arose in pure reliance on price guides. This suggested a possible study of more general decentralized systems e.g., systems which employ other devices besides prices for coordination.

In the final section of the paper we studied an operational scheme for determining price guides. We introduced a hierarchy in terms of a central staff coordinating unit and various divisional subunits. The central staff relied on activity proposals of divisional managers in order to determine price guides. Divisional Managers, in turn, relied on price guides supplied by central staff to determine their tentative proposals.

This paper has explored decentralized decision making within the firm. We have presented a discussion of present practices in order to suggest topics for theoretical research. We have also developed several theoretical models which should suggest areas for improvement in present practices.

APPENDIX

We first state and prove a series of lemmas.

Lemma I. A closed bounded convex set has extreme points in every supporting hyperplane.<sup>1/</sup>

Lemma II. A closed bounded convex set is the closure of the convex hull of its extreme points.<sup>2/</sup>

Let  $X$  be the set and  $E$  the set of extreme points.  $H(E)$  will be the convex hull of  $E$  and  $\overline{H(E)}$  the closed convex hull.

- (1)  $X \supset E$
- (2)  $X \supset H(E)$
- (3) Since  $X$  is closed  $X \supset \overline{H(E)}$

Suppose that the theorem is false, then

$\overline{H(E)} \supset X$  and there exists a point

$x^* \in X$  and  $x^* \notin \overline{H(E)}$

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<sup>1/</sup> See Blackwell and Girshick, Theory of Games and Statistical Decisions, John Wiley and Sons, p. 37.

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<sup>2/</sup> In its most elegant form this is the Krein-Milman theorem. See Dunford and Schwartz, Linear Operators, Part I, Interscience Press. We give a proof for the finite dimensional case.

By the theorem of supporting hyperplane<sup>1/</sup> there exists a hyperplane  $ux = c$  with  $ux^* > c$ ,  $u\tilde{x} < c$  for  $\tilde{x} \in \overline{H(E)}$ . Let  $c_1 = \max_{x \in X} (ux)$ .

$ux = c_1$  is a supporting hyperplane of  $X$  which cannot contain any points of  $\overline{H(E)}$  and a fortiori any extreme points. This contradicts previous Lemma I. Thus we must have  $\overline{H(E)} \supset X$  and therefore  $\overline{H(E)} = X$ .

**Theorem I.** A closed bounded convex set is the convex hull of its extreme points.

Proof: We proceed by induction. Assume it is true for  $n - 1$  dimensional convex sets and prove it is true for  $n$  dimensional sets. The theorem is obviously true in 1 dimensional space. Let  $X$  be the convex set in  $n$  dimensional space and  $E$  the set of extreme points. We know from the above lemma that the closure of the convex hull  $\overline{H(E)} = X$ . Thus if  $\overline{H(E)} = H(E)$  we are through. Assume the theorem is false. Then since  $\overline{H(E)} \supset H(E)$  there exists at least one point  $x \in \overline{H(E)}$   $x \notin H(E)$ . Note that  $x$  is a boundary point of  $X$ . Then there exist at the point  $x$  a plane of support (supporting hyperplane)  $L$ . Consider the set  $X' = L \cap X$ . Let  $E'$  be the set of extreme points of  $X'$ . By induction hypothesis  $H(E') = X'$ .

We now show  $E' \subset E$ . Suppose not, then there is a point  $y \in E'$  and  $y \notin E$ . Then

$$y = \lambda y_1 + (1 - \lambda)y_2 \quad y_1, y_2 \in X \text{ but } y_1, y_2 \notin X'.$$

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<sup>1/</sup> Von Neumann and Morgenstern, Theory of Games and Economic Behavior, Princeton University, 1953, p. 134.

But this is impossible since

$$c \cdot uy = \lambda u y_1 + (1 - \lambda)u y_2 < C \text{ since } y_1 y_2 \notin X'$$

where  $uy = C$  is the extreme support at  $y$  and the set

$X = Y = \{y | uy \leq c\}$ . Thus  $E' \subset E$ . Then  $H(E') \subset H(E)$ . But by inductive hypothesis  $x \in H(E') \subset H(E)$  therefore a contradiction.

q.e.d.

We wish to show that the nonlinear programming problem we have presented can be transformed into a linear programming problem. While we may formally represent the problem we actually have used both the nonlinear parts and the linear programming representation to indicate a way of solving the problem. Of course our main emphasis is on ways of developing schemes of decentralized decision making and decomposing large programming models, not on developing methods of solving nonlinear programming problems per se.

Consider the set of constraints for the  $j^{\text{th}}$  division

$$g_q^j(x_1^j \dots x_{i_j}^j) \leq R_q^j \quad q = 1 \dots m_j$$

and the set  $X^j = \{x^j = (x_1^j \dots x_{i_j}^j) | g_1^j(x_1^j \dots x_{i_j}^j) \leq R_1^j$

$$g_{m_j}^j(x_1^j \dots x_{i_j}^j) \leq R_{m_j}^j \quad x_1^j \geq 0 \dots x_{i_j}^j \geq 0\}$$

Since each  $g_i^j$  is convex the set  $X^j$  is convex. (We assume the constraints are consistent.) The set  $X^j$  is closed by construction.

We also assume that the function  $g_q^j$  considered, have the property that the set  $X^j$  is bounded.

Thus the set of possible solutions from the viewpoint of division  $j$  is a closed bounded convex set. Since the constraints are nonlinear the set of extreme points is not necessarily finite. The usual theorem based on polyhedral convex sets does not apply. However Theorem I in the Appendix may be used. Let  $E^j$  be the set of extreme points of  $X^j$ . Then any point  $x_1^j \in X^j$  can be represented as

$$x^j = \sum_{\eta \in I_j} \lambda_{\eta}^j e_{\eta}^j \quad I_j \text{ is some index set}$$

$$\sum \lambda_{\eta}^j \geq 0 \quad \lambda_{\eta}^j \in E^j$$

Assume that for each  $j$  the set  $E_j$  of extreme points is known. Then model A may be equivalently formulated as

$$\text{Max } \sum_j \sum_i \sum_{\eta \in I_j} \lambda_{\eta}^j c_i^j e_{\eta i}^j$$

subject to

$$\sum_j \sum_i \sum_{\eta \in I_j} \lambda_{\eta}^j a_{zi}^j e_{\eta i}^j \leq b_z \quad z = 1 \dots \ell$$

$$\sum_{\eta \in I_j} \lambda_{\eta}^j = 1 \quad j = 1 \dots m$$

$$\lambda_{\eta}^j \geq 0$$

For known extreme points this is a linear programming problem to

determine the optimal set of  $\lambda_{\eta}^j$ . Since the index set  $I_j$  is, in general, infinite, the programming problem as stated has an infinite number of columns.<sup>1/</sup>

Essentially, the method developed here introduces extreme points and their related columns by solving a series of nonlinear programming problems to determine new extreme points. Note that the criterion function for the divisional nonlinear programming problem is linear, which implies that a solution to such a problem will always be at an extreme point.

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<sup>1/</sup> This has been referred to as semi-infinite programming problems by Charnes, Cooper and Kortanek. See their report "Duality in Semi-Infinite Programs," O.N.R. Report No. 55, Northwestern University, Evanston, Illinois. They reduce an arbitrary convex programming problem by characterizing the space of solutions as the intersection of an arbitrary number of linear inequalities. They then use a duality condition to obtain a form similar to the equivalence used here.