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MODELS OF THE JOINT DEMAND FOR CASH AND FOR AN INTEREST-BEARING ASSET

PART ONE

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PART ONE

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1. Presentation of the problem

The purpose of this paper is to analyze the distribution of a given wealth between two financial assets: cash and interest-bearing Z-assets. Most of the time we will associate the Z-assets with bonds. We will first study the distribution at a given set of initial conditions and next investigate the implications of changes in some of the conditions, i.e., we want to study the shape of the demand functions for cash and an interest-bearing asset.

The analysis will be carried out with three simple models. All three models assume that there are only the two assets, and the planning horizon is only "one period." The alternative models will differ only in the formulation of the budget equation. The same utility function, cf. (1.1), will be used in all three models. This utility function and the first budget equation, cf. (1.2), were previously considered by Professor Haavelmo in [1].*

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*[1] Dynamisk Pristeori (Dynamic Price Theory), memorandum from the Institute of Economics at the University of Oslo, April 28, 1952. These are mimeographed reports by Bjorn Thalberg of lectures given by Professor Trygve Haavelmo, 1951-52. The specific utility function and budget equation (cf. (1.1)-(1.2)), is used in chapter 5: Ettersporsel etter plaseringsobjekter (Demand for speculation assets).

Haavelmo uses the two relations for the determination of the first order conditions for optimal distribution, cf. (2.1), but he does not analyze the model's implications for the shape of the demand functions. Model I extends Haavelmo's analysis by exploring these implications.

We are considering a decision maker [1] who at time t has a given wealth and who faces the problem of how to distribute this wealth between the two financial assets. It is further assumed that the decision maker has to keep the Z-assets for a period of ω units of time [2] and that total wealth and the holdings of cash at $t+\omega$ are the only arguments of his utility function.

For the analysis of this selection problem we define the following variables:

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[1] The decision maker in question might be a household group, an enterprise or a financial institution. For the sake of convenience, we will often refer to the decision maker as "he."

[2] In a more elaborate model, the length of the period for which a decision maker sticks to a specific distribution of his wealth should be considered as an endogenous variable and not-as here-be introduced as a datum.

F_t = total wealth at t (initial wealth).

$F_{t+\omega}$ = total wealth at $t+\omega$.

m_t = cash holdings at t (initial cash holdings).

$m_{t+\omega}$ = cash holdings at $t+\omega$. This is also the holdings of cash implied in the optimal distribution at point of time t, i.e.: the interest income (positive, zero or negative) does not enter into $m_{t+\omega}$.

Z_t = number of units of Z-assets at t (initial holdings of Z-assets). This item may be positive, zero or negative. A positive figure indicates that the decision maker is lending money, a negative figure indicates that he is issuing the assets in question, i.e., that he is borrowing money.

$Z_{t+\omega}$ = Number of Z-assets at $t+\omega$. More precisely: number of Z-assets after the redistribution has taken place at t. This number is assumed constant during the period t till $t+\omega$. As for the sign of $Z_{t+\omega}$, cf. the remarks on Z_t.

P_t = actual price of one Z-asset at t.

$P_t \cdot i$ = interest income of one Z-asset during the period t till $t+\omega$, (i.e., the rent period is assumed identical to ω units of time.) 1/

1/ This definition of interest income is specific for Model I. The presentation of the Models II and III, including definitions of interest income, will be given in later chapters.

$P_{t+\omega}$ = price of one Z-asset at $t+\omega$.

The decision maker is assumed to have the following utility function:

$$(1.1) \qquad U = U\,(F_{t+\omega}\,,\,m_{t+\omega}) \quad .$$

We assume that (1.1) is twice continuously differentiable. Our study of the shape of the demand functions will first be carried out under the assumption that $\frac{\partial^2 U}{\partial F_{t+\omega}\,\partial m_{t+\omega}} = 0$ everywhere. This assumption implies that utility is assumed unique up to an increasing linear transformation. Later on we will drop the special assumption about $\frac{\partial^2 U}{\partial F_{t+\omega}\,\partial m_{t+\omega}}$ and assume that utility is unique up to any increasing transformation.

The budget equation of Model I is defined as follows:

The wealth at $t+\omega$ equals the cash holdings at $t+\omega$ plus the value of the Z-assets at $t+\omega$ plus the interest income of Z-assets in the period from t till $t+\omega$, i.e.:

$$(1.2) \qquad F_{t+\omega} = m_{t+\omega} + \left(Z_t + \frac{m_t - m_{t+\omega}}{P_t} \right) \left(P_{t+\omega} + P_t\, i \right) \quad .$$

The item $\frac{m_t - m_{t+\omega}}{P_t}$ is the total number of Z-assets the decision maker purchases at t . If $m_{t+\omega} > m_t$, he is actually selling Z-assets.

We also have the following definitional relations:

$$F_t = m_t + P_t\ Z_t \tag{1.3}$$

$$Z_{t+\omega} = Z_t + \frac{m_t - m_{t+\omega}}{P_t} \quad . \tag{1.4}$$

The "sign matrix," Table (1.5), reveals the alternative combinations of signs (including zero) for Z_t and $m_t - m_{t+\omega}$. To each combination is attached a number which identifies corresponding interpretation below.

Table (1.5) Sign matrix for Z_t and m_t-$m_{t+\omega}$.

		$m_t-m_{t+\omega} = P_t\ (Z_{t+\omega} - Z_t)$		
		-	0	+
Z_t	-	1	2	3
	0	4	5	6
	+	7	8	9

Comments on the squares of the sign matrix (1.5):

1)-3) are all referring to a decision maker who initially is a borrower, and who:

1) increases his debt

2) keeps the debt constant

3) decreases his debt. If $(-Z_t) < \frac{m_t - m_{t+\omega}}{P_t}$ he switches to a lender.

4)-6) are all characterizing a decision maker who initially holds no Z-assets, and who

4) borrows

5) neither borrows nor lends

6) lends.

7)-9) are all characterizing a decision maker who initially is a lender, and who

7) decreases his holdings of Z-assets.

If $Z_t < - \frac{m_t - m_{t+\omega}}{P_t}$, he switches to a borrower.

8) keeps his positive amount of Z-assets constant.

9) increases his holdings of Z-assets.

If the future price, $P_{t+\omega}$, is known with certainty, the fixation of $m_{t+\omega}$ will by (1.2) also determine a specific value of $F_{t+\omega}$. The selection problem will then be solved by the maximisation of (1.1) under the side condition (1.2). During this maximization procedure the decision maker will consider P_t and $P_{t+\omega}$ as well as his initial amount of cash and Z-assets as data.

If $P_{t+\omega}$ is supposed to be unknown, the selection problem becomes more complicated. One approach is to consider $P_{t+\omega}$ as a stochastic variable for which the probability distribution -- or some moments of the probability distribution -- are assumed given. The "parameter of action" for our decision maker, i.e., $m_{t+\omega}$, will still be a non-stochastic item in the sense that the decision maker may realize any value of it he wants to within the feasible range:[1/]

$$(1.6) \qquad 0 \leqq m_{t+\omega} \leqq \frac{P_{t+\omega} + i\ P_t}{P_{t+\omega} + (i-1)\ P_t} \cdot F_t \quad .$$

[1/] The upper bound on $m_{t+\omega}$ is obtained from (1.2) by claiming that $F_{t+\omega}$ shall be non-negative. Some further comments on this range are given at the end of this chapter.

By (1.2), however, $F_{t+\omega}$ will now be a stochastic variable. Given the stochastic properties of $P_{t+\omega}$ and given the values of the other data: m_t , Z_t , P_t , i and the value of his "action parameter,"

$m_{t+\omega}$, it is possible to develop the stochastic properties of $F_{t+\omega}$. Further, now U will be a stochastic variable and with a specification of (1.1) it is in principle possible to determine the probability distribution of U. We may say that his decision problem now is the choice of a probability distribution for U. The set of possible probability distributions of U is defined by the distribution function of $P_{t+\omega}$ and the values of m_t, Z_t, P_t, i and the shape of the utility function.

Some axioms on choice under uncertainty yield the following solution of the decision problem now considered: Choose the value of $m_{t+\omega}$ which yields the maximum expected value of U.

In this paper we will, however, assume that the decision maker is basing his decision on a specific value of $P_{t+\omega}$. The analysis under this assumption is relevant in two alternative cases:

1) $P_{t+\omega}$ is actually known with certainty.

2) $P_{t+\omega}$ is a stochastic variable, but the decision maker treats the random variable in a somewhat crude way, namely by basing his decision only upon a "point forecast" of the price. The point forecast may for instance be the expected value of $P_{t+\omega}$ $(= P^*_{t+\omega})$ and the decision maker is then planning as if $P^*_{t+\omega}$ and the corresponding expected value of $F_{t+\omega}$ (i.e., the value of $F_{t+\omega}$ obtained from (1.2) by

inserting $P^*_{t+\omega}$ for $P_{t+\omega}$) would be realized.[1]

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[1] This is the case which is assumed in the model of Haavelmo referred to above. $F_{t+\omega}$ is there termed planned wealth on $t+\omega$ and denoted by $\bar{F}_{t+\omega}$. Similarly for $m_{t+\omega}$.

It may easily be shown that this method emerges as a special case of the "maximization of the expected utility"-approach when the utility function is linear.

We will make some remarks on the introducing of one specific component of wealth, namely cash, into the utility function. One reason for this is the so-called transaction motive. The only transactions introduced explicitly in this model is the redistribution of the given wealth at t , but we may consider our model as a kind of a partial model ("sub-model") and assume that our decision maker is participating in other transactions. More specifically we will assume that he has a total of receipts (besides his receipts from the net sale of Z-assets and his net interest income) which -- for the period in question -- equal his total outlays, but that the lack of synchronization between the two streams creates the need for liquidity.

If the time distribution of these receipts and outlays are known with certainty by the decision maker, he would also know the "maximum cumulated deficit"[1] during the period, and by keeping an amount of cash equal to the maximum cumulated deficit he would not run into liquidity problems. In order to rationalize the introducing of money in the utility function (under the "heading" of the transaction motive) we will make the same assumption as Don Patinkin in [2], namely that the time distribution of receipts and outlays during the period may be considered as produced by a stochastic process. By holding a given amount of cash the decision maker will now achieve

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[1] We introduce the following flow concepts:

$R(\tau)$ = receipts per unit of time at point of time τ .

$O(\tau)$ = outlays per unit of time at point of time τ .

The cumulated deficit at point of time Ω $(t < \Omega \leqq t + \omega)$ is then defined as: $\int_t^{\Omega} [O(\tau) - R(\tau)]\, d\tau$. According to our assumption the value of this integral is zero when $\Omega = t+\omega$. The maximum cumulated deficit is the maximum value (with respect to Ω) of the integral above.

[2] Money, Interest and Prices. (Row, Peterson and Company, 1956). See particular chapter VII: "The Nature of the Demand for Money," and the Mathematical Appendix 6: "The probability distribution generated by the random payment process".

The models used by Don Patinkin are more general than ours in so far as all transactions are endogenous variables, i.e., he does not have to make assumptions about "additional transactions."

a certain security for avoiding liquidity problems. If the distribution function for the maximum cumulated deficit can be derived, we may determine the security level corresponding to a given amount of cash and vice versa -- if the security level is given we may determine the corresponding amount of cash. In this case we may say that the "true" variable to be introduced in the utility function is not cash but "security against liquidity problems."[1] This security will, however, be an increasing function of cash, such that cash may be a useful indicator of the "true" variable.

This rationalization for introducing money as a specific variable in the utility function implies that there is a point of saturation for money. The maximum level of security, i.e., the situation where the decision maker is quite sure that he will not get into any liquidity problems is obtained by holding an amount of cash equal to the maximum maximori of the maximum cumulated deficits. This maximum, denoted $m_{t+\omega}^{max}$, would be realized if all the outlays had to be made before the flow of receipts even started.[2] The marginal utility of cash is then zero when $m_{t+\omega} = m_{t+\omega}^{max}$ and zero or negative when $m_{t+\omega} > m_{t+\omega}^{max}$.

An illustration of the indifference curves is given in Figure (1.7), the lines of short dashes representing the alternative

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[2] $$m_{t+\omega}^{max} = \int_{t}^{t+\omega} R(\tau)d\tau = \int_{t}^{t+\omega} O(\tau)d\tau .$$

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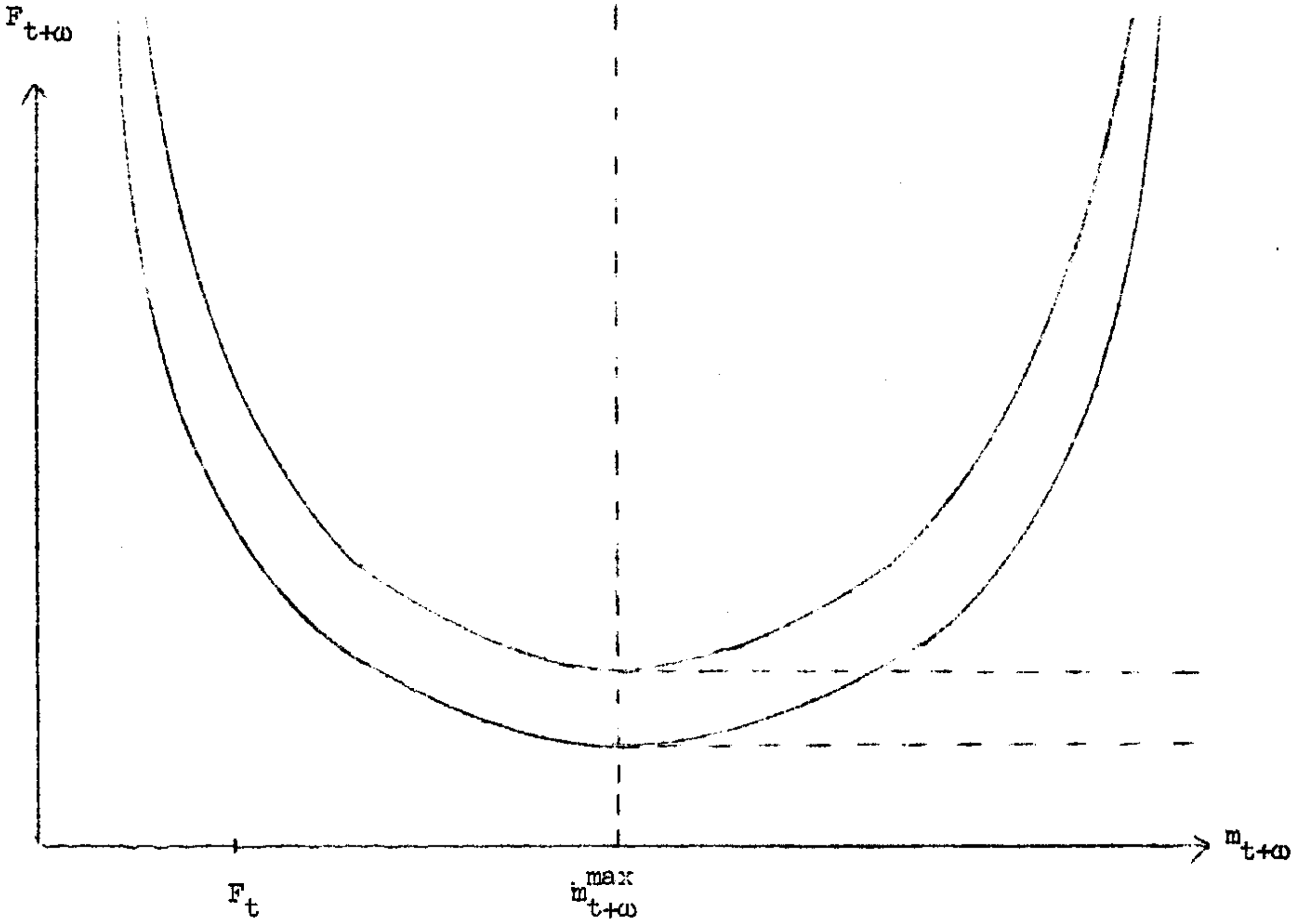
[1] One workable approach for a closer study of the preference for being liquid should be to explore the consequences of not being liquid and further the decision maker's preference (or "aversion") towards these consequences.

$\frac{\partial U}{\partial m_{t+\omega}} = 0$ (when $m_{t+\omega} > m_{t+\omega}^{max}$) .

Figure (1.7)



If the initial wealth of the decision maker, $F_t = P_t Z_t + m_t$, is equal to or larger than $m_{t+\omega}^{max}$, he will never become a borrower, i.e., we have $Z_t \geqq 0$. In Figure (1.7) we have assumed that $m_{t+\omega}^{max} > F_t$. Some further comments on $m_{t+\omega}^{max}$ will be given in connection with the respective models.

An alternative for using the nominal values of cash and total wealth is to deflate them by some price index. As far as the cash is concerned, one alternative is to use a price index for the commodities which are involved in the "other transactions." We will assume, however, that the prices of these commodities are constant. As far as the total wealth is concerned, we have made no assumptions about what kind of "commodities" the decision maker is going to purchase at $t+\omega$. One alternative is the commodities involved in the "other transactions." Another alternative is that he will continue his "speculation," i.e., again make a distribution of his wealth between cash and Z-assets. If so, the price index used for deflating of $F_{t+\omega}$ should be a compromise of $P_{t+\omega}$ and the prices of the "other" goods. This kind of deflating would, of course, lead to other results for the effect of changes in $P_{t+\omega}$ than those obtained when using the undeflated variables.

Our utility function should be considered as being less autonomous than utility functions usually are supposed to be.[1/]

[1/] The concept of autonomy is treated in "Autonomy of Economic Relations," memorandum from the Institute of Economics at the University of Oslo, November 6, 1948. Authors: R. Frisch, T. Haavelmo, T. C. Koopmans, J. Tinbergen.

We will not try to give a precise definition of this concept, but content ourselves with merely saying that it is supposed to express how much one relation can "stand" of changes in other relations. The concept of "other relations" should here be interpreted quite widely. In order to "measure" the autonomy of a relation it is necessary to go "behind" it and try to construct a sort of super-structure, from which the specific relation is derived.

First, important variables are omitted. An eventual change in the constellation of these variables may therefore violate our utility function.

Second, one of the goods introduced, $m_{t+\omega}$, is only a proxy for another, namely "security against liquidity problems." The relation between "security" and cash is given by the distribution function for the maximum cumulated deficit which depends upon several factors. Changes in these factors may therefore change the form of the utility function (1.1).

With a more complete model, including a more general utility function and an explicit formulation of the mechanism explaining the probability distribution for the maximum cumulated deficit, it would have been possible to find out which assumptions are necessary in order to derive the special utility function (1.1). Further we might have been able to appraise the realism of -- including the "probabilities" for having changes in -- these assumptions and finally tracing the consequences of such changes on the shape of (1.1).

Without having this more complete model it is difficult to say something more precise about the "degree of autonomy" of the utility function (1.1)[1] In this paper we will, however, use (1.1) even if it

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[1] Comparisons between actual observations and the results derived from our models may serve as an indication of the validity of (1.1).

probably is quite a confluent relation.[1]

Another reason for introducing cash in the utility function may rise if $P_{t+\omega}$ is considered as a stochastic variable and we only emphasize the expected value of $P_{t+\omega}$ and the corresponding expected value of $F_{t+\omega}$. In this case the appearance of $m_{t+\omega}$ in the utility function to some extent modifies the crude way of treating the uncertainty elements involved in the model. Because the decision maker is aware of the uncertainty concerning the realization of the expected (or another special) value of the future price he has specific preferences regarding the safe asset, cash.[2]

Roughly we may say that this will diminish the consequences of differences between the realized and the expected value of the price. If

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[1] "Any relation that is derived by combining two or more relations within a system, we call a confluent relation. Such a confluent relation has, of course, usually a lower degree of autonomy (and never a higher one) than each of the relations from which it was derived, and all the more so the greater the number of different relations upon which it depends." (From "Autonomy of Economic Relations," p. 28).

[2] This specific reason for introducing cash in the utility function implies that $\frac{\partial U}{\partial m_{t+\omega}} = 0$ when $m_{t+\omega} = F_t$ and $\frac{\partial U}{\partial m_{t+\omega}} < 0$ when $m_{t+\omega} > F_t$, i.e., the decision maker would never become a borrower.

$m_{t+\omega}$ is omitted from the utility function, the decision maker would keep all his wealth either in Z-assets or in cash, dependent upon whether the Z-assets are supposed to yield gain or loss. With $m_{t+\omega}$ as a specific variable in the utility function, he will always (or at least as long as the actual constellation of data are within a certain region) keep some of his wealth in cash, which reduces his potential gain as well as his potential loss.

If we abstract from the transaction motive and if we treat the stochastic aspects in a more elaborate way (than merely introducing cash into the utility function) we may consider utility as a function only of wealth. The selection problem may then be solved by the "maximization of expected utility" — approach. If this is being done, it may be possible to appraise the crude treatment of the random variable mentioned above. For instance by making comparisons between the portfolio selections emerging from the two methods, eventually under alternative assumptions about the data involved.

Finally we will comment on the feasible range of $m_{t+\omega}$, cf. (1.6).

A negative amount of cash would in this model imply that the decision maker is borrowing money -- without paying any interest -- and using the money for purchase of interest bearing Z-assets. This would enable him to increase his wealth without bound. Graphically this means that his budget line is extended into the second quadrant. The upper limit on $m_{t+\omega}$ is obtained from (1.2) by claiming that $F_{t+\omega}$ shall be non-negative, i.e., that the decision maker shall be solvent at the point of time $t+\omega$.

Reasons for introducing boundary restraints (1.6) are:

1) It may be a market condition imposed on the decision maker from "outside," stating respectively that he has not the opportunity to borrow at an interest rate equal to zero and that he is bound to have a portfolio composition which is supposed to make him solvent at $t+\omega$ .1/ We assume that the market in question imposes these conditions on the decision maker.

2) Perhaps also his utility function (1.1) only is valid in the first quadrant (including the borders), implying that his aversion to "low" liquidity on the one hand as well as becoming insolvent on the other hand is so strong that he prefers not to violate (1.6) whatever the market conditions are.

In either of these cases the inequality (1.6) would complicate the optimization procedure, at least when we are using an analytical approach. Alternatively with 2), we may assume that the shape of the utility function is such that when the decision maker acts as a quantity adjuster (i.e., price taker), (1.6) will never be violated, whatever

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1/ Since the solvency of a borrower is dependent upon the future price, there may be some disagreement as to his solvency.

the constellation of the other data may be. This strong assumption implies that the indifference curves will never intersect the axes, but for instance approaching towards the axes asymptotically.

In our analysis we will use an analytical technique which does not allow for an automatic inclusion of the bounds (1.6). This implies that the analysis is valid either

1) under the strong assumption about the shape of the utility function referred to above, or -- if this is not the case --

2) only for that region of the data, m_t , Z_t , P_t , $P_{t+\omega}$, and i , which yields optimal points not violating (1.6). Without specifying the utility function, we are, however, not able to derive explicitly the feasible region for the data in question.

We have introduced two upper bounds on the actual demand for money; $m_{t+\omega}^{max}$ (equal to the level of the exogenous transactions assumed to be carried out during the period) and the right hand side of (1.6). While we are not going to study the effects of changes in $m_{t+\omega}^{max}$, the alternative upper bound depends upon some of the "variable data." The ranking of the two bounds may therefore be changed under variation in the data.

Summary

In this paper we will deal with the problem of portfolio selection under the assumption that the future price of Z-assets are known with certainty or, if this is not so, that the decision maker is basing his decision only on a point forecast. In Model I and partly in Model II we will assume that this point forecast is constant and independent of P_t. As a specific part of Model II we will introduce a function, which relates the forecast to the present value of the price. This will be done in Part Two.

We will first study the selection problem under a given constellation of data and next analyze the effect of changes in some of the data. This will be done under three alternative assumptions regarding the budget equation (Models I-III). The purpose of this investigation is first and foremost to see if the usual (Keynesian) assumptions regarding the shape of the demand functions for cash and bonds are valid when we use this utility approach.

We will also study the shape of the relation between total wealth and the data in question. For matter of convenience this relation will also be referred to as a demand function.

Besides utilizing the general utility function (1.1) we will introduce a more specified function which enables us to obtain explicit expressions for the demand functions.

The investigation will mainly be based on an analytical approach: first determining the first order conditions for the optimal selection by means of the method of Lagrange and next differentiating these conditions implicitly with respect to the data in question in order to study the shape of the demand functions.

We will use diagrams to show the construction of the budget lines and how they are altered by changes in the relevant data and to illustrate some of the results obtained analytically.

Part One will only deal with the Models I-II.

MODEL I

2. The optimal portfolio selection under a given constellation of data

The selection problem may be stated as follows: Find the values of $F_{t+\omega}$ and $m_{t+\omega}$ which satisfy (1.2) and maximize (1.1). This problem is solved by maximizing the following Lagrange-expression:[1]

$$H\left(F_{t+\omega},\ m_{t+\omega}\right) = U\left(F_{t+\omega},\ m_{t+\omega}\right)$$

$$-\ \lambda\left[F_{t+\omega} - m_{t+\omega} - \left(Z_t^o + \frac{m_t^o - m_{t+\omega}}{P_t}\right)\left(P^*_{t+\omega} + P_t\ i\right)\right].$$

This leads to the following relation:

$$\text{(2.1)} \qquad \frac{\partial U}{\partial m_{t+\omega}} = \frac{\partial U}{\partial F_{t+\omega}}\left(\frac{P^*_{t+\omega}}{P_t} + i - 1\right).$$

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[1] From now on we use the symbol $P^*_{t+\omega}$ for the specific value of the future price on which the decision is made. We further introduce F_t^o, m_t^o and Z_t^o for the initial holdings of total wealth, cash and Z-assets.

(2.1) is a condition of the Gossen type: The utility of the last dollar spent on the alternative goods shall equal. The left hand side of (2.1) is the utility of the last dollar used for cash and the right hand side is the utility of the last dollar spent on Z-assets. The item within the parenthesis gives namely the net increase in wealth per dollar spent on Z-assets. By multiplying this item with the marginal utility of wealth we get the utility of the last dollar spent on Z-assets.

We assume that $\frac{P^*_{t+\omega}}{P_t} + i - 1$ is positive. If this item was zero or negative the only reason for keeping positive holdings of Z-assets should be a zero or negative marginal utility of cash.

(1.2) and (2.1) give us two relations for the determination of $m_{t+\omega}$ and $F_{t+\omega}$. Formally we may write the demand functions:

$$(2.2) \qquad m_{t+\omega} = f_1 \ (m^o_t \ , Z^o_t \ , P_t \ , P^*_{t+\omega} \ , i) \ ^{1/}$$

$$(2.3) \qquad F_{t+\omega} = f_2 \ (m^o_t \ , Z^o_t \ , P_t , P^*_{t+\omega} \ , i) \ .$$

1/ The demand function for $Z_{t+\omega}$ is found by inserting the demand function for $m_{t+\omega}$ into (1.4).

We may notice -- when P_t is assumed given and constant -- that only the total initial wealth (and not its distribution between m_t^o and Z_t^o) is of importance for the actual demand.

In the next chapter we will study the shape of the demand functions with respect to changes in the future price.

3. The effects of changes in the future price of Z-assets

When $P^*_{t+\omega}$ increases the item within the parenthesis of the right hand side of (2.1) increases. By unchanged distribution of the initial wealth between Z-assets and cash the wealth on $t+\omega$ will increase.[1] Whether the product on the right hand side will increase or decrease is depending upon how the marginal utility of the wealth is varying with the wealth. If, for instance, the marginal utility of wealth is constant, the right hand side of (2.1) becomes larger than the left hand side. In order to establish the equality of (2.1) one would have to reduce cash holdings (i.e., increase the holdings of Z-assets). This reasoning implies that the marginal utility of cash is decreasing and that $F_{t+\omega}$ and $m_{t+\omega}$ are independent in the utility sense $\left(\frac{\partial^2 U}{\partial F_{t+\omega} \, \partial m_{t+\omega}} = 0 \right)$.

If the marginal utility of the wealth is decreasing with increasing wealth, the first item on the right hand side of (2.1) will decrease by the increase in $P^*_{t+\omega}$, (as long as $m_{t+\omega}$ and $Z_{t+\omega}$ is unchanged). Thus we get that the one factor on the right hand side increases, while the other factor decreases. If the product decreases, the left hand side is too large. In order to establish the equality, cash must be increased.

Before attacking the question analytically we will show

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[1] In this tentative reasoning we assume that $Z_{t+\omega}$ is positive.

the graph of the budget line and how it is altered by partial changes in $P^*_{t+\omega}$. We firstrewrite (1.2) in the following way:

$$(3.1)\quad F_{t+\omega} = -\left(\frac{P^*_{t+\omega}}{P_t} + i - 1\right) m_{t+\omega} + \left(\frac{P^*_{t+\omega}}{P_t} + i\right)\left(P_t\ Z^o_t + m^o_t\right).$$

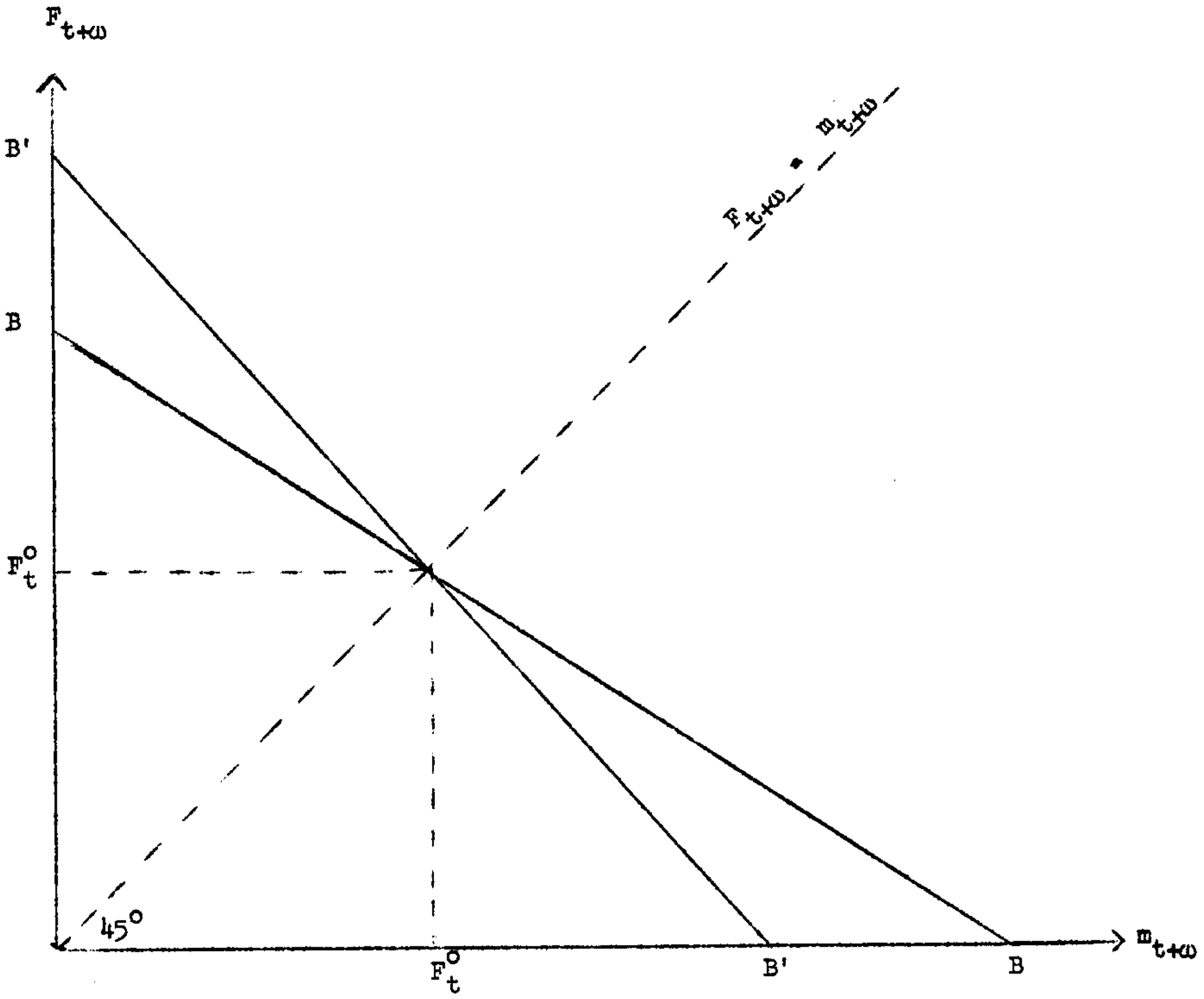
By utilizing (1.3) we obtain:

$$(3.2)\quad F_{t+\omega} = -\left(\frac{P^*_{t+\omega}}{P_t} + i - 1\right) m_{t+\omega} + \left(\frac{P^*_{t+\omega}}{P_t} + i\right) F^o_t ,$$

i.e., the budget equation will be represented in a $m_{t+\omega}$, $F_{t+\omega}$ -- diagram by a straight line with the slope $-\left(\frac{P^*_{t+\omega}}{P_t} + i - 1\right)$ and which cuts a piece equal to $\left(\frac{P^*_{t+\omega}}{P_t} + i\right) F^o_t$ of the $F_{t+\omega}$ - axes. By inserting $m_{t+\omega} = F^o_t$ in (3.2) it is easily seen that all the lines which are generated by changes in $P^*_{t+\omega}$, will pass through the point $m_{t+\omega} = F_{t+\omega} = F^o_t$. When all data, except $P^*_{t+\omega}$, are kept constant, the initial wealth, F^o_t , is also constant. We assume that $F^o_t < m^{max}_{t+\omega}$, i.e., that the decision maker may be a borrower as well as a lender.

The line BB in Figure (3.3) represents the budget line at a given value of $P^*_{t+\omega}$, and the line B'B' represents the budget line at a higher value of $P^*_{t+\omega}$.

Figure (3.3)



By introducing different sets of "acceptable" indifference maps in Figure (3.3) it will be seen that the effect of changes in $P^*_{t+\omega}$ are not clear, even if we only are concerned with the sign of the changes in $F_{t+\omega}$ and $m_{t+\omega}$.

Note on the type of decision maker

On the basis of Figure (3.3) we will make a small digression concerning the type of decision maker for which this model is applicable. If the decision maker is choosing a point on the budget line where $F_{t+\omega} = m_{t+\omega}$, he keeps all his wealth in cash. If he is choosing a point (on the budget line) northwest to the line $F_{t+\omega} = m_{t+\omega}$ he is a lender $(F_{t+\omega} > F_t^o > m_{t+\omega})$, and if he is choosing a point southeast to the line $F_{t+\omega} = m_{t+\omega}$, he is a borrower $(F_{t+\omega} < F_t^o < m_{t+\omega})$.

Which of these three cases will actually be realized depends on his indifference map and the actual constellation of the data. Given his indifference map he will -- in most cases -- switch from the one "main" adaptation to the other by "large" changes in the data. This fact may restrict the types of decision makers for which this analysis is applicable, namely, if the right of issuing Z-assets is legally limited to certain types of decision makers. For the other types of decision makers we would then have to impose a non-negativity condition on their holdings of Z-assets, $Z_\tau \geqq 0$. This case will be analogous -- from an analytical point of view -- to the introducing of rationing of commodities, for instance when dealing with the theory of consumer's demand.

The same applies, of course, if there are upper bounds on the amount of Z-assets to be issued, (or: a negative lower bound on the holdings of Z-assets), i.e., $Z_\tau \overset{=}{>} - \underline{Z}_\tau$, where $\underline{Z}_\tau$ is strictly positive. Both these cases of "credit rationing" may very well violate the assumption of quantity adaptation (i.e., that the decision maker behaves as a price taker). In the following we will assume that neither of these cases of credit rationing occur.

For further investigation into the question of the sign of the changes in $F_{t+\omega}$ and $m_{t+\omega}$ we will calculate $\frac{\partial F_{t+\omega}}{\partial P^*_{t+\omega}}$ and $\frac{\partial m_{t+\omega}}{\partial P^*_{t+\omega}}$ by implicit derivation of (1.2) and (2.1). We first introduce two new variables T and A_1 :

$$(3.4) \qquad T = t+\omega$$

$$(3.5) \qquad A_1 = \frac{P^*_{t+\omega}}{P_t} + i - 1$$ (= net increase in wealth per dollar spent on Z-assets, assumed positive) .1/

1/ A_1 is a kind of an interest rate. It might have been termed the effective rate of interest for the period t till T .

We will further assume that the marginal utilities are positive and decreasing in those $F_{t+\omega}$, $m_{t+\omega}$ points satisfying (1.2) and (2.1) and that the two "commodities" F_T and m_T are independent in the utility sense:

$$\frac{\partial U}{\partial F_T} > 0 , \frac{\partial U}{\partial m_T} > 0 , \frac{\partial^2 U}{\partial F_T^2} < 0 , \frac{\partial^2 U}{\partial m_T^2} < 0 , \frac{\partial^2 U}{\partial m_T \partial F_T} = 0 .$$ [1]

We rewrite (1.2) as (3.6) and (2.1) as (3.7):

$$F_T = m_T + Z_T \left(P_T^* + P_t\, i\right) \tag{3.6}$$

$$\frac{\partial U}{\partial m_T} = \frac{\partial U}{\partial F_T} \quad A_1 . \tag{3.7}$$

Derivation of (3.6) with respect to P_T^* and utilizing (1.4) yield:

$$\frac{\partial F_T}{\partial P_T^*} = \frac{\partial m_T}{\partial P_T^*} + Z_T + \left(P_T^* + P_t\, i\right)\left(-\frac{1}{P_t}\right) \frac{\partial m_T}{\partial P_T^*} ,$$

$$\frac{\partial F_T}{\partial P_T^*} = - \left[\frac{P_T^*}{P_t} + i - 1 \right] \frac{\partial m_T}{\partial P_T^*} + Z_T$$

i.e.,

$$\frac{\partial m_T}{\partial P_T^*} = \frac{Z_T - \dfrac{\partial F_T}{\partial P_T^*}}{A_1} . \tag{3.8}$$

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[1] Later on we will relax the assumptions about the utility indicator, cf. pp. 35 - 35h.

By derivation of (3.7) with respect to P_T^* , we get:

$$(3.9) \quad \frac{\partial^2 U}{\partial m_T^2} \; \frac{\partial m_T}{\partial P_T^*} = A_1 \frac{\partial^2 U}{\partial F_T^2} \; \frac{\partial F_T}{\partial P_T^*} + \frac{1}{P_t} \; \frac{\partial U}{\partial F_T} \quad .$$

By inserting the expression for $\frac{\partial m_T}{\partial P_T^*}$ from (3.8) into (3.9), solving the equation thus obtained with respect to $\frac{\partial F_T}{\partial P_T^*}$ and utilizing (3.7), we get:

$$(3.10) \quad \frac{\partial F_T}{\partial P_T^*} = \frac{\frac{1}{P_t} \frac{\partial U}{\partial m_T} + Z_T \left(- \frac{\partial^2 U}{\partial m_T^2} \right)}{- \left[A_1^2 \; \frac{\partial^2 U}{\partial F_T^2} + \frac{\partial^2 U}{\partial m_T^2} \right]} \quad .$$

Inserting (3.10) into (3.8) and rearranging the terms, we obtain for $\frac{\partial m_T}{\partial P_T^*}$:

$$(3.11) \quad \frac{\partial m_T}{\partial P_T^*} = \frac{- \frac{1}{P_t} \; \frac{\partial U}{\partial F_T} + A_1 \; Z_T - \frac{\partial^2 U}{\partial F_T^2} \Bigr)}{- \left[A_1^2 \; \frac{\partial^2 U}{\partial F_T^2} + \frac{\partial^2 U}{\partial m_T^2} \right]} \quad .$$

Under our assumption the denominator in (3.10) and (3.11) is positive.[1] The numerators in both (3.10) and (3.11) may be a sum of a positive and a negative term.

Further comments on $\frac{\partial F_T}{\partial P_T^*}$: Case α_1: $Z_T \geqq 0$, i.e., the decision maker is not a borrower at the given price. In this case his wealth will increase. This conclusion is valid even if he increases his holdings of cash (i.e., decreases his holdings of Z-assets). This result reminds of a conclusion from the classical analysis of the supply of labor. At an increase in the wage rate, the total income will always increase even if the individual is working less.

Case α_2: $Z_T < 0$, i.e., the decision maker is actually a borrower at the given price. The condition $Z_T \geqq 0$ is a sufficient, but not necessary condition for arriving at the conclusion of increase in

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[1] This follows from our assumption of decreasing marginal utilities.

The second order condition for a regular maximum of U under the side condition (1.2) takes the form

$$A_1^2 \frac{\partial^2 U}{\partial F_T^2} + \frac{\partial^2 U}{\partial m_T^2} - 2A_1 \frac{\partial^2 U}{\partial F_T \partial m_T} < 0 ,$$

which, under the assumption of independent commodities, reduces to

$$-\left[A_1^2 \frac{\partial^2 U}{\partial F_T^2} + \frac{\partial^2 U}{\partial m_T^2} \right] > 0 .$$

wealth. In order to arrive at the opposite result, the debt must be "large," more precisely:

$$(3.12) \quad \frac{\partial F_T}{\partial P_T^*} \lesseqgtr 0 \quad \text{if} \quad Z_T \lesseqgtr \frac{\frac{1}{P_t} \frac{\partial U}{\partial m_T}}{\frac{\partial^2 U}{\partial m_T^2}} .$$

Case α_3: $\frac{\partial^2 U}{\partial m_T^2} = 0$. If we leave one of our assumptions, $\frac{\partial^2 U}{\partial m_T^2} < 0$, and consider the very special case of a utility function with constant marginal utility of money, the decision maker will always gain on an increase in P_T^*, even if he is a borrower.

Further comments on $\frac{\partial m_T}{\partial P_T^*}$: [1]

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[1] From (1.4) we obtain:

$$\frac{\partial Z_T}{\partial P_T^*} = - \frac{1}{P_t} \frac{\partial m_T}{\partial P_T^*}$$

, i.e., as for the sign, the effect on Z_T will be opposite that on m_T.

Case β_1: $Z_T \leqq 0$, i.e., the decision maker is not a lender. In this case his demand for cash will decrease.

Case β_2: $Z_T > 0$, i.e., the decision maker is actually a lender. In this case the numerator of (3.11) is a sum of a positive and a negative term. If the holdings of Z-assets are "large," he will increase his holdings of cash, more precisely:

$$(3.13)\quad \frac{\partial m_T}{\partial P_T^*} \geqq 0 \quad \text{if} \quad Z_T \geqq \frac{\frac{1}{P_t}\ \frac{\partial U}{\partial F_T}}{A_1\left(-\ \frac{\partial^2 U}{\partial F_T^2}\right)} \quad .$$

By combining the cases $\beta_1 - \beta_2$ we get the following picture of the relation between m_T and P_T^* and Z_T and P_T^*, assuming the other data constant:

If, for a given value of $P_T^* \left(= P_T^{*o}\right)$, Z_T is negative, case β_1, the decision maker reduces his cash, i.e., he reduces his debt $\left(Z_T < 0 ,\ \frac{\partial Z_T}{\partial P_T^*} > 0 \right)$. As P_T^* increases, it will therefore reach a value $(= \bar{P}_T^*)$ for which his holdings of cash equals his total wealth, i.e., $Z_T = 0$.[1] We still have

$$\frac{\partial m_T(\bar{P}_T^*)}{\partial P_T^*} < 0, \quad \text{or} \quad \frac{\partial Z_T(\bar{P}_T^*)}{\partial P_T^*} > 0 \quad ,$$

i.e., he switches from being a borrower to become a lender.

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[1] It may be that the shape of the utility function and the given constellation of the other data are such that Z_T is increasing without reaching zero, for instance approaching asymptotically to a negative value. The reasoning above assumes that this does not take place.

As P_T^* (and thereby Z_T) increases, it may occur that, for a finite value of $P_T^* \left(= \bar{\bar{P}}_T^* \right)$, $\frac{\partial m_T \, (\bar{\bar{P}}_T^*)}{\partial P_T^*}$ equals zero, cf. (3.13), and $\frac{\partial m_T}{\partial P_T^*} > 0$ when $P_T^* > \bar{\bar{P}}_T^*$, i.e., the demand for Z-assets will decrease. As we have $\frac{\partial Z_T}{\partial P_T^*} > 0$, when $Z_T = 0$, Z_T will, however, never be reduced to zero when P_T^* increases. This implies that even if the case (3.13) should occur, the decision maker will never become a borrower when $P_T^* > \bar{P}_T^*$.

<u>Case β_3</u> : $\frac{\partial^2 U}{\partial F_T^2} = 0$. If the marginal utility of wealth is constant, the decision maker will always reduce his demand for money by an increase in P_T^*.

Introducing the term abnormal for the results $\frac{\partial F_T}{\partial P_T^*} \leqq 0$ and $\frac{\partial m_T}{\partial P_T^*} \geqq 0$, we may formulate the following conclusions:

When imposing the abnormal results there emerge boundary restraints on the actual demand for Z-assets. The restraint corresponding to $\frac{\partial F_T}{\partial P_T^*} \leqq 0$ is an upper bound which is negative, cf. (3.12). The restraint corresponding to $\frac{\partial m_T}{\partial P_T^*} \geqq 0$ is a lower bound which is positive, cf. (3.13). This implies that both the abnormal results will not occur simultaneously.

What the demand for Z-assets actually is depends upon the shape of the utility function and the other data: the initial holdings of cash and Z-assets, the present price and interest and finally the value of the "variable" datum, the future price.

More general assumptions about the utility indicator

So far we have studied the shape of the demand functions under the special assumption that wealth and money are independent in the utility sense, i.e., that $\frac{\partial^2 U}{\partial F_T \partial m_T} = 0$. One rationalization for this assumption is the following: Let us assume that the reason for introducing money in the utility function is uncertainty regarding the time distribution of receipts and outlays during the period, cf. the reasoning pp. 9-12. Let us further assume that the Z-assets are completely illiquid 1/ and that there is no possibility for obtaining transaction credit. If this is the case, an increase in total wealth while the amount of money is kept constant, will not yield any higher security against illiquidity. In other words, the marginal utility of money is independent of wealth, $\frac{\partial^2 U}{\partial F_T \partial m_T} = 0$.

The assumption that $\frac{\partial^2 U}{\partial F_T \partial m_T} = 0$ is, however, not invariant with

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1/ Even if the Z-assets had "some degree" of liquidity this would only be of help to a decision maker who has positive holdings of them (i.e., a lender).

respect to any increasing non-linear transformation of the utility indicator.[1] If therefore $\frac{\partial^2 U}{\partial F_T \partial m_T}$ is assumed equal to zero with respect to one utility indicator, it will quite certainly be different from zero with respect to an alternative indicator which is obtained by an increasing non-linear transformation of the former. We will now drop the special assumption that $\frac{\partial^2 U}{\partial F_T \partial m_T} = 0$ and only assume that utility is unique up to any arbitrary increasing transformation of the utility indicator and -- as before -- that the second order condition for a regular maximum is fulfilled.

Under this more general assumption we obtain for $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$:

$$(3.14) \qquad \frac{\partial F_T}{\partial P_T^*} = \frac{\frac{1}{P_t} \frac{\partial U}{\partial m_T} + Z_T \left[A_1 \frac{\partial^2 U}{\partial F_T \partial m_T} - \frac{\partial^2 U}{\partial m_T^2} \right]}{2 A_1 \frac{\partial^2 U}{\partial F_T \partial m_T} - A_1^2 \frac{\partial^2 U}{\partial F_T^2} - \frac{\partial^2 U}{\partial m_T^2}}$$

$$(3.15) \qquad \frac{\partial m_T}{\partial P_T^*} = \frac{- \frac{1}{P_t} \frac{\partial U}{\partial F_T} + Z_T \left[\frac{\partial^2 U}{\partial F_T \partial m_T} - A_1 \frac{\partial^2 U}{\partial F_T^2} \right]}{2 A_1 \frac{\partial^2 U}{\partial F_T \partial m_T} - A_1^2 \frac{\partial^2 U}{\partial F_T^2} - \frac{\partial^2 U}{\partial m_T^2}} \quad .$$

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[1] The sign of the second order derivatives -- if they are assumed different from zero -- may also be changed by an increasing non-linear transformation of the utility indicator.

For typographical reasons we introduce the following variables:

$$(3.16) \qquad U'_X = \frac{\partial U}{\partial X_T} \ , \quad U''_{XY} = \frac{\partial^2 U}{\partial X_T \partial Y_T} \qquad \left(\begin{array}{l} X = F, m \\ Y = F, m \end{array} \right)$$

$$(3.17) \qquad N_{1F} = A_1 U''_{Fm} - U''_{mm}$$

$$(3.18) \qquad N_{1m} = U''_{Fm} - A_1 U''_{FF}$$

$$(3.19) \qquad D_1 = 2 A_1 U''_{Fm} - A_1^2 U''_{FF} - U''_{mm} \quad .$$

The signs of N_{1F} , N_{1m} and D_1 are all unchanged by any arbitrary increasing transformation of the utility indicator. We assume that D_1 is strictly positive. This is identical to assuming that the second order condition for a regular maximum is fulfilled. The algebraic values of both $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$ are unchanged by any arbitrary increasing transformation of the utility indicator.

<u>Proof of the transformation properties mentioned above.</u>

If the utility indicator $U(X_1 , X_2)$ is transformed into $W = W\,[U(X_1 , X_2)]$, where $\frac{\partial W}{\partial U}$ is assumed strictly positive, we obtain the following relations between the derivatives in the two systems:

$$(3.20) \qquad \frac{\partial W}{\partial X_i} = \frac{\partial W}{\partial U} \frac{\partial U}{\partial X_i} \qquad (i = 1, 2)$$

$$(3.21) \qquad \frac{\partial^2 W}{\partial x_i \partial x_j} = \frac{\partial W}{\partial U} \frac{\partial^2 U}{\partial x_i \partial x_j} + \frac{\partial^2 W}{\partial U^2} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \qquad \begin{pmatrix} i = 1, & 2 \\ j = 1, & 2 \end{pmatrix} .$$

By utilizing (3.21) we obtain the following relation between $N_{1F}^{[W]}$ and $N_{1F}^{[U]}$ (i.e., the value of the variable N_{1F}, defined by (3.17), in respectively the W - and the U - system):

$$(3.22) \qquad N_{1F}^{[W]} = \frac{\partial W}{\partial U} N_{1F}^{[U]} + \frac{\partial^2 W}{\partial U^2} U'_m \left[A_1 U'_F - U'_m \right] .$$

The terms within the bracket parenthesis is zero, cf. (3.7), and thereby:

$$(3.23) \qquad N_{1F}^{[W]} = \frac{\partial W}{\partial U} N_{1F}^{[U]} .$$

Expressions of the type (3.20) and (3.23), i.e., that the variable in the W - system equals the corresponding variable in the U - system multiplied by $\frac{\partial W}{\partial U}$, is also obtained for N_{1m} and D_1. This further implies that the algebraic values ofthe derivatives (3.14)-(3.15) are unchanged by any arbitrary increasing transformation of the utility indicator.

We will now discuss the expressions (3.14)-(3.15) under alternative assumptions about the signs of N_{1F} and N_{1m}.[1/] When the product of Z_T and respectively N_{1F} and N_{1m} equals zero, we always obtain what we have termed the normal result: $\frac{\partial F_T}{\partial P_T^*} > 0$ and $\frac{\partial m_T}{\partial P_T^*} < 0$. This implies that a necessary condition for getting the abnormal result is, also in the more general case now studied, that the decision maker does not keep all his

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[1/] Our previous study of the shape of the demand functions, cf. pp. 29-34, is based on a set of assumptions which is sufficient to make N_{1F} and N_{1m} strictly positive.

wealth in cash. It further implies that when respectively N_{1F} and N_{1m} is equal to zero, the abnormal result will never emerge whatever Z_T may be.

In the following we assume that the products referred to above are different from zero.

Comments on $\frac{\partial F_T}{\partial P_T^*}$.

(i) $Z_T N_{1F} > 0$ always yields the normal result, $\frac{\partial F_T}{\partial P_T^*} > 0$.

(ii) $Z_T N_{1F} < 0$. The numerator of (3.14) now is a sum of a positive and a negative term. By imposing the abnormal result we obtain:

$$\frac{\partial F_T}{\partial P_T^*} \lesseqq 0 \quad \text{if}$$

$$(3.24) \qquad Z_T \lesseqq \frac{-\frac{1}{P_t} U_m'}{N_{1F}} \qquad \text{when } N_{1F} > 0$$

or if

$$(3.25) \qquad Z_T \gtreqq \frac{\frac{1}{P_t} U_m'}{-N_{1F}} \qquad \text{when } N_{1F} < 0 .$$

When $N_{1F} > 0$, we get an upper bound -- which is negative -- on the actual demand for Z-assets, cf. also (3.12). The assumption that $N_{1F} < 0$ implies that the abnormal result will occur if the decision maker is a "large" lender.

Comments on $\frac{\partial m_T}{\partial P_T^*}$.

(i) $Z_T N_{1m} < 0$ always yields the normal result, $\frac{\partial m_T}{\partial P_T^*} < 0$.

(ii) $Z_T N_{1m} > 0$. The numerator of (3.15) now is a sum of a negative and a positive term. When imposing the abnormal result, we obtain:

$$\frac{\partial m_T}{\partial P_T^*} \gtreqqless 0 \qquad \text{if}$$

$$(3.26) \qquad Z_T \gtreqqless \frac{\frac{1}{P_t} \quad U_F'}{N_{1m}} \qquad \text{when} \quad N_{1m} > 0$$

or if

$$(3.27) \qquad Z_T \lesseqqgtr \frac{\frac{1}{P_t} \quad U_F'}{N_{1m}} \qquad \text{when} \quad N_{1m} < 0 .$$

When $N_{1m} > 0$, we get a lower bound -- which is positive -- on the actual demand for Z-assets, cf. also (3.13). The assumption that $N_{1m} < 0$ implies that the abnormal result will be brought about if the decision maker has a "large" debt.

We will further study the feasibility of the alternative combinations of signs for N_{1F} and N_{1m} . From (3.17)-(3.19) we obtain:

$$(3.28) \qquad N_{1F} + A_1 N_{1m} = D_1 .$$

Since A_1 and D_1 both are assumed to be strictly positive, (3.28) implies that at least one of the two items N_{1F} and N_{1m} has to be strictly positive. In Table (3.29) is shown the alternative combinations of signs for N_{1F} and N_{1m}. The non-feasible combinations are indicated by drawing the diagonals of the squares in question. The feasible combinations are indicated by "O.K."

Table (3.29). Sign Matrix for N_{1F} and N_{1m}.

		N_{1m}		
		-	0	+
N_{1F}	-	╳	╳	O.K. if $N_{1F} > - A_1 N_{1m}$
	0	╳	╳	O.K.
	+	O.K. if $N_{1F} > - A_1 N_{1m}$	O.K.	O.K.

Under the combinations in the lower left and the upper right corner of Table (3.29) the abnormal result may occur for both demand functions simultaneously. From (3.8), which is derived from the budget equation, it is easily seen:

1. The conditions $\frac{\partial m_T}{\partial P_T^*} \geqq 0$ and $Z_T < 0$ -- which always are fulfilled in the case (3.27) -- are sufficient to yield $\frac{\partial F_T}{\partial P_T^*} < 0$.

2. The conditions $\frac{\partial F_T}{\partial P_T^*} \leqq 0$ and $Z_T > 0$ -- which always are fulfilled in the case (3.25) -- are sufficient to yield $\frac{\partial m_T}{\partial P_T^*} > 0$.

We therefore obtain the abnormal result for both demand functions simultaneously if:

$$Z_T \leqq \frac{\frac{1}{P_t} U'_F}{N_{1m}} \qquad \begin{pmatrix} \text{when} & N_{1F} > 0 \\ \text{and} & N_{1m} < 0 \end{pmatrix} \tag{3.30}$$

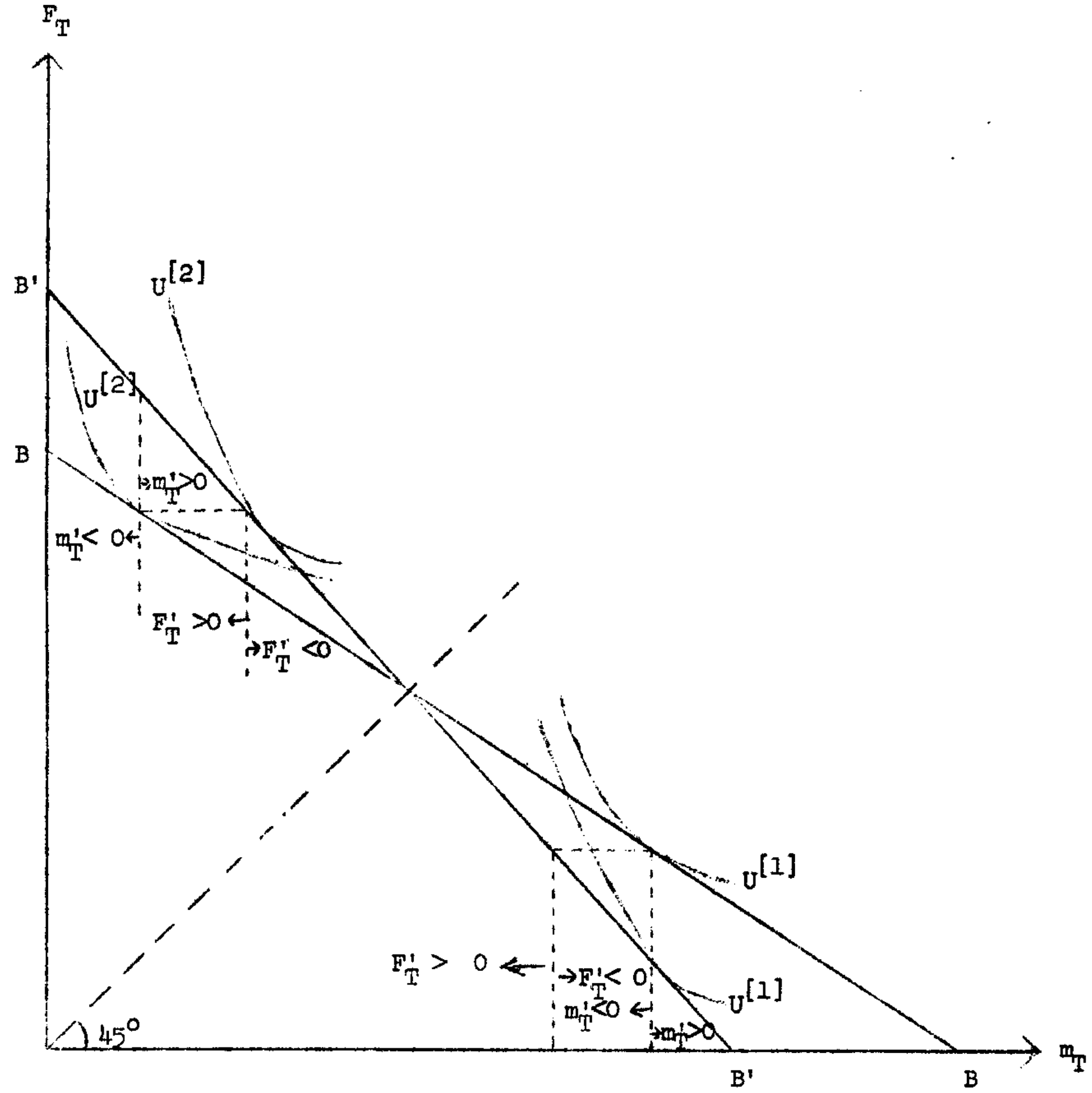
or if

$$Z_T \geqq \frac{\frac{1}{P_t} U'_m}{- N_{1F}} \qquad \begin{pmatrix} \text{when} & N_{1F} < 0 \\ \text{and} & N_{1m} > 0 \end{pmatrix} . \tag{3.31}$$

Graphical illustrations of these two cases are given in Figure (3.32). As in Figure (3.3) the line B'B' shows the budget equation at the higher price. The indifference curves marked $U^{[1]}$ belong to a preference map where -- at least in the relevant region -- $N_{1F} > 0$ and $N_{1m} < 0$, cf. (3.30), and the curves marked $U^{[2]}$ belong to an alternative indifference map where $N_{1F} < 0$ and $N_{1m} > 0$, cf. (3.31).

In the diagram is further given the signs of the derivatives $\frac{\partial F_T}{\partial P_T^*}$ $(= F'_T)$ and $\frac{\partial m_T}{\partial P_T^*}$ $(= m'_T)$ in the different intervals of the new budget line B'B' . The diagram indicates that a decision maker with the $U^{[1]}$-curves would require quite a high value of $A_1 \left(= \frac{P_T^*}{P_t} + i - 1 \right)$ in order to become a lender, and a decision maker with the $U^{[2]}$- curves would require quite a low value of A_1 in order to become a borrower.

Figure (3.32)


F_T
B'
B
$U^{[2]}$
$U^{[2]}$
$\rightarrow m_T' > 0$
$m_T' < 0 \leftarrow$
$F_T' > 0 \leftarrow$
$\rightarrow F_T' < 0$
$U^{[1]}$
$U^{[1]}$
$F_T' > 0 \leftarrow$
$\rightarrow F_T' < 0$
$m_T' < 0 \leftarrow$
$\rightarrow m_T' > 0$
45°
B'
B
m_T

The effect on U

We will also study the effect of an increase in P_T^* on the total utility. Considering F_T and m_T as functions of P_T^*, we get by differentiating (1.1) with respect to P_T^* :

$$(3.33) \qquad \frac{\partial U}{\partial P_T^*} = \frac{\partial U}{\partial m_T} \frac{\partial m_T}{\partial P_T^*} + \frac{\partial U}{\partial F_T} \frac{\partial F_T}{\partial P_T^*} .$$

By utilizing (3.7) - (3.8), (3.33) is reduced to:[1/]

$$(3.34) \qquad \frac{\partial U}{\partial P_T^*} = \frac{\partial U}{\partial F_T} Z_T \text{ , i.e.,}$$

$$(3.35) \qquad \frac{\partial U}{\partial P_T^*} \gtreqless 0 \text{ according as } Z_T \gtreqless 0 .$$

When P_T^* increases, the total utility will increase, be unchanged or decrease, according as the decision maker at the given price is a lender, neither a lender nor a borrower, or a borrower. This conclusion is also indicated by the effect of an increase in P_T^* on the budget line, cf. Figure (3.3).

We may consider the effect (3.34) as a kind of an income effect or a capital gain effect. When P_T^* is increased by one dollar, the future wealth is increased by one dollar multiplied with the total

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[1/] The sign of $\frac{\partial U}{\partial P_T^*}$ is unchanged by any arbitrary increasing transformation of the utility indicator and is further independent of the signs of N_{1F} and N_{1m} .

number of Z-assets. By multiplying this item with the marginal utility of wealth, we get the increase in utility. This reasoning assumes that the distribution between cash and Z-assets is unchanged, or -- in other words -- the substitution effect is not taken into account. Roughly, we may say that the substitution effect is of another dimension than the income effect.

We may further notice that -- when $N_{1m} > 0$ -- the income effect has to be "large" if the decision maker shall increase his demand for cash, cf. (3.26). When Z_T fulfills the inequality of (3.26) and $N_{1F} > 0$, the income effect is in fact so large that total wealth as well as holdings of cash will increase. When $N_{1m} < 0$, the algebraic value of the income effect has to be below a negative item if the demand for cash shall be increased, cf. (3.27).

The value of P_T^* $(= \bar{P}_T^*)$ which makes the decision maker keep all his wealth in cash $(F_T = m_T = F_t^0 , Z_T = 0)$ yields the minimum value of U. This is due to the fact that whatever P_T^* is, the decision maker is always able to keep his given (and constant) initial wealth in cash, and when $P_T^* \gtrless \bar{P}_T^*$ other points on the budget line will yield a higher value of U.

In Table (3.36) we have summarized the effects of an increase in P_T^* on the demand for wealth, cash and on utility under alternative assumptions about the signs of Z_T, N_{1F} and N_{1m}.

Table (3.36)

The effect of an increase in P_T^* on the demand for total wealth, cash and on utility under alternative assumptions about the signs of Z_T , N_{1F} and N_{1m}

		$N_{1F} = A_1 U''_{Fm} - U''_{mm}$			$N_{1m} = U''_{Fm} - A_1 U''_{FF}$			Effect on U
		-	0	+	-	0	+	
Z_T	-	$\frac{\partial F_T}{\partial P_T^*} > 0$	$\frac{\partial F_T}{\partial P_T^*} > 0$	$\frac{\partial F_T}{\partial P_T^*} \lesseqgtr 0$ according as $Z_T \lesseqgtr - \frac{U'_m}{P_t N_{1F}}$	$\frac{\partial m_T}{\partial P_T^*} \gtreqless 0$ according as $Z_T \lesseqgtr \frac{U'_F}{P_t N_{1m}}$	$\frac{\partial m_T}{\partial P_T^*} < 0$	$\frac{\partial m_T}{\partial P_T^*} < 0$	$\frac{\partial U}{\partial P_T^*} < 0$
	0	$\frac{\partial F_T}{\partial P_T^*} > 0$	$\frac{\partial F_T}{\partial P_T^*} > 0$	$\frac{\partial F_T}{\partial P_T^*} > 0$	$\frac{\partial m_T}{\partial P_T^*} < 0$	$\frac{\partial m_T}{\partial P_T^*} < 0$	$\frac{\partial m_T}{\partial P_T^*} < 0$	$\frac{\partial U}{\partial P_T^*} = 0$
	+	$\frac{\partial F_T}{\partial P_T^*} \gtreqless 0$ according as $Z_T \lesseqgtr \frac{U'_m}{- P_t N_{1F}}$	$\frac{\partial F_T}{\partial P_T^*} > 0$	$\frac{\partial F_T}{\partial P_T^*} > 0$	$\frac{\partial m_T}{\partial P_T^*} < 0$	$\frac{\partial m_T}{\partial P_T^*} < 0$	$\frac{\partial m_T}{\partial P_T^*} \lesseqgtr 0$ according as $Z_T \lesseqgtr \frac{U'_F}{P_t N_{1m}}$	$\frac{\partial U}{\partial P_T^*} > 0$

4. The effects of changes in the rate of interest

By considering the budget equation (1.2) and the adjustment condition (2.1), one may notice that the effects of partial changes in i are equal to the effects of partial changes in P_T^* as far as only the sign of effects are concerned. The analysis in chapter 3 is therefore also applicable to an analysis of the effects of changes in the interest rate, i . For the sake of completeness we present the expressions for $\frac{\partial F_T}{\partial i}$ and $\frac{\partial m_T}{\partial i}$ below:

$$(4.1) \qquad \frac{\partial F_T}{\partial i} = \frac{\frac{\partial U}{\partial m_T} + P_t Z_T \left[A_1 \frac{\partial^2 U}{\partial F_T \partial m_T} - \frac{\partial^2 U}{\partial m_T^2} \right]}{2 A_1 \frac{\partial^2 U}{\partial F_T \partial m_T} - A_1^2 \frac{\partial^2 U}{\partial F_T^2} - \frac{\partial^2 U}{\partial m_T^2}}$$

$$(4.2) \qquad \frac{\partial m_T}{\partial i} = \frac{- \frac{\partial U}{\partial F_T} + P_t Z_T \left[\frac{\partial^2 U}{\partial F_T \partial m_T} - A_1 \frac{\partial^2 U}{\partial F_T^2} \right]}{2 A_1 \frac{\partial^2 U}{\partial F_T \partial m_T} - A_1^2 \frac{\partial^2 U}{\partial F_T^2} - \frac{\partial^2 U}{\partial m_T^2}} .$$

The inequalities (3.24)-(3.27) remain the same except that i is substituted for P_T^* in respectively $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$.

On the basis of this model we thus obtain as one result that the demand for cash may increase at an increase in the rate of interest, cf. (3.26) - (3.27). This conclusion is the opposite of the liquidity preference theory of Keynes.[1]

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[1] We obtain the relation between demand for money and the interest rate as a result of our model. We will not discuss here to which extent the liquidity preference of Keynes is a "direct" assumption or emerges as a result from more basic assumptions.

The alternative asset to money in Keynes' theory is bonds. For these bonds there exists an effective price. The effective rate of interest which these bonds yield may be considered as a function of, among other things, the effective price. If we associate the Z-assets in Model I with bonds, we may consider P_t as the effective price at t and i as the effective rate of interest. If this is done, we can't any more take P_t and the rate of interest as independent variables, but we have to take into consideration that there exists one relation between them. This will be done in the Models II and III.

Before we present Model II, we will, however, introduce a more specified utility function in order to derive the explicit expressions for the demand functions This is done for illustration purposes.

5. Explicit expressions for the demand functions under stronger specification of the utility function

We introduce the following utility function:

$$U = a_1 \ln F_T + a_2 \ln m_T + a_o \quad , \tag{5.1}$$

where a_1 and a_2 are assumed positive. (5.1) yields positive and decreasing marginal utilities and independence between wealth and cash. (5.1) does not, however, satisfy the assumptions (cf. also Figure (1.7)):

$$\frac{\partial U(F_T \, , \, m_T^{max} \,)}{\partial m_T} = 0 \quad \text{(for all values of } F_T\text{) and} \tag{5.1a}$$

$$\frac{\partial U(F_T \, , \, m_T)}{\partial m_T} \leqq 0 \quad \text{(for all values of } F_T \, , \, m_T > m_T^{max} \text{) .}$$

We will therefore consider (5.1) as a kind of an approximation which only is acceptable for those constellations of the other data yielding

$$m_T \leqq m_T^{max} \quad \text{- a positive constant.} \tag{5.1b}$$

It is seen from the indifference curves in Figure (1.7) that the budget line has to be horizontal if the decision maker should choose $m_T = m_T^{max}$. This corresponds to $A_1 = 0$, cf. Figure (3.3), and the corresponding comments. We will not make any attempt at an explicit determination of the positive constant of (5.1b) and the corresponding feasible region for the data. In the graphs of the demand functions emerging from (5.1) we will, however, give an indication of their feasibility.

We will first study the shape of the indifference curves corresponding to (5.1). We denote the constant value of U by $\overline{U}$ and solve (5.1) with respect to $a_1 \ell n\, F_T$:

$$a_1\ \ell n\, F_T = -\ a_2\ \ell n\, m_T + \overline{U} - a_o\ ,\ \text{i.e.:}$$

$$\ell n\, F_T^{a_1} = \ell n\ \frac{e^{\overline{U}-a_o}}{m_T^{a_2}}\ ,$$

which further yields:

$$F_T^{a_1} = \frac{e^{\overline{U}-a_o}}{m_T^{a_2}}\ . \tag{5.2}$$

By taking the a_1^{th} root on both sides of (5.2), we obtain:

$$F_T = \frac{e^{\frac{\overline{U}-a_o}{a_1}}}{m_T^{\frac{a_2}{a_1}}}\ . \tag{5.3}$$

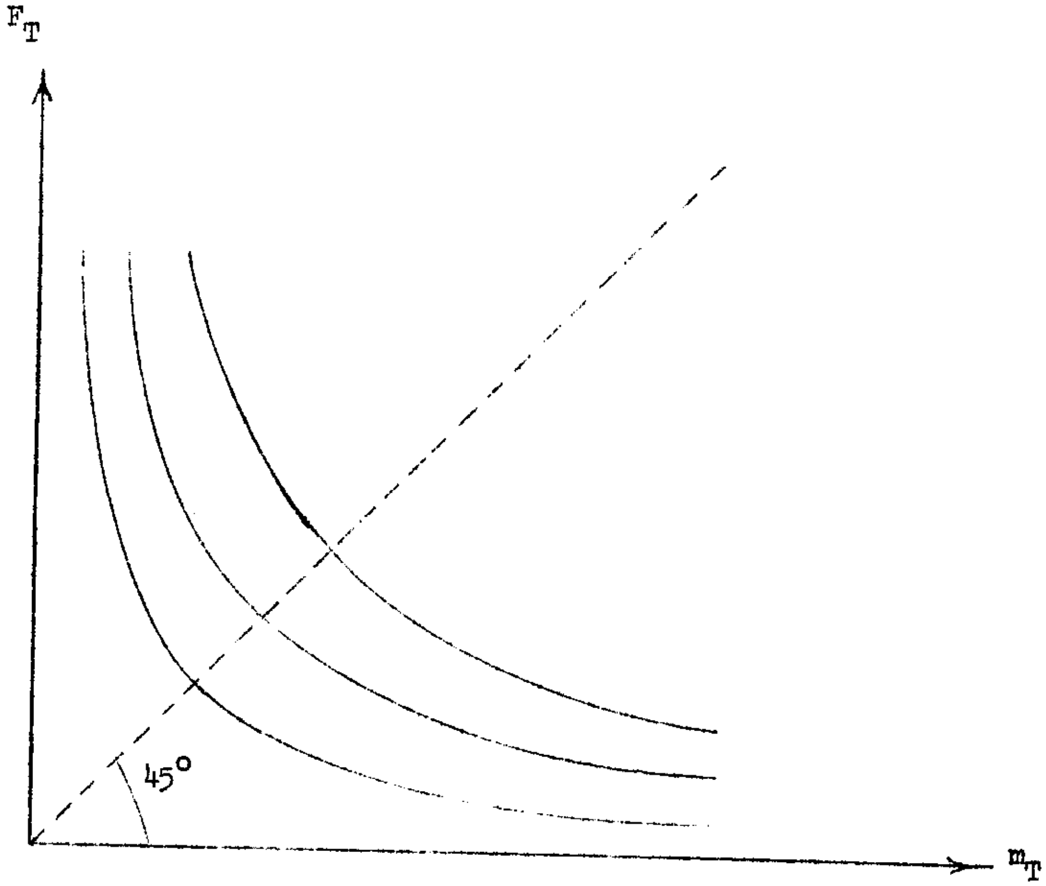
When m_T increases from zero to infinity, F_T decreases from plus infinity to zero and $\left(\frac{\partial F_T}{\partial m_T}\right)_{U = \text{constant}}$ increases from minus infinity to zero. This implies that the indifference curves are approaching the axes asymptotically. Thus the utility function (5.1) belongs to the class of utility functions which fulfill the strong assumption mentioned on pp. 17-18.

By assuming $a_2 = a_1$ -- which implies that the marginal utility of an amount of wealth equals the marginal utility of the same amount of cash -- and -- for the sake of convenience -- putting $a_o = 0$, (5.3) becomes:

$$(5.4) \qquad F_T = \frac{e^{\frac{\bar{U}}{a_1}}}{m_T} \text{, i.e., } F_T\, m_T = e^{\frac{\bar{U}}{a_1}} = \text{constant.}$$

(5.4) is a regular hyperbola whose axes coincide with the m_T , F_T axes. The graph of (5.4) is given in Figure (5.5) for three values of $\bar{U}$.

Figure (5.5)



From (5.1) we get for the marginal rate of substitution:

$$(5.5a) \qquad \left(\frac{\partial F_T}{\partial m_T} \right)_{U = \text{constant}} = - \frac{a_2}{a_1} \frac{F_T}{m_T} .$$

When $F_T = m_T$ (i.e., the point of intersection between the indifference curves and the 45^o - line), the marginal rate of substitution equals minus $\frac{a_2}{a_1}$ (= - 1 when $a_2 = a_1$ as assumed in Figure (5.5)). When $\bar{U}$, a_o and a_1 are assumed constant, all the indifference curves which are generated by variation in a_2 will pass through the point $m_T = 1$, $F_T = e^{\frac{\bar{U}-a_o}{a_1}}$, cf. (5.3). When $a_2 < a_1$, the indifference curves will be situated below the corresponding one in Figure (5.5) as long as $m_T < 1$ and above it when $m_T > 1$.

Before deriving the explicit expressions for the demand functions, we will see if the abnormal results, cf. (3.12) and (3.13), may occur. We now have:

$$\frac{\partial U}{\partial m_T} = \frac{a_2}{m_T} , \qquad \frac{\partial^2 U}{\partial m_T^2} = - \frac{a_2}{m_T^2} ,$$

and (3.12) takes the form:

$$(5.6) \qquad \frac{\partial F_T}{\partial P_T^*} \leqq 0 \text{ if } Z_T \leqq - \frac{m_T}{P_t} .$$

The total wealth of the decision maker "just after" he has made the redistribution is -- of course -- the same as his initial wealth, i.e., (cf. also (1.3) - (1.4)):

$$F_t^o = m_t^o + P_t Z_t^o = m_T + P_t Z_T . \tag{5.7}$$

We will now assume that his initial wealth is strictly positive, $F_t^o > 0$, whatever the present price, P_t, may be:

$$F_t^o > 0 \text{ (for all } P_t) . \tag{5.8}$$

This assumption will be called the strong assumption of solvency. The word strong is applied because of (5.8) is due whatever a "bad" (= high) price a debtor ($Z_t^o < 0$) is facing and not because we don't allow for the zero value of the wealth in this definition of solvency.[1]

(5.7) - (5.8) lead to the following inequality:

$$Z_T > - \frac{m_T}{P_t} \text{ , i.e.,} \tag{5.9}$$

the abnormal result, (5.6), will never emerge. When P_T^* increases,

---

[1] In model II we will give some further remarks on the assumption (5.8) and, to some extent, also study the implication of a non-positive F_t^o .

F_T will always increase.

(3.13) may now be written:

$$(5.10) \qquad \frac{\partial m_T}{\partial P_T^*} \geqq 0 \text{ if } Z_T \geqq \frac{1}{A_1} \frac{F_T}{P_t} .$$

By substituting the expression (3.5) for A_1, (5.10) gives the following upper bound for F_T:

$$(5.11) \qquad F_T \leqq \left(\frac{P_T^*}{P_t} + i - 1 \right) P_t Z_T .$$

From (3.6) we obtain for m_T:

$$(5.12) \qquad m_T = F_T - (P_T^* + P_t\, i)\, Z_T ,$$

and by using the non-negativity condition on m_T -- cf. the left hand side of (1.6) -- we obtain the following lower bound on F_T:

$$(5.13) \qquad F_T \geqq \left(\frac{P_T^*}{P_t} + i \right) P_t Z_T .$$

Since $\left(\frac{P_T^*}{P_t} + i \right) > \left(\frac{P_T^*}{P_t} + i - 1 \right)$ the case (5.11) -- or its equivalent (5.10) -- will never occur, i.e.: When P_T^* increases, m_T will always decrease, which further implies that Z_T will always increase.

Next we will derive explicitly the demand functions. (3.7) may now be written:

$$(5.14) \qquad \frac{a_2}{m_T} = \frac{a_1}{F_T} A_1 .$$

We rearrange (5.14) as (5.15) and rewrite (3.2) as (5.16), utilizing (3.4) - (3.5):

$$(5.15) \qquad a_2 F_T - a_1 A_1 m_T = 0$$

$$(5.16) \qquad F_T + A_1 m_T = \left(\frac{P_T^*}{P_t} + i \right) F_t^o .$$

The solution of (5.15) - (5.16) with respect to F_T and m_T yields:

$$(5.17) \qquad F_T = \left(\frac{P_T^*}{P_t} + i \right) a F_t^o$$

$$(5.18) \qquad m_T = \frac{\frac{P_T^*}{P_t} + i}{\frac{P_T^*}{P_t} + i-1} (1-a)F_t^o = \frac{1}{1 - \frac{1}{\frac{P_T^*}{P_t} + i}} (1-a)F_t^o$$

where

$$(5.19) \qquad a = \frac{a_1}{a_1 + a_2} \quad , \quad 0 < a < 1 .$$

By substituting (5.18) for m_T in (1.4) we obtain for Z_T :

$$Z_T = \frac{a\left(\frac{P^*_T}{P_t} + i\right) - 1}{\frac{P^*_T}{P_t} + i - 1} \cdot \frac{F^o_t}{P_t} \tag{5.20}$$

$$= \frac{a - \frac{1}{\frac{P^*_T}{P_t} + i}}{1 - \frac{1}{\frac{P^*_T}{P_t} + i}} \cdot \frac{F^o_t}{P_t} \; .$$

The demand functions (5.17)-(5.18) and (5.20) are invariant with respect to any increasing transformation of (5.1).

6. The demand for wealth, cash and Z-assets as functions of the future price

We have seen that when the utility function (5.1) (or any increasing transformation of (5.1)) is utilized, the demand for wealth will always increase and the demand for cash will always decrease (i.e. the demand for Z-assets will always increase) when the future price of Z-assets increases. We will now study somewhat more detailed the explicit demand functions, assuming P_t and i (and m_t^o and Z_t^o) constant. Under this study we will assume that m_t^o is strictly positive. We will use the notations $F_T(P_T^*)$, $m_T(P_T^*)$ and $Z_T(P_T^*)$ for the demand functions in question. The technique applied will be to determine the extreme values of F_T, m_T and Z_T, i.e., the values obtained for these variables when P_T^* assumes its minimal and maximal values, and further evaluate the first and the second order derivatives of the demand functions (5.17), (5.18) and (5.20).

We have assumed that $\frac{P_T^*}{P_t} + i - 1 > 0$. We will now consider 0 as a limit value:

$$\frac{P_T^*}{P_t} + i - 1 \geqq 0 \text{ , i.e.}$$

$$\frac{P_T^*}{P_t} + i \geqq 1 \quad \text{or} \quad P_T^* \geqq (1-i)\, P_t \text{ .} \tag{6.1}$$

Infinity will be considered as the maximal value of P_T^* .

(i) $F_T = F_T(P_T^*)$.

From (5.17) we obtain:

$$(6.2) \quad \lim_{\left(\frac{P_T^*}{P_t} + i\right) \to 1} F_T = a \, F_t^o$$

$$(6.3) \quad \frac{\partial F_T}{\partial P_T^*} = a \, \frac{F_t^o}{P_t}$$

$$(6.4) \quad \frac{\partial^2 F_T}{\partial {P_T^*}^2} = 0 \, .$$

(6.2) - (6.3) imply that total wealth will be a linear function of the future price, starting off from a positive value and increase towards infinity.

(ii) $m_T = m_T \, (P_T^*)$.

From (5.18) we obtain:

$$(6.5) \quad \lim_{\left(\frac{P_T^*}{P_t} + i\right) \to 1} m_T = + \infty$$

(6.6) $$\lim_{P_T^* \to \infty} m_T = (1-a)F_t^o$$

(6.7) $$\frac{\partial m_T}{\partial P_T^*} = - \frac{(1-a)}{\left(\frac{P_T^*}{P_t} + i-1\right)^2} \cdot \frac{F_t^o}{P_t} \quad \text{(negative)}$$

(6.8) $$\frac{\partial^2 m_T}{\partial P_T^{*2}} = \frac{2\ (1-a)}{\left(\frac{P_T^*}{P_t} + i-1\right)^3} \cdot \frac{F_t^o}{P_t^2} \quad \text{(positive)} .$$

The demand for cash will decrease from plus infinity to a finite value, $(1-a)\ F_t^o$, its first order derivative increasing from minus infinity to zero. (The function (5.18) is a hyperbola with axes $P_T^* = (1-i)\ P_t$ and $m_T = (1-a)\ F_t^o$) .

(iii) $Z_T = Z_T\ (P_T^*)$.

From (5.20) we obtain:

(6.9) $$\lim_{\left(\frac{P_T^*}{P_t} + i\right) \to 1} Z_T = - \infty$$

(6.10) $$\lim_{P_t^* \to \infty} Z_T = a\ \frac{F_t^o}{P_t}$$

$$(6.11) \qquad \frac{\partial Z_T}{\partial P_T^*} = - \frac{1}{P_t} \frac{\partial m_T}{\partial P_T^*} \quad \text{(positive)}$$

$$(6.12) \qquad \frac{\partial^2 Z_T}{\partial P_T^{*2}} = - \frac{1}{P_t} \frac{\partial^2 m_T}{\partial P_T^{*2}} \quad \text{(negative)}.$$

The demand for Z-assets will increase from minus infinity to a positive finite value, $a \frac{F_t^o}{P_t}$, its first order derivative decreasing from plus infinity to zero. (The function (5.20) is a hyperbola with axes $P_T^* = (1-i)\ P_t$ and $Z_T = a \frac{F_t^o}{P_t}$).

The value of P_T^* $(= \bar{P}_T^*)$ for which the decision maker doesn't demand any Z-assets (i.e., $F_T = m_T = F_t^o$) is found by equating the right hand side of (5.20) to zero. This yields:

$$(6.13) \qquad \bar{P}_T^* = (\frac{1}{a} - i)\ P_t\ .$$

We may notice that the initial holdings of cash and Z-assets are of no significance in the determination of $\bar{P}_T^*$. From (6.13) is seen that $\frac{\partial \bar{P}_T^*}{\partial a}$ as well as $\frac{\partial \bar{P}_T^*}{\partial i}$ are negative, i.e., the more the decision maker prefers wealth to cash and/or the higher the interest rate is, the lower is the value of the future price which makes him switch from borrowing to lending. In Table (6.14) we have -- for some combinations of a and i -- given the expected price increase, calculated as a percentage, at which the decision maker keeps all his wealth in cash.

TABLE (6.14)

Expected price increase $\left[= \left(\frac{1}{a} - i - 1\right) 100\right]$ which makes the decision maker demand no Z-assets under alternative assumptions about a and i.

a \ i	0	0.05	0.10	0.15	0.20	0.25	0.30
0.9	11	6	1	- 4	- 9	- 14	- 19
0.8	25	20	15	10	5	0	- 5
0.7	43	38	33	28	23	18	13
0.6	67	62	57	52	47	42	37
0.5	100	95	90	85	80	75	70

When a is as "low" as 0.5, which yields the indifference curves of Figure (5.5), the decision maker has to expect quite a high price increase in order to be a lender. If he expects no change in the price, i.e., $P_T^* = P_t$, he will demand no Z-assets if the right hand side of (6.13) equals P_t, which implies

$$\frac{1}{a} - i = 1 \,. \qquad (6.15)$$

In Table (6.16) we have listed some a , i - combinations satisfying (6.15).

TABLE (6.16)

Combinations of a and i yielding $\bar{P}^*_T = P_t$

i	0	0.05	0.10	0.15	0.20	0.25	0.30
a	1	0.95	0.91	0.87	0.83	0.80	0.77

Figure (6.17) - (6.18) give the graphs of the demand functions for total wealth, cash and Z-assets. In Figure (6.17) we have -- for illustration purposes -- plotted a value of m^{max}_T larger than F^o_t , and further assumed that $P^*_T \geqq \underline{P}^*_T$ is the feasible P^*_T -- range when taking into account that (5.1) only is an approximation. The position of the demand functions in the non-feasible P^*_T - range is indicated by short dashes. The diagrams are drawn under the assumptions that $a = 0.8$ and $i = 0.25$, which is one of the combinations yielding $\bar{P}^*_T = P_t$.

Figure (6.17) The demand functions for wealth and cash (a = 0.8, i = 0.25).

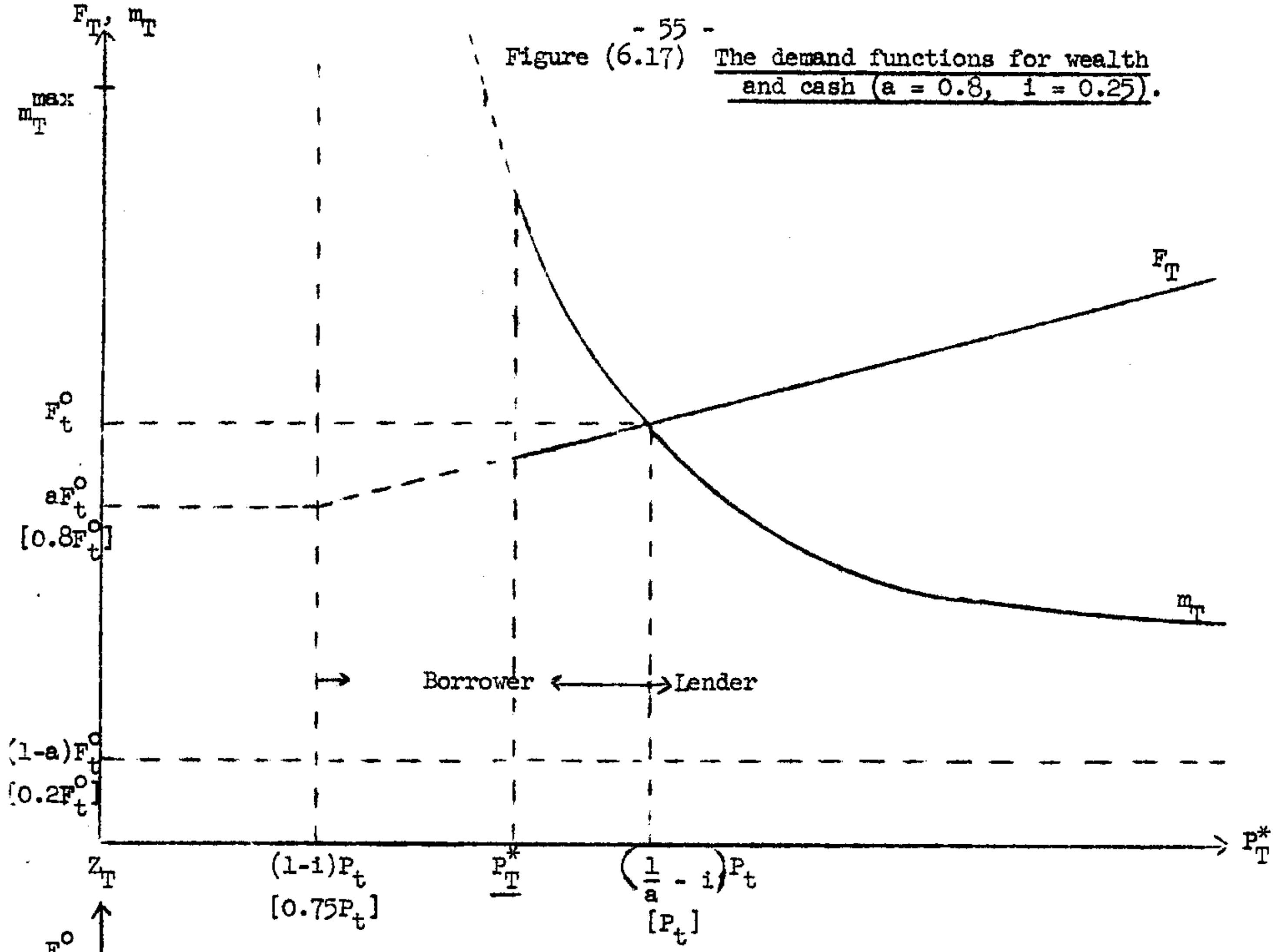
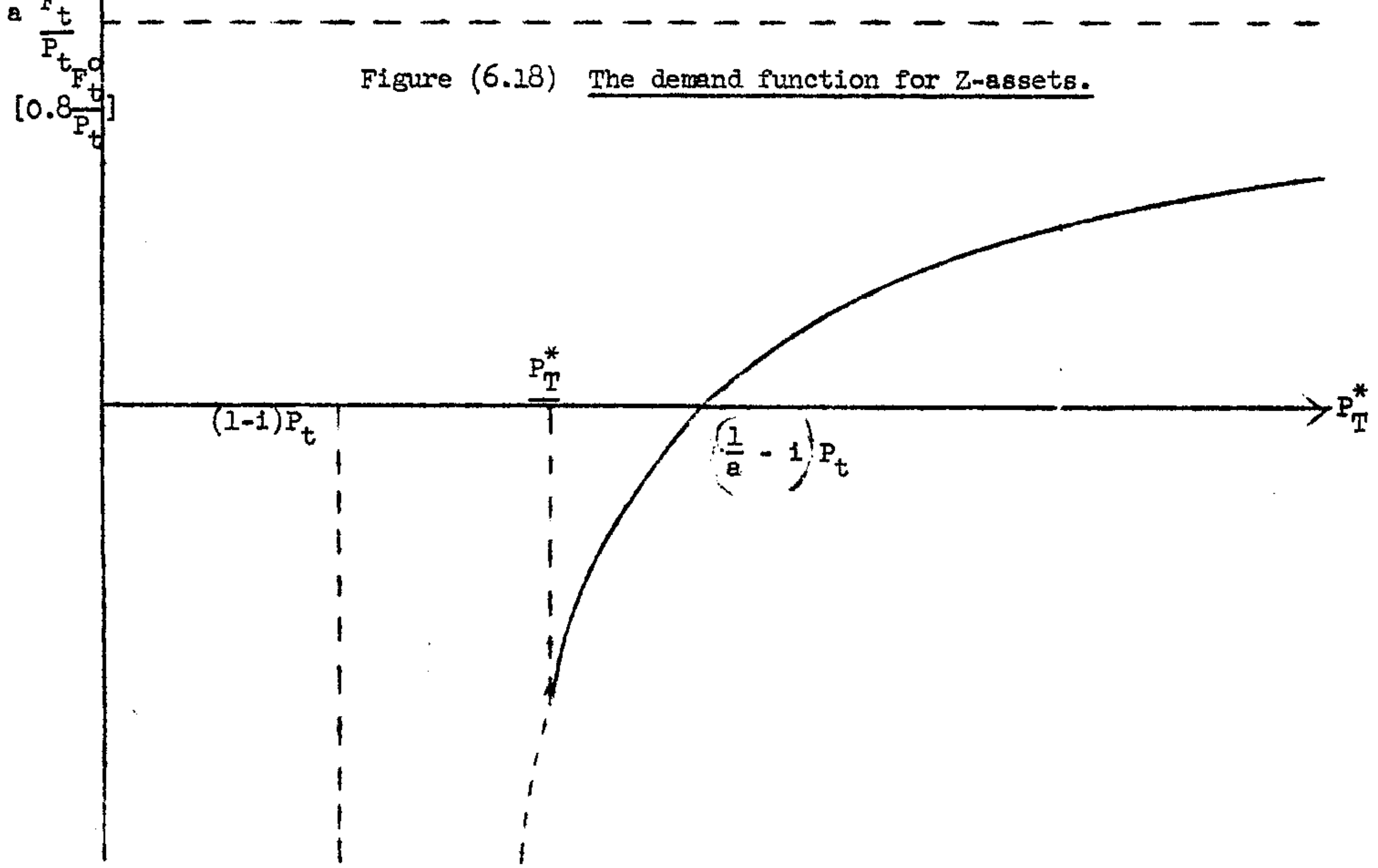


Figure (6.18) The demand function for Z-assets.



We may finally notice the following: From (5.17) follows that if two decision makers have the same utility function (i.e., the same value of a) and make the same price forecast, the one who is richest in the initial position (i.e., largest F_t^o) will stay the richest whatever P_T^* may be.[1] The richest will also demand more cash at any value of P_T^*, cf. (5.18), and the numerical value of his holdings of Z-assets will always be the largest, except when $P_T^* = (\frac{1}{a} - i) P_t$, when they both demand no Z-assets. The ranking of people according to their wealth in the initial situation may, however, be changed by changes in P_t, which has been assumed constant.

---

[1] If the price forecast, P_T^*, turns out to be wrong, this conclusion may be violated.

MODEL II

7. The budget equation

In Model II we will utilize a definition of the Z-assets which should be applicable to quite a wide class of bonds. We now define a Z-asset as an asset which -- after "one period" -- yields an increase in the wealth equal to a constant (and known) interest income plus the increase in the price (capital gain). The given interest income equals the nominal (face) value of the asset multiplied by the nominal rate of interest.

If we denote the constant interest income received in each period by b , we have:

(7.1) Increase in wealth per Z-asset: $P^*_T + b - P_t$.

The budget equation in this case is obtained from (1.2) by replacing $P_t \cdot i$ (the expression for interest income per Z-asset in Model I) with b . We then obtain:[1]

$$F_T = m_T + \left(Z^o_t + \frac{m^o_t - m_T}{P_t} \right) \left(P^*_T + b \right) . \tag{7.2}$$

---

[1] The definitional relations (1.3) - (1.4) are also valid for Model II.

We may further ask for the effective rate of interest, i. If the duration of the asset in question is definite, say n periods, and we denote the nominal (= redemption) value of the asset by b_o, we get the following relation between P_t, i, b and b_o:[1]

$$P_t = \frac{b}{1+i} + \frac{b}{(1+i)^2} + \ldots + \frac{b}{(1+i)^n} + \frac{b_o}{(1+i)^n} \text{, i.e.,}$$

$$P_t = \frac{b}{i}\left[1 - \frac{1}{(1+i)^n}\right] + \frac{b_o}{(1+i)^n} . \tag{7.3}$$

From (7.3) and the equation in the footnote above follow that the effective rate of interest, i, depends on the effective price, the nominal price and the nominal rate of interest and the duration of the asset.

An annuity bond, i.e., a bond which bears no maturity date, the interest continuing indefinitely may be considered as a special case of the class of Z-assets yielding the budget equation (7.2). When n increases towards infinite, we obtain from (7.3):

---

[1] If we denote the nominal rate of interest with i_o, we may write: $b = b_o \cdot i_o$.

The variable A_2, defined by (8.2) may also be considered as a kind of an effective rate of interest.

$$(7.4) \qquad P_t = \frac{b}{i} \quad , \text{ or}$$

$$(7.5) \qquad i = \frac{b}{P_t} \quad .$$

We may notice that the budget equation (7.2) emerges from (1.2) by replacing i in the latter with the expression (7.5).

Neither (7.3) nor the specification (7.4) is necessary for the following analysis. We may say that these relations leave us free as to choose between P_t and i as variables in our model. In the following we will use P_t as the independent variable and use the budget equation in the form (7.2). The relations (7.3) - (7.4) may be considered as "secondary" relations which make it possible to compute corresponding values of i and P_t for given values of b, b_o and n .

Whatever n (the duration of the Z-assets) may be, we will always have that the effect of changes in i is opposite the effects of changes in P_t as far as we only deal with the sign of the changes in the relevant variables: F_T , m_T , Z_T . This stems from the fact that $\frac{dP_t}{d\, i}$, derived from (7.3) , is negative, whatever n may be (provided that $n > 0$). By differentiating (7.3) with respect to i , we obtain:

$$(7.6) \quad \frac{dP_t}{d\, i} = \frac{b}{i^2(1+i)^{n+1}} \left[1 + i + ni - (1+i)^{n+1} \right] - \frac{nb_o}{(1+i)^{n+1}}$$

Putting $n = 0, 1, 2$, we obtain for the terms in the bracket:

$\underline{n = 0}$, $[\ -\ -\ -\ ] = 1 + i - (1+i) = 0$

$\underline{n = 1}$, $[\ -\ -\ -\ ] = 1 + 2\,i - 1 - 2\,i - i^2 = - i^2 < 0$

$\underline{n = 2}$, $[\ -\ -\ -\ ] = 1 + 3\,i - 1 - 3\,i - 3\,i^2 - i^3 = - 3\,i^2 - i^3 < 0$.

The terms within the bracket will always yield a negative factor when $n \geqq 1$. The factor before the bracket is strictly positive.

8. The optimal portfolio selection under a given constellation of data

The problem of the optimal portfolio selection is now solved by maximizing the following Lagrange-expression:

$$H(F_T \; , \; m_T) = U \; (F_T \; , \; m_T) \; - \lambda \left[F_T - m_T - \left(Z_t^o + \frac{m_t^o - m_T}{P_t} \right) (P_T^* + b) \right] .$$

This leads to the following relation:

$$(8.1) \qquad \frac{\partial U}{\partial m_T} = \frac{\partial U}{\partial F_T} \left(\frac{P_T^*}{P_t} + \frac{b}{P_t} - 1 \right) .$$

This condition may be interpreted in the same way as (2.1): Each dollar spent on Z-assets results in a net increase in the future wealth equal to $\frac{P_T^* + b - P_t}{P_t}$.

By multiplying this item with the marginal utility of wealth we obtain the utility of the last dollar spent on Z-assets, which -- in order to achieve an optimal distribution -- shall equal the marginal utility of the other good, namely cash. We will use the symbol A_2 for the expression within the parenthesis of the right hand side of (8.1):[1]

$$(8.2) \qquad A_2 = \frac{P_T^* + b}{P_t} - 1$$

(= net increase in wealth per dollar spent on Z-assets, A_2 is assumed positive).

---

[1] A_2 -- like A_1 , cf. (3.5) and the corresponding footnote -- may be considered as an alternative definition of the effective rate of interest.

(7.2) and (8.1) give us two relations for the determination of F_T and m_T .

We will again study the effect of changes in data. By comparing the relations (1.2) and (2.1) on the one hand with the relations (7.2) and (8.1) on the other hand it is easily seen that P_T^* enters analogous in the two models. This implies that the effects of changes in P_T^* in Model II will be the same as those in Model I. In the expressions for $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$, cf. (3.14)-(3.15) in the general case and (3.10)-(3.11) in the special case where $\frac{\partial^2 U}{\partial F_T \partial m_T} = 0$, we only have to substitute $\frac{b}{P_t}$ for i , or -- what amounts to the same -- A_2 for A_1 . Doing this we obtain in the general case:

$$(8.3) \qquad \frac{\partial F_T}{\partial P_T^*} = \frac{\frac{1}{P_t} \frac{\partial U}{\partial m_T} + Z_T \left[A_2 \frac{\partial^2 U}{\partial F_T \partial m_T} - \frac{\partial^2 U}{\partial m_T^2} \right]}{2 A_2 \frac{\partial^2 U}{\partial F_T \partial m_T} - A_2^2 \frac{\partial^2 U}{\partial F_T^2} - \frac{\partial^2 U}{\partial m_T^2}}$$

$$(8.4) \qquad \frac{\partial m_T}{\partial P_T^*} = \frac{- \frac{1}{P_t} \frac{\partial U}{\partial F_T} + Z_T \left[\frac{\partial^2 U}{\partial F_T \partial m_T} - A_2 \frac{\partial^2 U}{\partial F_T^2} \right]}{2 A_2 \frac{\partial^2 U}{\partial F_T \partial m_T} - A_2^2 \frac{\partial^2 U}{\partial F_T^2} - \frac{\partial^2 U}{\partial m_T^2}} .$$

The comments given on $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$ in Model I are therefore also valid in Model II.

We may further notice that P_T^* and b enter completely analogous in the two relations (7.2) and (8.1). This implies that the effect of an increase in the interest income is the same as the effect of an increase of the same size in the future price, cf. (8.3) - (8.4), when P_t in either case is assumed constant.[1]

In the next chapter we will study the effects of an increase in the present price of Z-assets. When only dealing with a qualitative analysis this is the same as studying the effects of a decrease in the effective rate of interest, whether the interest rate is defined by (7.3), (7.5) or (8.2).

---

[1] Assuming P_t constant, an increase in b implies an increase in the effective rate of interest.

9. The effects of changes in the present price of Z-assets

We will first study the effect of an increase in P_t on the budget line. We rewrite (7.2) as (9.1):

$$(9.1) \qquad F_T = - \left(\frac{P_T^* + b}{P_t} - 1 \right) m_T + \left(Z_t^o + \frac{m_t^o}{P_t} \right) \left(P_T^* + b \right) .$$

As far as the slope of the budget line is concerned, we see from (9.1) that it becomes less steep when P_t increases. Whatever the value of P_t may be, the decision maker is, of course, always able to keep amounts of cash and Z-assets equal to his initial holdings. By inserting $m_T = m_t^o$ in (9.1) we obtain:

$$(9.2) \qquad F_T = m_t^o + (P_T^* + b) Z_t^o ,$$

i.e.: the budget line will pass through the point $m_T = m_t^o$, $F_T = m_t^o + (P_T^* + b) Z_t^o$, whatever the present price is. This firm [1] point is located below, on or above the 45^o - line according as Z_t^o is negative, zero or positive.

We assume that the initial amount of cash is non-negative;

$$(9.3) \qquad m_t^o \geqq 0 .$$

---

[1] Firm as long as the data m_t^o , Z_t^o and $(P_T^* + b)$ is constant.

For a decision maker who initially is a non-borrower, i.e., $Z_t^o \geqq 0$, the right hand side of (9.2) will never be negative. We may consider $(P_T^* + b)$ as a maximum value of P_t . (Actually P_t has to be smaller than $(P_T^* + b)$ in order to obtain $A_2 > 0$, cf. (8.2)). The right hand side of (9.2) is therefore the minimum value of the initial wealth of a decision maker who initially is a borrower, i.e., $Z_t^o < 0$. We use the notation $F_t^o \,(P_T^* + b)$ for this value of the initial wealth.[1]

---

[1] Generally: $F_t^o \,(P_t') = m_t^o + P_t' \, Z_t^o$.

By assuming

$$(9.4) \qquad F_t^o \,(P_T^* + b) \geqq 0 ,$$

the decision maker will be solvent when he enters the market at t whatever the (feasible) value of P_t may be.[2]

---

[2] Cf. also (5.8), where $F_t^o \,(P_T^* + b)$ was assumed strictly positive, and the corresponding comments.

The assumptions (9.3) - (9.4) implies that the point of rotation for the budget line always will be located in the first quadrant of a m_T , F_T - diagram, including the borders. We will further assume that the equality sign does not occur simultaneously in (9.3) and (9.4), i.e., $m_t^o + F_t^o \,(P_T^* + b) > 0$. This implies that the point of rotation never will coincide with origin.

We summarize our results about the situation of the budget line and how it is affected by an increase in P_t in the following points:

(i) The budget line becomes less steep when P_t increases.

(ii) All the lines which are generated when P_t varies pass through the point $m_T = m_t^o$, $F_T = F_t^o (P_T^* + b) = m_t^o + (P_T^* + b) Z_t^o$.

(iii) The point of rotation for the budget line is located

(iiia) on the F_T - axis in the special case $m_t^o = 0$,

(iiib) on the m_T - axis in the special case $F_t^o (P_T^* + b) = 0$.

(iv) The point of rotation for the budget line is located in the interior of the first quadrant when $m_t^o > 0$ and $F_t^o (P_T^* + b) > 0$ and below, on or above the 45^o - line according as Z_t^o is negative, zero or positive.

The graph of the budget equation for two alternative values of P_t is given in Figure (9.5) - (9.9). In all these diagrams the line BB represents the budget line at an "initial" value of P_t and the line B'B' represents the budget line at a higher value of P_t . The diagrams are otherwise based on the alternative assumptions about m_t^o , $F_t^o (P_T^* + b)$ and Z_t^o mentioned above.

In accordance with Don Patinkin, 1/ , we will refer to the

1/ Money, Interest and Prices.

difference between actual demand for and initial holdings of a commodity as excess demand. In the rotation point we have $Z_T = Z_t^o$ and thereby the excess demand for Z-assets equal to zero. The points situated on the budget line to the left of the rotation point are characterized by $Z_T > Z_t^o$, i.e., a positive excess demand, while points to the right of the rotation point imply $Z_T < Z_t^o$, i.e., a negative excess demand for Z-assets.1/ Points situated on the budget line between the rotation point and the point of intersection between the budget line and the 45° - line involves that the decision maker

1) is a borrower when $Z_t^o < 0$. His debt, however, is smaller than in the initial situation, i.e.,

$$(9.10) \qquad Z_t^o < Z_T < 0 ,$$

2) is a lender when $Z_t^o > 0$. His claim, however, is smaller than in the initial situation, i.e.,

$$(9.11) \qquad 0 < Z_T < Z_t^o .$$

The location of the rotation point for the budget line indicates -- as when dealing with changes in P_T^*, cf. Figure (3.3) and the

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1/ We might as well have considered the excess demand for money which has the opposite sign of that for Z-assets.

Figure (9.5) $m_t^o = 0 , F_t^o (P_T^* + b) > 0$

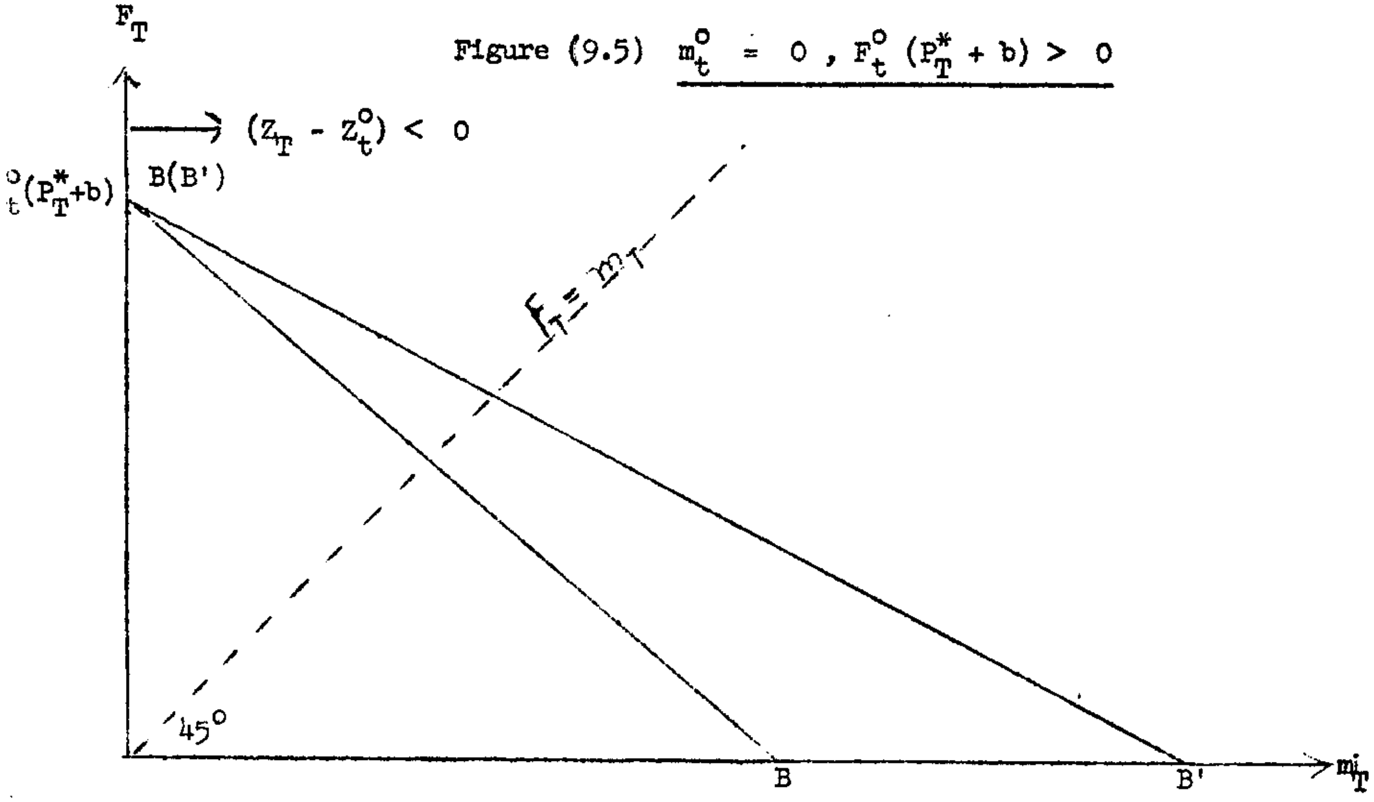


Figure (9.6) $m_t^o > 0, F_t^o (P_T^* + b) = 0$

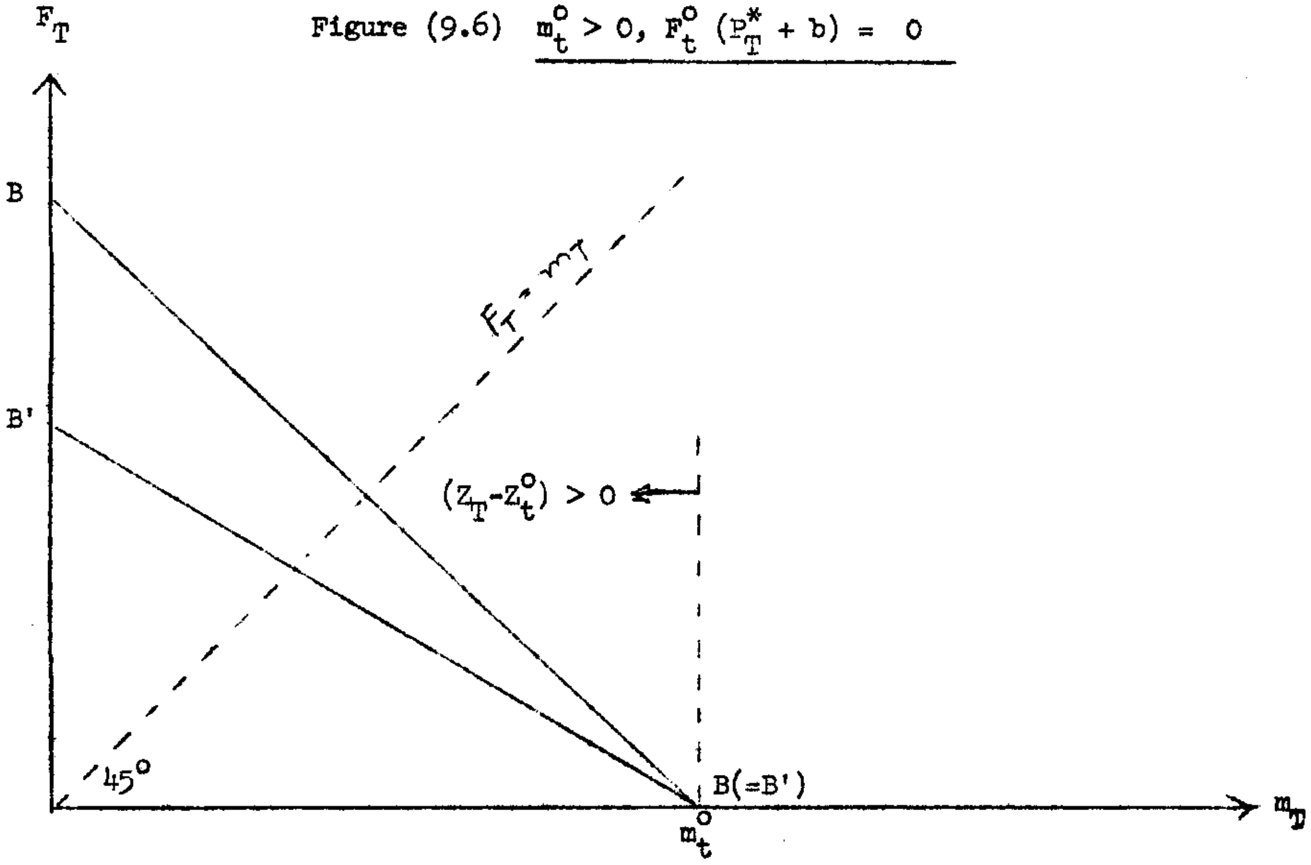


Figure (9.7). $m_t^o > 0, F_t^o(P_T^* + b) > 0, Z_t^o < 0$

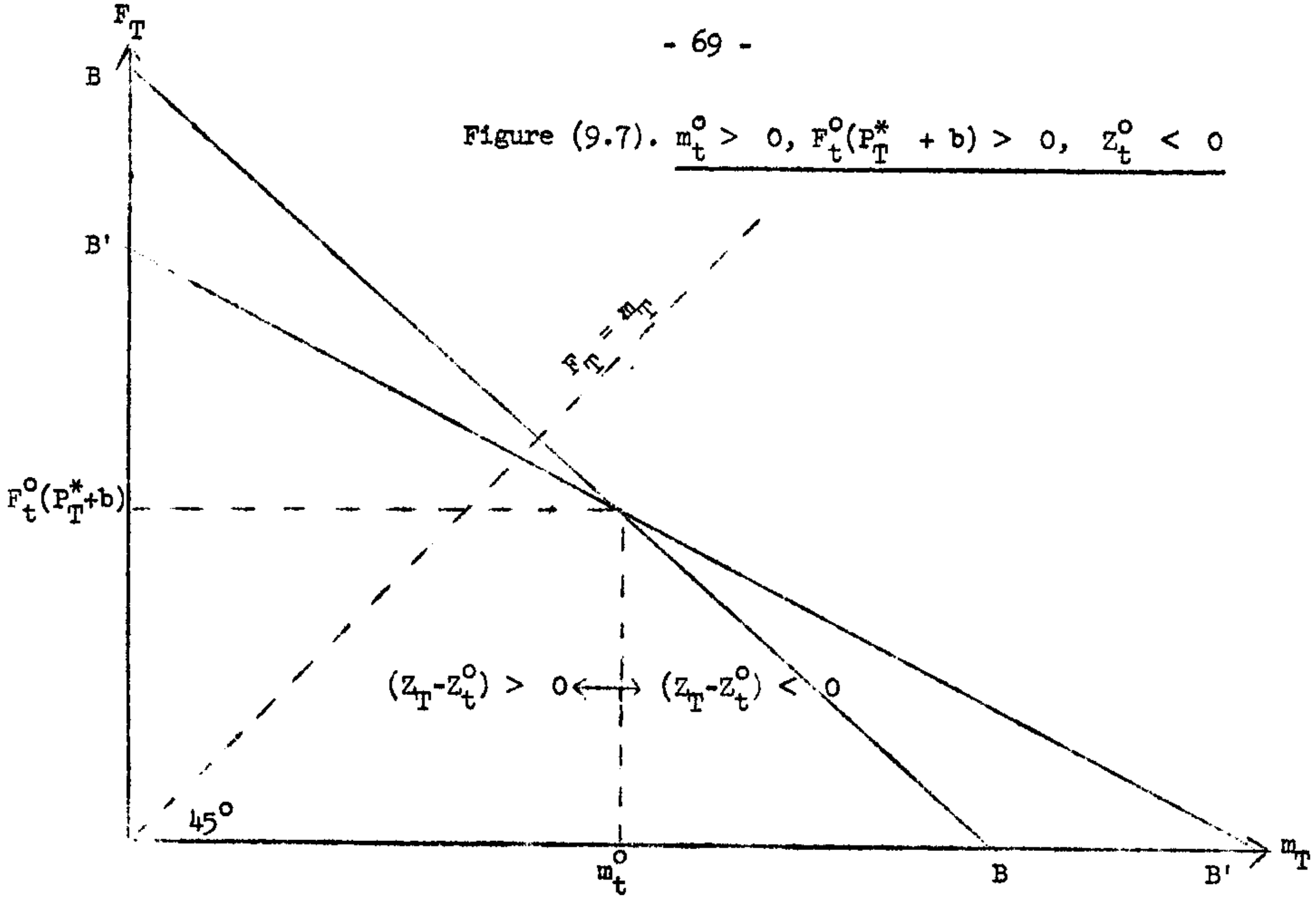


Figure (9.8). $m_t^o > 0 , Z_t^o = 0$

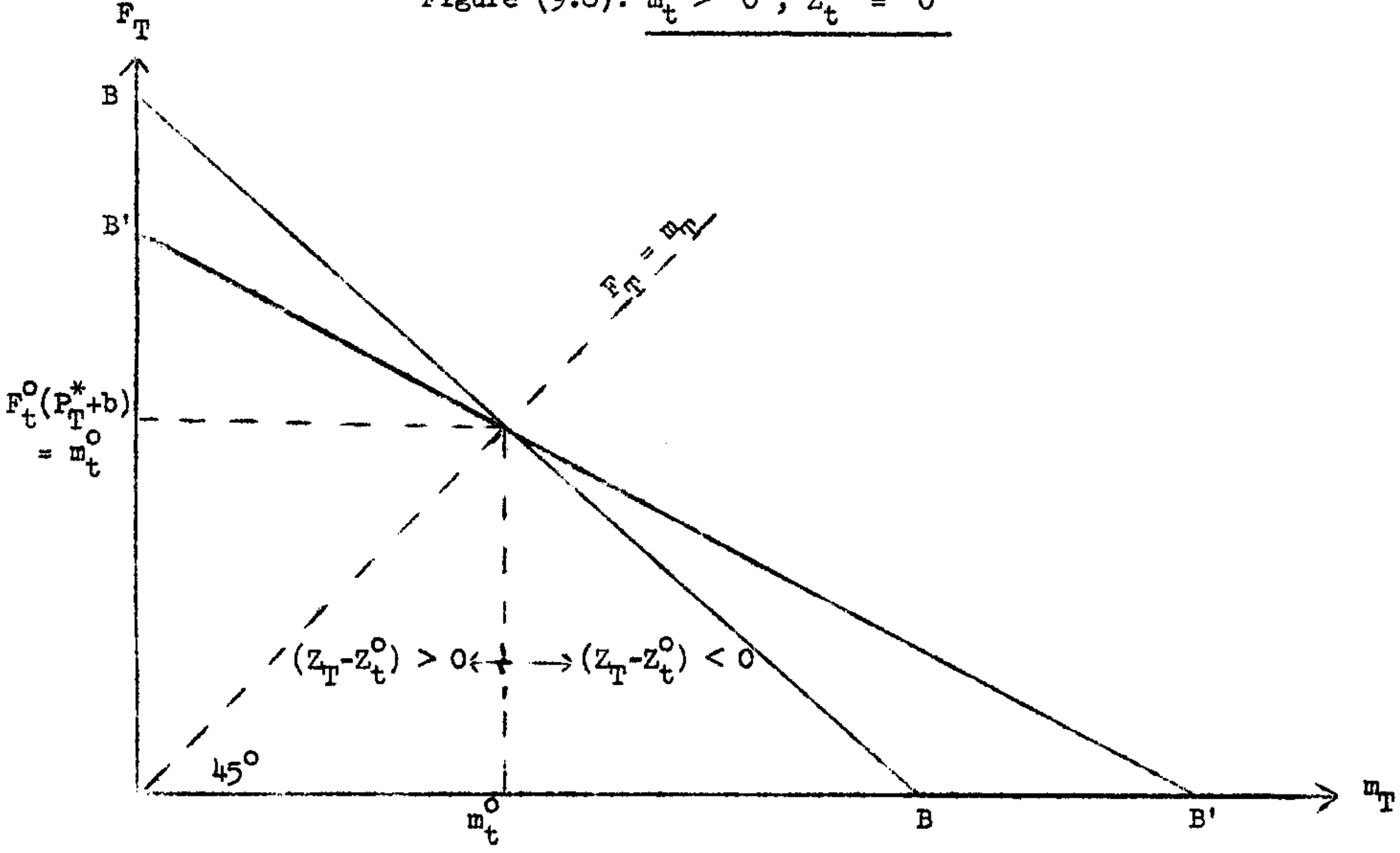
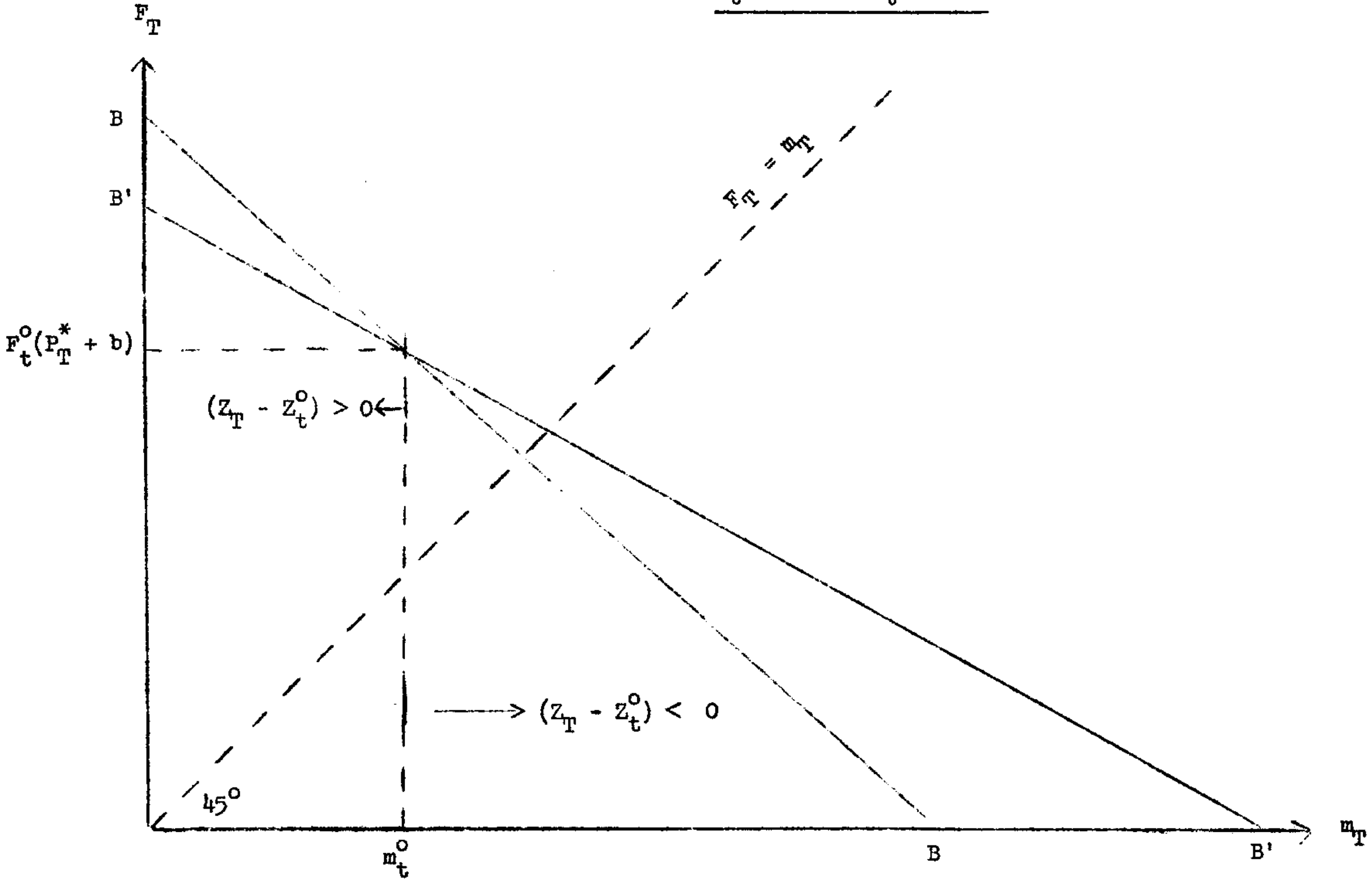


Figure (9.9) $m_t^o > 0$, $Z_t^o > 0$



corresponding remarks -- that the effect of changes in P_t on demand for wealth, cash and Z-assets are not clear, even if we only are concerned with the signs of the changes. For further investigation into this question we will derive $\frac{\partial F_T}{\partial P_t}$, $\frac{\partial m_T}{\partial P_t}$ and $\frac{\partial Z_T}{\partial P_t}$. We first assume that $\frac{\partial^2 U}{\partial F_T \partial m_T} = 0$.

We rewrite (9.1) and (8.1) as respectively (9.12) and (9.13).

$$(9.12) \qquad F_T = -\left(\frac{P_T^* + b}{P_t} - 1\right) m_T + \left(Z_t^o + \frac{m_t^o}{P_t}\right)\left(P_T^* + b\right)$$

$$(9.13) \qquad \frac{\partial U}{\partial m_T} = \frac{\partial U}{\partial F_T}\left(\frac{P_T^* + b}{P_t} - 1\right) .$$

Derivation of (9.12) with respect to P_t yields:

$$\frac{\partial F_T}{\partial P_t} = -\left(\frac{P_T^* + b}{P_t} - 1\right)\frac{\partial m_T}{\partial P_t} + \frac{P_T^* + b}{P_t^2} m_T - \frac{P_T^* + b}{P_t^2} m_t^o .$$

By introducing $\frac{m_t^o - m_T}{P_t} = Z_T - Z_t^o$, cf. (1.4), and utilizing (8.2), we obtain:

$$(9.14) \qquad \frac{\partial F_T}{\partial P_t} = - A_2 \frac{\partial m_T}{\partial P_t} - \frac{P_T^* + b}{P_t} (Z_T - Z_t^o) .$$

By differentiating (9.13) with respect to P_t we get:

$$(9.15) \qquad \frac{\partial^2 U}{\partial m_T^2} \frac{\partial m_T}{\partial P_t} = - \frac{\partial U}{\partial F_T} \frac{P_T^* + b}{P_t^2} + A_2 \frac{\partial^2 U}{\partial F_T^2} \frac{\partial F_T}{\partial P_t} .$$

By multiplying each side of (9.14) with $\frac{\partial^2 U}{\partial m_T^2}$ we obtain:

$$(9.16) \quad \frac{\partial^2 U}{\partial m_T^2} \frac{\partial F_T}{\partial P_t} = - A_2 \frac{\partial^2 U}{\partial m_T^2} \frac{\partial m_T}{\partial P_t} - \frac{P_T^* + b}{P_t} \frac{\partial^2 U}{\partial m_T^2} (Z_T - Z_t^o) \text{ .}$$

By further inserting (9.15) into (9.16), rearranging the terms (under which utilizing (9.13)) and finally solving for $\frac{\partial F_T}{\partial P_t}$, we obtain:

$$(9.17) \quad \frac{\partial F_T}{\partial P_t} = \frac{- \frac{P_T^* + b}{P_t} \left[\frac{1}{P_t} \frac{\partial U}{\partial m_T} + \left(- \frac{\partial^2 U}{\partial m_T^2} \right) \left(Z_T - Z_t^o \right) \right]}{- \left[A_2^2 \frac{\partial^2 U}{\partial F_T^2} + \frac{\partial^2 U}{\partial m_T^2} \right]} \text{ .}$$

By inserting (9.17) for $\frac{\partial F_T}{\partial P_t}$ in (9.14), rearranging the terms and solving with respect to $\frac{\partial m_T}{\partial P_t}$ we obtain:

$$(9.18) \quad \frac{\partial m_T}{\partial P_t} = \frac{\frac{P_T^* + b}{P_t} \left[\frac{1}{P_t} \frac{\partial U}{\partial F_T} + A_2 \frac{\partial^2 U}{\partial F_T^2} (Z_T - Z_t^o) \right]}{- \left[A_2^2 \frac{\partial^2 U}{\partial F_T^2} + \frac{\partial^2 U}{\partial m_T^2} \right]} \text{ .}$$

By differentiating

$$(9.19) \quad Z_T = Z_t^o + \frac{m_t^o - m_T}{P_t}$$

with respect to P_t we get:

$$(9.20) \quad \frac{\partial Z_T}{\partial P_t} = \frac{P_t\left(-\frac{\partial m_T}{\partial P_t}\right) - \left(m_t^o - m_T\right)}{P_t^2},$$

or, by utilizing (9.19):

$$(9.21) \quad \frac{\partial Z_T}{\partial P_t} = -\frac{1}{P_t}\left[\frac{\partial m_T}{\partial P_t} + (Z_T - Z_t^o)\right].$$

By inserting (9.18) for $\frac{\partial m_T}{\partial P_t}$ in (9.21) and rearranging the terms, we obtain:

$$(9.22) \quad \frac{\partial Z_T}{\partial P_t} = \frac{-\frac{P_T^* + b}{P_t^2}\frac{\partial U}{\partial F_T} - \left(Z_T - Z_t^o\right)\left[A_2\frac{\partial^2 U}{\partial F_T^2} + \left(-\frac{\partial^2 U}{\partial m_T^2}\right)\right]}{-P_t\left[A_2^2\frac{\partial^2 U}{\partial F_T^2} + \frac{\partial^2 U}{\partial m_T^2}\right]}.$$

By assuming that the decision maker has no initial holdings of Z-assets, $Z_t^o = 0$, and comparing the expressions (9.17) - (9.18) with (3.10) - (3.11), we notice a "high degree of similarity" between $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial F_T}{\partial P_t}$ and between $\frac{\partial m_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_t}$. Putting $Z_t^o = 0$, we may consider $\frac{\partial F_T}{\partial P_t}$, (9.17), as being obtained from $\frac{\partial F_T}{\partial P_T^*}$, (3.10), by multiplying the latter with minus $\frac{P_T^* + b}{P_t}$ and replace A_1 with A_2 . By applying the same calculation to the expression (3.11) for $\frac{\partial m_T}{\partial P_T^*}$ we obtain the expression (9.18) for $\frac{\partial m_T}{\partial P_t}$. We may say that this similarity also is indicated by our study of the budget line. When $Z_t^o = 0$, the rotation point is situated on the 45° - line, cf. Figure (9.8), which is always the case under variation in P_T^* , cf. Figure (3.3).

If $Z_t^o \neq 0$, we may still apply the formula above for the transformation of the expressions for $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$ to the expressions for $\frac{\partial F_T}{\partial P_t}$ and $\frac{\partial m_T}{\partial P_t}$ if we replace Z_T in the former with $\left(Z_T - Z_t^o\right)$.

As regards the different possible signs of $\frac{\partial F_T}{\partial P_t}$ and $\frac{\partial m_T}{\partial P_t}$ we may utilize the results derived from the discussion of $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$, cf. pp. 31-35. The only change we have to do is to replace the assumptions about the sign of the demand for Z-assets, Z_T , with assumptions about the sign of the excess demand, $Z_T - Z_t^o$, and finally reverse the sign of the conclusions arrived at.[1] We will however, write out the comments on $\frac{\partial F_T}{\partial P_t}$ and $\frac{\partial m_T}{\partial P_t}$ explicitly.

The denominators of the expressions (9.17) - (9.18) and (9.22) are all positive.

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[1] In a specific sense Z_T in (3.10) - (3.11) and $(Z_T - Z_t^o)$ in (9.17) - (9.18) measure the same: the deviation between the actual demand for Z-assets (= Z_T) and the demand for Z-assets which corresponds to the rotation point. Under partial variation in P_T^* the rotation point implies an amount of cash equal to the initial wealth, i.e., an amount of Z-assets equal to zero. Under variation in P_t the rotation point is characterized by an amount of Z-assets equal to the initial amount, Z_t^o .

Comments on $\frac{\partial F_T}{\partial P_t}$.

Case α_1: $(Z_T - Z_t^o) \geqq 0$,[1] i.e., the actual demand for Z-assets is not less than the initial holdings of Z-assets, or -- in other words -- the excess demand is not negative. This will always be the case when $F_t^o(P_T^* + b) = 0$, cf. Figure (9.6). In this case the total wealth will always decrease when the present price of Z-assets is increasing. This conclusion holds true either the decision maker initially is a borrower ($Z_t^o < 0$) or a lender ($Z_t^o > 0$) and either he is decreasing or increasing his demand for Z-assets when their present price is increasing.

This form of the demand function for wealth (which implies that total wealth is increasing at an increase in the effective rate of interest) will in the following be referred to as the "normal."

Case α_2: $\left(Z_T - Z_t^o\right) < 0$,[2] i.e., the actual demand for Z-assets is less than

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[1] As regards the alternative combinations of signs for Z_t^o and $(Z_T - Z_t^o)$, cf. the sign matrix (1.5).

[2] This case implies that the decision maker chooses a point to the right of the rotation point, where $m_T = m_t^o$. This further implies that the amount of money corresponding to the saturation point, m_T^{max} , has to be larger than m_t^o . If the decision maker shall become a borrower, i.e., choose a point to the right of the 45° - line, it is necessary (but not sufficient) that $F_t^o < m_T^{max}$. F_t^o is a function of P_t, and we have:

$$F_t^{o(max)} = F_t^o\ (0) = m_t^o \quad \text{(for a borrower, i.e., } Z_t^o < 0)$$

$$F_t^{o(max)} = F_t^o\ (P_T^* + b) = m_t^o + (P_T^* + b)\ Z_t^o \quad \text{(for a lender, i.e., } Z_t^o > 0) .$$

the initial holdings of Z-assets, or -- in other words -- the excess demand is negative. In this case the bracket parenthesis of the numerator of (9.17) is the sum of a positive and a negative term. If the numerical value of the excess demand is "small," we still get the normal result. In order to arrive at the abnormal result, the excess demand must be "large negative," more precisely:

$$(9.23) \quad \frac{\partial F_T}{\partial P_t} \geqq 0 \ \text{ if } \ (Z_T - Z_t^o) \leqq - \frac{\frac{1}{P_t}\frac{\partial U}{\partial m_T}}{\left(-\frac{\partial^2 U}{\partial m_T^2}\right)}$$

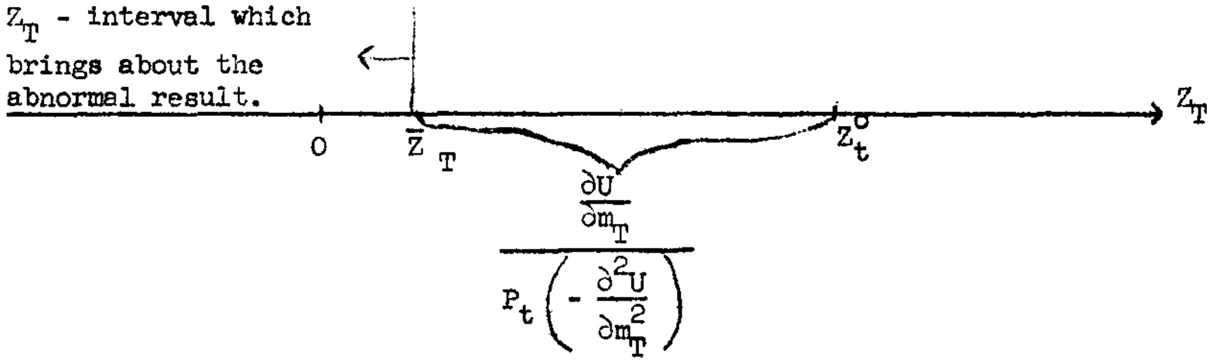
i.e. (when expressed as a bound on Z_T):

$$(9.24) \quad Z_T \leqq Z_t^o - \frac{\frac{1}{P_t}\ \frac{\partial U}{\partial m_T}}{\left(-\frac{\partial^2 U}{\partial m_T^2}\right)} .$$

In Figure (9.25) is illustrated the mutual location of Z_t^o and $\bar{Z}_T$, the right hand side of (9.24), and the Z_T - range which brings about the abnormal result. In Figure (9.25) we are assuming that

$$Z_t^o > \frac{\frac{\partial U}{\partial m_T}}{P_t\left(-\frac{\partial^2 U}{\partial m_T^2}\right)} .$$

Figure (9.25)



Case α_3: $\frac{\partial^2 U}{\partial m_T^2} = 0$. In the special case of a utility function with constant marginal utility of money, the decision maker will always lose on an increase in P_t , whatever the sign of $(Z_T - Z_t^o)$ may be.

Comments on $\frac{\partial m_T}{\partial P_t}$.

Case β_1: $(Z_T - Z_t^o) \leqq 0$, i.e., the excess demand for Z-assets is not positive. This will always be the case when $m_t^o = 0$. In this case we get $\frac{\partial m_T}{\partial P_t} > 0$, i.e., the demand for money will increase when the present price of Z-assets increases. Since the effective interest rate of Z-assets is reduced when their present price is increased, cf. (7.6), the above result implies that the demand for money is reduced when the effective rate of interest is increased. This result -- which coincides with the usual assumptions, cf. for instance the liquidity preference theory of Keynes -- will be referred to as the normal.

Case β_2: $(Z_T - Z_t^o) > 0$, i.e., the excess demand for Z-assets is positive. The assumption of a non-positive excess demand, case β_1 , is a sufficient, but not necessary condition for obtaining the normal result. Even when the excess demand is positive, we may have $\frac{\partial m_T}{\partial P_t} > 0$. In order to arrive at the opposite result, the excess demand has to be "large," more precisely:

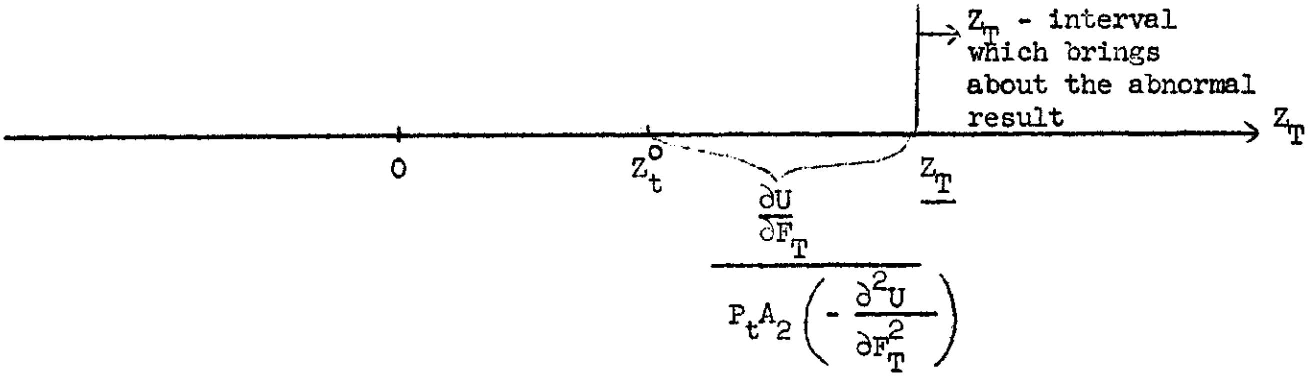
$$(9.26) \qquad \frac{\partial m_T}{\partial P_t} \leqq 0 \quad \text{if} \quad (Z_T - Z_t^o) \geqq \frac{\frac{1}{P_t}\,\frac{\partial U}{\partial F_T}}{A_2\left(-\frac{\partial^2 U}{\partial F_T^2}\right)}$$

i.e. (when expressed as a bound on Z_T):

$$(9.27) \qquad Z_T \geqq Z_t^o + \frac{\frac{1}{P_t}\frac{\partial U}{\partial F_T}}{A_2\left(-\frac{\partial^2 U}{\partial F_T^2}\right)} \; .$$

In Figure (9.28) is illustrated the mutual location of Z_t^o and $\underline{Z_T}$ (the right hand side of (9.27)) and the Z_T - range which brings about the abnormal result, Z_t^o is assumed positive.

Figure (9.28).



<u>Case β_3 : $\dfrac{\partial^2 U}{\partial F_T^2} = 0$.</u> In the special case when the marginal utility of wealth is constant, the decision maker will always increase his demand for cash by an increase in P_t , i.e., the normal result will always come out whatever the sign of the excess demand may be.

Comments on $\frac{\partial Z_T}{\partial P_t}$.

As far as (the effect of) an increase in P^*_T is concerned, the effect on Z_T will always be opposite of the effect on m_T, $\frac{\partial Z_T}{\partial P^*_T} = -\frac{1}{P_t}\frac{\partial m_T}{\partial P^*_T}$. From (9.21) is seen that this does not necessarily apply to the effects of an increase in P_t. This implies that even if we obtain the normal result as regards the demand for cash, we may obtain the abnormal result when it comes to the demand for Z-assets, $\frac{\partial Z_T}{\partial P_t} \geqq 0$, and vice versa: the abnormal result as regards demand for cash may be combined with the normal result for the demand for Z-assets.

The actual combinations of signs of $\frac{\partial m_T}{\partial P_t}$ and $\frac{\partial Z_T}{\partial P_t}$ depend upon the sign of the excess demand, $Z_T - Z^o_t$. The "relation" between the signs of $(Z_T - Z^o_t)$, $\frac{\partial m_T}{\partial P_t}$ and $\frac{\partial Z_T}{\partial P_t}$ is revealed in Table (9.29).

TABLE (9.29)

The connections between signs of $\left(Z_T - Z_t^o \right)$, $\frac{\partial m_T}{\partial P_t}$ and $\frac{\partial Z_T}{\partial P_t}$.

		$Z_T - Z_t^o$		
		− (negative excess demand)	0 (zero excess demand)	+ (positive excess demand)
$\frac{\partial m_T}{\partial P_t}$	− (abnormal)	~~$\frac{\partial Z_T}{\partial P_t} > 0$ (abnormal)~~	~~$\frac{\partial Z_T}{\partial P_t} > 0$ (abnormal)~~	$\frac{\partial Z_T}{\partial P_t} \gtreqless 0$
	0 (abnormal)	~~$\frac{\partial Z_T}{\partial P_t} > 0$ (abnormal)~~	~~$\frac{\partial Z_T}{\partial P_t} = 0$ (abnormal)~~	$\frac{\partial Z_T}{\partial P_t} < 0$ (normal)
	+ (normal)	$\frac{\partial Z_T}{\partial P_t} \gtreqless 0$	$\frac{\partial Z_T}{\partial P_t} < 0$ (normal)	$\frac{\partial Z_T}{\partial P_t} < 0$ (normal)

From Table (9.29) is seen that the combinations above the diagonal between the lower left and the upper right corner yield the abnormal result, $\frac{\partial Z_T}{\partial P_t} > 0$, the combinations below that diagonal lead to the normal result, $\frac{\partial Z_T}{\partial P_t} < 0$, and the combinations along that diagonal may lead to the normal as well as the abnormal result (except for the combination in the center of the diagonal, which is yielding $\frac{\partial Z_T}{\partial P_t} = 0$). Table (9.29) is, however, based only on the definitional relation (9.19). According to case β_1 the abnormal result $\frac{\partial m_T}{\partial P_t} \leqq 0$ will never emerge when $(Z_T - Z_t^o) \leqq 0$. This implies that we have to omit the four combinations which establish the upper left two times two matrix of Table (9.29). This is indicated by drawing the diagonals of the sub-matrix in question. This omission reduces the possibility of obtaining the abnormal result,

$\frac{\partial Z_T}{\partial P_t} \geqq 0$.

We denote the terms within the bracket of the numerator in (9.22) with K :

$$(9.30) \qquad K = A_2 \frac{\partial^2 U}{\partial F_T^2} + \left(- \frac{\partial^2 U}{\partial m_T^2} \right).$$

By imposing the abnormal result and assuming alternatively K negative, equal to zero and positive, we obtain from (9.22):

$$\frac{\partial Z_T}{\partial P_t} \geqq 0 \text{ if}$$

$$(9.31) \qquad \left(Z_T - Z_t^o\right) \gtreqless \frac{\frac{P_T^* + b}{P_t^2} \quad \frac{\partial U}{\partial F_T}}{- K} \qquad \text{when } K < 0$$

$$(9.32) \qquad - \frac{P_T^* + b}{P_t^2} \quad \frac{\partial U}{\partial F_T} \gtreqless 0 \qquad \text{when } K = 0$$

$$(9.33) \qquad (Z_T - Z_t^o) \lesseqgtr \frac{- \frac{P_T^* + b}{P_t^2} \quad \frac{\partial U}{\partial F_T}}{K} \qquad \text{when } K > 0 .$$

In the special case $\underline{K = 0}$, which may emerge for one or "a few" values of P_t, the abnormal result will never be brought about, cf. (9.32). When $\underline{K \neq 0}$, there has to be an excess demand if the abnormal result shall emerge. When $K < 0$, the excess demand has to be positive and when $K > 0$, the excess demand has to be negative. The numerical value of the excess demand has to satisfy a lower bound.

<u>Conclusions from the discussion about obtaining the abnormal results, i.e., $\frac{\partial F_T}{\partial P_t} \gtreqless 0$, $\frac{\partial m_T}{\partial P_t} \lesseqgtr 0$ and $\frac{\partial Z_T}{\partial P_t} \gtreqless 0$.</u>

When imposing the abnormal results there emerge boundary restraints on $\left(Z_T - Z_t^o\right)$, implying that the excess demand for Z-assets has to be different from zero(i.e., if the actual demand for Z-assets for instance equals the initial holdings,/neither of the abnormal results will be brought about).

In order to obtain $\frac{\partial F_T}{\partial P_t} \geqq 0$, the excess demand for Z-assets has to be lower than a negative item, cf. (9.23), while $\frac{\partial m_T}{\partial P_t} \leqq 0$ requires that the excess demand for Z-assets has to be above a positive item, cf. (9.26). These results imply that the combination $\frac{\partial F_T}{\partial P_t} \geqq 0$ and $\frac{\partial m_T}{\partial P_t} \leqq 0$ will never be brought about when $\frac{\partial^2 U}{\partial F_T \, \partial m_T} = 0$, $\frac{\partial^2 U}{\partial F_T^2} < 0$ and $\frac{\partial^2 U}{\partial m_T^2} < 0$.

In order to bring about the abnormal result $\frac{\partial Z_T}{\partial P_t} \geqq 0$, the excess demand for Z-assets has to be above a positive item when K -- defined by (9.30) -- is negative, cf. (9.31), and below a negative item when K is positive, cf. (9.33).

The actual demand for Z-assets, and thereby the actual value of the excess demand, $(Z_T - Z_t^o)$, depends upon the actual price and the constant data: the utility function, the initial amounts of cash and Z-assets, the future price and the interest income. The bounds on Z_T , which emerge when imposing the abnormal results, depend upon the same data. No attempt will be made here at deriving explicitly the class(es) of utility functions which -- for some region of the other data -- will yield Z_T - values satisfying the boundary restraints referred to above.

More general assumptions about the utility indicator

As in Model I, cf. pp. 35-35h, we will now discuss the shape of the demand functions when no special assumption is made about the sign of $\frac{\partial^2 U}{\partial F_T \partial m_T}$. We now obtain for the derivatives of the demand functions:

$$(9.34) \qquad \frac{\partial F_T}{\partial P_t} = \frac{- \frac{P_T^* + b}{P_t} \left[\frac{1}{P_t} U'_m + (Z_T - Z_t^o) N_{2F} \right]}{D_2}$$

$$(9.35) \qquad \frac{\partial m_T}{\partial P_t} = \frac{\frac{P_T^* + b}{P_t} \left[\frac{1}{P_t} U'_F - (Z_T - Z_t^o) N_{2m} \right]}{D_2}$$

$$(9.36) \qquad \frac{\partial Z_T}{\partial P_t} = \frac{- \left[\frac{P_T^* + b}{P_t^2} U'_F + (Z_T - Z_t^o) N_{2Z} \right]}{P_t D_2}$$

where N_{2F}, N_{2m} and D_2 are obtained from (3.17)-(3.19) by replacing A_1 in the latter with A_2,[1] and:

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[1] The rule given on pp. 74-75 for transforming the expressions for $\frac{\partial F_T}{\partial P_T^*}$ and $\frac{\partial m_T}{\partial P_T^*}$ to the expressions for $\frac{\partial F_T}{\partial P_t}$ and $\frac{\partial m_T}{\partial P_t}$ also holds good under the more general assumptions about the utility function.

(9.37) $$N_{2Z} = N_{2F} - N_{2m} ,$$

or -- by inserting the expressions for N_{2F} and N_{2m}:

(9.38) $$N_{2Z} = (A_2 - 1)\, U''_{Fm} + A_2\, U''_{FF} - U''_{mm} .$$

The signs of D_2 , N_{2F} , N_{2m} and N_{2Z} and the algebraic values of $\frac{\partial F_T}{\partial P_t}$, $\frac{\partial m_T}{\partial P_t}$ and $\frac{\partial Z_T}{\partial P_t}$ are invariant with respect to any arbitrary increasing transformation of the utility indicator. We assume that D_2 is strictly positive, i.e., that the second order condition for a regular maximum is fulfilled.

We will now discuss the signs of (9.34)-(9.36) under alternative assumptions about the signs of N_{2F} , N_{2m} and N_{2Z} . In Table (9.39) we have given the feasible sign combinations for N_{2F} and N_{2m} -- identical, of course, to those revealed in Table (3.29) -- and the resulting sign for N_{2Z} . Also N_{2Z} may be negative as well as zero and positive. When N_{2F} and N_{2m} both are positive, i.e., the lower right square of Table (9.39) the sign of N_{2Z} is not clear.

Table (9.39)

Sign Matrix for N_{2F} , N_{2m} and N_{2Z}

		N_{2m}		
		-	0	+
N_{2F}	-	╳	╳	O.K. if $N_{2F} > - A_2 N_{2m}$ $N_{2Z} < 0$
	0	╳	╳	O.K. $N_{2Z} < 0$
	+	O.K. if $N_{2F} > - A_2 N_{2m}$ $N_{2Z} > 0$	O.K. $N_{2Z} > 0$	O.K. $N_{2Z} \gtrless 0$

When the second item within the bracket parenthesis in the numerators of (9.34)-(9.36), i.e., the product of $(Z_T - Z_t^o)$ and respectively N_{2F} , N_{2m} and N_{2Z} , equals zero, we always obtain what we have termed the normal result: $\frac{\partial F_T}{\partial P_t} < 0$, $\frac{\partial m_T}{\partial P_t} > 0$ and $\frac{\partial Z_T}{\partial P_t} < 0$. This implies that a necessary condition for obtaining the abnormal result is that the excess demand for Z-assets is different from zero, i.e., $Z_T \gtrless Z_t^o$.

In the following we assume that the products referred to above are different from zero.

Comments on $\frac{\partial F_T}{\partial P_t}$.

(i) $(Z_T - Z_t^o) \; N_{2F} > 0$ always yields the normal result, $\frac{\partial F_T}{\partial P_t} < 0$.

(ii) $(Z_T - Z_t^o) \; N_{2F} < 0$. The numerator of (9.34) now is a sum of a negative and a positive term. By imposing the abnormal result we obtain:

$$\frac{\partial F_T}{\partial P_t} \geqq 0 \quad \text{if}$$

$$(9.41) \qquad Z_T \leqq Z_t^o - \frac{\frac{1}{P_t} \; U_m'}{N_{2F}} \qquad \text{when} \quad N_{2F} > 0$$

or if

$$(9.42) \qquad Z_T \geqq Z_t^o + \frac{\frac{1}{P_t} \; U_m'}{- N_{2F}} \qquad \text{when} \quad N_{2F} < 0 .$$

Comments on $\frac{\partial m_T}{\partial P_t}$.

(i) $(Z_T - Z_t^o) \; N_{2m} < 0$ always yields the normal result, $\frac{\partial m_T}{\partial P_t} > 0$.

(ii) $(Z_T - Z_t^o) \; N_{2m} > 0$. The numerator of (9.35) now is a sum of a positive and a negative term. By imposing the abnormal result, we obtain:

$$\frac{\partial m_T}{\partial P_t} \lessgtr 0 \quad \text{if}$$

$$(9.43) \qquad Z_T \geqq Z_t^o + \frac{\frac{1}{P_t} \quad U_F'}{N_{2m}} \qquad \text{when} \quad N_{2m} > 0$$

or if

$$(9.44) \qquad Z_T \leqq Z_t^o - \frac{\frac{1}{P_t} \quad U_F''}{- N_{2m}} \qquad \text{when} \quad N_{2m} < 0 \,.$$

Comments on $\frac{\partial Z_T}{\partial P_t}$.

(i) $(Z_T - Z_t^o) \; N_{2Z} > 0$ always yields the normal result, $\frac{\partial Z_T}{\partial P_t} < 0$.

(ii) $(Z_T - Z_t^o) \; N_{2Z} < 0$. The numerator of (9.36) now is a sum of a negative and a positive term. By imposing the abnormal result we obtain:

$$\frac{\partial Z_T}{\partial P_t} \geqq 0 \quad \text{if}$$

$$(9.45) \qquad Z_T \leqq Z_t^o - \frac{\frac{P_T^* + b}{P_t^2} \, U_F'}{N_{2Z}} \qquad \text{when} \quad N_{2Z} > 0$$

or if

$$(9.46) \qquad Z_T \geqq Z_t^o + \frac{\frac{P_T^* + b}{P_t^2} \, U_F'}{- N_{2Z}} \qquad \text{when} \quad N_{2Z} < 0 \,.$$

The abnormal result may occur for all three demand functions simultaneously. From (9.14), which is derived from the budget equation, it is easily seen:

1. The conditions $\frac{\partial m_T}{\partial P_t} \leqq 0$ and $(Z_T - Z_t^o) < 0$ -- which always are fulfilled in the case (9.44) -- are sufficient to yield $\frac{\partial F_T}{\partial P_t} > 0$.
2. The conditions $\frac{\partial F_T}{\partial P_t} \geqq 0$ and $(Z_T - Z_t^o) > 0$ -- which always are fulfilled in the case (9.42) -- are sufficient to yield $\frac{\partial m_T}{\partial P_t} < 0$.

The budget equation may be written as:

$$F_T = m_T + (P_T^* + b)\ Z_T \text{ , i.e.:}$$

$$\frac{\partial F_T}{\partial P_t} = \frac{\partial m_T}{\partial P_t} + (P_T^* + b)\ \frac{\partial Z_T}{\partial P_t} \quad .$$

From the last equation is seen that the simultaneous occurrence of the abnormal result for the demand for wealth and cash, $\frac{\partial F_T}{\partial P_t} \geqq 0$ and $\frac{\partial m_T}{\partial P_t} \leqq 0$, is a sufficient condition for obtaining the abnormal result as regards the demand for Z-assets, $\frac{\partial Z_T}{\partial P_t} \geqq 0$.

We summarize:

The abnormal result will occur for all three demand functions simultaneously if

$$(9.46a) \qquad Z_T \leqq Z_t^o - \frac{\frac{1}{P_t}\ U_F'}{-\ N_{2m}} \qquad \left(\begin{array}{l}\text{when } N_{2m} < 0 \text{ ,} \\ \text{i.e.,} N_{2F} > 0 \quad \text{and} \quad N_{2Z} > 0\end{array}\right)$$

or if

$$(9.46b) \qquad Z_T \geqq Z_t^o + \frac{\frac{1}{P_t}\ U_m'}{-\ N_{2F}} \qquad \left(\begin{array}{l}\text{when } N_{2F} < 0 \text{ ,} \\ \text{i.e.,} N_{2m} > 0 \quad \text{and} \quad N_{2Z} < 0\end{array}\right).$$

The effect on U

Considering F_T and m_T as functions of P_t, we get by differentiating (1.1) with respect to P_t:

$$(9.47) \qquad \frac{\partial U}{\partial P_t} = \frac{\partial U}{\partial m_T} \frac{\partial m_T}{\partial P_t} + \frac{\partial U}{\partial F_T} \frac{\partial F_T}{\partial P_t} .$$

By utilizing (9.13) - (9.14) and (8.2), (9.47) is reduced to: [1]

$$(9.48) \qquad \frac{\partial U}{\partial P_t} = -(Z_T - Z_t^o) \frac{P_T^* + b}{P_t} \frac{\partial U}{\partial F_T} \text{, i.e.}$$

$$(9.49) \qquad \frac{\partial U}{\partial P_t} \gtreqless 0 \text{ according as } (Z_T - Z_t^o) \lesseqgtr 0 .$$

When the price of Z-assets increases, the total utility will increase, be unchanged or decrease, according as the decision maker has a negative, zero or positive excess demand.

We may consider the effect (9.48) as a kind of an income effect. Instead of using the concept of excess demand we will now rather use the concept of net sale of Z-assets,[2] defined as $(Z_t^o - Z_T)$.

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[2] Or -- if wanted -- excess supply.

We may then say that if the decision maker has chosen a point on his

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[1] The sign of $\frac{\partial U}{\partial P_t}$ is unchanged by an arbitrary increasing transformation of the utility indicator and is further independent of the signs of N_{2F}, N_{2m} and N_{2Z}.

budget line which implies that he actually is selling Z-assets (i.e., reducing his initial amount of Z-assets), $(Z_t^o - Z_T)$ is positive and he will be better off when P_t increases.

(9.48) may more detailed be interpreted as follows:

When P_t is increased by one dollar, the income of the net sale of Z-assets is increased by $1 \cdot (Z_t^o - Z_T)$ which -- assuming m_T constant -- yields an additional number of Z-assets equal to $\frac{Z_t^o - Z_T}{P_t}$. This additional number of Z-assets yields further an increase in F_T equal to $\frac{Z_t^o - Z_T}{P_t} (P_T^* + b)$ and by multiplying this item with the marginal utility of wealth we obtain the increase in U per dollar increase in P_t. As when dealing with the effect of an increase in P_T^*, cf. (3.34), the substitution effect is left out.

The value of P_t at which the decision maker demands an amount of Z-assets equal to his initial holdings, $Z_T = Z_t^o$, yields the minimum value of U. Since the point $Z_T = Z_t^o$, i.e., $m_T = m_t^o$, is the point of rotation for the budget line, the reasoning at the end of Chapter 3 may be applied also to the case now studied.

In Tables (9.50)-(9.51) we have summarized the effects of an increase in P_t on the demand for wealth, cash and Z-assets and on utility under alternative assumptions about the signs of $(Z_T - Z_t^o)$, N_{2F}, N_{2m} and N_{2Z}. In these tables are given the boundary restraints on $(Z_T - Z_t^o)$, which are obtained by moving Z_t^o in (9.41)-(9.46) over to the left hand side.

Table (9.50)

The effect of an increase in P_t on the demand for total wealth and cash.

		$N_{2F} = A_2 U''_{Fm} - U''_{mm}$			$N_{2m} = U''_{Fm} - A_2 U''_{FF}$		
		-	0	+	-	0	+
$Z_T - Z_t^o$	-	$\frac{\partial F_T}{\partial P_t} < 0$	$\frac{\partial F_T}{\partial P_t} < 0$	$\frac{\partial F_T}{\partial P_t} \gtreqless 0$ according as $(Z_T - Z_t^o) \lesseqgtr -\frac{U'_m}{P_t N_{2F}}$	$\frac{\partial m_T}{\partial P_t} \lesseqgtr 0$ according as $(Z_T - Z_t^o) \lesseqgtr \frac{U'_F}{P_t N_{2m}}$	$\frac{\partial m_T}{\partial P_t} > 0$	$\frac{\partial m_T}{\partial P_t} > 0$
	0	$\frac{\partial F_T}{\partial P_t} < 0$	$\frac{\partial F_T}{\partial P_t} < 0$	$\frac{\partial F_T}{\partial P_t} < 0$	$\frac{\partial m_T}{\partial P_t} > 0$	$\frac{\partial m_T}{\partial P_t} > 0$	$\frac{\partial m_T}{\partial P_t} > 0$
	+	$\frac{\partial F_T}{\partial P_t} \lesseqgtr 0$ according as $(Z_T - Z_t^o) \lesseqgtr \frac{-U'_m}{P_t N_{2F}}$	$\frac{\partial F_T}{\partial P_t} < 0$	$\frac{\partial F_T}{\partial P_t} < 0$	$\frac{\partial m_T}{\partial P_t} > 0$	$\frac{\partial m_T}{\partial P_t} > 0$	$\frac{\partial m_T}{\partial P_t} \gtreqless 0$ according as $(Z_T - Z_t^o) \lesseqgtr \frac{U'_F}{P_t N_{2m}}$

Table (9.51)

The effect of an increase in P_t on the demand for Z-assets and utility

		$N_{2Z} = (A_2 - 1)\, U''_{Fm} + A_2\, U''_{FF} - U''_{mm}$			Effect on
		-	0	+	U
$Z_T - Z_t^o$	-	$\frac{\partial Z_T}{\partial P_t} < 0$	$\frac{\partial Z_T}{\partial P_t} < 0$	$\frac{\partial Z_T}{\partial P_t} \gtreqless 0$ according as $(Z_T - Z_t^o) \lesseqgtr - \frac{(P_T^* + b)U'_F}{P_t^2 N_{2Z}}$	$\frac{\partial U}{\partial P_t} > 0$
	0	$\frac{\partial Z_T}{\partial P_t} < 0$	$\frac{\partial Z_T}{\partial P_t} < 0$	$\frac{\partial Z_T}{\partial P_t} < 0$	$\frac{\partial U}{\partial P_t} = 0$
	+	$\frac{\partial Z_T}{\partial P_t} \lesseqgtr 0$ according as $(Z_T - Z_t^o) \lesseqgtr \frac{(P_T^* + b)U'_F}{- P_t^2 N_{2Z}}$	$\frac{\partial Z_T}{\partial P_t} < 0$	$\frac{\partial Z_T}{\partial P_t} < 0$	$\frac{\partial U}{\partial P_t} < 0$

In Figures (9.52a)-(9.52d) we have given a slightly different survey of our results. In these figures are indicated the signs of $\frac{\partial F_T}{\partial P_t}$ (= F') , $\frac{\partial m_T}{\partial P_t}$ (= m') and $\frac{\partial Z_T}{\partial P_t}$ (= Z') in different $(Z_T - Z_t^o)$ - intervals under alternative assumptions about the signs of N_{2F} , N_{2m} and N_{2Z} , all now assumed different from zero. Z_t^o is supposed to be a given constant, which may be negative as well as zero and positive. Some brief explanatory comments on the limits of the intervals along the $(Z_T - Z_t^o$)-axes are given below.

When N_{2F} , N_{2m} and N_{2Z} all are positive, cf. Figure (9.52a), the values of $(Z_T - Z_t^o)$ representing the interval-limits are obtained from the bounds (9.41), (9.43) and (9.45). The lower bound on $(Z_T - Z_t^o)$ obtained from (9.43) is strictly positive. The upper bounds on $(Z_T - Z_t^o)$ obtained from (9.41) and (9.45) are both negative. In order to compare the algebraic values of these two bounds we first rewrite the second term of the right hand side of (9.45). By utilizing (8.1)-(8.2) we obtain:

$$(9.53) \qquad \frac{(P_T^* + b)\ U_F'}{P_t^2\ \ N_{2Z}} = \frac{U_F'}{\frac{P_t\ N_{2Z}}{1 + A_2}} = \frac{U_m'}{\frac{P_t\ A_2\ N_{2Z}}{1 + A_2}} \ .$$

By forming the difference between N_{2F} and $\frac{A_2}{1 + A_2}$ N_{2Z} and utilizing (9.37) and $D_2 = N_{2F} + A_2\ N_{2m}$, we obtain:

$$(9.54) \qquad N_{2F} - \frac{A_2}{1 + A_2}\ N_{2Z} = \frac{D_2}{1 + A_2} \ ,$$

which is strictly positive. This implies that the (algebraic) value of the upper bound on $(Z_T - Z_t^o)$ obtained from (9.45) is smaller than the (algebraic) value of the upper bound on $(Z_T - Z_t^o)$ obtained from (9.41).[1]

When $N_{2F} > 0$, $N_{2m} > 0$ and $N_{2Z} < 0$, cf. Figure (9.52b), the problem is to compare the lower bound (9.43) to the lower bound (9.46), both being strictly positive when interpreted as bounds on $(Z_T - Z_t^o)$. In this case we have, cf. also the term in the middle of (9.53):

$$(9.55) \qquad N_{2m} - \left(- \frac{N_{2Z}}{1 + A_2} \right) = \frac{D_2}{1 + A_2} \quad ,$$

which implies that the value of the lower bound (9.46) is larger than the value of the lower bound (9.43).[2]

When $N_{2F} > 0$, $N_{2m} < 0$ and $N_{2Z} > 0$, cf. Figure (9.52c), the bound (9.44) is the strongest of the upper bounds in question, cf. also (9.46a). The comparison between the two other upper bounds, (9.41) and (9.45), was made above, cf. (9.54) and Figure (9.52a).

When $N_{2F} < 0$, $N_{2m} > 0$ and $N_{2Z} < 0$, cf. Figure (9.52d), the bound (9.42) is the strongest of the lower bounds in question, cf. also (9.46b).

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[1] This result is more easily obtained in the following way: From (9.37) and our assumptions about the signs of N_{2F} , N_{2m} and N_{2Z} follow that $N_{2F} > N_{2Z}$ and since we have $\frac{A_2}{1 + A_2} < 1$, $P_t \, N_{2F}$ is larger than the denominator of the item to the extreme right in (9.53).

[2] From (9.37) and our assumptions about the signs of N_{2F} , N_{2m} and N_{2Z} follow that $N_{2m} > - N_{2Z}$, which is a sufficient condition to determine which of the two bounds in question is the strongest.

The comparison between the two other lower bounds, (9.43) and (9.46), was made above, cf. (9.55) and Figure (9.52b).

In the Figures (9.52a)-(9.52d) the limits of the intervals along the $(Z_T - Z_t^o)$-axes are denoted by $\overline{F}$, $\overline{m}$ and $\overline{Z}$, defined as:

(9.56) $$\overline{F} = - \frac{U_m'}{P_t N_{2F}}$$

(9.57) $$\overline{m} = \frac{U_F'}{P_t N_{2m}}$$

(9.58) $$\overline{Z} = - \frac{\left(P_T^* + b\right) U_F'}{P_t^2 \; N_{2Z}} \quad .$$

Figures (9.52a)-(9.52d).

The signs of $\frac{\partial F_T}{\partial P_t} \left(= F' \right)$, $\frac{\partial m_T}{\partial P_t} \left(= m' \right)$ and $\frac{\partial Z_T}{\partial P_t} \left(= Z' \right)$ in different $\left(Z_T - Z_t^o \right)$- intervals under alternative assumptions about the signs of N_{2F} , N_{2m} and N_{2Z} .

Figure (9.52a). $N_{2F} > 0$, $N_{2m} > 0$, $N_{2Z} > 0$.

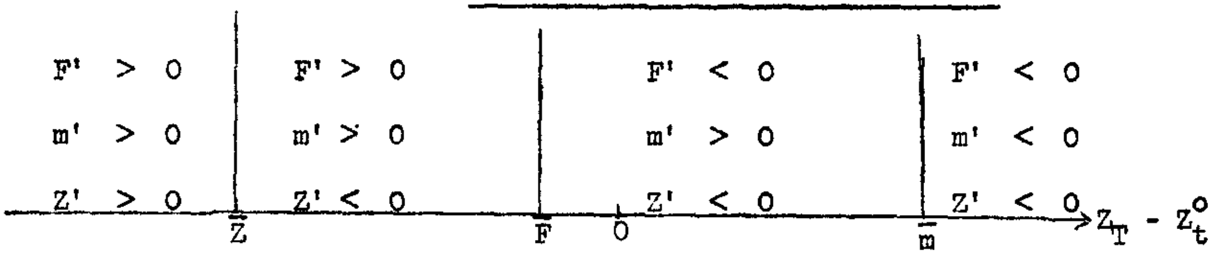


Figure (9.52b). $N_{2F} > 0$, $N_{2m} > 0$, $N_{2Z} < 0$.

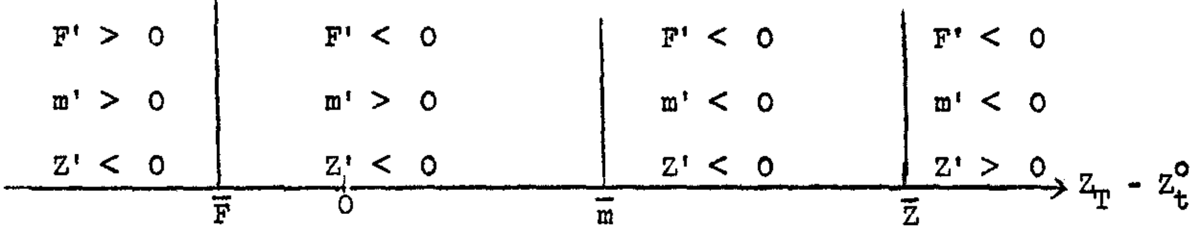


Figure (9.52c). $\underline{N_{2F} > 0\ ,\ N_{2m} < 0\ ,\ N_{2Z} > 0\ .}$

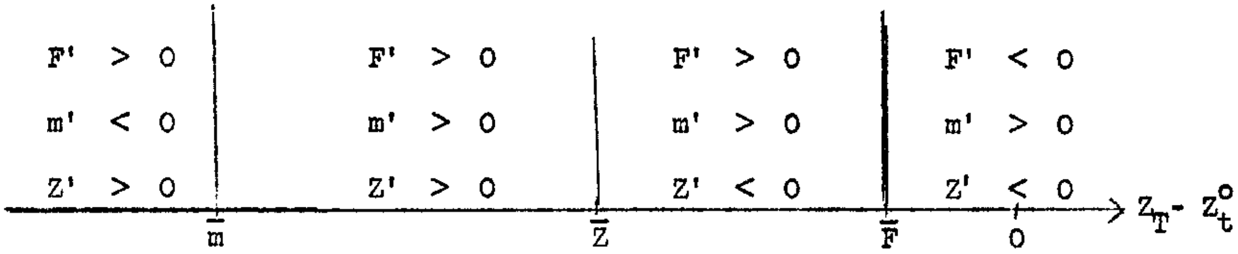
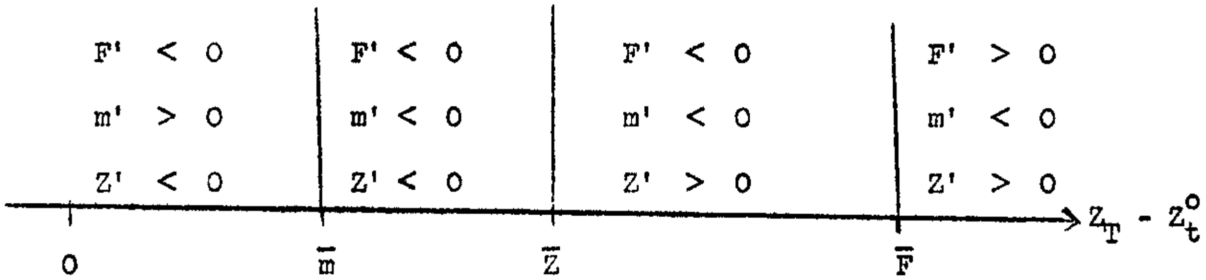


Figure (9.52d). $\underline{N_{2F} < 0\ ,\ N_{2m} > 0\ ,\ N_{2Z} < 0\ .}$



APPENDIX

In Model I we defined the variables D_1 , N_{1F} and N_{1m} , cf. (3.19) and (3.17)-(3.18). The corresponding variables in Model II are obtained by replacing A_1 in (3.17)-(3.19) with A_2 . In either model we have $A_i = \frac{U'_m}{U'_F}$ (i = 1,2), cf. (3.7) and (8.1)-(8.2). We now insert $\frac{U'_m}{U'_F}$ for A_1 in (3.17)-(3.19) and denote the variables obtained with D , N_F and N_m :

$$D = 2\frac{U'_m}{U'_F} U''_{Fm} - \left(\frac{U'_m}{U'_F}\right)^2 U''_{FF} - U''_{mm} \tag{1}$$

$$N_F = \frac{U'_m}{U'_F} U''_{Fm} - U''_{mm} \tag{2}$$

$$N_m = U''_{Fm} - \frac{U'_m}{U'_F} U''_{FF} \quad . \tag{3}$$

In this appendix we will give a graphical interpretation of the signs of (1)-(3).

If we denote the slope of the indifference curve, i.e., the derivative of F_T with respect to m_T , U assumed constant, with $\frac{\partial F_T}{\partial m_T}$, we have:

$$\frac{\partial F_T}{\partial m_T} = - \frac{U'_m}{U'_F} \qquad (U = \text{constant}) . \tag{4}$$

We will now find the change in $\frac{\partial F_T}{\partial m_T}$ with respect to the following variations:

a) Increase in m_T , U assumed constant, i.e., a move along the indifference curve.

b) Increase in m_T , F_T assumed constant, i.e., a move from a point on one indifference curve to a point on another indifference curve, the latter point being located straight east of the former.

c) Increase in F_T , m_T assumed constant, i.e., a move from a point on one indifference curve to a point on another indifference curve, the latter point being located straight north of the former.

The derivatives which express the change in $\frac{\partial F_T}{\partial m_T}$ with respect to the variations defined above will be denoted respectively:

$$\text{a}') \quad \frac{\partial^2 F_T}{\partial m_T^2}\,, \qquad \text{b}') \quad \frac{\partial\left(\frac{\partial F_T}{\partial m_T}\right)}{\partial m_T} \qquad \text{c}') \quad \frac{\partial\left(\frac{\partial F_T}{\partial m_T}\right)}{\partial F_T} \quad .$$

We obtain for these derivatives:

$$(5) \qquad \frac{\partial^2 F_T}{\partial m_T^2} = \frac{D}{U'_F}$$

$$(6) \qquad \frac{\partial\left(\frac{\partial F_T}{\partial m_T}\right)}{\partial m_T} = \frac{N_F}{U'_F}$$

(7) $$\frac{\partial\left(\frac{\partial F_T}{\partial m_T}\right)}{\partial F_T} = - \frac{N_m}{U'_F} .$$

The algebraic values of (5)-(7) are invariant with respect to any arbitrary increasing transformation of the utility indicator. Since U'_F is positive, we obtain from (5)-(7):

The sign of D gives the sign of the change in the slope of the indifference curves, i.e., the sign of the change in

$\frac{\partial F_T}{\partial m_T}$, defined by (4), with respect to the variation defined by a).

Similarly, the sign of N_F and the sign of $- N_m$ gives the sign of the change in $\frac{\partial F_T}{\partial m_T}$ with respect to the variations defined by respectively b)-c) above. 1/

1/ The right hand side of (5) equals the right hand side of (6) minus the product of $\frac{U'_m}{U'_F}$ and the right hand side of (7), cf. also (3.28).

Some supplementary examples on the graphical interpretation of the signs of D , N_F and N_m are given below:

a") Assuming $D > 0$ is identical to assume that the indifference curves are becoming less steep when m_T is increased or, in other words, that the numerical value of the marginal rate of substitution, $\frac{U'_m}{U'_F}$, is decreasing along the indifference curves.

b") Assuming $N_F > 0$ implies that the numerical value of the marginal rate of substitution is decreased by a move in straight easterly direction in the m_T , F_T - diagram, cf. point b) above.

This is the case with the indifference curves in Figure (5.5). In Figure (3.32) the $U^{[1]}$ - curves imply that $N_F > 0$, and the $U^{[2]}$ - curves imply that $N_F < 0$, at least in the vicinity of the optimal points.

c") When $N_m > 0$, the numerical value of the marginal rate of substitution is increased by a move in straight northerly direction in the m_T , F_T - diagram, cf. point c) above. This is the case for the indifference curves in Figure (5.5). In Figure (3.32) the $U^{[1]}$ - curves are characterized by $N_m < 0$ and the $U^{[2]}$ - curves by $N_m > 0$, at least in the vicinity of the optimal points.

In Model II we defined the variable N_{2Z} , cf. (9.37). We now leave out the first subscript and obtain:

(8) $$N_Z = N_F - N_m ,$$

where N_F and N_m are defined by (2) - (3). (6)-(8) imply that $\frac{N_Z}{U_F'}$ is the derivative of $\frac{\partial F_T}{\partial m_T}$ with respect to m_T (or F_T) when $dF_T = dm_T$. The sign of N_Z therefore gives the sign of the change in $\frac{\partial F_T}{\partial m_T}$ with respect to a move in north-easterly direction in the m_T , F_T-diagram. For example, when using the utility indicator (5.1) -- or any increasing transformation of (5.1) -- we obtain, cf. also (5.5a):

(9) $$\frac{N_Z}{U_F'} = \frac{a_2}{a_1} \frac{F_T - m_T}{m_T^2} .$$

(9) implies that in any point on the 45°-line we have $N_Z = 0$, i.e., the marginal rate of substitution is constant along the 45°-line. Above the 45°-line we have $N_Z > 0$, and below this line we have $N_Z < 0$.