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Substitution, Fixed Proportions, Growth and Distribution

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Two opposing concepts of capital pervade contemporary models of economic growth. The fixed-proportions school treats the labor requirements of capital goods as rigid, not subject to choice. The neoclassical school imagines that capital is like putty; it can be continuously reshaped to accommodate any supply of labor. There may be some truth in both concepts. This paper presents a model which incorporates elements of both.

In the present model, only new capital is putty. Before their installation, machines can be designed to utilize any desired amount of labor. But once this putty takes shape, it turns to hard-baked clay. The labor requirements of machines are fixed forever at the time of construction. The utilization of these machines may change over time but that is a different matter.

One of the products of this model is a theory of the operating life and labor intensity of capital goods. A machine is retired here when rising wages have absorbed all its revenues. Therefore a machine will operate longer the smaller its labor intensity. The labor intensity of the optimal type of new machine depends upon the anticipated course of wages and the rate of interest.

These relationships introduce a new dimension to the connection between investment and the growth of productivity. An increase in thrift lowers the

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* The author is grateful to Edwin Mansfield and T. N. Srinivasan for discussions with them of this subject.

rate of return on capital, reduces the labor intensity of new machines and thus ultimately lengthens the operating life of all machinery. We call this process "capital lengthening" to distinguish it from capital deepening, which denotes here the multiplication of machines without any change of their longevity. Increased thrift affects productivity through both the lengthening and deepening of capital.

It follows that an increase in thrift, far from modernizing the capital stock, except temporarily, must eventually increase the average age of machinery. But is maturity of the capital stock a bad thing? It is shown that the capital lengthening effect, on balance, acts to reinforce the capital deepening effect of increased thrift upon productivity so long as there is so little capital that the rate of interest exceeds the long-run rate of growth. This condition appears to be the rule at least in technologically progressive economies. Thus investment may be a more effective growth agent in these economies than it has been judged to be on the basis solely of its capital deepening effect.

Another product of the model is a theory of factor shares. Since labor's relative share of aggregate output is a weighted average of its relative share of the output of every machine and that share is normally higher at old machines than new, the average age of machines is seen to affect aggregate shares. The average age will tend to be greater and the share of wages (quasirents) in total output smaller (larger) the thriftier is the economy. This may help to explain the positive relation observed between saving and profits (as shares of income) across economies.

In neoclassical models, capital's share equals the capital elasticity of output. Solow and others have used the former as an estimate of the latter in order to assess the importance of investment as a source of productivity growth in the American economy.* In the present model, capital's share falls

* Robert M. Solow, "Technical Change and the Aggregate Production Function," Review of Economics and Statistics, 39, (August 1957).

short of its neoclassical level, so the procedure indicated would understate the importance of investment, wherever thrift is insufficient to drive the interest rate down to (or below) the rate of growth.** Therefore, in the

** In the model here, "capital" and the "capital elasticity" do not exist but an appropriate substitute is the investment elasticity of additions to capacity. The sentence above states that capital's share is smaller than this elasticity under the condition given.

United States, where this condition is satisfied, it may be that the effectiveness of capital deepening has been underestimated (quite apart from the matter of capital longevity discussed above). Thus the results here encourage a certain amount of optimism toward the utility of investment as a means to increase productivity.

Finally, some brief acknowledgments of the related literature: The notion that labor can be combined with new investment in variable proportions but with existing capital only in fixed proportions was introduced by Johansen.*

* Leif Johansen, "Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: A Synthesis," Econometrica, 27, (April 1959).

A model along similar lines recently appeared by Massell.** This paper owes much to them. Its differences from them arise from its capital-theoretic treatment of the longevity of machinery as a dependent variable, rather than a parameter.

** Benton F. Massell, "Investment, Innovation and Growth," RAND P-2149 (revised), (November 1961).

A third paper that takes a point of view nearer the one here is by Solow.*** His paper is directed toward capital theory rather than growth theory, as here. However it has been useful at a number of places in the present paper, especially in section 3.

*** Robert M. Solow, "Substitution and Fixed Proportions in the Theory of Capital," Review of Economic Studies (forthcoming, 1962).

1. The scene.

The scene is an economy or industry producing a single good by means of two scarce inputs, machinery and labor time.* While there is an infinite variety of machines with respect to their labor requirements, their durability cannot be varied: all capital lasts forever.

A basic notion of the model is the "capacity" of a machine. This is defined as the maximum output rate obtainable from it by means of increasing the amount of labor employed on it. $\bar{Q}(v, t)$ shall denote the capacity output at time t of all machines built at time v .

To produce their capacity output at time t , machines of vintage v require a certain (minimum) amount of labor time, denoted $\bar{N}(v, t)$.

All investment consists of the purchase of new machines. Existing machines cannot be modified in any way.

While certain patterns of efficiency loss through wear and tear of equipment would be easy to introduce, it is simplest to assume that the capacity of a machine remains constant throughout its life. The labor requirement at capacity is also assumed to be fixed. Thus

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* The "economy" interpretation raises the question of the source of machinery. One can think of the single good as homogeneous putty of unchanging quality in consumption. At the same time, engineers find increasingly efficient ways to shape this putty into machines. We ignore the machine-producing sector by assuming that no scarce inputs are required to mold putty into machinery.

(1.1) $$\overline{Q}(v, t) = \overline{Q}(v, v) \qquad \text{for all } t \geq v$$

(1.2) $$\overline{N}(v, t) = \overline{N}(v, v) \qquad \text{for all } t \geq v$$

Next we suppose that the producer neglects any possibility of Keynesian underutilization of capacity when he buys a new machine. He assumes he can sell whatever output he can produce. Pure competition prevails.

Further, a downward sloping relation between capacity utilization and unit variable (labor) costs is postulated making it optimal (profit maximizing) for the machine to produce at capacity if it is preferable to produce at all.*

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* This assumes the absence of escapable overhead (labor) costs. All wage costs are variable costs and all variable costs are wage costs.

Therefore it is optimal to produce at capacity if revenues cover capacity labor costs with some quasirent left over; otherwise it is optimal to shut down the machine. Thus

(1.3) $$Q^*(v,t) = \begin{cases} \overline{Q}(v,v) & \text{if } w(t)\,\overline{N}(v,v) < \overline{Q}(v,v) , \\ 0 & \text{otherwise} . \end{cases}$$

when $w(t)$ denotes the wage rate at t and $Q^*(v,t)$ denotes the optimal output rate of vintage v machines at time t.

Under these conditions the producer-investor who buys a new machine will expect to operate it at capacity as long as he expects it to be profitable to operate. The producer is supposed to predict the future course of the wage rate with complete confidence and to predict that the wage will rise at a constant relative rate. On these conditions he will expect to operate the machine continuously (at capacity) up to the date on which he expects the machine to cease to earn positive quasirents. He will expect to retire the machine permanently at that time.

Let $\hat{w}(u,t)$ denote the wage rate expected at time $\underline{t}$ to prevail at time $\underline{u}$, $u \geq t$. Of course, $\hat{w}(t,t) = w(t)$. And let $\omega(t)$ denote the constant relative rate of increase in the wage rate which is expected by producers at time $\underline{t}$.

$$(1.4) \qquad \hat{w}(u,t) = e^{[\omega(t)](u-t)} w(t)$$

Thus a machine with capacity $\bar{Q}(t,t)$ and labor requirement $\bar{N}(t,t)$ would be expected when new -- that is, at $\underline{t}$ -- to produce and yield quasirent for $\hat{z}(t)$ years where $\hat{z}(t)$ is determined by the relation

$$(1.5) \qquad \hat{w}(t + \hat{z}(t),t)\, \bar{N}(t,t) = \bar{Q}(t,t)$$

which, using (1.4), reduces to

$$(1.6) \qquad e^{\omega(t)\hat{z}(t)}\, w(t)\, \bar{N}(t,t) = \bar{Q}(t,t) .$$

It is evident that the prospective lifetime of new machines may vary through time so that $\hat{z}(t)$ is not a constant. Variations in $\omega(t)$, for example, will clearly produce changes in $\hat{z}(t)$.* In section 4 a growth model

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* Also, note that it is only when we ignore gestation periods, as we do, that the initial wage rate that a new machine owner would have to pay, $w(t)$, can be taken as a datum. If a gestation period were introduced, the investor would choose a machine for time u, $u \geq t$, on the basis of the wage rate expected to prevail then, $\hat{w}(u,t)$.

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is presented in which $\omega(t)$ is made a dependent variable instead of a parameter.

In order to select the optimal type of new machinery the producer needs a discount rate to compare prospective quasirents occurring at different times. Let $\hat{r}(u,t)$ denote the rate of return which the firm expects at time $\underline{t}$ to earn on investments of time $\underline{u}$. Of course, the firm can calculate the rate of return which they anticipate earning on current investment: this is denoted $r(t)$. It will be assumed that producers expect the rate of return on new capital to remain at its current level, even though past rates of return may have differed from the present rate. Thus

$$(1.7) \qquad \hat{r}(u,t) = r(t) \qquad \text{for all } u \geq t$$

Of course, producers may be wrong and $r(t)$ may in fact change through time.

Producers have to determine the scale and labor intensity of new machines in the light of these expectations and the technological possibilities before them. To eliminate the scale decision we take as exogenous the constant-dollar level of expenditures on new machinery at time $\underline{v}$, $I(v)$ and derive the implied $r(v)$. With $I(v)$ given, the problem reduces to the question of "labor intensity": Shall those I dollars be spent on a type of machinery requiring much or little labor (at capacity)? Presumably an engineer who is hired to design a machine costing I dollars can offer one having greater capacity the greater is the amount of labor which the machine can utilize. Labor-using machinery which does not produce more than labor-saving machinery is inefficient and never leaves the drawing board. We shall assume that the relations among capacity, labor requirement and cost of new (efficient) plants at time $\underline{v}$ are given by the familiar Cobb-Douglas function:

$$\bar{Q}(v,v) = B(v)\, I(v)^{\alpha}\, \bar{N}(v,v)^{\beta}, \quad B'(v)>0,\ 0<\alpha<1,\ 0<\beta<1. \tag{1.8}$$

The function $B(v)$ indicates the state of the technology at time $\underline{v}$. The assumption of competition requires that $\alpha + \beta \leq 1$ and through much of the paper it is required that $\alpha + \beta = 1$ (constant returns to scale).*

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* On the economy interpretation, $I(v)$ can be measured in the same units as $Q(t,t)$, e.g., pounds of putty. On the industry interpretation, (1.8) effectively assumes that the prices of all machine types move equiproportionately so that investment outlays can be deflated without any index number problem arising.

2. The labor intensity of new machinery.

Since the labor requirement (for capacity output) of any type of machinery is immutable, once chosen, it is impossible to buy a machine now which uses the optimal amount of labor at all times during its lifetime. The machine having a labor requirement such that it yields the greatest possible flow of quasirent in the near future will not be the machine yielding the greatest possible flow of quasirent later when the wage rate is higher. The optimal machine type at time t has the labor requirement, $\bar{N}(t,t)$, which maximizes the sum, denoted by U , of the expected discounted quasirents over its expected lifetime:

$$(2.1) \qquad U = \int_t^{t+\hat{z}(t)} R(u,t)\, e^{-[r(t)](u-t)}\, du$$

where $R(u,t)$ is the flow of quasirent expected as of t to accrue at time u .

By the assumptions made above,

$$(2.2) \qquad R(u,t) = \bar{Q}(t,t) - w(t)\, e^{[\omega(t)](u-t)}\, \bar{N}(t,t)$$

Equations (2.1) and (2.2) yield

$$(2.3) \qquad U = \bar{Q}(t,t) \int_t^{t+\hat{z}(t)} e^{-[r(t)](u-t)} du - w(t)\, \bar{N}(t,t) \int_t^{t+\hat{z}(t)} e^{[\omega(t)-r(t)](u-t)}\, du ,$$

which is to be maximized with respect to $\bar{N}(t,t)$, subject to the production function (1.8) and the operating lifetime function (1.6).

To find the optimal $\overline{N}(t,t)$ we take the derivative $\frac{\partial U}{\partial \overline{N}}$, letting both $\hat{z}(t)$ and $\overline{Q}(t,t)$ vary with $\overline{N}(t,t)$, and equate it to zero. This yields

$$(2.4)\quad \frac{\partial \overline{Q}}{\partial \overline{N}} \int_t^{t+\hat{z}(t)} e^{-[r(t)](u-t)} du - w(t) \int_t^{t+\hat{z}(t)} e^{[\omega(t)-r(t)](u-t)} du + \frac{\partial \hat{z}}{\partial \overline{N}} e^{-r(t)\hat{z}(t)} \left[\overline{Q}(t,t) - w(t)\overline{N}(t,t)\, e^{\omega(t)\hat{z}(t)} \right] = 0$$

The expression on the left has to be evaluated at the optimal $\hat{z}(t)$ if the solution of (2.4) for $\overline{N}(t,t)$ is to be optimal. Recalling (1.6) we note that the bracketed expression must equal zero. The fact that a slightly smaller labor requirement will increase slightly the prospective lifetime of the plant is irrelevant to the choice of machinery because quasirent toward the end of the life of the plant is zero!

Since the integrands in (2.4) are exponential functions an additional simplification is possible and we obtain:

$$(2.5)\quad \frac{\partial \overline{Q}(t,t)}{\partial \overline{N}(t,t)} = c_t(\hat{z})\, w(t)$$

where $$c_t(\hat{z}) \equiv \frac{r(t)}{r(t) - \omega(t)} \cdot \frac{1 - e^{-[r(t)-\omega(t)]\hat{z}(t)}}{1 - e^{-r(t)\,\hat{z}(t)}}$$

Thus the marginal product of the capacity labor requirement is equated to the current wage rate multiplied by some constant, $c_t(\hat{z})$, which is a reflection of expectations.

Assuming that $r(t) > \omega(t)$, it is shown in Appendix A that $c_t(0) = 1$. But of course $\hat{z}(t) > 0$. It is shown also that $c_t(\hat{z}) > 1$ and moreover that $c_t(\hat{z})$ is monotonically increasing in $\hat{z}(t)$, for all $\hat{z}(t) > 0$. It is obvious from (2.5) that $c_t(\infty) = \frac{r(t)}{r(t) - \omega(t)}$, which is the upper limit on $c_t(\hat{z})$ and which is approached asymptotically.

The assumption that $r(t) > \omega(t)$ seems reasonable in view of the relation between actual rates of return and actual growth rates of real wage rates in progressive economies. It is clear from (2.5) that as $\omega(t)$ approaches $r(t)$ the labor intensity of new machinery approaches zero. The growth model of section 4 shows that the operating life of machinery would become infinite and the productivity of labor would vanish in this process. Therefore we can safely rule out the case where $\omega(t) \geq r(t)$.

Combining the marginal productivity formula from the production function (1.8)

$$(2.6) \qquad \frac{\partial \bar{Q}(t,t)}{\partial \bar{N}(t,t)} = \beta \frac{\bar{Q}(t,t)}{\bar{N}(t,t)}$$

with (2.5) yields

$$(2.7) \qquad \frac{\bar{Q}(t,t)}{\bar{N}(t,t)} = \frac{c_t(\hat{z})\, w(t)}{\beta}$$

Equations (2.7) and (1.6), which can be writte

$$(1.6a) \qquad \frac{\bar{Q}(t,t)}{\bar{N}(t,t)} = e^{\omega(t)\hat{z}(t)}\, w(t) ,$$

constitute two equations in two unknowns: the prospective life of the new machine, $\hat{z}(t)$, and the machine's optimal "labor intensity", as defined by the ratio $\frac{\bar{N}(t,t)}{\bar{Q}(t,t)}$. These two equations are graphed in Figure 1.

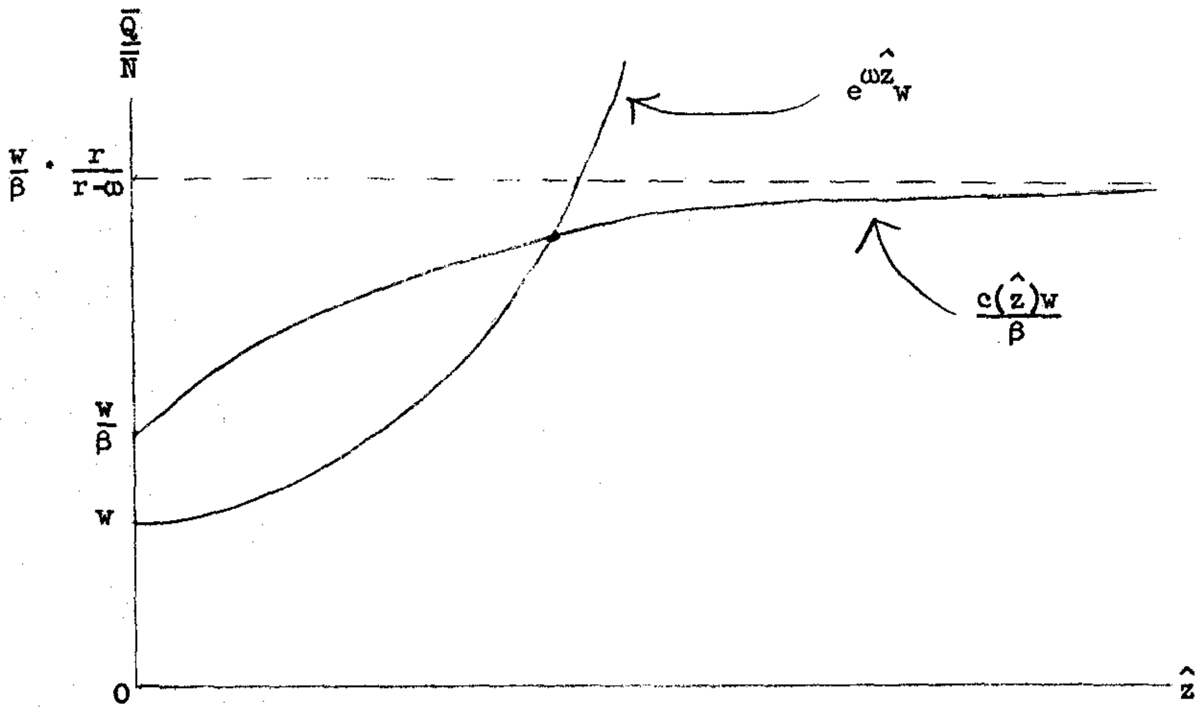

Figure 1

The intersection of the curves marks the optimal values of $\frac{\bar{Q}}{\bar{N}}$ and $\hat{z}$. By virtue of the assumptions $0 < \omega(t) < r(t)$ and $\beta < 1$ a unique solution exists.

The values of $\bar{N}(t,t)$ and $\bar{Q}(t,t)$ depend upon the amount to be invested, $I(t)$, which we take as exogenous. The production function (1.8) and (2.7) imply

$$(2.8) \qquad \bar{Q}(t,t) = \beta^{\frac{\beta}{1-\beta}} \left[\frac{B(t)}{(c_t(\hat{z})w(t))^{\beta}} \right]^{\frac{1}{1-\beta}} I(t)^{\frac{\alpha}{1-\beta}}$$

and

$$(2.9) \qquad \bar{N}(t,t) = \left[\frac{\beta\, B(t)}{c_t(\hat{z})w(t)} \right]^{\frac{1}{1-\beta}} I(t)^{\frac{\alpha}{1-\beta}}$$

The rate of return, $r(t)$, remains to be determined. This is defined by the relation

$$(2.10) \qquad \int_t^{t+\hat{z}} \frac{\partial R(u,t)}{\partial I(t)} e^{-[r(t)](u-t)} du = 1$$

Why should this rate of return be expected to prevail forever? If our competitive producers are continuously maximizing profits they must be investing at a rate such that the rate of return is equated to the rate of interest charged in the credit market. (If there are constant returns to scale, the interest rate must do the adjusting.) Thus the assumption of a constant $r(t)$ means that producers expect the rate of interest to remain constant.

From (2.2) we have $\frac{\partial R(u,t)}{\partial I(t)} = \frac{\partial \bar{Q}(t,t)}{\partial I(t)}$.

Making this substitution in (2.10) and evaluating the integral, one finds

$$(2.11) \qquad \frac{\partial \bar{Q}(t,t)}{\partial I(t)} = \frac{r(t)}{1-e^{-r(t)\hat{z}(t)}}$$

which, upon differentiating the production function (1.8), yields

$$(2.12) \qquad \frac{I(t)}{Q(t,t)} = \alpha \left[\frac{r(t)}{1 - e^{-r(t)\hat{z}(t)}} \right]$$

Equations (2.8) and (2.12) determine $r(t)$. Of course $r(t)$ depends upon $w(t)$. If there are constant returns to scale $(\alpha + \beta = 1)$ then $r(t)$ is independent of the extent of current investment, $I(t)$. (But $w(t)$ will depend upon the total past investment.)

Equations (2.7) and (2.12) yield the least-cost capital-labor ratio. If there are constant returns they imply

$$(2.13) \qquad \frac{I(t)}{\bar{N}(t,t)} \cdot \frac{\beta}{\alpha} = w(t) \left[\frac{1 - e^{-[r(t)-\omega(t)]\hat{z}(t)}}{r(t) - \omega(t)} \right]$$

The righthand side is the marginal rate of substitution. In the neo-classical case, in which capital is continuously reborn, the future drops out and the marginal rate of substitution is equated simply to $w(t)/r(t)$. Here the future has to be taken into account.

3. Equilibrium aggregate output and employment.

The equilibrium or "capacity" output of the entire industry (or economy), denoted $Q^*(t)$, is defined as the sum of the optimal outputs of the constituent firms. $N^*(t)$ shall denote the associated level of employment. By (1.3), $Q^*(t)$ is equal to the sum of the capacities of those plants which are currently profitable to operate.

As indicated by (1.3), the vintages of those existing machines which are profitable are those for which $w(t)\bar{N}(v,v) < \bar{Q}(v,v)$. Denote the set of such vintages by $V(t, w(t))$. Then aggregate equilibrium output is*

* This notion of the set of profitable vintages is borrowed from the related paper by Solow cited above.

$$(3.1) \qquad Q^*(t) = \int_{v \in V(t,w(t))} \bar{Q}(v, v)\, dv$$

and aggregate equilibrium employment is

$$(3.2) \qquad N^*(t) = \int_{v \in V(t,w(t))} \bar{N}(v, v)\, dv$$

Substituting (2.8) into (3.1) and (2.9) into (3.2) yields

$$(3.3) \qquad Q^*(t) = \beta^{\frac{\beta}{1-\beta}} \int_{v \in V(t,w(t))} \left[\frac{B(v)}{(c_v(\hat{z})w(v))^{\beta}}\right]^{\frac{1}{1-\beta}} I(v)^{\frac{\alpha}{1-\beta}}\, dv$$

and

$$(3.4) \qquad N^*(t) = \beta^{\frac{1}{1-\beta}} \int_{v\epsilon V(t,w(t))} \left[\frac{B(v)}{c_v(\hat{z})w(v)} \right]^{\frac{1}{1-\beta}} I(v)^{\frac{\alpha}{1-\beta}} dv$$

Equation (3.4) provides a demand function for labor. At a sufficiently high current wage no existing machine can cover wage costs so none operate and the amount of labor demanded is zero. As the wage rate falls, eventually the least labor intensive machines become profitable. The amount invested in that vintage determines the amount of labor demanded by these machines as shown by (2.9). The smaller the wage the larger is the set of machines (or vintages) that can operate and earn quasirents (up to the point where all machines are operating). Thus $N^*(t)$ is a decreasing function of $w(t)$.

If the industry or economy being modeled is small, the real wage might reasonably be treated as a parameter. If we want to make the wage an endogenous variable, the simplest way to determine its level is to assume that the wage rate equates labor demand, $N^*(t)$, to a perfectly wage-inelastic supply of labor, denoted $L(t)$. Then, in equilibrium

$$(3.5) \qquad L(t) = \beta^{\frac{1}{1-\beta}} \int_{v\epsilon V(t,w(t))} \left[\frac{B(v)}{c_v(\hat{z})w(v)} \right]^{\frac{1}{1-\beta}} I(v)^{\frac{\alpha}{1-\beta}} dv$$

(3.5) determines how large $V(t,w(t))$ must be, and thus how low $w(t)$ must be, in order to employ the available labor supply. We assume that a non-negative equilibrium wage rate exists.*

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* This is Solow's "nonredundancy assumption" (*op. cit.*) .

With $V(t, w(t))$ known, aggregate output can be computed from (3.3). A point of some methodological interest is the absence of $L(t)$ from the output equation. In Appendix B it is shown that it is not possible in general to express output as a function of labor and "capital." The reason appears to be that, loosely speaking, labor is not allocated over machines in any systematic way (according to a set of rules which apply at all times, e.g., "equalize labor's marginal productivity on all machines") but rather according to historical accident. Certain special histories do admit a production function. If the wage rate is stationary and expected to be so $(\omega(t) = 0)$ then aggregate output can be written as a Cobb-Douglas function of aggregate labor and capital. If the wage rate is constant it makes no difference whether capital is putty or clay.

Summarizing: Given the history of investment and employment -- thus an inventory of machines and their labor requirements -- we can determine potential output and employment at various wage rates. Given the current labor supply, the equilibrium wage rate, output and employment are determinate. From the wage rate and the current rate of investment we can determine the labor requirement and capacity output of new machines, provided we know the expected rate of increase in the real wage. Thus the whole future course of output and the wage rate is determined.

As a model of growth, the above is incomplete in that it treats investment and wage expectations as exogenous. There is however one special case -- the famous "golden age" of exponential growth -- in which these problems can be solved simply if not entirely satisfactorily. These simplifications suggest certain short-cuts which might be taken in practical application of the model. In particular the set $V(t,w(t))$ can be characterized quite easily. Also, to the degree that the exponential case approximates actual experience, the analysis may aid in the understanding of long-term growth and distribution.

4. Exponential growth.

Exponential or golden-age growth may be defined as an equilibrium in which labor, investment and output all grow at constant relative rates with the latter two growth rates equal.

In many models this equilibrium will be approached asymptotically upon the following conditions:

$$(4.1) \qquad I(t) = s\,Q^*(t)\,, \qquad 0 < s < 1$$

$$(4.2) \qquad B(t) = B_o\, e^{\lambda t}\,, \qquad \lambda > 0$$

$$(4.3) \qquad L(t) = L_o\, e^{\gamma t}\,, \qquad \gamma > 0$$

That is presumably true of the model here but we are unable to show the necessity or inevitability of this exponential equilibrium. We are able to find a golden-age solution to these equations. The difficulty lies in showing that it is the only asymptotic solution possible.

The solution found has the following properties. First of all, output and investment grow exponentially. Thus

$$(4.4) \qquad Q^*(t) = e^{gt}\, Q^*(0)$$

Second, there is an age level, $z(t)$, such that all machines at time <u>t</u> which are older than $z(t)$ are too labor-intensive to be profitable to operate while all newer machinery is profitable to operate; and this age level is constant through time. Thus

$$(4.5) \qquad z(t) = z$$

Of course the values of g, $Q^*(0)$ and z have to be determined.

Our procedure for showing that this is a solution to the model is to adopt (4.4) and (4.5) as assumptions. Then it is shown that associated with this trial solution is a time path of the real wage and interest rate which will sustain this equilibrium.

The first step in finding the long-run golden-age solution is to find the limiting distribution of employment over the operating vintages of machines. In Appendix C it is shown, by virtue of the equation (derived from (3.2), (4.2) and (4.3)),

$$(4.6) \qquad L_o\, e^{\gamma t} = \int_{t-z}^{t} \bar{N}(v,v)\, dv \quad ,$$

that, in long-run equilibrium, the amount of labor assigned to new plants (and also to plants of any age $x \leq z$) must grow exponentially at the same rate γ. Therefore, the "equilibrium" plant-age distribution of labor requirements is "exponential." The labor assigned to new machines is related to the total labor supply as follows:

$$(4.7) \qquad \bar{N}(t,t) = L(t) \left[\frac{\gamma}{1-e^{-\gamma z}} \right] = L_o\, e^{\gamma t} \left[\frac{\gamma}{1-e^{-\gamma z}} \right]$$

Substituting into the production function, (1.8), the expression for $\bar{N}(t,t)$ in (4.7), I(t) as given in (4.1) and B(t) as given in (4.2) yields:

$$(4.8) \qquad \bar{Q}(t,t) = B_o\, e^{(\lambda+\beta\gamma)v}\, s^{\alpha}\, Q^{*}(t)^{\alpha}\, L_o^{\beta}\, e^{\beta\gamma t} \left[\frac{\gamma}{1-e^{-\gamma z}} \right]^{\beta}$$

Noting that, by the constant z assumption

$$(4.9) \qquad Q^*(t) = \int_{t-z}^{t} \bar{Q}(v,v)\, dv$$

we obtain

$$(4.10) \qquad Q^*(t) = B_o\, s^{\alpha}\, L_o^{\beta} \left[\frac{\gamma}{1-e^{-\gamma z}}\right]^{\beta} \int_{t-z}^{t} e^{(\lambda+\beta\gamma)v} Q^*(v)^{\alpha}\, dv$$

Little seems to be known about the asymptotic behavior of the solution(s) of non-linear integral equations like (4.10). One solution is the exponential growth of $Q^*(t)$.

Retreating behind the exponential growth assumption, (4.4), we can write

$$(4.11) \qquad Q^*(t) = B_o\, s^{\alpha}\, L_o^{\beta}\left[\frac{\gamma}{1-e^{-\gamma z}}\right]^{\beta} \int_{t-z}^{t} e^{(\lambda+\beta\gamma+\alpha g)v}\, Q^*(0)^{\alpha}\, dv$$

It follows easily that $Q^*(t)$ grows at the rate $\lambda + \beta\gamma + \alpha g$; but also at the rate g by definition. Equating these we find that $Q^*(t)$ grows at the rate $\frac{\lambda+\beta\gamma}{1-\alpha}$.

Next, solving for the "level," at some arbitrary $t = 0$, of the exponential time path of output, $Q^*(0)$, one finds

$$(4.12) \qquad Q^*(0) = s^{\frac{\alpha}{1-\alpha}} \left\{B_o L_o^{\beta} \left[\frac{1-e^{-gz}}{g}\right] \left[\frac{\gamma}{1-e^{-\gamma z}}\right]^{\beta}\right\}^{\frac{1}{1-\alpha}}$$

where $g = \frac{\lambda+\beta\gamma}{1-\alpha}$.

Solutions resembling (4.12) have been obtained by Johansen (op. cit.) but our model differs from his in that the operating life of machines, $z(t)$, is a variable decided by economic considerations instead of a fixed parameter.

We turn now to the remaining unknowns. Capital's relative share, its operating life and its rate of return have to be solved simultaneously.

One of the links between the wage and the operating life of machinery is the *ex post* analogue to the expectational equation (1.5):

$$(4.13) \qquad w(t)\,\bar{N}(t-z,\ t-z) = \bar{Q}(t-z,\ t-z)$$

If z is constant and $\bar{Q}(v,v)$ and $\bar{N}(v,v)$ grow exponentially at rates g and γ respectively, then $w(t)$ must grow exponentially at the rate $g-\gamma$.

Recalling the exponential-growth relation between $\bar{N}(t,t)$ and $L(t)$ in (4.7) and the output analogue

$$(4.14) \qquad \bar{Q}(t,t) = Q^*(t)\left[\frac{g}{1-e^{-gz}}\right]$$

we find the equilibrium wage rate as a function of productivity:

$$(4.15) \qquad w(t) = b(z)\,e^{-(g-\gamma)z}\,\frac{Q^*(t)}{L(t)}$$

where

$$b(z) = \left[\frac{g}{\gamma}\cdot\frac{1-e^{-\gamma z}}{1-e^{-gz}}\right].$$

This equation is essentially (1.6a) of section 2 and it is the first of three equations we need.

Another equation necessary for determining the operating life of machinery involves the rate of return on new investments. We suppose that investors are able in golden-age equilibrium to predict accurately the rate of return, the rate of increase of the wage $(\omega(t) = g - \gamma)$ and the lifetime of new machinery $(\hat{z}(t) = z)$. Then from the golden-age relations (4.1), (4.7), (4.14) and equation (2.12) it follows that

$$\frac{\alpha}{s} = f(z)$$

where

$$f(z) = \left[\frac{r}{g} \cdot \frac{1-e^{-gz}}{1-e^{-rz}} \right] \tag{4.16}$$

Since g, α, and s are constants, r (and $f(z)$) must be constant through time if z is constant.

The third and last equation for determining distribution and the operating life of machinery recognizes that the labor intensity of the economy's productive processes (whence also the operating life of machinery) is the product of investor decisions and is thus a function of the rate of return and the real wage. Turning back to (2.7) and combining this with (4.7) and (4.14) yields

$$\frac{Q^*(t)}{L(t)} = \frac{w(t)}{\beta} \quad \frac{c(z)}{b(z)} \tag{4.17}$$

where

$$c(z) = \left[\frac{r}{r - \omega} \cdot \frac{1-e^{-(r-\omega)z}}{1-e^{-rz}} \right]$$

Of course, $c(z)$ is $c_v(z)$ of section 2 without the time subscript. Since r and ω are constant over time, the function $c(z)$ is independent of time.

The three equations (4.15), (4.16) and (4.17) contain three unknowns, the operating life of machinery, the rate of return and the ratio of the wage rate to output per unit of labor (i.e., labor's relative share).

To solve for r and z, write

$$\frac{W(t)}{Q^*(t)} = b(z)\, e^{-(g-\gamma)z} \tag{4.15a}$$

and

$$\frac{W(t)}{Q^*(t)} = \beta\, \frac{b(z)}{c(z)} \tag{4.17a}$$

where $W(t)$ denotes the wage bill, $w(t)\, L(t)$.

Equating the two expressions yields

$$c(z) = \beta\, e^{(g-\gamma)z} \tag{4.18}$$

(4.18) together with (4.16) constitute two equations in two unknowns, r and z . These equations are graphed in the upper quadrant of Figure 2 below. The diagram shows that a solution exists for almost all values of s .

Readers who wish to pursue this further may consult the accompanying footnote.*

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* The relation between z and r in (4.18) can be derived from the lower quadrant of Figure 2. As r is increased, (e.g., from r_2 to r_1) c(z) pivots downward around the vertical intercept at c(0) = 1 . This causes the intersection of c(z) and $\beta e^{(g-\gamma)z}$ to move downward and leftward along the latter curve, thus reducing z . As r approaches infinity z tends to a positive lower limit, $\check{z}$, where $\beta e^{(g-\gamma)\check{z}} = 1$. On the other hand, as r falls and approaches $\omega = g - \gamma$, c(z) shifts upward toward infinity, causing the intersection with $\beta e^{(g-\gamma)z}$ to move upward and rightward without limit. Thus a finite z requires $r > \omega$. The relation between z and r in (4.18) is therefore inverse and asymptotic to these two lower limits.

(4.16) also implies an inverse relation if $s < \alpha$. Then $r > g$ so that, relying once again on Appendix A, f(z) is increasing in z with upper limit $\frac{r}{g}$. As r is increased, f(z) pivots upward around the vertical intercept where f(0) = 1 . This moves the intersection of f(z) with $\frac{\alpha}{s} > 1$ to the left, thus reducing z . As r approaches infinity z tends to zero. As r falls and approaches $\frac{\alpha}{s} g$, f(z) pivots downward, approaching horizontality, so that the intersection with $\frac{\alpha}{s}$ moves to the right without limit, so that z approaches infinity. Since $\frac{\alpha}{s} g > g > \omega$, the inverse relation between r and g implied by (4.16) must intersect the relation implied by (4.18).

If $s = \alpha$, $r = g$ independently of z .

If $s > \alpha$ then $r < g$ and f(z) is decreasing in z with lower limit $\frac{r}{g}$ so that r and z are positively related rather than inversely. As r approaches zero so does z . As r approaches $\frac{\alpha}{s} g$, z approaches infinity. Unless this upper limit to r , $\frac{\alpha}{s} g$, from (4.16) exceeds the lower limit to r , which is ω , from (4.18), the curves do not cross and there is no solution. Evidently a solution requires that $\frac{\alpha}{s} g > \omega$ or $\frac{s}{\alpha} < \frac{g}{\omega} = 1 + \frac{\beta\gamma}{\lambda}$. In progressive economies $r > g$ seems to be the rule so that $s < \alpha$ and this problem does not really arise.

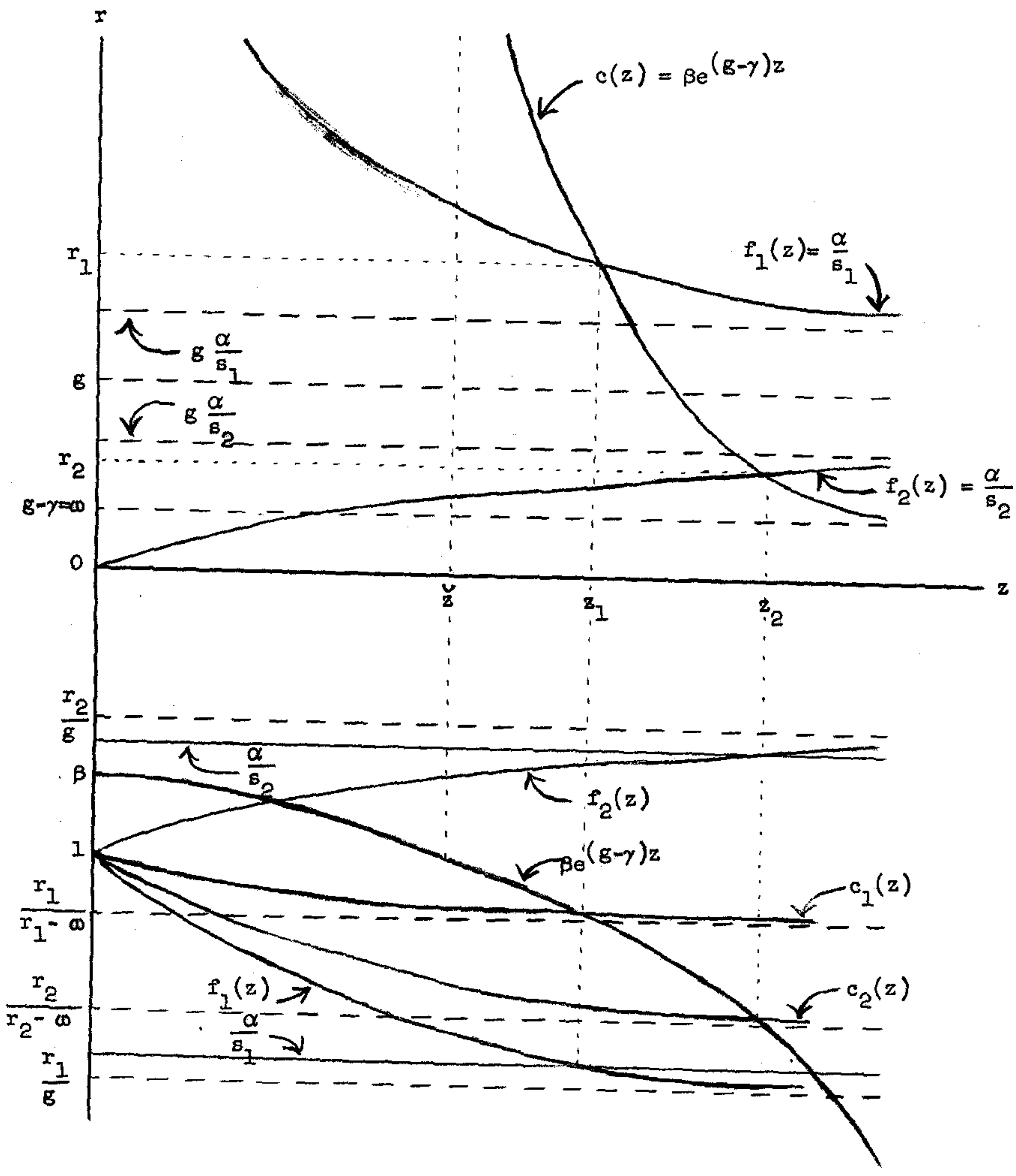
r
c(z) = βe^(g-γ)z
r1
f1(z)= α/s1
g α/s1
g
g α/s2
r2
f2(z) = α/s2
g-γ=ω
0
z
ž
z1
z2
r2/g
α/s2
β
f2(z)
1
βe^(g-γ)z
r1/(r1-ω)
c1(z)
r2/(r2-ω)
c2(z)
f1(z)
α/s1
r1/g

Figure 2

The graph indicates the effect of a change of the investment ratio from $s_1 < \alpha$ to $s_2 > \alpha$. The rate of return is decreased from $r_1 > g$ to $r_2 < g$ and the operating life of machinery is increased from z_1 to z_2.

What effect has a change in thrift upon labor's share of output? (4.15a) indicates that thrift influences labor's share through the operating life of capital. Appendix D shows that the righthand side of (4.15a) is decreasing in z and approaches zero as a lower limit. (See Figure 3.) Therefore increased thrift, by increasing the operating life of machines, decreases the share of wages in total output.

What is the relation of labor's share, thus determined, to the share labor would earn in this model's neoclassical analogue in which capital is putty? In the neoclassical version labor receives a share equal to β, the labor elasticity of output, of the produce of every machine throughout its operating life. Then labor's aggregate share is also equal to β. There is no such tendency in the present model where labor receives its marginal product on a given machine only for an instant during its operation.

Figure 3, which graphs equations (4.15a) and (4.17a), shows the relation between $\frac{W(t)}{Q^*(t)}$ and β. If $s = \alpha$, then $r = g$ and $\frac{b(z)}{c(z)} = 1$ for all z in which case $\frac{W(t)}{Q^*(t)} = \beta$. The number of "young" machines yielding a share to labor below β is balanced by the number of "old" machines on which labor earns a share exceeding β. If $s < \alpha$ then $r > g$ and z is smaller so that

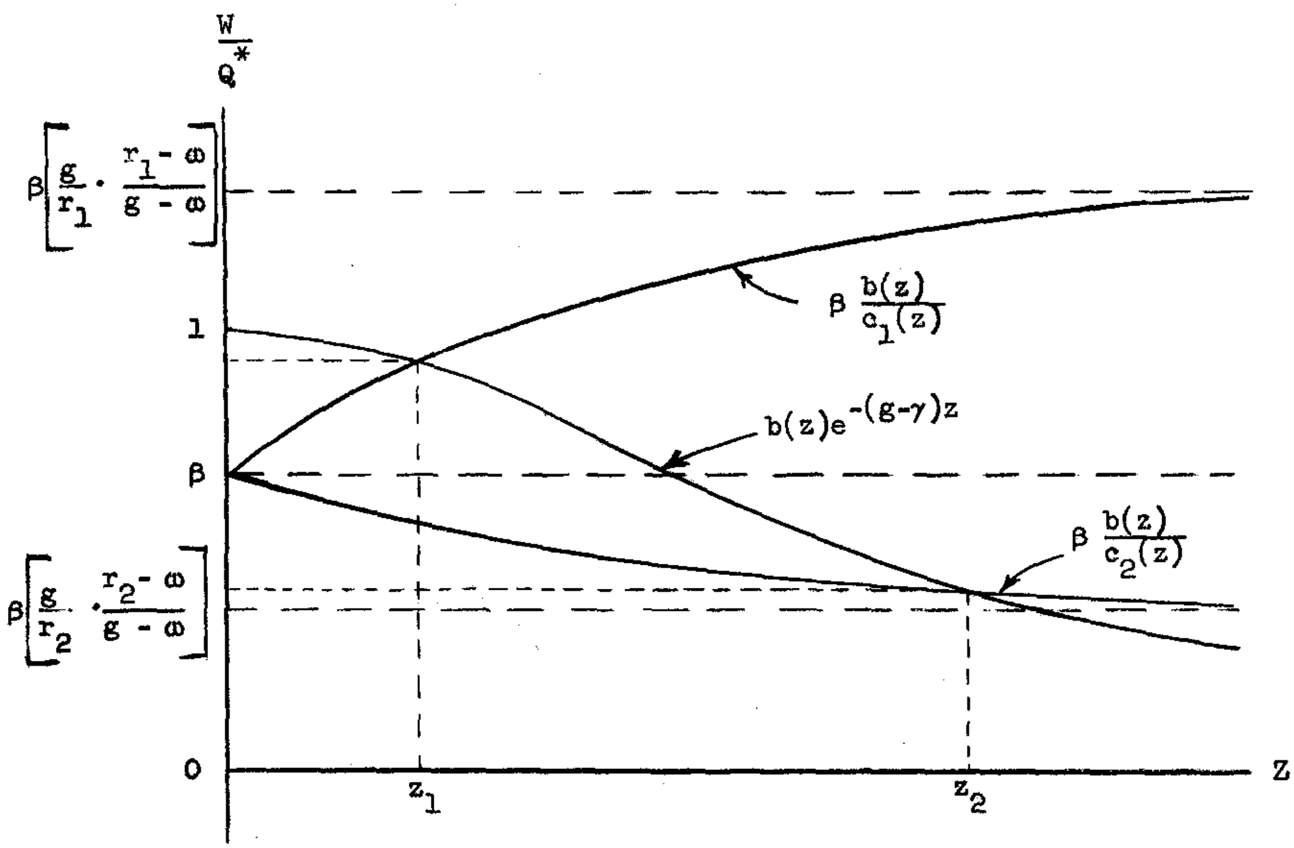

Figure 3

$\frac{W(t)}{Q^*(t)} > \beta$. The "average" machine is more labor intensive as reflected in the smaller z and higher r . Obversely, if $s > \alpha$ then $r < g$ and z is larger so that $\frac{W(t)}{Q^*(t)} < \beta$. It takes a long time, in this highly capital intensive case, for a machine to grow old and yield labor a share of its output greater than β ; the relative scarcity of old machines depresses labor's share below β.*

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* By (4.17a) if $g = \gamma$ (or $\omega = 0$) then labor's share equals β independently of g and z . Labor earns its marginal product on all machines at all times and the neoclassical result is obtained. But the economy of the model is progressive $(\lambda > 0)$ so that $g - \gamma = \omega > 0$. However the example suggests that the scope for possible divergence between β and labor's share is greater the more progressive is the economy.

5. Productivity and thrift.

These latter results cast new light on the question of the historical importance and future utility of investment in raising productivity. The degree of importance, by almost any measure, is a function of the capital elasticity of output, α. (This might better be called the investment elasticity of new capacity, gross of retirements.) Recent practice has been to take the relative share of capital income in total output as a measure of this elasticity. This is correct, under the neoclassical assumption that old and new capital are both putty. But if old capital is brittle (and presumably if only old capital is less malleable than new capital) then capital's share underestimates the capital elasticity in those economies where the rate of return exceeds the rate of growth. As a rule, progressive and industrialized economies do exhibit a growth rate well under the rate of return. In the U.S.A., for example, the latter might plausibly be put anywhere between 8% and 20% (averaging over all capital goods) but it is surely greater than 4%, the approximate American secular growth rate.

This finding is encouraging because if α is larger than has been thought then so too are the opportunities for higher productivity through increased thrift. This follows from the solution for output as a function of s and z in (4.12).

The relation between thrift, as measured by s, and the equilibrium exponential output path, Q^* is interesting. First there is the direct (capital-deepening) effect upon productivity of an increase in the investment ratio. A one per cent increase in s will increase by one per cent the number of machines of every age in the new equilibrium. The magnitude of this direct effect upon productivity is measured by the partial elasticity (holding $\underline{z}$ fixed) of Q^* with respect to $\underline{s}$ in (4.12). It equals $\frac{\alpha}{1-\alpha}$ which is increasing in α; for this reason a high α is favorable.

Second, there is an additional <u>indirect</u> (capital-lengthening) effect of an increase in the investment ratio. As s increases so does z, in the manner described by Figure 2. How does a change in the operating life of machinery affect productivity?

One can imagine an economy in which the operating life of machinery is so small that productivity suffers from the crowding of labor around brand-new machines. If the operating life were increased, productivity would rise because the available labor would have more, albeit less modern, machines on which to work. Continued lengthening of machinery's operating life would increase productivity without limit were it not that the progressively older machines being dusted off and assigned a portion of the labor force are progressively less "efficient" (more labor using) than the competing new machines.* Eventually

* Could a machine be found sufficiently old to absorb the whole labor force? No, because the labor force was never as large as now so none such machine would have been built!

a finite operating life is reached, call it $\bar{z}$, such that any further lengthening produces a net decline in productivity: The effect on productivity of spreading workers over more (already existing) machines is more than offset by the resulting decline in the average modernity and "efficiency" of machinery.

What is the typical position of the progressive economy with respect to this indirect effect? If thrift is sufficiently little that $z < \bar{z}$, then the indirect effect of an increase in thrift will reinforce the direct effect, both working to raise productivity. If thrift is so great that $z > \bar{z}$, then the capital lengthening effect works against the capital deepening effect of increased thrift. We show now that progressive economies are typically in the former situation.

To find the algebraic sign of the indirect effect we take the logarithmic partial derivative of Q^* with respect to z in (4.12). Assuming constant return to scale, this equals

$$\frac{g}{e^{gz}-1} - \beta \frac{\gamma}{e^{\gamma z}-1}$$

which is positive for $z < \bar{z}$ and negative for $z > \bar{z}$. A little manipulation shows that the derivative is positive if

$$b(z)\, e^{-(g-\gamma)z} > \beta$$

in the notation of the previous section.*

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* The result was first obtained by Massell, op. cit.

The familiar lefthand expression is none other than labor's relative share, $\frac{W}{Q^*}$ (see 4.15a). Therefore, the indirect effect is positive if labor's share exceeds β. As argued earlier, the latter condition is the rule in progressive economies.

It follows that the indirect effect of an increase in thrift supports the direct effect up to the point where s reaches α, whence $r = g$ and $\frac{W}{Q^*} = \beta$. Further increases in s will cause $\frac{W}{Q^*} < \beta$ so that the indirect capital-lengthening effect will work against the capital deepening effect. Thus there are increasing, then decreasing, marginal returns to thrift.

The proper objective of investment policy is not maximum productivity but an optimal path of consumption. The area of investment policy is well beyond the scope of the present paper. However it may be of interest to many readers that the policy of equating investment to profits (quasirents) which was proved to be a quasi-optimal policy for certain neoclassical models is also a quasi-optimal policy here.*

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* Edmund Phelps, "The Golden Rule of Accumulation," American Economic Review, 51, (September 1961).

The investment ratio corresponding to the highest attainable exponential time path of consumption, $C(t)$, will be said to be quasi-optimal. Along this maximal path the total derivative of $C = (1 - s)Q^*$ with respect to s will therefore be zero. This derivative is the sum of a direct and indirect effect:

$$\frac{\partial}{\partial s}[(1-s)Q^*] + \frac{\partial z}{\partial s}\frac{\partial}{\partial z}[(1-s)Q^*] = 0$$

If the same value of s should happen to equate both $\frac{\partial C}{\partial s}$ and $\frac{\partial C}{\partial z}$ to zero this value would be quasi-optimal, for it would make the derivative zero independently of $\frac{\partial z}{\partial s}$. In fact, such a solution occurs.

First,

$$\frac{\partial C}{\partial s} = - Q^* + (1 - s) \frac{\partial Q^*}{\partial s} = 0$$

implies

$$\frac{s}{1-s} = \frac{\partial Q^*}{\partial s} \frac{s}{Q^*} = \frac{\alpha}{1-\alpha}$$

Hence, $\frac{\partial C}{\partial s} = 0$ if $s = \alpha$.

But if $s = \alpha$ then $\frac{W}{Q^*} = \beta$ so that $\frac{\partial C}{\partial z} = 0$ simultaneously.

Therefore $s = \alpha$ is quasi-optimal. This policy equates investment to quasirents.

6. Concluding remarks.

Undoubtedly the reader can think of many desirable generalizations and modifications of the model.

The Cobb-Douglas function was adopted out of convenience rather than any evidence of its validity. Possibly it can be replaced by a more general function.

A more serious restriction may be the assumption that existing machines cannot be renovated. A renovation function applying to old machines is needed alongside the production function which applies to new machines. The investor then has to allocate abstract new capital between new and old machines. And he must consider the extent to which renovations can extend the lifetime of old machines.

This is a "pure obsolescence" model in which the choice of physical durability of capital goods is elided. If less durable machines cost less (contrary to the assumption here) then it might pay to buy machines that "expire" before becoming completely obsolete. (Exponential decay is easy to introduce.)

Finally it may be important to introduce a second, machine building sector. Then an increase of the rate of saving will shift resources to this sector, changing the structure of the economy. This can be expected to introduce many complications and possibly to change some results.

Appendix A

Let $f(z) = \frac{a}{b} \cdot \frac{1-e^{-bz}}{1-e^{-az}}$. Assume $a > b > 0$.

Evaluating $f(z)$ at $z = 0$ by l'Hopital's rule shows that $f(0) = 1$.

Clearly $f(z) \to \frac{a}{b}$ as $z \to \infty$. Thus $f(\infty) > 1$.

Differentiating,

$$f'(z) = \frac{e^{-(a+b)z}}{(1-e^{-az})^2} \left[b\, e^{az} - ae^{bz} + (a-b) \right] \tag{A.1}$$

The algebraic sign of $f'(z)$ is the sign of the bracketed terms. Let $\Phi(z)$ denote the value of these terms. Note that $\Phi(0) = 0$. Therefore if we can show that $\Phi'(z) > 0$ for all z then $\Phi(z) > 0$, whence $f'(z) > 0$.

Differentiating,

$$\Phi'(z) = a\, b\, (e^{az} - e^{bz}) \tag{A.2}$$

This is positive if $a > b$.

Appendix B

The total labor requirement of the economy is

(B.1) $$N^*(t) = \int_{v \in V(t,w(t))} \bar{N}(v,v)\, dv$$

This together with (2.9) of the text yield

(B.2) $$N^*(t) = \beta^{\frac{1}{1-\beta}} G^*(t)$$

where $$G^*(t) = \int_{v \in V\left(t,w(t)\right)} \left[\frac{B(v)}{c_v(\hat{z})w(v)}\right]^{\frac{1}{1-\beta}} I(v)^{\frac{\alpha}{1-\beta}} dv$$

Therefore, (2.9) can be written

(B.3) $$\bar{N}(v,v) = \frac{N^*(t)}{G^*(t)} \quad I(v)^{\frac{\alpha}{1-\beta}} \left[\frac{B(v)}{c_v(\hat{z})w(v)}\right]^{\frac{1}{1-\beta}}$$

which, together with the production function (1.8) implies

(B.4) $$\bar{Q}(v,v) = B(v)\, I(v)^{\alpha} \frac{N^*(t)^{\beta}}{G^*(t)^{\beta}} I(v)^{\frac{\alpha\beta}{1-\beta}} \left[\frac{B(v)}{c_v(\hat{z})w(v)}\right]^{\frac{\beta}{1-\beta}}$$

which simplifies to

(B.4a) $$\bar{Q}(v,v) = \left[\frac{B(v)}{(c_v(\hat{z})w(v)\ )^{\beta}}\right]^{\frac{1}{1-\beta}} I(v)^{\frac{\alpha}{1-\beta}} N^*(t)^{\beta}\, G^*(t)^{-\beta}$$

Integrating to obtain aggregate output, we find

(B.5) $$Q^*(t) = H^*(t)\, G^*(t)^{-\beta}\, N^*(t)$$

where $$H^*(t) = \int_{v\in V(t,w(t))} \left[\frac{B(v)}{c_v(\hat{z})w(v))^{\beta}} \right]^{\frac{1}{1-\beta}} I(v)^{\frac{\alpha}{1-\beta}}\, dv$$

It is apparent that $G^*(t)$ and $H^*(t)$ are capital-like variables. Unfortunately they differ in the exponent over $(c_v(\hat{z})w(v))$. Therefore they cannot be merged, for all time paths of $I(v)$, unless $c_t(\hat{z})w(t)$ is constant over time. Then $G^*(t) = H^*(t)$ and output can be written as a function of "effective capital," $J^*(t)$:

$$Q^*(t) = J^*(t)^{1-\beta}\ N^*(t)^{\beta}$$

where $$J^*(t) = \int_{v\in V(t,w(t))} \left[\frac{B(v)}{c\ w} \right]^{\frac{1}{1-\beta}} I(v)^{\frac{\alpha}{1-\beta}}\, dv$$

This is essentially the production function obtained by Solow in his extension of the neoclassical model to the case of investment-embodied technical progress.*

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* "Investment and Technical Progress", in <u>Mathematical Methods in the Social Sciences</u>, (Stanford, 1959).

It is interesting to notice that we can simply sum investments to obtain "capital" only if $\alpha + \beta = 1$. If $\frac{\alpha}{1-\beta} = 2$ (increasing returns) then investments must be squared before being summed.

Appendix C

Differentiating

$$(C.1) \qquad L(t) = \int_{t-z}^{t} \bar{N}(v,v)\,dv$$

one obtains, for all v ,

$$(C.2) \qquad \bar{N}(v,v) = \dot{L}(v) + \bar{N}(v-z, v-z)$$

Therefore

$$(C.3) \qquad \bar{N}(v,v) = \dot{L}(v) + \dot{L}(v-z) + \dot{L}(v-2z) + \dots + \dot{L}(v-nz+z) + \bar{N}(v-nz,v-nz)$$

Since $\bar{N}(v,v) \leq L(v)$ by (C.1) and since $L(v)$ vanishes as $v \to -\infty$ by virtue of its exponential growth, $\bar{N}(v-nz, v-nz)$ goes to zero as n goes to infinity. Therefore, as n approaches infinity we obtain

$$(C.4) \qquad \bar{N}(v,v) = \dot{L}(v) \left\{1 + e^{-\gamma z} + e^{-2\gamma z} + \dots \right\}$$

whence, writing $\dot{L}(v) = \gamma L(v)$,

$$(C.5) \qquad \bar{N}(v,v) = L(v)\left[\frac{\gamma}{1-e^{-\gamma z}}\right] .$$

Appendix D

Let $g(z) = f(z)\, e^{-(a-b)z} = \frac{a}{b} \cdot \frac{e^{bz} - 1}{e^{az} - 1}$,

where $f(z)$ is defined in Appendix A and $a > b > 0$ is assumed.

By l'Hopital's rule, $g(0) = 1$.

The limit, $g(\infty)$, equals the product of the limits of $f(z)$ and $e^{-(a-b)z}$, which is zero.

Differentiating,

$$g'(z) = \frac{e^{(a+b)z}}{(e^{az}-1)^2}\left[ae^{-bz} - be^{-az} + (b-a) \right] \tag{D.1}$$

The proof that $g'(z) < 0$ parallels the one for $f'(z)$ in Appendix A .