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UNILATERAL MARKETS

Part I: Price-Quantity Markets

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INTRODUCTION

In this paper we begin a study of a class of markets widely employed in contemporary economies: markets in which the scope of action of customers and sellers is confined by conventional or technological limitations in such a way that members of one group (usually the sellers) always make offers -- by publishing a price or submitting a bid -- and members of the other group (usually the buyers) always choose among alternative offers. Because the initiative always comes from one group we shall call such markets unilateral markets, in contrast to bilateral markets in which both customers and sellers sometimes make offers, sometimes select among offers and can interchange roles freely. Thus automobiles, television sets, soap, computers and machine tools, for example, are typically sold in unilateral markets; whereas securities, (organized) labor, wheat and houses are typically sold in bilateral markets.^{1/}

Outline

Section 1 formulates a theory of the structure of unilateral markets, beginning with a detailed description of markets in extensive form (Sl.1), then proposing a more compact representation in normalized form (Sl.2). A structural postulate is introduced (Sl.3) placing certain restrictions on the behavior of customers and sellers. We then derive the fundamental structural equations (Sl.4) of unilateral markets which describe the relationship between the actions of the sellers and their resulting sales and profits,

showing that the postulate is a necessary and sufficient condition for these structural relationships.

Section 2 investigates the behavior of competing sellers within the structural framework developed in the preceding section. We first define the strategy space of the sellers (S2.1) and the concept of a non-cooperative equilibrium point in this space (S2.2) and impose two additional restrictions on market structure, linearity and inelasticity (S2.3). We then derive the principal quantitative results showing the explicit conditions determining entry and exit of sellers at equilibrium and the surplus obtained by customers (S2.4) and the equilibrium prices, market shares and profits of the sellers (S2.5). Wherever possible the economic implications of the quantitative results are interpreted and discussed.

In Section 3 we relax certain of the restrictions. We show first that elastic markets can be treated by extension to inelastic markets (S3.1). Then certain useful properties of non-linear markets are presented (S3.2). Finally, we relax the assumption that sellers are interested only in their own profits and show that more general equilibria may be treated by the methods already introduced.

Price and quantity are the only competitive variables treated in the present paper. All other influences are taken as given. We intend to extend the study in subsequent work to cover additional aspects of unilateral markets: advertising, selling effort, and product design and the effect of customer memory (Part II), non-linear cost functions and resource constraints (Part III), investment and resource planning (Part IV), and finally, the efficiency of unilateral markets and its welfare and policy implications (Part V).

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1. MARKET STRUCTURE

We shall define a market as a collection, \mathcal{C} , of customers who have certain needs, a collection, \mathcal{S} , of sellers who sell means of satisfying these needs and a collection, \mathcal{B} , of behavioral processes which govern interactions between customers and sellers. The elementary events of a market will be called transactions. We will restrict our analysis to markets in which either (a) the number of customers is large compared to the number of sellers and a transaction consists of a single customer selecting among alternative sellers, or (b) the number of sellers is large compared to the number of customers and a transaction consists of a single seller selecting among alternative customers (oil lease bidding, for example).^{2/} Markets satisfying either condition (a) or condition (b) we will call unilateral markets since one group always makes offers and the other group always selects among offers. For convenience we will develop the analysis in terms of type (a) markets. The results may easily be reinterpreted for type (b) markets.

Borrowing, as Shubik (11) has suggested, a fundamental and extremely useful concept from game theory, we will distinguish two forms in which a market may be described: the extensive form, a detailed specification of \mathcal{C} , \mathcal{S} , and \mathcal{B} , and the normalized form, a minimal representation identifying only essential components of \mathcal{C} , \mathcal{S} , and \mathcal{B} . We will call the functional relationship between the normalized components the structure of a market. The purpose of this section is to propose a normalized form for price-quantity markets and, under a single behavioral postulate, to formulate a theory of market structure.

1.1 Extensive Form

First we will examine the market in extensive form with some care in order to provide a meaningful framework for normalization. The customer collection, C , is an arbitrary assembly of individuals, families, groups, institutions, etc. It may include every potential customer in the entire economy or it may be restricted to a particular class of customers (e.g., grey-haired residents of Bolinas, California, or investor-owned electric utilities). We will let k designate an individual customer belonging to C .

The seller collection, S , is similarly an arbitrary assembly of individuals, institutions, etc. Each seller is the agent for one alternative which may be purchased by customers. Notice that this definition is narrower than the customary notion of a firm. A typical firm (say, General Motors) may contain, under our definition, many sellers who present competing alternatives to a given collection of customers. Nor does S correspond necessarily to the customary notion of an industry. The competition that Chevrolet convertibles, for example, face from trips to Europe and backyard swimming pools may be just significant as the competition from Ford convertibles, and we may wish to define a market containing sellers of all four of these alternatives.

We will treat the alternative offered by each seller as an indivisible entity (a television set or a computer) which, in any single transaction, the customer simply accepts or rejects.^{3/} Thus, if a firm sells sugar in 1, 5, and 20 pound containers we will separate the firm into three sellers. A customer can buy two or more units of a given seller's alternative only by participating in two or more transactions. We will let N designate the number of sellers in S ,

and use i ($1, 2, \dots, N$) to identify the i^{th} alternative and S_i to identify the seller of the i^{th} alternative. Since i and S_i are in one-to-one correspondence we will also use \mathcal{A} to designate the set of alternatives.

The collection, \mathcal{B} , of behavioral processes contains five basic processes which we will call B_1, \dots, B_5 . We will define each process indirectly by describing its role in an individual transaction, defining, at the same time, the variables, generated by these processes, which constitute a transaction.

Definition 1. B_1 , the transaction generator, selects k , a single customer, from \mathcal{C} .

Each transaction involves one customer who chooses at most one alternative from \mathcal{A} . B_1 determines the relative frequency of each customer's participation in transactions. But sequence is not implied by B_1 . All transactions occur simultaneously. Thus, B_1 should be thought of as the underlying random process which determines the probability that any particular customer, k , will participate in a transaction.

Definition 2. B_2 , the customer evaluation process, determines W_i , the worth of i to k , for each i in \mathcal{A} .

B_2 is a random process through which k considers each i separately (ignoring all other alternatives present in this market) and determines, in terms of his utility function, his resources, his external (to this market) alternatives,

his knowledge of i , the availability of i , etc., the lowest price at which he would rather go without than buy i . This threshold price we call W_i , the worth of i to k . Since future transactions are, under Definition 1, external to the market the customer's beliefs concerning the future availability and price of an alternative, together with the disutility he attaches to postponement and reenactment of the transaction process place an upper bound on W_i . If k is unaware of i , then $W_i = 0$.

Definition 3. B_3 , the seller pricing process, determines P_i , the price of i to k , for each i in \mathcal{I} .

B_3 is a random process through which each seller selects a price to announce to k . P_i constitutes the entire net payment k must make if he selects i including discounts, allowances, and all other price-equivalent modifications. If k is unaware of i , then $P_i = 0$.

Definition 4. B_4 , the customer selection process, determines δ_i , the quantity of i purchased by k , for each i in \mathcal{I} .

From Definition 2 it follows that k will select i only if $W_i - P_i > 0$.

B_4 is a random process through which k selects for purchase one among the alternatives satisfying this inequality. If no i satisfies the inequality, k rejects all alternatives. Since each alternative is indivisible and k may purchase at most one unit, we may write

$$(1.1) \quad \delta_i = \begin{cases} 1 & \text{if } k \text{ selects } i \\ 0 & \text{otherwise} \end{cases}$$

Definition 5. B_5 , the seller technology, determines C_i , the cost to S_i of producing i for k , for all i in \mathcal{I} .

B_5 is a random process generating the entire cost to S_i , including production, storage, packaging, delivery, etc., of making i available to k . But C_i includes only those costs which S_i would not have had if k had not purchased i . Thus all fixed costs are excluded from C_i .

While the four variables defined above, W , P , δ , and C (each N -dimensional), are sufficient to describe a transaction, two additional variables are useful to summarize the total quantity purchased and the total surplus obtained by k .

Definition 6. q : the total quantity purchased by k .

Recalling the restrictions discussed in Definitions 2 and 4 we may write

$$(1.2) \quad q = \sum_{i=1}^N \delta_i = \begin{cases} 1 & \text{if } \max_{1 \leq i \leq N} (W_i - P_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 7. λ : the surplus obtained by k .

Following Marshall (8), we shall call the difference, $W_i - P_i$, between the worth of alternative i to k and the price k would have to pay for i

the surplus k can obtain by purchasing i . If k rejects all alternatives then $\lambda = 0$. Combining Definitions 2, 3, 4, and 6, we may describe the surplus actually obtained by k by

$$\begin{aligned} \lambda &= \sum_{i \in \mathcal{S}} \delta_i (W_i - P_i) \\ (1.3) \quad &> 0 \quad \text{if } q = 1 \\ &= 0 \quad \text{if } q = 0 . \end{aligned}$$

1.2 Normalized Form

We have shown that the extensive form description of a market requires detailed specification of customers and sellers and of five behavioral processes which govern individual transaction variables: worth, price, quantity and cost. While the extensive form is useful conceptually a more parsimonious representation is obviously essential if we are to deal with more than the most fragile idealizations of real markets. To by-pass this difficulty we will abandon the individual transaction and investigate, instead, the statistical behavior of the transaction process, just as kinetic theory avoids the complexity of individual molecular interactions by limiting its attention to the statistical behavior of the transaction process (temperature, pressure, etc.). Thus we are led to a formulate description of the market in terms of the mathematical expectations of certain transaction variables. We will call this representation the normalized form description of the market.

Each of the unconditional expectations introduced below, Definitions 8 - 12, is defined in the usual way and taken over every possible transaction weighted by

the occurrence probability governed by B_1 . The conditional expectations, Definitions 13 - 15, are taken over the successful transactions of individual sellers with appropriate redefinition of the occurrence probabilities.

Definition 8. $Q = E(q)$: Expected total quantity sold per transaction.

Since $q = 1$ if a sale is made and $q = 0$ otherwise, Q is the probability that a randomly selected transaction will result in a sale. We will call a market closed if $Q = 1$, open if $0 < Q < 1$, and empty if $Q = 0$. Since the behavior of empty markets is trivial we will always assume $Q > 0$.

Definition 9. $\sigma_i = E(\delta_i)$: Expected quantity sold by S_i per transaction.

Since $\delta_i = 1$ if a sale is made by S_i and $\delta_i = 0$, otherwise, σ_i is the probability that S_i will make a sale in a randomly selected transaction.

Since, from (1.2),

$$\begin{aligned} \sum_{j=1}^N \sigma_j &= E \left(\sum_{j=1}^N \delta_j \right) \\ &= E(q) \end{aligned}$$

it follows that

$$(1.4) \quad \sum_{j=1}^N \sigma_j = Q.$$

If $\sigma_i = 0$ we will say alternative i is dominated. If $\sigma_i > 0$ alternative i is undominated.

Definition 10. $x_i = E(\delta_i(W_i - P_i))$: Expected competitive expense of S_i per transaction.

x_i is the expected value of a random variable that has value zero whenever alternative i is rejected and equals the difference between worth and price whenever i is accepted. If the sellers were engaged in a perfect conspiracy with discriminatory pricing we would expect the difference between worth, the customer's refusal threshold, and price to approach zero at least for the successful seller. Thus x_i is an indirect summary of the pricing strategy of S_i , measuring the expected amount S_i reduces price to attract customers away from competing alternatives.

Definition 11. $\Lambda = E(\lambda)$: Expected customer surplus per transaction.

From Definition 7,

$$\begin{aligned} E(\lambda) &= E \left\{ \sum_{j=1}^N \delta_j (W_j - P_j) \right\} \\ &= \sum_{j=1}^N E(\delta_j (W_j - P_j)) \end{aligned}$$

thus

$$(1.5) \quad \Lambda = \sum_{j=1}^N x_j .$$

The expected customer surplus equals the sum of the competitive expenditures of the sellers, as we might expect. Thus x_i may be interpreted as the average contribution S_i makes to total customer surplus.

Definition 12. $\Pi_i = E(\delta_i(P_i - C_i))$: Expected profit of S_i .

Since C_i includes only those costs which S_i would not have if no sale were made, we do not include fixed costs in this definition of profit. An alternative expression might be "contribution to fixed costs."

Definition 13. $\bar{W}_i = E(W_i | \delta_i = 1)$: Expected secured worth of i .

\bar{W}_i is a conditional expectation taken over only those transactions in which alternative i is selected by a customer. We will extend the definition to include dominated alternatives by stipulating $\bar{W}_i = 0$ if $\sigma_i = 0$. Since

$$\begin{aligned} E(\delta_i W_i) &= E(W_i | \delta_i = 1) \cdot \Pr\{\delta_i = 1\} + E(0 | \delta_i = 0) \cdot \Pr\{\delta_i = 0\} \\ &= E(W_i | \delta_i = 1) \cdot E(\delta_i) \end{aligned}$$

we observe that

$$(1.6) \quad E(\delta_i W_i) = \sigma_i \bar{W}_i$$

Definition 14. $\bar{P}_i = E(P_i | \delta_i = 1)$: Expected secured price of i .

As above we stipulate $\bar{P}_i = 0$ if $\sigma_i = 0$ and note that

$$(1.7) \quad E(\delta_i P_i) = \sigma_i \bar{P}_i .$$

Definition 15. $\bar{C}_i = E(C_i | \delta_i = 1)$: Expected secured cost of i .

Again, $\bar{C}_i = 0$ if $\sigma_i = 0$ and

$$(1.8) \quad E(\delta_i C_i) = \sigma_i \bar{C}_i .$$

1.3 Structural Postulate

The normalized form variables introduced above provide a domain in which the statistical behavior of the transaction process may be described. We now postulate a restriction on the relationship between certain of the normalized variables which we believe it is reasonable to expect most markets to meet and which we will subsequently show is a necessary and sufficient condition for the structural relationships that underlie our theory.

We first state a sufficient condition in the extensive domain which is stronger than we require and then weaken it to a condition in the normalized domain that is also necessary:

$$(\alpha_0) \quad \delta_i \text{ and } \lambda \text{ are } \underline{\text{independent}} \text{ if } \lambda > 0$$

or, in terms of conditional probabilities:

$$P_r \left\{ \delta_i = \delta, \lambda = \lambda_0 | \lambda > 0 \right\} = P_r \left\{ \delta_i = \delta | \lambda > 0 \right\} \cdot P_r \left\{ \lambda = \lambda_0 | \lambda > 0 \right\} .$$

Postulate (α_0) asserts that in those transactions in which the customer selects an alternative $(\lambda > 0)$ the probability that a particular seller is

selected is independent of the probability that the customer obtains a given amount of surplus. In Bayesian language (α_0) asserts that knowing whether or not a particular seller is selected in a given transaction (but nothing else) would not improve our a priori estimate of the probability that the customer will obtain some level of surplus, or equivalently, knowing the amount of surplus obtained by the customer (but nothing else) would not improve our a priori estimate of the probability that a given seller will be selected.

Observe that if (α_0) is not satisfied then at least one of the sellers, say S_1 , wins more frequently (relative to the other sellers) when a large surplus is obtained by the customer than when a smaller surplus is obtained. This suggests that either S_1 must inefficiently be bidding low prices too frequently or the customer must inefficiently be selecting alternatives that offer less surplus than S_1 . Thus, if (α_0) is not satisfied there is evidence of systematic bias in either the price strategies of the sellers, the selection process of the customer, or both. If (α_0) (or (α_1) below) is satisfied we will call the market unbiased.

Since the analysis that follows is concerned only with the normalized domain, we need only assert the somewhat weaker condition

(α_1) δ_1 and λ are uncorrelated if $\lambda > 0$

or, in terms of conditional expectations:

$$(1.9) \quad E(\delta_1 \lambda | \lambda > 0) = E(\delta_1 | \lambda > 0) \cdot E(\lambda | \lambda > 0)$$

which may be restated in terms of unconditional expectations if we observe, first, that

$$\begin{aligned}
 E(\delta_1 \lambda) &= E(\delta_1 \lambda | \lambda > 0) \cdot P_r\{\lambda > 0\} + E(\delta_1 \lambda | \lambda = 0) \cdot P_r\{\lambda = 0\} \\
 &= E(\delta_1 \lambda | \lambda > 0) \cdot E(q) + 0 \\
 &= Q \cdot E(\delta_1 \lambda | \lambda > 0).
 \end{aligned}$$

Thus

$$E(\delta_1 \lambda | \lambda > 0) = \frac{E(\delta_1 \lambda)}{Q}$$

Similarly

$$\begin{aligned}
 E(\delta_1 | \lambda > 0) &= \frac{\sigma_1}{Q} \\
 E(\lambda | \lambda > 0) &= \frac{\Delta}{Q}
 \end{aligned}$$

Substituting these three equations in (1.9) and multiplying by Q gives

$$(1.10) \quad E(\delta_1 \lambda) = \sigma_1 \frac{\Delta}{Q} .$$

Thus (α_1) asserts that the expected product of the quantity sold by S_1 and the customer's surplus equals the expected sales of S_1 times the expected surplus of customers divided by the expected total sales.

An alternative condition, which may give greater insight into the nature of the restriction (α_1) imposes on the market is

$$(\alpha'_1) \quad \bar{P}_1 - \bar{P}_j = \bar{W}_1 - \bar{W}_j \quad \text{for all } i, j \text{ such that } \sigma_i \cdot \sigma_j > 0$$

that is, the difference between the secured prices of any two undominated sellers equals the difference in their secured worths. In the special case of undifferentiated products ($\bar{W}_i = \bar{W}_j$) (α_1^i) asserts that either the secured prices will be identical or one or both of the sellers must be dominated. Thus if a market with undifferentiated products fails to meet (α_1^i) customers are systematically paying one seller more than another for an equivalent good. This seems implausible when we recall that W is defined to include considerations of accessibility and other extrinsic aspects of an alternative's usefulness.

If the worths, W_i , of the various alternatives are of substantially different magnitude the status of (α_1^i) is less clear. If we assume that the difference $W_i - P_i$ is sufficient to determine the ordering of customer preference among the alternatives we are implicitly assuming the customer's money utility is constant over the range of expenditures defined by the alternative prices. Hicks (6) points this out in an appraisal of Marshall's formulation of the doctrine of consumer surplus and goes on to suggest that even if the transaction represents only a small fraction of the customer's total budget (as Marshall assumed) there is no guarantee that the marginal utility of money is independent of the size of the expenditure. He acknowledges, however, that the effect diminishes as the relative significance of the expenditure declines. Thus (α_1^i) appears (to us) to impose a reasonable restriction on the relationship between price and worth so long as the alternatives included in the market are not too dissimilar in scale and large relative to the customer's total budget.

To prove the equivalence of (α_1) and (α'_1) we first show

$$(1.11) \quad (\alpha_1) \text{ implies } (\alpha'_1)$$

by observing that

$$(1.12) \quad E(\delta_i \lambda) = E\left(\delta_i (W_i - P_i)\right)$$

since, from (1.3), $\lambda = W_i - P_i$ whenever $\delta_i = 1$ and otherwise $\delta_i = 0$.

Thus, from (1.6) and (1.7)

$$(1.13) \quad E(\delta_i \lambda) = \sigma_i (\bar{W}_i - \bar{P}_i)$$

which, combined with (1.10), gives

$$(1.14) \quad \sigma_i (\bar{W}_i - \bar{P}_i) = \sigma_i \frac{\Delta}{Q} .$$

Hence, if $\sigma_i > 0$ and $\sigma_j > 0$

$$(1.15) \quad \bar{W}_i - \bar{P}_i = \bar{W}_j - \bar{P}_j = \frac{\Delta}{Q}$$

which establishes the necessity of (α_1) .

To show

$$(1.16) \quad (\alpha_1) \text{ is implied by } (\alpha'_1)$$

we note first that (α_1') implies

$$(1.17) \quad \sigma_i (\bar{W}_i - \bar{P}_i) = \sigma_i H$$

where H is some positive constant.

From (1.12) and (1.13) ,

$$\sigma_i (\bar{W}_i - \bar{P}_i) = E (\delta_i \lambda) .$$

Thus

$$(1.18) \quad E (\delta_i \lambda) = \sigma_i H \quad i = 1, \dots, N$$

Summing (1.18) over i yields

$$\sum_{j=1}^N E (\delta_j \lambda) = H \sum_{j=1}^N \sigma_j$$

which, after summing inside the expectation and combining with (1.4) and Definition 11 gives

$$\Lambda = H \cdot Q .$$

After substitution of $H = \Lambda/Q$ (1.18) is identical to (1.10) which we have shown is merely an unconditional restatement of (α_1) . This establishes the sufficiency of (α_1) .

1.4 Structural Equations

Now we will display the structural implications of (α_1) . From (1.12) and Definition 10 it follows that

$$x_i = E(\delta_i \lambda)$$

which, with (1.10) yields

$$x_i = \sigma_i \frac{\Lambda}{Q},$$

but, from (1.5),

$$\Lambda = \sum_{j=1}^N x_j$$

thus

$$(1.19) \quad x_i = \sigma_i \frac{\sum_{j=1}^N x_j}{Q}$$

and

$$(1.20) \quad \sigma_i = Q \frac{x_i}{\sum_{j=1}^N x_j}.$$

Since (1.19) may be written

$$E(\delta_1 \lambda) = \sigma_1 \frac{\Delta}{Q}$$

which (1.10), a restatement of (α_1) , it follows that (1.20) both implies and is implied by (α_1) .^{4/}

Thus we have proved that the expected sales of each seller equal the expected total sales of the market times the ratio of the seller's competitive expense to the total competitive expense of all sellers if and only if the market is unbiased.

We derive the relationship between profit and competitive expense by observing, from Definition 12, that

$$\Pi_1 = E(\delta_1(P_1 - C_1))$$

which may be written

$$\Pi_1 = E(\delta_1(W_1 - C_1) - \delta_1(W_1 - P_1)) .$$

But, from (1.6) and (1.7),

$$E(\delta_1(W_1 - C_1)) = (\bar{W}_1 - \bar{C}_1) \sigma_1$$

and, from Definition 10

$$E(\delta_1(W_1 - P_1)) = x_1 .$$

Thus

$$\Pi_i = (\bar{W}_i - \bar{C}_i) \sigma_i - x_i$$

which, after substitution of (1.20), becomes

$$(1.21) \quad \Pi_i = QK_i \frac{x_i}{\sum_{j=1}^N x_j} - x_i$$

where $K_i = \bar{W}_i - \bar{C}_i$.

Thus, in unbiased markets the profit, Π_i , of seller i equals the total quantity sold in the market, Q , times the difference, K_i , between the seller's secured worth and secured cost times the ratio of the seller's competitive expense, x_i , to the total competitive expense of all sellers minus the seller's competitive expense. We shall call K_i the strength of seller i , reflecting its physical significance (secured worth minus secured cost) and its role in the profit equation.

Equations (1.20) and (1.21) are the fundamental structural equations of unbiased markets.

2. COMPETITIVE BEHAVIOR IN LINEAR, INELASTIC MARKETS

In the previous section we proposed a normalized description of the transaction process and showed that in unbiased markets the surplus obtained by customers and the sales and profits of the individual sellers are completely determined by (1) the competitive expenditures of each seller, (2) the strength of each seller and (3) the total quantity purchased. Within this framework we now formulate a theory of the behavior of competing sellers in an unbiased market under certain simplifying assumptions concerning seller strength and quantity purchased.

2.1 Canonical Representation of Seller Strategies

We have shown that the competitive expenditures of the sellers, (x_1, x_2, \dots, x_N) , measure the average contribution each seller makes to the total surplus obtained by customers, and that in unbiased markets they determine the relative sales of each seller. Indeed, from (1.20) it follows that for any pair of undominated sellers, (i, j) ,

$$\frac{\sigma_i}{\sigma_j} = \frac{x_i}{x_j} .$$

That is, for any undominated pair the relative sales of each seller equals the seller's relative expense. These properties suggest that $X \equiv (x_1, \dots, x_n)$, the space of competitive expenditures, is the fundamental action domain or canonical strategy space of the sellers.

Other variables have been proposed. Cournot (3), in the classical analysis of unilateral markets, assumed that the sellers use quantity as their basic decision variable and thus that the quantity space, $\Sigma \equiv (\sigma_1, \dots, \sigma_N)$, is the strategy space of the sellers. Bertrand (1) rejected Cournot's approach arguing that it is unreasonable to assume the σ_i are selected and proposed instead that prices are the basic decision variables and thus that the price space, $\underline{P} \equiv (P_1, \dots, P_N)$, is the strategy space of the sellers. Others have pointed out that Bertrand's treatment is just as unreasonable since it tacitly assumes that prices are selected independently.^{5/}

The communications systems, production technologies, and physical storage and distribution systems of sellers in contemporary markets further weaken the arguments for either quantity or price as the fundamental decision variable. Price information is communicated rapidly and inventories are normally large enough to absorb short-term differences between production and demand. Thus it is increasingly doubtful that either the quantity brought to the market or the price named can meaningfully serve as the basic framework of seller strategies.

But does the competitive expenditure space, X , suffer from the same weakness of interdependent components as the price and quantity spaces? This is a crucial question and we shall attempt to answer it with some care. First we will show that it is possible for sellers to determine their competitive expenditures independently, then we will argue that it is reasonable to expect they might do so.

We have already shown that

$$x_i = \sigma_i(\bar{w}_i - \bar{p}_i)$$

that is, competitive expense equals quantity sold times secured worth minus secured price. This suggests that if over a long sequence of transactions $t (1, 2, \dots)$ seller i wishes to have an average competitive expense $x_i = x_i^0$, the following pricing policy might be used:

$$P_{it} = \hat{W}_{it} - \frac{x_i^0}{\hat{\sigma}_{it}} \quad t > t_1$$

where

$$\hat{W}_{it} = \frac{\sum_{\tau=t_1}^{t-1} \delta_{i\tau} W_{i\tau}}{\sum_{\tau=t_1}^{t-1} \delta_{i\tau}}$$

$$\hat{\sigma}_{it} = \frac{1}{t-t_1} \sum_{\tau=t_1}^{t-1} \delta_{i\tau}$$

$$\delta_{it} = \begin{cases} 1 & \text{if } i \text{ is selected in transaction } t \\ 0 & \text{otherwise} \end{cases}$$

t_1 = time of first transaction in which i was selected.

This policy may be summarized as follows: in transaction t seller i names a price, P_{it} , equal to estimated average worth as of t , \hat{W}_{it} , minus target expense level, x_i^0 , divided by estimated average sales. Where worth is averaged over all transactions prior to t in which seller i was selected and sales are averaged over all prior transactions. Thus the policy does not require that seller i anticipate the random variable W_{it} but simply that he record it

after he is selected.^{6/} We shall also assume that the seller begins with some history of transactions and that this history includes at least one sale. All averages are formed from the time of the first sale.

Consider first the special case of $x_i^0 = 0$. In this case the price set by i equals average secured worth over past sales. Since customers never buy unless price is less than worth it follows that each new sale (if there are any) must have a worth higher than the prior average. Thus \hat{W}_{it} must steadily increase if $x_i^0 = 0$. Since the random variable W_{it} is obviously bounded (if the customers have finite resources) it seems reasonable to conclude that we can make the average sales of i as arbitrarily small by continuing the process for a long enough time. We conclude, then, that the policy will insure that average expense approaches zero when target expense is zero, regardless of the strategies of the other sellers.

Now consider the more general case of $x_i^0 > 0$. Notice that P_{it} and $\hat{\sigma}_{it}$ move in the same direction if x_i^0 is positive. Seller i , following this policy, lowers his price after each unsuccessful transaction and raises his price after each successful transaction unless the new worth, W_{it} , lowers \hat{W}_{it+1} more than the corresponding increase in $x_i^0 / \hat{\sigma}_{it+1}$. Thus if a run of transactions occurs in which i is never selected i 's price will steadily decline. Eventually P_{it} will become so low that i will be selected, no matter what strategies are followed by the other sellers (barring extremely pathological exceptions in which prices approach $-\infty$). This argument suggests the somewhat stronger conclusion that i 's average sales level will approach some positive limit, σ_i , in the

sense that no matter what happens initially and no matter what strategies are followed by the other sellers we can make i 's sales arbitrarily close to σ_i by continuing the process for a long enough time. Similarly, it seems reasonable to expect that i 's average secured worth will approach some finite limit if the process is continued indefinitely. These assumptions may be summarized as follows:

$$\begin{aligned} \lim_{t \rightarrow \infty} \hat{\sigma}_{it} &= \sigma_i > 0 \\ \lim_{t \rightarrow \infty} \hat{W}_{it} &= \bar{W}_i < \infty \end{aligned} \quad x_i^0 > 0$$

It is possible to prove that these rather weak conditions are sufficient to insure that average expense approaches target expense, no matter what strategies are employed by the other sellers. The proof is given in Section 2.6. Thus we have demonstrated the feasibility of independent determination of the competitive expense levels by the individual sellers.

The policy we have just investigated makes no fixed determination of either price or quantity. It determines, instead, how the seller will respond in price to the history of sales. There is little doubt that seller behavior in contemporary markets is of this character. Strategies determine how to respond, not how much to charge or how much to produce. And we have shown that competitive expense is more than an arbitrary parameter governing the response curve of the seller. It is directly measurable as the product of units sold times average successful price cut and has direct physical interpretation as the average contribution the seller makes to the total surplus obtained by customers. Moreover, in unbiased markets it completely governs the manner in which customers allocate sales among sellers.

Thus we conclude that competitive expense provides the most meaningful framework within which to represent the actions of the competing sellers.

2.2 Non-Cooperative Equilibria

Now we come to one of the central questions of any theory of market behavior. Given that the space, X , of competitive expenditures is the strategy space of the sellers, how does each seller decide what his competitive expense will be? The profit equation (1.21) immediately suggests that we might look upon the market as a vast, iterated contest in which each seller selects a competitive expense level and then receives a "prize" which depends jointly on his expense level and the sum of the expenses of all sellers, then selects the next expense level, receives the next prize, etc. Such an interpretation is fraught with intriguing possibilities: threats, bluffs, signals, second-guesses, counter measures, counter-counter measures, etc. Marvelously intricate and ingenious conceptions of competition have been cultivated (or plowed under) in this rich ground.

In bilateral markets with only a few buyers and a few sellers where face-to-face bargaining is the principal interaction process the relevance of exotic conceptions of strategy is unquestionable. But in unilateral markets as the number of customers grows larger the very apparatus required to serve the customers -- salesmen, catalogues, distributors, retailers, technical representatives, warehouses, contracts, credit policies, trade shows, etc. -- make the seller's activities so massive and his knowledge of the market so fragmentary, ambiguous and unreliable that one finds it increasingly difficult to envision subtle, n^{th} -order plays and counter-plays. Recalling that markets may include competing sellers who may not

even be aware of each other (convertibles and backyard swimming pools) the "battle of wits" notion becomes even more unlikely.

Thus we are prompted to postulate a more prosaic conception of the mechanism which underlies the sellers' choices of strategies. Specifically, we shall assume (1) each seller makes periodic adjustments in his competitive expense level (or some equivalent proxy variable), (2) the seller perceives, perhaps erroneously, a relationship between these adjustments and changes in profits (or some proxy) and (3) the seller tends to reinforce adjustments which are perceived to have caused profit increases and to abandon adjustments which are perceived to have caused profit decreases. Notice that the seller is not required to have any direct knowledge of the customers, the other sellers or the underlying structure of the market. For the present we shall assume that each seller perceives his own profit (or its proxy) correctly. This assumption will be relaxed in Section 3.3 as will the implicit assumption that the seller is indifferent to the outcomes of the other sellers.

Thus we picture the sellers as restlessly searching for improved profit levels through trial-and-error adjustment of competitive expense. The dynamics of this collective search process and the important question of stability will not be considered here. We shall assume the process is stable and investigate the equilibrium configurations, if any, in which the simultaneous adjustments of the sellers are mutually counteractive. If such configurations exist each must have the property that no individual seller can improve his profit by increasing or decreasing his competitive expense level. We will call such configurations non-cooperative equilibrium points, following Nash (10), who showed that such points always exist in finite^{7/} games. Our assumptions concerning the strategy space of the sellers and

the mechanism they employ to select strategies may now be summarized as

(α_2) The market is in equilibrium if and only if the strategies of the sellers constitute a non-cooperative equilibrium point in X , the space of competitive expenditures.

2.3 Linearity and Inelasticity

To simplify our discussion of the properties of equilibrium points in unbiased markets two restrictions will be imposed on the structure of the market.

We will assume

$$(\alpha_3) \quad \frac{\partial \bar{W}_i}{\partial \sigma_i} = \frac{\partial \bar{C}_i}{\partial \sigma_i} = 0 \quad i = 1, \dots, N$$

that is, the secured worth and secured cost of each seller are independent of his sales level. Observe that we do not require that \bar{W}_i and \bar{C}_i be constant. They may be functions of various exogenous factors such as population growth, inflation, etc. The condition $\partial \bar{C}_i / \partial \sigma_i = 0$ is commonly labeled constant marginal cost and obviously implies that the seller's total cost function is linear. The condition $\partial \bar{W}_i / \partial \sigma_i = 0$ might analogously be labeled constant marginal worth and similarly implies that the total worth is a linear function of sales level. Thus, when (α_3) is satisfied we will say the market is linear.

Since a seller's secured worth and secured cost are, by definition, independent of the sales levels of other sellers it follows that

$$\frac{\partial \bar{W}_i}{\partial \sigma_j} = \frac{\partial \bar{C}_i}{\partial \sigma_j} = 0 \quad j \neq i$$

and since a seller's strength, K_i , is defined as secured worth minus secured cost, it follows that if (α_3) is satisfied, then

$$\begin{aligned} \frac{\partial K_i}{\partial x_i} &= \frac{\partial \bar{W}_i}{\partial x_i} - \frac{\partial \bar{C}_i}{\partial x_i} \\ &= \sum_{j=1}^N \frac{\partial \bar{W}_i}{\partial \sigma_j} \frac{\partial \sigma_j}{\partial x_i} - \sum_{j=1}^N \frac{\partial \bar{C}_i}{\partial \sigma_j} \frac{\partial \sigma_j}{\partial x_i} \\ &= 0 \end{aligned}$$

that is, a seller's strength is independent of his competitive expense in linear markets. In a linear market if $K_i = 0$ it follows from (1.21) that the seller's profit will be negative unless his competitive expense is zero, in which case both profit and sales (from (1.20)) will be zero. It follows that only sellers with positive strength will participate in the market at equilibrium. We can, therefore, exclude from equilibrium analysis all sellers but those with positive strength without loss of generality.

We will assume

$$(\alpha_4) \quad \frac{\partial Q}{\partial x_i} = 0 \quad i = 1, \dots, N$$

that is, the total industry sales level is independent of the competitive expenditures of any seller. Following common usage, if (α_4) is satisfied we will say the market is inelastic. To simplify notation we will normalize the sales of sellers in inelastic

by setting $Q = 1$ and will use the familiar term market share to refer to normalized sales levels.

Restrictions (α_3) and (α_4) will be relaxed in Sections 3.1 and 3.2.

But one can think of many markets that directly satisfy (α_3) and (α_4) , at least approximately. Consider, for example, competitive bidding markets in which the customer requests bids from a collection of sellers with the explicit or implicit understanding that bids above some level (W_1) are unacceptable. If the customers have approximately constant evaluations of each seller's product in successive transactions, if the sellers have approximately linear cost functions and if the prices bid are such that at least one seller's bid is acceptable in each transaction, then (α_3) and (α_4) are both satisfied. Markets with these characteristics are not uncommon. Note also that even markets which are non-linear over large changes in sales levels or elastic over large changes in competitive expense levels, so long as \bar{W} and \bar{C} are continuous in σ and Q is continuous in x , linearity and inelasticity are at least local properties over small enough changes in sales and competitive expense levels. Thus the properties of equilibrium points in linear, inelastic markets which we will now investigate, give insight into at least the local behavior of a large class of unbiased markets.

2.4 Entry, Exit and Customer Surplus

Conditions (α_2) , (α_3) and (α_4) may be summarized as follows: If X^* is an equilibrium point, then either

$$\begin{array}{l}
 (2.1) \quad x_i^* > 0 \quad \text{and} \quad \left. \frac{\partial \pi_i}{\partial x_i} \right|_{X=X^*} = 0 \\
 \text{or} \\
 (2.2) \quad x_i^* = 0 \quad \text{and} \quad \left. \frac{\partial \pi_i}{\partial x_i} \right|_{X=X^*} \leq 0
 \end{array}
 \left. \vphantom{\begin{array}{l} (2.1) \\ (2.2) \end{array}} \right\} i = 1, \dots, N$$

where

$$(2.3) \quad \left. \frac{\partial \pi_i}{\partial x_i} \right|_{X=X^*} = K_i \frac{\sum_{j=1}^N x_j^* - x_i^*}{\left(\sum_{j=1}^N x_j^* \right)^2} - 1$$

That is, at an equilibrium point, X^* , no undominated seller can improve his profit by any independent change in competitive expense and no dominated seller can do better than to spend nothing. Both Q and the $\{K_i\}$ are assumed constant and independent of X and, for convenience, we let $Q = 1$.

In the special case of $N = 1$, monopoly, we shall assume x_1^* approaches zero (from the right) but is positive, for $x_1^* = 0$ would imply, from the definition of worth, that no sale would ever be made. Similarly, for $N > 1$ we shall assume that at least one component of X^* is positive. Thus the degeneracy

of $x_i / \sum_{j=1}^N x_j$ where $\sum_{j=1}^N x_j = 0$ is avoided. Since the monopoly equilibrium point has already been defined we shall always assume $N \geq 2$ unless monopoly is explicitly mentioned. Also, equations involving a single seller, i , are to be understood to hold for all $i(1, \dots, N)$ unless some other condition is explicitly stated.

Sufficient conditions for an equilibrium point are

$$\frac{\partial^2 \pi_i}{\partial x_i^2} < 0$$

and since

$$\frac{\partial^2 \pi_i}{\partial x_i^2} = -2K_i \frac{\sum_{j=1}^N x_j - x_i}{\left(\frac{N}{\sum_{j=1}^N x_j} \right)^3}$$

it is clear that sufficiency is guaranteed everywhere in the non-negative orthant (to which X^* is constrained by (2.1) and (2.2)) so long as at least two sellers are undominated. We shall see that for $N \geq 2$ this requirement is always satisfied.

Let the sellers now be labeled in decreasing order of strength, so that

$$(2.4) \quad K_1 \geq K_2 \geq \dots \geq K_N$$

Under this ordering if $i < j$ we will say that seller i is stronger than seller j or that j is weaker than i , and if $K_i > K_j$ we will say that

seller i is strictly stronger than seller j or that j is strictly weaker than i .

We first show that this ordering, (2.4), implies

$$(2.5) \quad x_1^* \geq x_2^* \geq \dots \geq x_N^*$$

That is, at equilibrium the ordering of competitive expenditures corresponds to the ordering of seller strength. And since, in Section 1, we have already shown that the quantity sold by each seller is proportional to the seller's competitive expense, this implies at equilibrium the ordering of sales corresponds to the ordering of strength.

This result is easily proved by contradiction, for suppose (2.5) were not true for a given pair

$$(x_i^*, x_{i+s}^* \mid s \geq 1) .$$

Then

$$(2.6) \quad x_i^* < x_{i+s}^* .$$

Since from (2.1) and (2.2) no x^* is negative it follows that

$$x_{i+s}^* > 0$$

and, from (2.1) and (2.3)

$$(2.7) \quad x_{i+s}^* = \frac{\sum_{j=1}^N x_j^*}{K_{i+s}} - \frac{\left(\sum_{j=1}^N x_j^* \right)^2}{K_{i+s}} .$$

But, from (2.1), (2.2) and (2.3)

$$(2.8) \quad x_i^* \geq \frac{\sum_{j=1}^N x_j^*}{K_i} - \frac{\left(\sum_{j=1}^N x_j^* \right)^2}{K_i}$$

from (2.4)

$$K_i \geq K_{i+s}$$

thus

$$\frac{\sum_{j=1}^N x_j^*}{K_i} - \frac{(\sum_{j=1}^N x_j^*)^2}{K_i} \geq \frac{\sum_{j=1}^N x_j^*}{K_{i+s}} - \frac{(\sum_{j=1}^N x_j^*)^2}{K_{i+s}}$$

Combined with (2.7) and (2.8) this gives

$$x_i^* \geq x_{i+s}^*$$

which contradicts (2.6). Thus, at equilibrium, no seller spends more than any stronger competitor.

Also, by exchanging the indices in (2.7) and (2.8) we see that

if $x_i^* > x_{i+s}^* \geq 0$ then

$$\frac{\sum_{j=1}^N x_j^*}{K_i} - \frac{(\sum_{j=1}^N x_j^*)^2}{K_i} > x_{i+s}^* \geq \frac{\sum_{j=1}^N x_j^*}{K_{i+s}} - \frac{(\sum_{j=1}^N x_j^*)^2}{K_{i+s}}$$

hence

$$K_i > K_{i+s}$$

which may be summarized as

$$(2.9) \quad x_i^* > x_{i+s}^* \quad \text{implies} \quad K_i > K_{i+s}$$

Thus at equilibrium one seller spends more than another only if it is strictly stronger than the other.

Now we define the Equilibrium Entry Condition (EEC):

$$(2.10) \quad K_i > \frac{i-2}{i-1} \frac{1}{\sum_{j=1}^i \frac{1}{K_j}} \quad i = 2, \dots, N$$

which requires that a seller's strength be greater than the number of stronger sellers minus one divided by the sum of the reciprocals of the strengths of all stronger sellers. We will show that

$$(2.11) \quad \text{a seller is undominated if and only if EEC is satisfied.}$$

Here we extend the meaning of undomination to convey that x_i is positive as well as σ_i . Since we have shown in Section 1 that σ_i is proportional to x_i the two meanings are obviously equivalent.

EEC is undefined for $i = 1$, but we have shown that a non-decreasing ordering of the $\{K_i\}$ implies a non-decreasing ordering of the $\{x_i^*\}$ and since, by assumption, at least one component of X^* is undominated it follows that seller 1 is always undominated.

From this it follows that seller 2 is also always undominated for

$$x_2^* = 0$$

implies, from (2.5),

$$x_j^* = 0 \quad j = 2, \dots, N$$

and thus

$$\sum_{j=1}^N x_j^* = x_1^*$$

which, through (2.3), implies

$$\begin{aligned} \left. \frac{\partial \Pi_i}{\partial x_i} \right|_{X=X^*} &= K_1 \frac{x_1^* - x_1^*}{x_1^*} - 1 \\ &= -1 \end{aligned}$$

which contradicts (2.1) since $x_1^* > 0$ by assumption. Thus $x_2^* > 0$.

Since, by assumptions K_2 is positive EEC is obviously satisfied for $i = 2$. Thus the equivalence of EEC and undomination need only be proved for sellers in the range $3 \leq i \leq N$.

First we prove a seller is undominated only if EEC is satisfied. Suppose $i > 2$ is undominated, then

$$x_i^* > 0$$

and, from (2.1),

$$(2.12) \quad \frac{\sum_{j=1}^N x_j^* - x_1^*}{\left(\begin{array}{c} N \\ \sum x_j^* \\ 1 \end{array} \right)^2} = \frac{1}{K_1} .$$

From (2.5) it follows that

$$x_h^* > 0 \quad h = 1, \dots, i - 1$$

so that

$$\frac{\sum_{j=1}^N x_j^* = x_h^*}{\left(\begin{array}{c} N \\ \sum x_j^* \\ 1 \end{array} \right)^2} = \frac{1}{K_h} \quad h = 1, \dots, i - 1$$

If we sum both sides of these $i - 1$ equations we get

$$(2.13) \quad \frac{(i-1) \sum_{j=1}^N x_j^* - \sum_{j=1}^{i-1} x_j^*}{\left(\begin{array}{c} N \\ \sum x_j^* \\ 1 \end{array} \right)^2} = \sum_{j=1}^{i-1} \frac{1}{K_j}$$

Now divide each side of (2.13) by the corresponding side of (2.12) and transpose to give

$$K_1 \sum_{j=1}^{i-1} \frac{1}{K_j} = \frac{(i-1) \sum_{j=1}^N x_j^* - \sum_{j=1}^{i-1} x_j^*}{\sum_{j=1}^N x_j^* - x_1^*}$$

which may be reduced to

$$(2.14) \quad K_i \sum_{j=1}^{i-1} \frac{1}{K_j} = i - 2 + \left[\frac{\sum_{j=i+1}^N x_j^* + (i-1)x_i^*}{\sum_{j=1}^N x_j^* - x_i^*} \right]$$

The term in brackets is obviously positive, thus

$$(2.15) \quad K_i > \frac{i - 2}{\sum_{j=1}^{i-1} \frac{1}{K_j}}$$

which establishes the necessity of EEC.

Note that through (2.5) we have also proved that

$$(2.16) \quad \text{if } x_i^* > 0 \quad \text{then EEC is satisfied by all } j \text{ in the range } 2 \leq j \leq i .$$

To prove that a seller is undominated if EEC is satisfied we will first show

$$(2.17) \quad \text{if EEC is satisfied by } i \geq 2 \text{ then EEC is satisfied by all } j \text{ in the range } 2 \leq j \leq i .$$

If i satisfies EEC then

$$K_{i-1} > \frac{i - 2}{\sum_{j=1}^{i-1} \frac{1}{K_j}}$$

since $K_{i-1} \geq K_i$ under (2.4).

Thus

$$K_{i-1} \sum_{j=1}^{i-1} \frac{1}{K_j} > i - 2$$

$$K_{i-1} \sum_{j=1}^{i-2} \frac{1}{K_j} + 1 > i - 2$$

$$K_{i-1} \sum_{j=1}^{i-2} \frac{1}{K_j} > i - 3$$

Hence

$$K_{i-1} > \frac{i - 3}{\sum_{j=1}^{i-2} \frac{1}{K_j}}$$

So $i - 1$ satisfies EEC and, by induction, (2.17) is established.

Now, suppose $i > 2$ satisfies EEC and $x_1^* = 0$. We have already shown $x_2^* > 0$. Then, from (2.5) there must be an h in the range $2 \leq h < i$ such that

$$(2.18) \quad x_\ell^* > 0 \quad \ell = 1, \dots, h$$

$$(2.19) \quad x_m^* = 0 \quad m = h+1, \dots, N$$

That is, seller h and every stronger seller is undominated and seller $h+1$ and every weaker seller is dominated.

Thus

$$\sum_{j=1}^N x_j^* = \sum_{j=1}^h x_j^*$$

and, from (2.1)

$$\frac{\sum_{j=1}^h x_j^* - x_l^*}{\left(\sum_{j=1}^h x_j^*\right)^2} = \frac{1}{K_l} \quad l = 1, \dots, h$$

Summing over l gives

$$\frac{(h-1) \sum_{j=1}^h x_j^*}{\left(\sum_{j=1}^h x_j^*\right)^2} = \sum_{j=1}^h \frac{1}{K_j}$$

Thus

$$(2.20) \quad \sum_{j=1}^h x_j^* = \frac{h-1}{\sum_{j=1}^h \frac{1}{K_j}}$$

But from (2.19) and (2.2)

$$K_{h+1} \frac{\sum_{j=1}^h x_j^*}{\left(\sum_{j=1}^h x_j^*\right)^2} \leq 1$$

hence

$$K_{h+1} \leq \sum_{j=1}^h x_j^*$$

Which, with (2.20), gives

$$K_{h+1} \leq \frac{h-1}{\sum_{j=1}^h \frac{1}{K_j}}$$

so that $h+1$ does not satisfy EEC.

But for $h < i$ this contradicts (2.17). Thus if i satisfies EEC, $x_1^* > 0$ establishing the sufficiency of EEC and proving, through (2.5)

$$(2.21) \quad \text{if } i \text{ satisfies EEC then } x_j^* > 0 \quad j = 1, \dots, i .$$

Together, (2.16) and (2.21) establish (2.11).

Now we define n^* , the effective number of sellers, by

$$(2.22) \quad n^* = \max_{1 \leq i \leq N} \left\{ i \mid i \text{ satisfies EEC} \right\}$$

that is, n^* is the index of the weakest seller satisfying EEC. Since n^* is the largest of a finite set of integers and EEC is satisfied for $i = 2$, n^* exists, is unique, to the ordering of the $\left\{ K_i \right\}$, and can be determined by simple enumeration. From (2.16) and (2.21) it follows that :

$$(2.23) \quad \begin{array}{ll} x_i^* > 0 & i = 1, \dots, n^* \\ x_i^* = 0 & i = n^*+1, \dots, N \end{array}$$

Since, from (2.9) and (2.23)

$$K_{n^*} > K_{n^*+1} \quad \text{if } n^* < N$$

it follows that n^* is unique to any ordering of K which preserves strict inequalities.

At equilibrium seller n^* and every stronger seller is undominated and seller n^*+1 and every weaker seller is dominated. We will call n^* the virtual number of sellers, in contrast to N , the real number of sellers. Thus, regardless of N , if $n^* = 2$ we will say the market is a virtual duopoly. Similarly, if $K_1 = K_2 = \dots = K_{n^*}$, we will say the market is virtually symmetrical, regardless of K_{n^*+1}, \dots, K_N .

We have shown that EEC completely prescribes the entry and exit of sellers when the market is at equilibrium. Several interesting consequences follow. First, we have observed that in markets with two sellers the weaker seller ($i = 2$) always participates. But this property does not extend to the weakest seller when $N = 3$ (or more). To see the source of this rather surprising special property of duopoly markets consider, for convenience, a market with undifferentiated alternatives ($\bar{W}_i = W$, all i) but unequal technologies. In such a market the stronger of two sellers has no incentive to price below the cost of the weaker seller but

given three sellers the strongest may well find it advantageous to price below the cost of the weakest seller, thus excluding him from the market. The same distinction applies more generally to competitive expenditures on differentiated alternatives.

Now consider symmetric markets in which each of the N sellers has identical strength. If K_j is replaced by K in (2.10), EEC becomes

$$K_1 = K > \frac{i-2}{i-1} K \quad i = 1, \dots, N$$

Thus EEC is satisfied by all sellers and consequently none is excluded, regardless of the size of N . Since most of the literature of oligopoly theory is devoted to these two special cases -- duopoly and symmetry -- it is not surprising that the problem of the exit and entry of sellers has proved so elusive in the past.

A third implication is that n^* can be large only if most of the sellers are very similar in strength. Consider the weakest seller not excluded. From (2.10)

$$K_{n^*} > \frac{n^* - 2}{n^* - 1} \frac{1}{\sum_{j=1}^{n^*} \frac{1}{K_j}}$$

which can be reduced to

$$(2.24) \quad K_{n^*} > \frac{n^* - 1}{n^*} \bar{K}$$

where

$$(2.25) \quad \bar{K} = \frac{n^*}{\sum_{j=1}^{n^*} \frac{1}{K_j}},$$

the harmonic mean of the strengths of the active sellers.

Since

$$K_{n^*} \leq K_j \quad j \leq n^*$$

it is readily shown that

$$K_{n^*} \leq \bar{K}$$

which may be combined with (2.25) and rearranged to give

$$(2.26) \quad 0 \leq \frac{\bar{K} - K_{n^*}}{\bar{K}} < \frac{1}{n^*}$$

Thus the relative deviation between the weakest seller's strength and the average^{8/} strength of all active sellers cannot exceed the reciprocal of the number of active sellers. In a market, for example, with 50 active sellers the weakest seller's strength must be within 2% of the average! Notice also that where n^* is large the survival of the weaker sellers is extremely tenuous. A relatively small increase in the strength of some sellers may exclude from the market a large number of weaker sellers.

Since (2.18) and (2.19) are satisfied if and only if $h = n^*$,

(2.20) becomes

$$(2.27) \quad \sum_{j=1}^{n^*} x_j^* = \frac{n^* - 1}{\sum_{j=1}^{n^*} \frac{1}{K_j}}$$

which, with (2.23), establishes the existence and uniqueness of $\sum_{j=1}^N x_j^*$.

Consequently (2.7) becomes

$$x_i^* = \frac{\frac{n^* - 1}{n^*}}{\sum_{j=1}^n \frac{1}{K_j}} \left(1 - \frac{\frac{n^* - 1}{n^*}}{K_i \sum_{j=1}^n \frac{1}{K_j}} \right) \quad i = 1, \dots, n^*$$

which, with (2.23), establishes the existence and uniqueness of X^* .^{9/}

Combined with (1.5) and (2.25), (2.27) yields one of the fundamental relationships of the market:

$$(2.28) \quad \Lambda^* = \frac{n^* - 1}{n^*} \bar{K} .$$

The total surplus obtained by customers at equilibrium depends only on the number of undominated sellers and their average strength. If $n^* = 1$ (which we have shown is possible only if $N = 1$), the customer obtains no surplus -- exactly the result we would expect of the price-discriminating monopolist. As n^* becomes large, on the other hand, the surplus obtained by customers rapidly approaches the average strength of the sellers -- exactly the result we would expect under pure competition.

2.5 Price, Market Share and Profit

Now we can describe the outcome to the active sellers at equilibrium: their prices, market shares and profit. Since the market is assumed inelastic ($Q = 1$), (1.15) becomes

$$(2.29) \quad \Lambda^* = \bar{W}_i - \bar{P}_i \quad i \leq n^*$$

which, combined with (2.28) gives

$$(2.30) \quad \bar{P}_i^* = \bar{W}_i - \frac{n^* - 1}{n^*} \bar{K} \quad i \leq n^*$$

Thus the \bar{P}_i^* , equilibrium secured price of the i^{th} active seller, depends on the seller's secured worth, the number of active sellers and their average strength. For fixed \bar{W}_i and \bar{K} , \bar{P}_i^* declines as n^* increases. The price effect of the entry of new sellers can be stated in a more meaningful way if we consider a hypothetical "average" seller, i_0 , for whom $K_{i_0} = \bar{K}$. Recalling, from (1.21), that $K_i = \bar{W}_i - \bar{C}_i$, we write

$$\bar{P}_{i_0}^* = \bar{W}_{i_0} - \frac{n^* - 1}{n^*} (\bar{W}_{i_0} - \bar{C}_{i_0})$$

which may be reduced to

$$(2.31) \quad \bar{P}_{i_0}^* = \bar{C}_{i_0} + \frac{\bar{W}_{i_0} - \bar{C}_{i_0}}{n^*} .$$

Thus, as n^* becomes large the average seller's price approaches cost, satisfying one of the traditional characteristics of pure competition. This result helps explain the tendency toward homogeneity of strength in many-seller markets which we discussed earlier.

Notice that (2.31) describes the price of every seller in a symmetrical market since $K_i = \bar{K}$ for all $i \leq n^*$. As we would expect, in the simplest symmetrical market, monopoly, secured price equals secured worth.

To determine the market share of each active seller we first note that in an inelastic market (1.20) may be written

$$(2.32) \quad \sigma_i = \frac{x_i}{\sum_{j=1}^N x_j}$$

$$\text{Thus } \frac{\partial \pi_i}{\partial x_i} = 0 \text{ implies, from (2.3),}$$

$$1 - \sigma_i^* = \frac{\sum_{j=1}^{n^*} x_j^*}{K_i} \quad i = 1, \dots, n^*$$

which, combined with (2.27) and (2.23) may be reduced to

$$(2.33) \quad \sigma_i^* = 1 - \frac{n^* - 1}{K_i \sum_{j=1}^{n^*} \frac{1}{K_j}} \quad i = 1, \dots, n^*$$

$$= 0 \quad i = n^*+1, \dots, N$$

which may also be written

$$(2.34) \quad \begin{aligned} \sigma_i^* &= 1 - \left(\frac{n^* - 1}{n^*} \right) \frac{\bar{K}}{K_i} & i = 1, \dots, n^* \\ &= 0 & i = n^*+1, \dots, N \end{aligned}$$

where \bar{K} is the harmonic mean of the strengths of undominated sellers, defined by (2.25).

Thus σ_i^* , the equilibrium market share of the i^{th} undominated seller, depends only on the number of undominated sellers, n^* , and the seller's relative strength K_i/\bar{K} . Dominated sellers have market shares of zero.

Under virtual symmetry the relative strength of every undominated seller is 1, and (2.34) becomes

$$\begin{aligned} \sigma_i^* &= 1 - \frac{n^* - 1}{n^*} & i = 1, \dots, n^* \\ &= \frac{1}{n^*} \end{aligned}$$

as we might expect.

Under virtual duopoly (2.33) can be reduced to

$$\sigma_i^* = \frac{K_i}{K_1 + K_2} \quad i = 1, 2$$

Thus under virtual duopoly and virtual symmetry the market share of the undominated sellers is proportional to their strength. But no corresponding simplification occurs for asymmetric markets in which $n^* > 2$. For example, if $n^* = 3$,

$$\sigma_1^* = \frac{K_1 K_2 + K_1 K_3 - K_2 K_3}{K_1 K_2 + K_1 K_3 + K_2 K_3}$$

and the closed form expressions for σ_i^* become even more complicated for larger n^* . Here again we see how unrepresentative duopoly and symmetry are of more general markets.

We can combine (2.34) and (2.30), eliminating both n^* and \bar{K} to yield

$$\sigma_i^* = 1 - \frac{\bar{W}_i - \bar{P}_i^*}{K_i} \quad i = 1, \dots, n^*$$

which, recalling that $K_i = \bar{W}_i - \bar{C}_i$, may be reduced to

$$(2.35) \quad \sigma_i^* = \frac{\bar{P}_i^* - \bar{C}_i}{\bar{W}_i - \bar{C}_i} \quad i = 1, \dots, n^*$$

Thus equilibrium market share equals the ratio of secured margin (secured price less secured cost) to strength. One consequence of (2.35) is

$$(2.36) \quad \bar{P}_i^* = (1 - \sigma_i^*) \bar{C}_i + \sigma_i^* \bar{W}_i \quad i = 1, \dots, n^*$$

Thus equilibrium price is a linear mixture of cost and worth and equilibrium market share is the coefficient governing the mixture. Suppose an undominated seller, i , makes investments that reduce \bar{C}_i and/or increase \bar{W}_i (through product improvement), and suppose that other sellers make corresponding changes so that K_i/\bar{K} and n^* remain constant. Then, from (2.34), σ_i^* remains constant

even though K_i increases. Let ΔK_i be the net increase in K_i and $\Delta(\bar{P}_i^* - \bar{C}_i)$ the change in secured margin. Since σ_i^* is unchanged, it follows from (2.36) that

$$(2.37) \quad \Delta(\bar{P}_i^* - \bar{C}_i) = \sigma_i^* \Delta K_i .$$

Thus the seller receives only a fraction of the net value of the cost reduction or product improvement and that fraction is its equilibrium market share. The remainder goes to the customers as increased surplus. This systematic evaporation of incremental seller margin may explain the persistent discrepancy between the gains that sellers anticipate when investments are being considered and the gains actually realized after the investments have been made. Of course, the assumed constancy of K_i/\bar{K} is essential to this argument. But product and process innovations are frequently made more or less concurrently by competing sellers. A unique innovation, on the other hand, obtains for the seller an incremental increase in margin that is much closer to the incremental change in strength. The derivative of secured margin with respect to seller strength is

$$\frac{\partial(\bar{P}_i^* - \bar{C}_i)}{\partial K_i} = 1 - \frac{(1 - \sigma_i^*)^2}{n^* - 1} \quad i = 1, \dots, n^*$$

Thus small increments in a single seller's strength will produce virtually equal increments of secured margin except for very weak sellers in highly concentrated markets.^{10/}

A second consequence of (2.35) is that

$$(2.38) \quad \bar{W}_i = \bar{C}_i + \frac{\bar{P}_i^* - \bar{C}_i}{\sigma_i^*} . \quad i = 1, \dots, n^*$$

Secured worth equals secured cost plus secured margin divided by equilibrium market share. Since worth is frequently more difficult to measure than price, cost or market share, (2.38) can be used to estimate \bar{W}_i under the assumption that the market is in equilibrium. Notice that, for given cost and margin, low market share implies high worth, which seems counter-intuitive but simply reflects the fact that sellers with low market share retain only a correspondingly small fraction of the difference between worth and cost.

To determine equilibrium profits we first recall that (since $Q = 1$) (1.21) may be written

$$\Pi_i = K_i \sigma_i - x_i$$

and, from (2.32),

$$x_i = \sigma_i \sum_{j=1}^N x_j$$

thus

$$(2.39) \quad \Pi_i = \sigma_i \left(K_i - \sum_{j=1}^N x_j \right)$$

Substituting (2.27) in (2.33) we find the following relationship between equilibrium market share, total competitive expense and seller strength:

$$\sigma_i^* = 1 - \frac{\sum_{j=1}^N x_j^*}{K_i}, \quad i = 1, \dots, n^*$$

Thus, at equilibrium,

$$K_i - \sum_{j=1}^N x_j^* = \sigma_i^* K_i \quad i = 1, \dots, n^*$$

and (2.39) now may be written

$$(2.40) \quad \begin{aligned} \Pi_i^* &= K_i (\sigma_i^*)^2 & i = 1, \dots, n^* \\ &= 0 & i = n^*+1, \dots, N \end{aligned}$$

At equilibrium the profit of the i^{th} undominated seller equals the seller's strength times the square of the seller's equilibrium market share. The dominated seller's profits are zero. We have shown that the ordering of equilibrium market share corresponds to the ordering of strength. From (2.40) it follows that the ordering of equilibrium profit corresponds to the ordering of strength.

The rapid degradation of profits as n^* increases is best demonstrated by returning to the hypothetical "average" seller, i_0 , whose strength is $K_{i_0} = \bar{K}$ and whose market share we have shown is $\sigma_{i_0}^* = 1/n^*$. From (2.4)

$$(2.41) \quad \Pi_{i_0}^* = \frac{K_{i_0}}{(n^*)^2}$$

Thus the equilibrium profits of the "average" seller are inversely proportional to the square of the number of sellers. Clearly, if n^* is large, the profits earned by weaker sellers is very small. Notice that (2.41) describes the equilibrium profits of all sellers under virtual symmetry. As n^* is increased the zero profit condition of pure competition is approached rapidly.

The sensitivity of profits to seller strength is best seen in the special case of virtual duopoly in which, as we showed earlier, market share is proportional to strength. If $n^* = 2$, (2.40) may then be written

$$\Pi_i^* = K_i \left(\frac{K_i}{K_1 + K_2} \right)^2 \quad i = 1, 2$$

thus

$$\frac{\Pi_1^*}{\Pi_2^*} = \left(\frac{K_1}{K_2} \right)^3 \quad \text{if } n^* = 2 .$$

that is, the relative profits of an undominated seller in a virtual duopoly equals the cube of the seller's relative strength.

2.6 Implementation of Competitive Expense Decisions

We now prove that the pricing policy described in Section 2.1 implements the decision $x_i = x_i^0$ regardless of the pricing strategies of the other sellers so long as $\sigma_i \equiv \lim_{t \rightarrow \infty} \hat{\sigma}_{it}$ exists and is positive and $\bar{W}_i \equiv \lim_{t \rightarrow \infty} \hat{W}_{it}$ is bounded.

Specifically, we prove that

$$x_i = \lim_{t \rightarrow \infty} \bar{x}_{it} = x_i^0$$

where \bar{x}_{it} is seller i 's average expense through t transactions, and x_i^0 is seller i 's target expense level.

For convenience we digress briefly to prove the following:

Lemma 1

If $t = t_1, t_2, \dots$ is a monotonic increasing sequence of positive integers satisfying

$$(2.42) \quad \lim_{N \rightarrow \infty} \frac{N}{t_N} = \sigma > 0$$

$$(2.43) \quad t_N \geq N$$

Then

$$(2.44) \quad \lim_{N \rightarrow \infty} \frac{1}{t_N} \sum_{h=1}^N \frac{t_h}{h} = 1$$

Proof:

Let

$$\Theta_N = \frac{1}{t_N} \sum_{h=1}^N \frac{t_h}{h}$$

Observe that for $N > M$

$$(2.45) \quad \Theta_N = \frac{1}{t_N} \sum_{h=M+1}^N \frac{t_h}{h} + \frac{t_M}{t_N} \Theta_M$$

From (2.42) for any ϵ in the range $0 < \epsilon < \sigma$ there exists an M_ϵ such that

$$\sigma - \epsilon < \frac{h}{t_h} < \sigma + \epsilon \quad \text{for all } h \geq M_\epsilon$$

hence

$$\frac{1}{\sigma+\epsilon} < \frac{t_h}{h} < \frac{1}{\sigma-\epsilon} \quad h \geq M_\epsilon$$

Let $M \geq M_\epsilon$. Then clearly

$$\frac{1}{\sigma+\epsilon} < \frac{1}{N-M} \sum_{h=M+1}^N \frac{t_h}{h} < \frac{1}{\sigma-\epsilon} \quad \text{for all } N > M$$

Combined with (2.45) this may be written

$$\frac{1}{\sigma+\epsilon} \frac{N-M}{t_N} + \frac{t_M}{t_N} \Theta_M < \Theta_N < \frac{1}{\sigma-\epsilon} \frac{N-M}{t_N} + \frac{t_M}{t_N} \Theta_M$$

Let $N \rightarrow \infty$, M constant. From (2.43)

$$t_N \rightarrow \infty.$$

Thus

$$\frac{t_M}{t_N} \Theta_M \rightarrow 0$$

and, from (2.42)

$$\frac{N-M}{t_N} \rightarrow \sigma$$

Hence

$$\frac{\sigma}{\sigma+\epsilon} < \lim_{N \rightarrow \infty} \Theta_N < \frac{\sigma}{\sigma-\epsilon}$$

Now let $\epsilon \rightarrow 0$, which completes the proof.

The pricing policy we wish to investigate is

$$(2.46) \quad P_t = \hat{W}_t - \frac{x^0}{\sigma_t} \quad t > t_1 > 0$$

where

$$(2.47) \quad \hat{W}_t = \frac{\sum_{\tau=t_1}^{t-1} \delta_\tau W_\tau}{\sum_{\tau=t_1}^{t-1} \delta_\tau}$$

$$(2.48) \quad \hat{\sigma}_t = \frac{1}{t-t_1} \sum_{\tau=t_1}^t \delta_\tau$$

$$\delta_t = \begin{cases} 1 & \text{if seller was selected in transaction } t \\ 0 & \text{otherwise} \end{cases}$$

$$t_1 = \text{time of first transaction in which seller was selected}$$

The subscript 1 has been omitted since the results are independent of the index of the seller. Also, to simplify the discussion we have limited the policy to transactions following t_1 , the first transaction in which the seller is selected.

We shall assume, for reasons discussed in Section 2.1, that

$$(2.49) \quad \lim_{t \rightarrow \infty} \hat{\sigma}_t = \sigma > 0$$

and

$$(2.50) \quad \lim_{t \rightarrow \infty} \hat{W}_t = \bar{W} < \infty .$$

Average competitive expense through the first t transactions, \bar{x}_t , is defined analogously to expected competitive expense (Definition 10).

$$(2.51) \quad \bar{x}_T = \frac{1}{T-t_1} \sum_{t=t_1+1}^T \delta_t \left(W_t - \frac{\sum_{\tau=t_1}^{t-1} \delta_\tau W_\tau}{\sum_{\tau=t_1}^{t-1} \delta_\tau} + \frac{x_0(t-1)}{\sum_{\tau=t_1}^{t-1} \delta_\tau} \right)$$

Let $t \equiv t_1, t_2, \dots$ be the ordered sequence of the transaction times when the seller was selected. Thus

$$\delta_t = \begin{cases} 1 & t = t_1, t_2, \dots \\ 0 & \text{otherwise} \end{cases}$$

and

$$t_1 < t_2 < \dots$$

Since $\delta_t = 0$ except at the times t_1, t_2, \dots , (2.52) may now be written

$$(2.53) \quad \bar{x}_{t_N} = \frac{1}{t_N - t_1} \sum_{h=2}^N \left[W_{t_h} - \frac{\sum_{l=1}^{h-1} W_{t_l}}{h-1} + \frac{x_0(t_h-1)}{h-1} \right]$$

It is easy to prove that (2.50) implies

$$\lim_{N \rightarrow \infty} \frac{1}{t_N - t_1} \sum_{h=2}^N \left[W_{t_h} - \frac{\sum_{l=1}^{h-1} W_{t_l}}{h-1} \right] = 0$$

Thus

$$(2.54) \quad \lim_{N \rightarrow \infty} \bar{x}_{t_N} = x_0 \quad \lim_{N \rightarrow \infty} \frac{1}{t_N - t_1} \sum_{h=2}^N \frac{t_h - 1}{h - 1}$$

In terms of the sequence t (2.48) may be written

$$\hat{\sigma}_{t_N} = \frac{N - 1}{t_N - 1} \quad N > 1$$

thus, from (2.49)

$$\lim_{N \rightarrow \infty} \frac{N}{t_N} = \sigma > 0$$

Thus condition (2.42) is satisfied and condition (2.43) is obviously satisfied by the sequence t . From Lemma 1 it then follows that

$$\lim_{N \rightarrow \infty} \frac{1}{t_N - t_1} \sum_{h=2}^N \frac{t_h - 1}{h - 1} = \lim_{N \rightarrow \infty} \frac{1}{t_N} \sum_{h=1}^N \frac{t_h}{h} = 1$$

Thus, from (2.54)

$$x = \lim_{N \rightarrow \infty} \bar{x}_{t_N} = x_0$$

which completes the proof.

3. GENERALIZATIONS

3.1 Linear, Elastic Markets

The theory of equilibrium behavior in inelastic markets is readily generalized to a large class of elastic markets. Suppose an inelastic market is at equilibrium with a set \mathcal{S}^* of n^* undominated sellers. Consider an arbitrary subset, \mathcal{S}' , of \mathcal{S}^* containing $m^* < n^*$ of these undominated sellers which we relabel with indices $i = 1, 2, \dots, m^*$. From (2.34) and (2.30) we have already shown that

$$\sigma_i^* = 1 - \frac{\bar{W}_i - \bar{P}_i^*}{K_i} \quad i = 1, \dots, m^*$$

and since, from (2.28) and (2.29)

$$\bar{W}_i - \bar{P}_i^* = \bar{W}_j - \bar{P}_j^* = \Lambda^*$$

for all pairs of undominated sellers we shall write

$$(3.1) \quad \sigma_i^* = 1 - \frac{\bar{W}_\cdot - \bar{P}_\cdot^*}{K_i} \quad i = 1, \dots, m^*$$

where the dummy index in \bar{W} and \bar{P}^* is understood to hold for any of the \underline{n}^* undominated sellers.

If we sum (3.1) over all sellers in \mathcal{S}' we obtain

$$(3.2) \quad \sigma_{(m^*)}^* = \sum_{j=1}^{m^*} \sigma_j^* = m^* - (\bar{W}_\cdot - \bar{P}_\cdot^*) \sum_{j=1}^{m^*} \frac{1}{K_j}$$

Thus (3.2) defines the total sales of an arbitrary subset of undominated sellers. Notice that no direct reference is made to the undominated sellers not in this subset except through the difference, $\bar{W}_\cdot - \bar{P}_\cdot^*$, between secured worth and secured price which prevails for all undominated sellers.

Now consider the complementary set $\mathcal{S}'' = \mathcal{S}^* - \mathcal{S}'$ of $n^* - m^*$ undominated sellers which we relabel with indices $i = m^*+1, \dots, n^*$. The total market was assumed inelastic and by convention we set

$$\sum_{j=1}^{n^*} \sigma_j^* = 1$$

It then follows that

$$\sum_{j=m^*+1}^{n^*} \sigma_j^* = 1 - \sum_{j=1}^{m^*} \sigma_j^*$$

$$(3.3) \quad \sum_{j=m^*+1}^{n^*} \sigma_j^* = (\bar{W}_\cdot - \bar{P}_\cdot^*) \sum_{j=1}^{m^*} \frac{1}{K_j} - (m^* - 1).$$

Thus the total sales of all undominated sellers not in \mathcal{S}' depend only on the prevailing difference between secured worth and secured price, the total (reciprocal) strength of sellers in \mathcal{S}' and the number of sellers in \mathcal{S}' . Equation (3.3) may be interpreted as the demand function of the set \mathcal{S}'' induced by the set \mathcal{S}' .

Recall that the non-cooperative equilibrium point is generated by the independent profit seeking adjustments of the sellers. Thus equilibrium behavior does not depend on which sellers we considered internal to the market and which we considered external. It is precisely this property which makes the non-cooperative

equilibrium concept so appropriate to unilateral markets embedded in large, complex economies. Now, suppose we had excluded all members of the arbitrary set \mathcal{S}' from our original specification of the market so that alternatives offered by sellers in \mathcal{S}' became part of the external alternatives considered by customers when determining the worth of internal alternatives. The resulting reduced market would be elastic since some transactions would result in sales to members of \mathcal{S}' and thus be external to the reduced market. But behavior would not change, as we just noted. Thus the total sales of the reduced market would be given by (3.3) and we would observe

$$(3.4) \quad Q = (\bar{W}_0 - \bar{P}_0^*) \sum_{j=1}^{m^*} \frac{1}{K_j} - (m^* - 1)$$

as the demand function of the reduced market.

Conversely, suppose we are given an elastic market with the linear demand function

$$(3.5) \quad \begin{aligned} Q &= 0 & \bar{W}_0 - \bar{P}_0 &\leq \beta/\alpha \\ &= \alpha(\bar{W}_0 - \bar{P}_0) - \beta & \beta/\alpha < \bar{W}_0 - \bar{P}_0 < (\beta + 1)/\alpha \\ &= 1 & \bar{W}_0 - \bar{P}_0 &\geq (\beta + 1)/\alpha \end{aligned}$$

Now consider the extension of this market to an inelastic market with m^* additional sellers, each with identical strength K_0 , where

$$(3.6) \quad K_0 = \frac{\beta + 1}{\alpha}$$

$$(3.7) \quad m^* = \beta + 1$$

From (3.3), the demand function of the original sellers induced in the inelastic extension by the additional sellers is

$$\begin{aligned} Q &= (\bar{W} - \bar{P}) \frac{m^*}{K_0} - (m^* - 1) \\ &= \alpha(\bar{W} - \bar{P}) - \beta \end{aligned}$$

and one can easily show that the limits, $Q = 0$ and $Q = 1$, are also satisfied as in (3.5).

It follows then that any elastic market with a linear demand function may be represented, by extension, as an inelastic market. Several interesting consequences follow.

First, it follows that we cannot distinguish between elasticity induced by unspecified sellers and elasticity induced by variations in taste or income by examining only the behavior of sales in an elastic market whose demand function is linear. One may postulate either interpretation with identical conclusions. Thus the underlying cause of elasticity is behaviorally indeterminate if the demand function is linear.

Second, all of the properties of inelastic markets presented in Section 2 are now extended to elastic markets with linear demand functions. Moreover, since all of the information required to extend the elastic market to an equivalent inelastic market is contained in the parameters of the demand function, it follows that knowledge of the demand function is sufficient to determine the equilibrium behavior of elastic markets with linear demand functions.

Consider the market whose demand function is given by (3.5). Suppose there are N sellers and let them be labeled in decreasing order of strength. We will consider only the special case where

$$(3.8) \quad K_1 < \frac{\beta + 1}{\alpha}$$

that is, the strongest seller is weaker than the upper bound of the demand curve. (The general case offers no difficulties but the equations are somewhat more cumbersome.) We now extend the market to an inelastic market with m^* additional sellers with identical strength K_0 , as defined in (3.6) and (3.7). Modifying (2.10) to acknowledge the new sellers, the Equilibrium Entry Condition becomes

$$K_i > \frac{i - 2 + m^*}{\sum_{j=1}^{i-1} \frac{1}{K_j} + \frac{m^*}{K_0}}$$

or, in terms of the parameters of the demand function

$$(3.9) \quad K_i > \frac{i - 1 + \beta}{\sum_{j=1}^{i-1} \frac{1}{K_j} + \alpha}$$

thus, as we might expect, β , the intercept of the demand function (with sign reversed), inhibits entry and α , the rate at which increased demand is generated by increased customer surplus, facilitates entry. Let n^* be the number of sellers in the unextended market satisfying (3.9). Then, from (2.27) and (1.5), we can write

$$\bar{W}_* - \bar{P}_*^* = \frac{n^* + m^* - 1}{\sum_{j=1}^{n^*} \frac{1}{K_j} + \frac{m^*}{K_0}}$$

for the extended market, and

$$(3.11) \quad \bar{W}_* - \bar{P}_*^* = \frac{n^* + \beta}{\sum_{j=1}^{n^*} \frac{1}{K_j} + \alpha}$$

for the original market. Equilibrium sales, q_i^* , and profits, Π_i^* , may now be determined from (2.35) and (2.40).

To see the effect the number of undominated sellers has on total market sales, consider the virtually symmetric market where $K_i = K$, $i = 1, \dots, n^*$.

Then (3.11) becomes

$$\begin{aligned} \bar{W}_* - \bar{P}_*^* &= \frac{n^* + \beta}{\frac{n^*}{K} + \alpha} \\ &= K \frac{n^* + \beta}{n^* + \alpha K} \end{aligned}$$

which, when substituted in (3.5), yields

$$q^* = \alpha K \left(\frac{n^* + \beta}{n^* + \alpha K} \right) - \beta$$

which may be reduced to

$$q^* = (\alpha K - \beta) \left(1 - \frac{\alpha K}{n^* + \alpha K} \right) .$$

Thus Q^* is smallest if $n^* = 1$ and approaches $\alpha K - \beta$ as n^* grows large, reflecting the well-known propensity of markets with few sellers to restrict output.

Further generalization is possible. It is easy to show that elastic markets whose demand functions are piece-wise linear and convex downward can always be represented by extension to an inelastic market with additional sellers. The breaks in the demand function correspond to exit or entry of one or more of the additional sellers whose strengths are exactly equal to the value of $\bar{W} - \bar{P}$ where the break occurs. Elastic markets whose demand functions are piece-wise linear and convex upwards require direct investigation (by the methods given above) of each linear segment since more than one equilibrium point may exist.^{11/}

3.2 Non-linear Markets

A formal analysis of non-linear markets will be deferred to Part III of this study. Here we shall only call attention to some properties of non-linear strength functions which provide, at least in principle, a key to the determination of equilibrium points under certain conditions. We will restrict the discussion to inelastic markets.

Recall from Section 2.4 that since both worth and cost of a given seller are, by definition, independent of the sales levels of other sellers, it follows that

$$\begin{aligned}
 \frac{\partial K_i}{\partial x_i} &= \frac{\partial}{\partial x_i} \left(\bar{w}_i - \bar{c}_i \right) \\
 &= \sum_{j=1}^N \frac{\partial}{\partial \sigma_j} \left(\bar{w}_j - \bar{c}_j \right) \cdot \frac{\partial \sigma_j}{\partial x_i} \\
 &= \frac{\partial K_i}{\partial \sigma_i} \cdot \frac{\partial \sigma_i}{\partial x_i}
 \end{aligned}$$

that is, the derivative of strength with respect to competitive expense equals the derivative of strength with respect to sales times the derivative of sales with respect to competitive expense.

The i^{th} seller's profit derivative may then be written

$$\begin{aligned}
 \frac{\partial \Pi_i}{\partial x_i} &= \frac{\partial}{\partial x_i} \left(K_i \sigma_i - x_i \right) \\
 (3.13) \quad &= \left(K_i + K_i' \sigma_i \right) \frac{\partial \sigma_i}{\partial x_i} - 1
 \end{aligned}$$

where

$$K_i' = \frac{\partial K_i}{\partial \sigma_i}$$

In inelastic markets,

$$\sigma_i = \frac{x_i}{\sum_{j=1}^N x_j}$$

thus

$$\frac{\partial \sigma_i}{\partial x_i} = \frac{\sum_{j=1}^N x_j - x_i}{\left(\sum_{j=1}^N x_j \right)^2}$$

which may be written

$$(3.14) \quad \frac{\partial \sigma_i}{\partial x_i} = \frac{1 - \sigma_i}{\sum_{j=1}^N x_j}$$

Combining (3.13) and (3.14) gives

$$(3.15) \quad \frac{\partial \Pi_i}{\partial x_i} = (K_i + K_i' \sigma_i) \frac{(1 - \sigma_i)}{\sum_{j=1}^N x_j} - 1$$

Recalling, from (1.5), that $\sum_{j=1}^N x_j$, the sum of competitive expenditures, equals Λ , the customer surplus, (3.14) may be written

$$\frac{\partial \Pi_i}{\partial x_i} = (K_i + K_i' \sigma_i) \frac{(1 - \sigma_i)}{\Lambda} - 1$$

and since, from (1.20), σ_i is proportional to x_i , the necessary conditions for a non-cooperative equilibrium point, (2.1) and (2.2), may now be written:

either

$$(3.16) \quad \left. \begin{array}{l} \sigma_i^* > 0 \quad \text{and} \quad (K_i + K_i' \sigma_i^*) (1 - \sigma_i^*) = \Lambda^* \\ \text{or} \\ (3.17) \quad \sigma_i^* = 0 \quad \text{and} \quad K_i \leq \Lambda^* \end{array} \right\} \quad i = 1, \dots, N$$

where K_i and K_i' are both understood to be functions of σ_i^* .

Thus seller i is excluded from the market if the prevailing customer surplus level Λ^* is not less than seller i 's limiting strength in the vicinity of zero sales. The expression

$$F_i(\sigma_i) = (K_i + K_i'\sigma_i)(1 - \sigma_i)$$

describes the surplus level that must prevail if a given positive sales level is an equilibrium sales level and might therefore be called the inverse supply function of seller i .

In general the equilibrium point, $\Sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_N^*)$, will not be unique. There may be many such points or -- in a sufficiently pathological case -- there may be none.

Two methods of finding equilibrium points suggest themselves. An iterative procedure might be used. Starting with some initial estimate of Σ , such as $\sigma_i = 1/N$, for all i , the corresponding local values of $K_i + K_i'\sigma_i$ could be computed and used as fixed values in the non-linear solution. This would produce new estimates of the $\{\sigma_i\}$ from which new estimates of $K_i + K_i'\sigma_i$ could be computed. Then a new set of $\{\sigma_i\}$ could be calculated, etc. If each K_i is a continuous, decreasing function of σ_i we would expect that the iterations would converge rapidly to an equilibrium point.

A second method involves constructing each seller's supply function, $G_i(\Lambda)$, by inverting $F_i(\sigma_i)$, deleting multiple values, if any, by some criterion such as

selecting the largest σ corresponding to a given Δ . If the resulting supply curves are continuous, decreasing functions of prevailing surplus, as we expect they would be in many markets, then the equilibrium point is unique and may be found by solving the equation

$$\sum_{j=1}^N G_j(\Delta^*) = 1$$

which forces the total sales of the market to equal one, as required in an inelastic market. The Δ^* which solves this equation is, of course, the equilibrium surplus level and the equilibrium sales levels are given by

$$\sigma_i^* = G_i(\Delta^*) .$$

If the K_i functions are of simple form, for example, linear (decreasing) or negative exponential, this second approach is particularly useful.

3.3 More General Equilibria

In Section 2 we investigated non-cooperative equilibrium points in linear, inelastic markets. Here we present a method of dealing with other kinds of behavior and examine some of the properties of more general equilibria. Specifically, we relax the assumption that each seller is concerned only with his own profit postulating, instead, a seller utility function of the form

$$(3.18) \quad U_i = \sum_{j=1}^N e_{ij} \Pi_j$$

that is, the i^{th} seller wishes to maximize some linear combination of the profits of all sellers.^{12/} If we assume that no seller is entirely indifferent to his own profit ($\Theta_{ii} \neq 0$) then there is no loss of generality in requiring the matrix Θ be normalized so that $\Theta_{ii} = 1$ for all i . In this normalized form we will call Θ the profit exchange matrix since Θ_{ij} then expresses the number of units of seller j profit seller i would trade for one unit of his own profit. We will say seller i is aggressive (with respect to j) if $\Theta_{ij} < 0$, cooperative if $\Theta_{ij} > 0$, and indifferent if $\Theta_{ij} = 0$.

Suppose we now examine non-cooperative equilibrium points under this enlarged definition of seller utility. One can show that generally such points exist and are unique (so long as the sellers are not too cooperative!) A number of interesting interpretations are possible. Notice that if $\Theta \equiv \mathbf{I}$, that is, if each seller is indifferent to every other seller, then the equilibrium solution to the enlarged process obviously corresponds to the equilibrium solution to the original process. If Θ is $+1$ everywhere, the enlarged solution (which exists only as a limit) corresponds to the joint maximal solution to the original process, since every seller is trying to maximize the total profits of all sellers. On the other hand, if Θ is $-1/N-1$ everywhere off the main diagonal, then the enlarged solution corresponds to a strictly competitive solution to the original process, since every seller is trying to maximize the difference between his profit and the average profit of all other sellers. Notice that in this latter case the market may be interpreted as a zero-sum game since
$$\sum_{j=1}^N U_j = 0.$$

Thus the extension from Π to U makes it possible to study a very large class of patterns of competition using only the tools of non-cooperative equilibrium analysis. We can study, for example, the effect of clusters of cooperative sellers which might be present in a market in which various sub-sets of the sellers are members of the same firms. Since the analysis of such special cases is readily carried out we shall not present it here. Instead we will restrict our attention to an interesting sub-class of profit exchange matrices which possesses a number of important properties. Specifically, let

$$(3.19) \quad \begin{aligned} \Theta_{ij} &= 1 && \text{if } j = i \\ &= \frac{\gamma_i}{K_j} && \text{otherwise} \end{aligned}$$

that is, the weight each seller places on every other seller's profit is inversely proportional to the other seller's strength. Notice that under (3.19) the seller attaches greatest weight to the weakest competitors and least weight to the strongest. Thus if γ_i is negative seller i acts like a bully, doing most damage to the weaker sellers. Similarly if γ_i is positive, seller i is a kind of commercial Robin Hood, helping the weak much more than the strong. This pattern of interaction is not so unlikely as it might seem, for (3.18) may now be written

$$U_i = \Pi_i + \gamma_i \cdot \sum_{j \neq i} \frac{\Pi_j}{K_j}$$

and, from (1.20) and (1.21)

$$\Pi_i = K_i \sigma_i - x_i$$

Thus

$$U_i = K_i \sigma_i - x_i + \gamma_i \sum_{j \neq i} \sigma_j - \gamma_i \sum_{j \neq i} x_j .$$

We will assume the market is inelastic, so that

$$\sum_{j=1}^N \sigma_j = 1$$

Hence

$$U_i = (K_i - \gamma_i) \sigma_i - x_i + \gamma_i \left\{ 1 - \sum_{j \neq i} x_j \right\}$$

Since the term in brackets is independent of x_i , it drops out when we differentiate U_i . Thus the weighting procedure we introduced in (3.19) equivalent to having seller i attach a value of $-\gamma_i$ to each unit of market share he obtains. Such extrinsic evaluation of market share is not uncommon in many contemporary firms. It is particularly significant that a seller may adopt a pattern of this type without knowing anything about the other sellers, even how many there are. Notice that γ_i simply produces a translation of K_i .

We will let

$$K_i' = K_i - \gamma_i$$

and call K_i' the effective strength of seller i , as distinguished from K_i , the true strength of seller i . Observe that if seller i is cooperative ($\gamma_i > 0$) his effective strength is less than his true strength and, conversely if he is aggressive ($\gamma_i < 0$) his effective strength is greater than his true strength. Thus cooperative sellers tend to understate their strength and aggressive sellers tend to overstate their strength.

We will call γ_i the competitive disguise of seller i . It is interesting that this disguise can spring from intentional distortion as in (3.19) or from incorrect estimation of K_i . The effect is the same.

It follows that every market of this class has a unique equilibrium point which can be determined by replacing K_i by K_i^i and applying the methods developed in Section 2. More significantly, for any K , every point in X^+ is an equilibrium point for some Γ , where $K = (K_1, \dots, K_N)$ denotes an arbitrary set of (positive) seller strengths, X^+ denotes the positive orthant of $X = (x_1, \dots, x_N)$, the space of competitive expenditures and $\Gamma = (\gamma_1, \dots, \gamma_N)$ denotes some point in the space of competitive disguises. In Section 2.4 we showed that if X^* is an equilibrium point, then

$$K_i = \frac{\left(\begin{array}{c} N \\ \sum_{j=1}^N x_j^* \end{array} \right)^2}{\sum_{j=1}^N x_j^* - x_i^*} \quad i = 1, \dots, N$$

for undominated sellers ($x_i > 0$) and since X^* is confined to the positive orthant all K_i must then satisfy this condition. It is easy to show that this is also sufficient condition and that for any K^0 , X^0 (in X^+) the competitive disguises

$$\gamma_i = K_i^0 - \frac{\left(\begin{array}{c} N \\ \sum_{j=1}^N x_j^0 \end{array} \right)^2}{\sum_{j=1}^N x_j^0 - x_i^0} \quad i = 1, \dots, N$$

produce a system whose unique equilibrium point is $X^* = X^0$.

An extremely important consequence follows. From X^* , the equilibrium behavior of the market, we can determine K^i , the effective strengths of the sellers, but it is not possible to separate K and Γ . Thus one cannot determine a seller's true strength from his behavior; only his effective strength. A seller who is strong and cooperative will behave precisely the same way as a seller who is weak and aggressive. Unless we can directly observe his true K there is no way of determining his competitive disguise. This raises serious questions as to the meaningfulness of threats and signals in markets in which the sellers do not know each other's costs.

It is interesting to observe that in such a market there are two very different ways a seller might interpret the behavior of his competitors. He might, on one hand, assume indifference ($\gamma_j = 0, j \neq i$) and estimate their "true" strengths under this assumption. On the other, he might assume symmetry ($K_j = K_i$), and estimate their apparent disguise. Using the first approach a seller would be able to provide a technological rationalization for any move his competitors made and would never question their intent. Using the second approach a relatively strong seller would conclude that his competitors were trying to act cooperatively and a relatively weak seller would conclude that his competitors are predatory, aggressive, uncooperative, and out to "get" him. We suspect there are strong analogies between these two ways of interpreting competitive actions in a market and similar perception and interpretation alternatives at the inter-personal and international level.

FOOTNOTES

1. Certain bilateral markets can be treated as simultaneous unilateral markets by adding various conservation restrictions. For example, in stock markets, stock specialists who "take a position" in a particular stock synthetically split the market into two components; one in which sellers make offers and buyers choose and the other in which buyers make offers and sellers choose. But the role of bargaining in bilateral markets creates serious complications. We will not attempt such generalizations here.
2. Actually, the number of choosers does not have to be large so long as convention produces the same effect. For example, in some competitive bidding markets a single military purchasing officer may be the only customer, but convention (and law) can preclude bilateral negotiation and force the customer simply to select among offers made by sellers.
3. Infinitely divisible alternatives could be handled with slightly more cumbersome notation, but since these are relatively rare in unilateral markets we will confine our analysis to indivisible alternatives.
4. Mills (9), Friedman (5), and many others have used (1.20) as an ad hoc assumption where x_i denoted either the selling effort (Mills) or the advertising expense (Friedman) of the i^{th} seller. We note, also, a structural similarity between (1.20) and the results of Luce (7) who investigated individual choice mechanisms which satisfy a single behavioral postulate (essentially that the probability an undominated alternative i in \mathcal{S} will be chosen equals the probability i would be chosen from a subset \mathcal{S}' -- i in \mathcal{S}' in \mathcal{S} -- times the probability some member of \mathcal{S}' would be chosen from \mathcal{S}). Luce showed that where this condition is met (with two minor restrictions) choice is governed by a ratio scale V , defined over each alternative, such that the probability i is selected is proportional to V_i .
5. A very lucid exposition of the ideas of Cournot, Bertrand, and other important contributors of oligopoly theory is given by Fellner (4).
6. Actually, it is not even necessary to measure W_{it} . With a slightly more elaborate argument one can show that the recursive policy

$$P_{it} = P_{it-1} - x_i^0 \cdot \left(\frac{1}{\sigma_{it}} - \frac{1}{\sigma_{it-1}} \right)$$

also implements the decision $x_1 = x_1^0 (> 0)$, regardless of the initial value P_{i0} . This follows easily from the assumption that $\lim_{t \rightarrow \infty} \hat{W}_{it} = \bar{W} < \infty$

since $\hat{W}_{it} - \hat{W}_{it-1}$ can then be made arbitrarily small by choosing a sufficiently large t , and if $\Delta \hat{W}_{it}$ is added to the right-hand side of this recursive policy the resulting equation is simply the first difference of the original policy. Note that x_1^0 can thus be interpreted as the response coefficient of the recursive policy.

7. I.e., games in which the strategies of the players are reducible to a finite set. Since x_1 may take on any non-negative real value the game defined by the profit equation (1.21) is not a finite game.
8. Here and throughout the paper when we speak of the "average" strength of a set of sellers the harmonic mean is implied.
9. Mills (9) has given a constructive proof of the existence and uniqueness of X^* . We have developed the proof in a somewhat different manner here to demonstrate additional properties of the equilibrium point such as the ordering of competitive expenditures, the undomination of the strongest and second-strongest seller, and the role of the Equilibrium Entry Condition.
10. A more complete investigation of the investment implications of the theory will be presented in Part IV of the study.
11. We acknowledge that Marshallian tradition would have demand functions be convex upwards. But such conclusions are based on a theory of individual consumption of infinitely divisible commodities by insatiable consumers who have limited consumption alternatives. We, on the other hand are interested in aggregate consumption in contemporary unilateral markets characterized by indivisible alternatives selected by satiable customers who have numerous consumption alternatives. We suspect the demand function is more likely to be convex downward in such markets but shall not attempt a more extensive argument here.
12. Bishop (2) proposed that inter-seller hostility might be described by adding to each seller's profit function a negatively-weighted linear combination of the profits of the other sellers' profits.

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