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A Note On Uncertainty, Bayesian Inference and Competitive Behavior

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by

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I. Of late several authors, e.g., Mills [2] Nelson [3] have discussed several aspects of a firm's output decision making process under uncertainty. In the current formulation of the problem relating to a competitive firm, uncertainty adheres only to the price at which the firm's output is to be disposed.

Thus price is regarded as a random variable whose density function and parameters -- at least the mean and possibly variance -- are assumed to be known by the firm.

Costs are supposed to be completely nonstochastic and fully known.

In this note we explore the implications of pushing uncertainty one step back by assuming that the firm does not know the distribution parameters. In this case we assume the firm to employ a Bayesian decision making approach and explore the manner in which it absorbs the information made available to it as a by-product of its market activities. We then examine a measure of the implicit valuation placed by the firm on information concerning the parameters of its price distribution. Finally we examine the implications of such behavior when all firms in a competitive industry behave in the manner above. Under certain conditions it is found that the output decision making process and the resulting price sequence are stable in a manner to be defined below.

II. Suppose a competitive firm obeying the axioms defining the Von Neumann-Morgenstern-cardinal-utility.

The firm is assumed to produce a nonstorable (perishable) commodity so that no inventories are held. There exists a lag between the decision to produce -- acquisition of inputs -- and the appearance of output. By appropriate definition of units we can make this lag of length one. The planning horizon of the firm is thus a one period horizon.

The price of its output at time t, i.e., $p(t)$, is regarded as a random variable and the observed price $p(t)$ at time t may then be regarded as a random sample of size one drawn from a population characterized by the density function $f(p(t) \mid \Theta)$ when Θ is the set of parameters. We imply here that Θ is not itself a function of time -- stationarity of distribution.

If the firm's utility is linear in "money" then by appropriate choice of units and origin of utility the firm, as is well known -- will behave as if it maximized the expected utility of its production gamble. In this case the utility function is simply

$$(1) \qquad U(p, X) = pX - \Phi(X)$$

where X is output, p price and $\Phi(X)$ is the cost function. In (1) we have simply the profit function. Expecting (1) with respect to the density function $f(p \mid \Theta)$ we find

$$(2) \qquad \int U(p, X) f(p \mid \Theta) d\Theta = \mu X - \Phi(X)$$

where μ is the mean of the distribution.

Now the firm is not assumed to know μ . Hence it cannot act with respect to the data given in (2). In dealing with the problem we shall adopt an approach developed in another context by Raiffa and Schlaifer [5].

It is shown there and also in Savage [6] that the notion of subjective probability can be given rigorous operational content.

We take this as a point of departure.

For concreteness of exposition suppose that price is distributed normally conditionally on its mean μ and its precision, inverse of variance, h .

Thus the density function is $f(p \mid \mu, h)$. Now let h be known. This is innocuous for most of the development since information on h will not be utilized.

The firm is further supposed to have some notions as to the probable behavior of μ . These we may summarize in a subjective-prior-distribution assigned to μ . Suppose it to be normal with mean m^o and precision hn^o , and denote it by:

$$\text{(2c)} \qquad s\ (\mu;\ m^o,\ hn^o)\ .$$

The problem of initial action is thus essentially solved. For now the firm merely has to complete the expectation of (2) with respect to the "random" variable μ . Thus:

(3) $$EU(\mu, X) = m^{o} X - \Phi (X)$$

From (3) we easily compute the optimal initial output of the firm

(4) $$X^{o}(0) = D(m^{o})$$

where $D(m^{o})$ is the inverse function of Φ', i.e., it is a solution of

(3a) $$m^{o} - \Phi' (X) = 0$$

We must of course assume that this inverse exists. For completeness we should also impose the condition that:

(4) $$m^{o} D(m^{o}) - \Phi[D(m^{o})] \geq 0$$

i.e., that expected profits are non-negative.

This will be assumed throughout. If (4) were violated then the firm would not produce.

Now the firm having produced the quantity implied by (3a) and having placed it on the market it disposes it, say, at price $p(1)$.

Following this the firm must decide on the output to produce for marketing in period 2.

But now it has an additional piece of information viz., the price $p(1)$. The firm may well ask: What is the (subjective) distribution of μ given $p(1)$? It is shown in Raiffa and Schlaifer [5] that under the assumption made here it is still normal with parameters:

(4) $$m^1 = \frac{n^o m^o + p(1)}{n^o + 1} \qquad \text{mean}$$

$$h(n^o + 1) \,. \qquad \text{inverse of variance}$$

This is obviously a procedure that may be iterated yielding at time t :

(4a) $$m^t = \frac{n^o m^o + t\,\bar{p}}{n^o + t} \qquad \text{mean}$$

$$h(n^o + t) \qquad \text{inverse of variances.}$$

We may interpret (4a) as follows:

The price expected at time t to prevail at time $(t+1)$, i.e., the price with respect to which the firm acts is simply m^t and the confidence with which it holds this expectation is indicated by $h(n^o + t)$; $\bar{p}$ is the average of observed prices up to period t .

One interesting aspect of the relations (4a) is that they give us an expression for expected price which is a simple version of the adoptive expectations variant discussed recently in the literature, e.g., in [1] and [4] .

Note that here "adaptive" expectations are not assumed. They are deduced from general principles of statistical reference.

The expressions (4a) also provide us with a meansof appraising, in some sense, the value of information to the firm.

Obviously we cannot hope to have perfect information on price. We can, however, have information on the mean, μ , of this random variable with varying degrees of accuracy, as follows:

If the firm does not hold price expectations with sufficient degree of certitude the firm would wish to sample from the population of $p(t)$ in order to obtain information on μ . What sampling would do is presumably to alter the firm's current beliefs as to the probable behavior of μ and the degree of accuracy it attaches thereto, to a set of posterior beliefs. Then in the light of the sample outcome the firm would in general revise its output decision.

Formally, let a sample of size n^* yield at time t yield the statistic

$$p^* = \left(\sum_{i=1}^{n^*} p^*(i) \right) / \, n^*$$

Then the resulting posterior distribution of μ at time t is also normal but with parameters

$$(5) \qquad m^{*t} = \frac{n^o m^o + t\,\bar{p} + n^* p^*}{n^o + t + n^*}$$

$$h(n^o + t + n^*) \; .$$

We may ask what is the expected value of sample information[1] (E.V.S.I.)?

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[1]The term expected value of sample information is due to Raiffa and Schlaifer [5].

But in the absence of sampling the firm would have chosen

(5a) $$X_o(t) = D(m^t)$$

with sampling it chooses

(5b) $$X_o^*(t) = D(m^{*t})$$

The difference in the expected profitability is given by:

(6) $$E\left\{p\,X_o^*(t) - \Phi[D(m^{*t})] - p\,X_o(t) + \Phi[D(m^t)]\right\}$$

where E is the expectation operator relative to the posterior density $f(p|\mu)\,S^*(\mu)$. It may be verified that (6) is nonnegative. While it is not possible to obtain an explicit expression in (6) without further specifications of $\Phi(X)$ it is clear that in general it is not the same for all t .

In the case we are considering -- known h -- if we introduce a quadratic cost function, i.e.,

(7) $$\Phi(X) = C_o + C_1 X + C_2 X^2$$

we easily find appraising (6):

$$\text{(7a)} \qquad E.V.S.I = \frac{1}{2\bar{c}_2} \frac{1}{h(n^o + t + n^*)}$$

Note that (7a) is a decreasing function of the length of market experience of this firm.

Similar results may be obtained if h is not assumed to be known, but at the cost of considerable formal complications.

Note, however, that because of the nature of the problem, sampling cannot take place. So the operations involved in (6) and (7a) are virtual operations that cannot in fact be carried out. The exposition, nonetheless, makes it clear that sampling information would be evaluated by the firm differently at different stages of its market participation.

III. It may be of interest to examine the implications of the assumptions stated in II when we apply them to all firms of a competitive industry.

Thus suppose the i^{th} firm has cost function

$$\text{(8)} \qquad C_i = \Phi_i (X) \qquad i = 1, 2, \ldots, s.$$

Let the industry demand function be the simple linear one

$$\text{(9)} \qquad X(t) = a - b\, p(t) + u(t)$$

where $a, b > o$ and $u(t)$ is, say, a normal random variable with mean 0 and precision h.

If every firm acts in accord with (3a) of Section II than we have for industry output:

$$(10) \qquad \sum_{i=1}^{s} X_{oi}(t) = \sum_{i=1}^{s} D_i(m_i^t) = Q(t)$$

Thus price is given by:

$$(11) \qquad p(t+1) = \frac{a - Q(t)}{b} + \frac{u(t+1)}{b}$$

Let us see now if some justification can be made for the assumption made in section II that the single firm views price as being a random normal variable.

Thus suppose that firms act on their preconceptions alone and refuse to learn from observation. This would give the constant output stream on the part of the i^{th} firm $D_i(m_i^o)$ and hence we could rewrite (11) as

$$(11a): \qquad p(t) = \frac{a - Q_o(0)}{b} + \frac{u(t)}{b}$$

But (11a) gives price as a random variable with mean $\frac{a - Q(0)}{b}$ and inverse of variance $b^2 h$. This provides some partial justification for the assumption made in section II, since this would be the situation materializing if we imagine all other firms' actions to be frozen and let only an individual firm try to adjust its actions optimally.

We may assume, as we must, that $\Phi_i'(X)$ is monotonic increasing in X, for all i.

What can we now say concerning the sequence of output decisions $Q(t)$ and of price realizations $p(t)$. Well from (3a), (4a) and (10) we easily deduce the following:

i. Although firms may begin with varying initial "conditions" -- i.e., different parameters for their initial priors, these differences are narrowed. This is eminently reasonable since if firms view what they all regard as a realization of a normal process, they "should" all arrive at the same conclusions after a sufficiently long period of observation.

ii. We also see that output will oscillate at least slightly. This is so since, from (4a), if the price observed at time t is higher than the average of prices observed up to time t, then the firm will produce more than it produced at time $(t-1)$. Since this is true for all firms the result follows by (10).

iii. Prices will also oscillate (in the mean) but in direction opposite to that of output. This may be seen easily from (11), if we confine ourselves to their conditional mean, i.e., their mean at time $(t+1)$ conditional upon the price sequence $\{p(1), \ldots, p(t)\}$.

Thus if $p(t)$ were higher than the average of past observed prices, output would be higher and by (11) the conditional mean would be lower, $Q(t)$ being an increasing function of $p(t)$ and $E\,p(t+1)$ being a decreasing function of $Q(t)$.

Nothing of course can be said with certainty about $p(t+1)$ since it depends on the random variable $u(t+1)$.

At any rate the sequence of adjustments just described is a stable one, in the sense that successive price deviations of equal magnitude from the average of observed (past) prices give rise to output adjustments which are of declining magnitudes.

To illustrate this consider the quadratic cost functions:

$$(12) \qquad C_i = C_{oi} + C_{1i}X + C_{2i}X^2 \qquad i = 1, 2, \ldots, s$$

Using (12) in (11) and (11a) we get:

$$(13) \qquad p(t+1) = \frac{a - Q(t)}{b} + \frac{u(t)}{b}$$

$$(13a) \qquad Q(t) = \sum_{i=1}^{s} \left[\frac{m_i^t - C_{1i}}{2\,C_{2i}} \right]$$

$$(13b) \qquad m_i^t = \frac{n_i^o\, m_i^o + t\bar{p}}{n_i^o + t}$$

where m_i^t is the price expected by the i^{th} firm at time t , i.e., the price with respect to which it makes its output decision. As before $\overline{p} = \frac{1}{t} \sum_{i=1}^{t} p\,(t)$

First note: For large t we have approximately

$$(14) \qquad m_i^t \approx \overline{p} \qquad\qquad i - 1, 2, \ldots, S$$

This says that given a sufficiently large period of operation, firms price expectations converge to the same parameters. This is reasonable since many decision makers observing a phenomenon whose stochastic nature is known to all "should" arrive at similar conclusions.

A consequence of (14) is that (13a) and (13) become

$$(15) \qquad Q(t) \approx \sum_{i=1}^{s} \left[\frac{\overline{p} - C_{1i}}{2\,C_{2i}}\right]$$

$$(15a) \qquad E\,p(t+1) \approx \frac{1}{b}\left[a - \sum_{i=1}^{s}\left(\frac{\overline{p} - C_{1i}}{2\,C_{2i}}\right)\right]$$

while it is true that (15a) has a "stationary" value, i.e., there exists a $\overline{p}$, viz.

$$(16) \qquad \overline{p} = \frac{a + \sum_{i=1}^{s} \frac{C_{1i}}{2\,C_{2i}}}{b + \sum_{i=1}^{s} \frac{1}{2\,C_{2i}}}$$

such that $E(p(t+1)) - \bar{p} = 0$, yet it should be clear that this is of no consequence from the point of view of the output decision of the firm. Even in (15) it is easily seen that $Q(t)$ depends on the average of past prices, which would then depend on $p(t+1)$ and not on $Ep(t+1)$.

Stability in this context cannot be meaningfully defined in terms of the persistence of some stationary value of the variable of the system. It seems reasonable to define it in terms of the magnitude of the response elicited by disturbances of equal "magnitude" at the different point in time. Thus we give:

Definition 1: A system of output adjustment decisions is stable if and only if

$$\frac{\partial\, Q(t+\tau)}{\partial\, p(t+\tau)} \quad \text{is monotonic in } \tau \text{ and } \quad \lim_{\tau\to\infty} \frac{\partial\, Q(t+\tau)}{\partial\, p(t+\tau)} = 0$$

Definition 2: A sequence of price realizations is said to be stable in the mean if and only if

$$\frac{\partial\, F\; p(t+\tau+1)}{\partial\, p(t+\tau)} \quad \text{is monotonic in } \tau \text{ and } \quad \lim_{\tau\to\infty} \frac{\partial\, E\; p(t+\tau+1)}{\partial\, p(t+\tau)} = 0$$

With the aid of these definitions we can prove the following simple theorems

Theorem 1: Under the assumption (9) and (12) the system of Bayesian output decision adjustments described in this section is stable:

Proof: Consider the price observation p at times t and $t+\tau$ and compute:

$$\frac{\partial\, Q(t)}{\partial\, p(t)} = \sum_{i=1}^{s} \left[\frac{1}{2\, C_{2i}(n_i^o + t)} \right] \tag{17}$$

$$\frac{\partial\, Q(t+\tau)}{\partial\, p(t+\tau)} = \sum_{i=1}^{s} \left| \frac{1}{2\, C_{2i}(n_i^o + t+\tau)} \right| \tag{17a}$$

It is apparent from (17a) that its limit as $\tau \to \infty$ is zero. q.e.d.
It is thus evident that observations which obey $|p(t)-\bar{p}| = K$ where K is same constant will have output repercussion whose magnitudes will be a declining function of time. Thus "odd" price observations will not sway the market much if entrepreneurs have already accumulated sufficient experience.

Theorem 2: Under the conditions of Theorem 1, the price sequence resulting is stable in the mean.

Proof: We again consider the effect of a price observation p at time t and $t+\tau$. We find:

$$\frac{\partial\, E\, p(t+1)}{\partial\, p(t)} = -\frac{1}{b} \sum_{i=1}^{s} \frac{1}{2\, C_{2i}(n_i^o + t)} \tag{18}$$

$$\frac{\partial\ E\ p(t+\tau+1)}{\partial\ p(t+\tau)} = -\ \frac{1}{b} \sum_{i=1}^{s} \frac{1}{2\ C_{21}(n_i^o + t+\tau)} \tag{18a}$$

From (18a) we obviously obtain that its limit as $\tau \to \infty$ is zero. q.e.d.

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