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On Devising Unbiased Estimators for the Parameters of A Cobb - Douglas Production Function

Phoebus J. Dhrymes

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On Devising Unbiased Estimators for the Parameters of

A Cobb* - Douglas Production Function**

by

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I. In many empirical investigations the geometric mean of observed factor shares serves as an estimator of the factor exponents in a Cobb-Douglas production function. This particular estimator is due to Klein, [7, p.194]. It is here shown that such an estimator is biased. Under certain conditions an alternative estimator is derived which is unbiased, sufficient, efficient and consistent.

II. The following description for the production process of a firm sector or economy is often employed:

$$Q(t) = A(t) \prod_{i=1}^{n} X_i^{\alpha_i}(t)\, e^{u(t)} \qquad t=1, 2, \ldots, T \tag{1}$$

where $Q(t)$, $X_i(t)$ are respectively the output and i^{th} input at time t, $A(t)$ is some positive function of time $\alpha_i > 0$ and

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*The name Cobb-Douglas traditionally given to such function is somewhat inappropriate. Wicksell derives these functions from elementary economic considerations in the course of his analysis of Akerman's problem. (Wicksell, K: Lectures on Political Economy, Vol. II, Routledge and Kegan Paul Ltd., London 1951, pp. 285-286)

$\sum_{i=1}^{n} \alpha_i = 1$, this latter preserving the condition imposed on (1) when $u(t) \equiv 0$, i.e., when it is nonstochastic, due to the requirements of competitive theory. In virtue of the latter assumption we may also write:

$$(2) \qquad P_i(t) = P_Q(t) \frac{\partial Q^*(t)}{\partial X_i(t)} e^{v_i(t)} \quad i = 1, 2, \ldots, n \quad t = 1, 2, \ldots, T$$

where $P_i(t)$, $P_Q(t)$ are respectively the price of the i^{th} factor, and output at time t and $Q^*(t) = Q(t)\, e^{-u(t)}$, i.e., it is the nonstochastic part of (1). A variant of this formulation is given e.g., by Wolfson [13]. What (2) implies is that decisions on factor employment are made on the basis of the anticipated output $Q^*(t)$ and not on the basis of the output actually materializing.

The term $u(t)$ in (1) and $v_i(t)$ in (2) are assumed to be serially uncorrelated and to be distributed [the latter marginally] normally with mean zero and variance σ^2 and $\sigma^2_{v_1}$ respectively. We may also assume that $u(t)$ and $v_i(t)$ are independent.

Manipulating (1) and (2) it is then an easy matter to show that:

$$(3) \qquad \frac{P_i(t)\, X_i(t)}{P_Q(t)\, Q(t)} = \alpha_i(t) = \alpha_i\, e^{w_i(t)} \quad i = 1, 2, \ldots, n \quad t = 1, 2, \ldots, T .$$

where $w_i(t) = u(t) - v_i(t)$. Under the assumption made above the $w_i(t)$ are (marginally) normal with mean zero and variance $\sigma_i^2 = \sigma^2 + \sigma^2_{v_i}$; they are also serially uncorrelated.

Now if the $\alpha_i(t)$ are interpreted as observed factor shares then the factor exponents in (1) can be estimated by first transforming into logarithms. Thus:

$$(4) \qquad \log \alpha_i(t) = \log \alpha_i + w_i(t) \quad i = 1, 2, \ldots, n \quad t = 1, 2, \ldots, T$$

From (4) minimizing the second moment of estimated disturbances we find:

$$(5) \qquad \widehat{\log \alpha_i} = \frac{1}{T} \sum_{i=1}^{T} \log \alpha_i(t) \qquad i = 1, 2, \ldots, n$$

where T is the sample size.

On the hypotheses maintained the estimators (5) are unbiased, efficient and consistent. This is due to the Markov theorem. Due to the (marginal) normality of the $w_i(t)$ (5) are also sufficient, since they are maximum likelihood estimators.

Now it is obvious that (5) imply a certain set of estimators for the α_i , <u>vit</u>.,

$$(6) \qquad \hat{\alpha}_i = \left[\prod_{t=1}^{T} \alpha_i(t) \right]^{1/T} \qquad i = 1, 2, \ldots, n.$$

Thus a rather convenient estimator is deduced which requires only readily available information, obviating or at least ameliorating the problems involved in obtaining accurate information concerning, in particular, capital inputs.

It is also occasionally claimed in the literature, e.g., Hoch [4, p. 567] and Walters [11, p. 26] that the estimators (6) are on the hypotheses maintained -- except for normality -- "best linear unbiased." This "result" is attributed to Klein [7, p. 193]. The ascription, however, is not correct in that Klein nowhere explicitly claims this.

The claim concerning the estimators (6) results presumably from the properties that the Markov theorem under the classical assumptions ascribed to (5) as estimators of $\log \alpha_i$.

Now while (6) are transforms of (5) under a homeomorphism -- i.e., a continuous one-one mapping whose inverse is also (one-one and) continuous -- yet not all properties ascribed to (5) can be ascribed to (6) as well.

It is a simple matter to show that (6) are sufficient because essentially of the Fisher-Neyman criterion for sufficient statistics [5, p.101] and the homeomorphism involved in obtaining (6) from (5).

Consistency could also be shown to be preserved using the following heuristic argument which can be made rigorous thus put.

(7) $\widehat{\log \alpha_i} \quad \hat{\alpha}_i = e^{\widehat{\log \alpha_i}} = f(\widehat{\log \alpha_i})$ Then we have:

$$\operatorname*{plim}_{T\to\infty} \hat{\alpha}_i = f(\operatorname*{plim}_{T\to\infty} \widehat{\log \alpha_i}) = f(\log \alpha_i) = \alpha_i$$

Unbiasedness, however, is not necessarily preserved under a homeomorphism; in fact since $f(\widehat{\log \alpha_i})$ obeys $f''(\widehat{\log \alpha_i}) > 0$ and is everywhere analytic we have by using the finite form of Taylor's theorem (with remainder after two terms about $\log \alpha_i$)

$$(8) \qquad E[f(\widehat{\log \alpha_i})] \geq f[E(\widehat{\log \alpha_i})] = f(\log \alpha_i) = \alpha_i \qquad i = 1, 2, \ldots, n .$$

In (8) equality holds if and only if $\operatorname{Var}(\widehat{\log \alpha_i}) = 0$

This suggests that the estimators (6) are asymptotically unbiased. In virtue of the asymptotic unbiasedness and sufficiency of the estimators (6) it follows that they are asymptotically efficient. The term efficient here means the property of being unbiased and of minimum variance within the class of unbiased estimators of the parameter in question.

The asymptotic efficiency of (6) is essentially due to the completeness[1]

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[1] For an elementary development of the notion of completeness see Hogg and Craig [5, p. 106]. For a full development see Lehmann and Scheffé [8, p. 305 ff.].

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of the density function of the mean of a sample derived from a normal parent.

In the present case we may show directly asymptotic unbiasedness by utilizing the fact that $E(\alpha_i) = \varphi_i(1)$ when $\varphi_i(1)$ is the moment generating function of $\log \hat{\alpha}_i$ with parameter 1:

$$(9) \qquad E(\hat{\alpha}_i) = \alpha_i \, e^{\frac{1}{2T}\sigma_i^2} > \alpha_i \qquad i = 1, 2, \ldots, n .$$

The result in (9) points out another difficulty with this particular formulation, vit.,

$$(10) \qquad E\left[\sum_{i=1}^{n} \hat{\alpha}_i\right] > \sum_{i=1}^{n} \alpha_i = 1$$

We may wish to preserve the requirement $\sum_{i=1}^{n} \alpha_i = 1$ so that the model would remain a true generalization of the nonstochastic model of production and distribution. This may be accomplished by treating one of the factors, say the n^{th}, asymumetrically by requiring its share to accrue residually.

This has the following consequences: In (2) we do not have n conditions, but rather $n-1$ so that optimization occurs only with respect to $n-1$ factors; in (5) and consequently in (6) we have only $n-1$ independent estimators so that we always put by definition:

$$(11) \qquad \hat{\alpha}_n \equiv 1 - \sum_{i=1}^{n-1} \hat{\alpha}_i$$

What follows then should be interpreted in the light of (11). From (9) it is easily seen that the inverse of the relative bias of (6) is given by:

$$(12) \qquad e^{-\frac{1}{2T}\sigma_i^2} = \sum_{k=o}^{\infty} \left(-\frac{\sigma_i^2}{2T}\right)^k \Big/ k! \qquad i = 1, 2, \ldots, n-1$$

Bearing this in mind we have:

<u>LEMMA 1</u>: There **exist unbiased** estimators for the α_i , i=1, 2, ..., n-1.

Proof: We proceed constructively: Since $\log \hat{\alpha}_i$ and S_i^2— the sample mean and variance of the logarithms of $\alpha_i(t)$— are distributed independently and since

$$(12a) \qquad E(S_i^{2k}) = \frac{2^k \sigma_i^{2k} \Gamma\left(\frac{T-1}{2} + k\right)}{T^k \Gamma\left(\frac{T-1}{2}\right)}$$

it will suffice to determine a function $f(S_i^2)$ such that $E[f(S_i^2)] = e^{-\frac{1}{2T}\sigma_i^2}$

Writing formally

$$(13) \qquad f(S_i^2) = \sum_{k=o}^{\infty} a_k S_i^{2k} \qquad i = 1, 2, \ldots, n-1$$

then expecting (13) with respect to the density function of S_i^2 and equating coefficient of like powers of σ_i^2 in (12) we find

$$(13a) \qquad f(S_i^2) = \sum_{k=o}^{\infty} \left(-\frac{1}{4}\right)^k \frac{B\left(\frac{T-1}{2}, k\right)}{\Gamma(k)\, k!} S_i^{2k} \qquad i = 1, 2, \ldots, n-1$$

It follows[2] that:

[2]Expectation of (13a) is justified since it is uniformly convergent for $S_i^2 < \infty$; further $f(S_i^2)$ is an analytic function of S_i^2, hence independent of $\hat{\alpha}_i$, so that $E[\hat{\alpha}_i f(S_i^2)] = E(\hat{\alpha}_i) E[f(S_i^2)] = \alpha_i$

$$(14) \qquad \tilde{\alpha}_i = \hat{\alpha}_i f(S_i^2) \qquad i = 1, 2, \ldots, n-1$$

are unbiased estimators of the α_i. q.e.d.

<u>LEMMA 2:</u> The estimators $\tilde{\alpha}_i$ in (14) are sufficient.

Proof: This follows from the fact that the $\tilde{\alpha}_i$ are simple functions of the (jointly) sufficient statistics $\log \hat{\alpha}_i$, S_i^2 alone, i.e., not involving unknown parameters.

LEMMA 3: The estimators $\tilde{\alpha}_i$ in (14) are efficient.

Proof: The joint density function of $\log \hat{\alpha}_i$ and S_i^2 is complete;[3]

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[3]For this fact see Hogg and Craig [5, p.117].

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due to a result by Rao [9, p. 86] if an efficient estimator exists it must be an explicit function of the (jointly) sufficient statistics $\log \hat{\alpha}_i$, S_i^2 alone. Thus if another continuous[4] estimator

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[4]Continuity is assumed only in order to disallow the possibility $\tilde{\alpha}_i \neq \rho_i$ on a set of measure zero.

$$(15) \quad \rho_i = \rho_i (\log \hat{\alpha}_i , S_i^2) \qquad i = 1, 2, \ldots, n-1$$

exists which is also unbiased, then:

$$(15a) \qquad E[\tilde{\alpha}_i - \rho_i] = 0$$

Due to the completeness of the joint density function of $\log \hat{\alpha}_i$, S_i^2 it follows that $\tilde{\alpha}_i \equiv \rho_i$. q.e.d.

LEMMA 4: The estimators $\tilde{\alpha}_i$ in (14) are consistent.

Proof: Had efficiency been defined by the Cramer-Rao equation, [2, p. 481] the proof would have been immediate. Since we have not employed this definition, it seems simpler to give a direct argument from the variance of the $\tilde{\alpha}_i$. Thus:

$$\text{(16)} \qquad \operatorname{Var}(\tilde{\alpha}_i) = e^{2 \log \alpha_i} \left[e^{\frac{2\sigma_i^2}{T}} V_i(f) - 1 \right] \quad i = 1, 2, \ldots, n-1$$

where

$$\text{(16a)} \qquad V_i(f) = E\,[f(S_i^2)]^2 \qquad i = 1, 2, \ldots, n-1$$

It can be shown that

$$\text{(16b)} \qquad \lim_{T \to \infty} V_i(f) = 1$$

Hence (16b) together with Lemma 1 and Chebyshev's inequality imply consistency q.e.d.

We may conclude this section by formally stating the theorem: Under the hypotheses (1) and (2) as modified by (11) the estimators (6) are biased; the estimators given in (14) are unbiased, sufficient, efficient and consistent.

III. CONCLUSION

We give below an indication of the size of the bias in estimating the α_i from share of capital data in various sectors of the postwar period (1945-1958) American Economy. The entries in Table 1 are computed values of $f(S_i^2)$ truncated after 8 terms.

Table 1

Sector	truncated $f(S_i^2)$
Manufacturing	.996
Transportation	.976
Services	.999

As is apparent from (9) the relative bias will be small if the variance of $\log \alpha_i(t)$ is also small.

We may be concerned with unbiasedness, however, if we wish to use estimators of the α_i in obtaining an estimator of a productivity parameter as follows.

It will be noted that the term $A(t)$ in (1) has not appeared in subsequent developments. If a specific functional form were to be employed, say, $A(t) = A\, e^{\lambda t}$, then using the estimators $\tilde{\alpha}_i$ we easily find:

$$Q(t) \prod_{i=1}^{n} X_i(t)^{-\tilde{\alpha}_i} = A\, e^{\lambda t} \prod_{i=1}^{n} X_i(t)^{-(\tilde{\alpha}_i - \alpha_i)} e^{u(t)} \qquad t = 1, 2, \ldots, T . \tag{17}$$

The right member of (17) involves the random term $\prod_{i=1}^{n} X_i(t)^{-(\tilde{\alpha}_i - \alpha_i)} e^{u(t)}$.

Hence we could derive an estimator for λ and study its properties. In this context it would be desirable to have unbiased estimators for the α_i . An application of this will be made in a subsequent paper.

This procedure gives a statistical counterpart to the method suggested by Solow [10, p. 312] and provides a rather simple means of testing a statistical hypothesis on the "rate of change" of productivity as between two economies, firms or sectors whose productive process is characterized by a relation such as (1).

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