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Output, Sales and Inventory Policy in the Competitive Firm and the Stability of a Competitive Market

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Output, Sales and Inventory Policy in the Competitive Firm and the Stability of a Competitive Market[1]

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1. Introduction

This chapter is concerned with the behavior of the firm and the industry in a perfectly competitive market. The following chapters are concerned with similar topics for an imperfectly competitive industry. For the purpose of this Chapter, a firm is said to be in a perfectly competitive (or "competitive" for short) market if and only if it expects, for whatever reason, to be able to sell as much as it wishes each period at whatever price happens to rule in that period. The notion of a market is taken to include uniformity of price among all firms in the same market. It follows from this definition that, although the firm may be uncertain as to prices in various future periods, the sales level at each possible price is a nonstochastic decision variable, completely under the control of the firm. On the other hand, when imperfect competition is considered in succeeding chapters it will be natural to assume that the firm is uncertain as to the sales which will result at each price set.

This distinction as to the way in which uncertainty enters is a fundamental one and permeates the analysis at every point. I originally thought that, with or without uncertainty, the problem of programming the perfectly competitive firm's activity was much simpler than that of the imperfectly competitive firm. My reason for believing this was the notion that a competitive market performs a considerable amount of computation which the firm would have to do otherwise. Specifically, it computes the prices at which, in one sense or another,

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[1] This paper constitutes a chapter from a forthcoming book by the author. References are taken from those for the book as a whole and are therefore not numbered consecutively.

supply equals demand. I am no longer convinced that this is an important simplification. In any case, it seems clear that there are important unsolved programming problems for firms in both kinds of markets. Some of these are described in some detail below.

What is certain is that the kinds of programming problems involved, and in particular the considerations which govern inventory policy, are quite different in the two cases. In imperfect competition, inventory policy is closely related to the desirability of hedging against a demand which, at the price set, is abnormally high. In perfect competition this factor cannot enter since there is no uncertainty concerning the amount demanded at any given price. Instead, the competitive firm's inventory policy is related to the "speculative" consideration that, by producing ahead when price is expected to rise, it can take advantage of lower production costs and hence increase its profits.

Thus, the first purpose of this chapter is to introduce inventories into a model of the output and sales decisions of a firm in a perfectly competitive market. In the absence of uncertainty, this is not terribly difficult. Surprisingly enough, in spite of this and of the interest by economists in perfectly competitive markets, no one ever seems to have presented a complete analysis of the optimum output, sales and inventory policy of a competitive firm on the rather general assumptions made in the next section. Most writers on the subject[1] have been motivated by the analogy between price

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[1] See [37], [57], and [59]

discrimination and multi-period planning and, while this analogy is certainly helpful, it suggests inessential restrictions. In the presence of uncertainty the programming problem is much more complicated and is largely unsolved. It is argued below in section 5, however, that in this case important insights can still be gained from the results established for the certainty case.

The second purpose of this chapter is to use the decision-making model developed below in order to investigate the dynamic stability of the market as a whole. Recent years have seen a resurgent and continuing interest in the questions of stability in economies characterized by perfectly competitive markets.[1] By the application of more rigorous and sophisticated

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[1] See [8] and the references therein.

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techniques it has been possible to establish many important new results which were inaccessible with simpler techniques employed in earlier investigations of stability. Now from the point of view of decision theory the interesting thing about these studies is the contrast between the decision processes assumed in the static and dynamic parts of the analysis. The static decision theory employed, based on the assumptions of profit maximization by firms and ordinal utility maximization by consumers, involves a careful and rigorous working-out of the implications of rational decision-making on the two sides of the market. Great care is taken to introduce explicit motivational assumptions and to deduce exactly what behavior they imply. The dynamic analysis, on the other

hand, is based on the assumption of simple, mechanical, non-rational behavior by buyers and sellers, for which no motivational assumptions have been introduced. Specifically, this is true in two respects.

The first has to do with the introduction of expections. When future expected prices are introduced into the model they are normally assumed to be generated by one of the simple autoregressive schemes criticized in Chapter III, section 3. It is then assumed that these expected prices have an effect on current demand and supply which is symmetrical with the effect of current prices. In other words, the same restrictions (mainly concerning sign) are placed on the coefficients relating current demand and supply to expected prices as are placed on the coefficients relating current demand and supply to current prices.[1] Now the assumption that expected future prices

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[1] Two examples are [11] and [21].

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have an effect on current behavior is presumably related to the possibility of inventory holding, at least on the part of firms. Otherwise, it is not easy to see why, in a competitive market, expected future prices should be relevant to firms' current output and sales policy. But the possibility of holding inventories introduces an important asymmetry into the effect of expected future prices. While a rise in future expected price may induce the firm to build up inventories, a fall in future expected price can induce it to unload inventory only if it already has some inventory. If expected prices are to be introduced it would seem desirable to do so within the framework of

an explicit theory of inventory holding.

The second mechanical element in studies of dynamic stability relates to the assumptions made concerning price formation. The assumption usually made is simply that the rate of change of price is an increasing function of the excess of demand over supply in the market in question. Here demand refers to nonspeculative demand and supply to the usual sum of firms' marginal production cost curves. Now if the price in question is below the equilibrium and demand exceeds supply, this means that some consumers are unable to buy the quantity they wish at this price unless firms hold inventories which they are willing to reduce. If the price is above equilibrium and supply exceeds demand, then some firms are unable to sell the quantity they wish at this price and, presumably, inventories accumulate. Thus, again there appears to be an implicit inventory consideration hiding in the background. Yet the accumulation of inventories at prices above equilibrium affects neither the production nor the sales of firms. In either case there is a breakdown of competition in the sense that someone is no longer free to buy or sell in unlimited quantities at the existing price. Thus, expectations are wrong. Yet firms (and consumers) are assumed to behave in a way (making decisions on the static profit and utility maximization principles) which is reasonable only if buying and selling in unlimited quantities are possible. This is an example of the practice, criticized in detail in section 3 of Chapter III, of assuming economic agents to behave in ways which are rational only if expectations are such as to be continuously proven wrong.

In the dynamic analysis of sections 4 and 5 these deficiencies are removed. Inventory policy is introduced in the model developed in the next section as an integral part of the firm's planning. In addition, in the spirit of the discussion in Chapters II and III, expectations are assumed to be correct in one sense or another. In particular, this requires that the firm's belief, that it will be able to sell what it wants at whatever price materializes in the future, be correct. However, in order for this to be true the price must adjust each period in such a way that firms' output and inventory plans are consistent with consumers' purchase plans.

Now it may be that a satisfactory and thorough investigation would have to assume the breakdown of competition in disequilibrium, i.e., the inconsistency of firms' and consumers' plans.[1] It may be that disequilibrium

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[1] This has been argued by Arrow in [2].

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is intimately related to incorrect expectations. If so, the nature of the breakdown, and the resulting behavior, should be analyzed more carefully than has been done in the past. Nevertheless, it seems worthwhile to follow the procedure adopted here and to retain the essential assumption of competition--that the price adjusts so as to clear the market in each period even in disequilibrium, and firms are therefore able to carry out their planned behavior.

The first justification for this is to show that dynamic analysis with determinate results is possible in this case; that a dynamic model, whose

stability properties can be established, can be developed in which firms' expectations are always correct, in either a numerical or a statistical sense. At the very least this should answer the question whether the stability, or instability, of such markets is necessarily associated with inaccurate expectations. The analysis in sections 4 and 5 suggests an answer to this question. It suggests that a competitive market in which expectations are correct is necessarily an unstable one. If this conclusion is accepted it suggests that the reasons for stability in such markets, if any, must be sought in behavior which, in disequilibrium, is based on systematically incorrect expectations.

The second reason for retaining the assumption that the price clears the market each period is that things become rather complicated when it is dropped; i.e., when it is assumed that firms can affect the price out of, but not in, equilibrium. Hence the retention of the competitive assumption, which greatly simplifies the analysis of sections 4 and 5, is desirable in this case, as in others, on grounds of convenience.

The final justification is that this assumption maintains the tradition in the analysis of competitive markets that one of the fundamental characteristics of such markets is that prices move quickly to the level which clears the market. On the other hand, this assumption has the disadvantage that it assumes a considerable amount of dynamic adjustment already to have occurred before prices and sales are recorded, and it leaves this adjustment unanalyzed.

The plan of the rest of the chapter is as follows: Section 2

considers the production, sales and inventory plan of a firm in a competitive market when its expectations are numbers rather than random variables. While Section 2 is concerned with the individual firm, sections 3 and 4 are concerned with the adjustment of the market as a whole. Section 3 investigates the implications, in a simple cobweb-type model, of the assumption that expectations are correct. This section is intended simply to illustrate the remarks above and in the previous chapter concerning the consistency of actual and expected events when the latter are generated by a simple expectational formula, the adaptive expectations equation in this example. In section 4 the stability is investigated of a market in which firms hold inventories and make plans according to the model developed in Section 2. In Section 5 the attempt is made to generalize these results to the case in which expectations are stochastic. It is argued that the programming problem is formidable, but that an approximation, analogous to the one used in Section 4, can be obtained for the stochastic case. This approximation is then shown to yield results which parallel those found in Section 4.

2. Production and inventory policy in a competitive firm.

The purpose of this section is to consider the optimum production, sales and inventory policy of a firm in a perfectly competitive market. Here it is assumed that expected prices are nonstochastic. The relevance of this model to the case in which the firm knows only the probability distribution of future prices is considered in Section 5. Although the model presented here has important nonlinearities which are not present in many similar

models which have been studied in the literature, the techniques employed bear a strong family resemblance to those employed by others who have worked on such problems. The resemblance is particularly great to techniques which have frequently been employed in nonstochastic production smoothing problems.[1]

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[1] See especially the investigation by Modigliani and HOhn in [48]

For notational simplicity the firm is assumed to have a finite horizon of N periods within which it does not discount the future. The introduction of discounting with either a finite or an infinite horizon makes no essential difference and the interested reader can easily rework the results with these changes. While prices may be expected to change in the future, the firm does not expect its production and inventory cost functions to charge over time. In addition, production costs in one period are assumed to be independent of the volume of production in other periods. Starting with period one, the firm's profits over the entire horizon can be written

$$\overline{\Pi}^{N}_{1} = \sum_{1}^{N} p^{e}_{n} x_{n} - \sum_{1}^{N} c(z_{n}) - \sum_{1}^{N} r(I_{n}), \qquad (1)$$

where p^{e}_{n} = price expected in period n, x_{n} = sales, z_{n} - production, both during the nth period, and I_{n} = inventory remaining at the end of the nth period. $c(z)$ is the total cost of producing z units in any period and $r(I)$ is

the total cost of storing I units between any two consecutive periods. Both $c(z)$ and $\pi(I)$ are assumed to be strictly convex functions of their arguments (marginal production and inventory costs increase with the amounts produced and stored respectively). Concavities in the two functions (decreasing marginal costs) greatly complicate the analysis and linearities (constant marginal costs) make the problem uninteresting. The reason for the latter statement is that if $r(I)$ were linear the firm would either store nothing or sell nothing each period. If $c(z)$ were linear the firm would either produce nothing or an infinite amount (or at least the amount which caused the breakdown of perfect competition)each period.

Before commencing the formal analysis, one other assumption should be discussed explicitly. In (2a) below, one of the constraints imposed on the firm's policy is $I_n \geq 0$. While this nonnegativity constraint is a natural and widely used one in the programming literature, it is often omitted in models of speculative behavior. There are three substantive interpretations which have been given to the notion of negative inventories: (i) A firm may be thought to have the possibility of buying from other firms as well as of producing, thus making it possible to sell more than the sum of its own production and initial inventory. While this may be possible for a single firm it is obivously, by definition, not possible for the market as a whole and therefore should be excluded from a study aimed at the analysis of the entire market. (ii) A firm (or the entire market) may be thought to have the possibility of postponing some deliveries

for one or more periods.[1] I have two objections to this assumption. First,

[1] The fact that Muth allows for this possibility is one of the main differences between his model in [49] and the one in this chapter.

it is inconsistent with the notion, discussed above, that the observed price should clear the market in a competitive industry. Second, delivery postponement is only profitable if price is expected to fall in the future. That consumers in a competitive market should not only be willing to accept postponed delivery, but also to pay a higher price than that ruling when delivery is made is an assumption which should not be made without explicit justification. In fact such behavior can probably only be rationalized within a model containing a futures market. Then this interpretation becomes a special case of the last one. (iii) A firm may be thought to make some of its sales on a futures market and some on a spot market. While this modifies the form of the constraints in (2) below, it also introduces other constraints and complications. Others, more competent in these matters [2]

[2] See Telser [61]

have recently undertaken extensive analysis of futures markets (although without most of the complications introduced in this chapter by the multi-period expectations horizon) and such markets are simply excluded from this analysis.

The firm's problem is to find the values z_n^* , x_n^* and I_n^* , of z_n , x_n and I_n respectively, $(n = 1, \ldots, N)$ which maximize (1) subject to the constraints

$$x_n \, , \, z_n \, , \, I_n \geq 0 \qquad (2a)^1 \qquad (n = 1, \ldots, N)$$

$$z_n - x_n = I_n - I_{n-1} \qquad (2b)$$

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[1](2b) could be written more symmetrically with (2a) as $z_n - x_n \geq I_n - I_{n-1}$, allowing for costless disposal, but this would never be profitable.

Stated in this way we have a nonlinear programming problem and, as is so often true, it is the non-negativity conditions (2a) which cause trouble. In fact, the special structure of the problem, implied by the conditions of perfect competition, permits a rather simple solution. It is convenient here, as in many programming problems, to find the solution when some of the non-negativity conditions are ignored and then to modify the solution thus found in such a way as to satisfy them. Eliminate the x_n from (1) by (2b) and equate to zero the partial derivatives of (1) with respect to $z_n (n = 1, \ldots, N)$ and $I_n (n = 1, \ldots, N-1)$. This gives us the following $2N-1$ equations:

$$c'(z_n) = p_n^e \qquad (n = 1, \ldots, N) \qquad (3a)$$

$$r'(I_n) = p_{n+1}^e - p_n^e \qquad (n = 1, \ldots, N-1) \qquad (3b)$$

The convexity assumptions ensure that each of these equations has at most one non-negative solution and that these solutions will be associated with a (local) maximum of (1) rather than with a minimum.

(3a) is the usual price-equal-marginal-cost condition. (3b) says that the firm should hold for sale in future periods the amount which makes marginal storage cost equal to the expected price change between the two consecutive periods. Both of these are intuitively appealing but, as will be shown below, neither is a correct necessary condition for an optimum program in any period.

It will now be convenient to introduce the concept of a fundamental solution to the firm's programming problem. The fundamental solution for period n consists of three values z_n^o, x_n^o and I_n^o of z_n, x_n, and I_n respectively. z_n^o and I_n^o are defined below. x_n^o is defined, using (2b), by

$$x_n^o = z_n^o - I_n^o + I_{n-1}$$

x_n^o is thus the sales volume implied by the production decision z_n^o, the inventory decision I_n^o and the arbitrary initial inventory I_{n-1}. The fundamental solution of the firm's programming problem consists of the set of fundamental solutions for all N periods.

z_n^o and I_n^o can now be defined. In any n for which non-negative solutions of (3a) and/or (3b) exist, those solutions constitute the values of z_n^o and I_n^o. For example, if $z_n = \epsilon_n \geq o$ is a solution of (3a), then $z_n^o = \epsilon_n$. Likewise for the inventory equations (3b).

However, it is clear that under the conditions stated solutions may fail to exist for any or all of the $2N-1$ equations (3a) and (3b). If no $z_n \geq o$ satisfies (3a) for some n, this means that there is no output whose marginal production cost is as low as expected price. If no $I_n \geq o$ satisfies (3b) for some n this means that there is no inventory whose marginal storage cost

is as small as the expected change in price. This will certainly be the case if $p^e_{r+1} < p^e_n$. In any n for which the solution to (3a) fails to exist, we put $z^o_n = 0$. Likewise, in any n for which the solution to (3b) fails to exist, we put $I^o_n = 0$. Logically, these are simply definitions of the fundamental solution when solutions to (3a) and (3b) fail to exist. Intuitively the motivation is as follows. If price is too low to justify any production, then nothing should be produced. If there is some inventory from the previous period this may be sold currently or stored for future use, or both, depending on what price is expected in the future. Likewise if price is not expected to rise enough to cover the cost of storing even the smallest amount, then both current production and existing inventory should be sold currently. While these intuitive arguments are appealing, it should be remembered that they are used merely to 'justify' the definition of the fundamental solution. This does not necessarily constitute the firm's optimum program.

To summarize: the fundamental solution is built up from the solutions of (3a) and (3b) when these exist. Otherwise zeroes are used. Once z^o_n and I^o_n have been thus found, x^o_n is the sales volume which they and I_{n-1} imply. It follows immediately that the fundamental solution always exists and that z^o_n and I^o_n are nonnegative for all n. Under what conditions will the fundamental solution be the optimum program? It is obvious that the only way in which the fundamental solution can fail to be optimum is if $x^o_n < o$ for at least some n. If $x^o_n \geq o$ for all n, the fundamental solution is said to be feasible and, together with the obvious terminal condition $I_N = o$, it then constitutes the

firm's optimum program.

It should be clear that nothing in the definition of the fundamental solution implies its feasibility. Consider the simple example in which

$$c'(z) = z, \quad r'(I) = I , \quad I_o = 0 , \quad N > 2 ,$$

$$p_1^e = p_2^e = 1 \quad \text{and} \quad p_3^e = 10 .$$

Then (3a) and (3b) give us

$$z_1^o = z_2^o = 1 , \quad z_3^o = 10 , \quad I_1^o = 0 , \quad I_2^o = 9 .$$

Hence,

$$x_1^o = z_1^o = 1 , \quad \text{but} \quad x_2^o = 1 - 9 = -8 .$$

Thus, the fundamental solution for period one is feasible, but that for period two is not. Hence the fundamental solution for the program as a whole is not feasible. The reason that non-feasibility occurs is that one rule, (3b), tells the firm to store more than the sum of what another rule, (3a), tells it to produce plus the amount available from last period's terminal inventory.

In order to see what the firm's optimum program is in case of nonfeasibility we must consider an optimization condition more fundamental than those expressed by (3a) and (3b). This is that it is never worthwhile to produce in the (n+1)st period an amount whose marginal cost of production is greater than the marginal cost of producing a unit in the previous period and storing it, i.e., whatever else is true of the optimum production and storage quantities

it certainly must be true that

$$c'(z_n^*) + r'(I_n^*) \geq c'(z_{n+1}^*) \qquad (4)$$

The inequality runs in the direction indicated because the firm always has the option of producing now for future use, but not of producing later for current use. In particular, (4) holds as an equality in the special case in which production and inventory are the solutions of (3a) and (3b) . (4) is, however, more fundamental and holds even when (3a) and (3b) do not constitute the optimum program.

We can now demonstrate the relevance of (4) in the case of nonfeasibility. Suppose that the fundamental solution is nonfeasible and choose a value j of n for which $x_j^o < o$. Now from (3a) and (3b)

$$c'(z_j^o) + r'(I_j^o) = c'(z_{j+1}^o)$$

But, since $x_j^o < o$, it follows that $z_j^o + I_{j-1} < I_j^o$ and hence[1]

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[1]The second inequality follows from the fact that z_n^* must be at least as great as z_n^o for any n . If $z_n^o = o$, this is obviously true. If z_n^o is the solution of (3a) then this is true because it would never pay to produce less than z_n^o . If less than z_n^o were produced profit could be increased by producing and selling a larger amount in n since marginal cost would be less than price.

$$c'(z_j^o) + r'(z_j^o + I_{j-1}) < c'(z_{j+1}^o) \leq c'(z_{j+1}^*)$$

Thus the maximum amount which can in fact be stored if z_j^o is produced violates the inequality (4) . Thus it must be worthwhile to produce more than z_j^o in period j . How much more? The answer to this is that enough should be produced in period j so that the marginal cost of production and storage in j equals the marginal cost of production in $j+1$. This establishes the following theorem:

I: If j is a value of n for which $x_j^o < o$, then

$z_j^* > z_j^o$ and z_j^* must satisfy

$$c'(z_j^*) + r'(z_j^* + I_{j-1}) = c'(z_{j+1}^*)$$

Before pointing out an important implication of this result it is worthwhile demonstrating a slightly more general theorem:

II: Whenever $z_n^* > z_n^o$, z_n^* must satisfy[1]

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[1]Theorem II implies Theorem I, but not vice versa. This is because $x_n^o < o$ implies $z_n^* > z_n^o$ but z_n^* may exceed z_n^o even if $x_n^o \geq o$. I am greatly indebted to F. H. Hahn and W. M. Gorman for setting me straight on this point.

$$c'(z_n^*) + r'(z_n^* + I_{n-1}) = c'(z_{n+1}^*) \qquad (5)$$

It is obvious that, if $z_n^* > z_n^o$, then (4) must hold as an equality, i.e.,

$$c'(z_n^*) + r'(I_n^*) = c'(z_{n+1}^*) \qquad (4a)$$

If it pays to produce more than the amount which equates marginal production cost and current (expected) price, this can only be because it is worthwhile to store some units for future sale. But since the firm always has the option of producing these units (those for future sale) during the next period, it will equate the cost at the margin of producing extra units in either of the two periods. This means that (4) holds as an equality, (4a). However, Theorem II says more than this. It says that, when $z_n^* > z_n^o$, not only does (4a) hold, but also $x_n^* = o$, (i.e., $I_n^* = z_n^* + I_{n-1}$). That this is correct can be seen as follows. Suppose that (4a) holds but that $x_n > o$. I now propose to show that by reducing x_n profit will be increased. If x_n is reduced, then either (i) I_n increases, or (ii) z_n is reduced. (i) is evidently not profitable since this would increase the left hand side of (4a) and make it a strict inequality. This makes extra units available next period, but they could more cheaply be produced next period than this period. (ii), however, is profitable. If x_n and z_n are both reduced by one unit, then revenue falls by p_n^e and cost by $c'(z_n)$. But, by hypothesis, $c'(z_n) > p_n^e$. Hence costs fall by more than revenue and profits go up. Thus, it pays to reduce x_n either to zero or to the point at which $z_n^* = z_n^o$. This establishes Theorem II.

We now have the key to the solution of the firm's programming problem. Since z_n^* cannot be less than z_n^o, either $z_n^* = z_n^o$, where z_n^o is determined as above, or z_n^* satisfies (5). Before showing how a complete solution can be built up from these results, an important corollary of Theorem II should be pointed out.

Corollary: Whenever optimum production z_n^* is other than the amount z_n^o implied by the fundamental solution, then optimum current sales x_n^* are zero.

This follows immediately from Theorem II, since (5) says that, when the premise is satisfied, $I_n^* = z_n^* + I_{n-1}$, or $x_n^* = z_n^* + I_{n-1}$, $-I_n^* = o$. This corollary means that whenever the firm violates the marginal-cost-equal-price rule (unless it does so in order to produce nothing), it will make no current sales. The converse is that if the firm is observed to make positive current sales then, unless current production is zero, it will obey the marginal-cost-equal-price rule. The case in which $x_n^* > o$ will be called the 'normal' case. In order to know whether the n^{th} period is normal or not, the firm must (except as noted below) consider expected prices over its entire horizon. However, provided n is a normal period, it follows that the firm's output and inventory policy can be 'explained' by taking into consideration only p_n^e and p_{n+1}^e. In other words, if n is a normal period the firm behaves 'as if' it were considering expected prices only one period in the future. Although this argument never seems to have been stated explicitly in the literature, it has presumably been in the backs of the minds of many writers

who have assumed that all future expected prices of a competitive firm can be represented by a single number.[1] The normal case appears to be closely related

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[1] See, for example, [11], and [21] .

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to the analysis of stability in competitive markets and this concept will be used extensively in the discussion in sections 4 and 5 .

We are now in a position to derive the firm's optimum N-period plan. Starting in period one, it has an arbitrary initial inventory I_o . The two essential elements in the solution of the firm's programming problem are the fundamental solution and the roots of sets of equations like (5) . However, over any interval of consecutive periods within the horizon it is extremely important to know which calculation yields the optimum program. The reason is that finding the roots of sets of equations like (5) involves a high degree of simultaneity if many such equations are involved. The fundamental solution for period n , on the other hand, can be found by considering only I_{n-1} , p_n^e and p_{n+1}^e . No problem of simultaneity is involved and the fundamental solution for the entire N periods can be found by considering consecutively periods 1, 2, ..., N . The natural question to ask in this situation is whether it is possible to break up the N periods into subintervals which can be considered separately. The answer is 'yes' in some circumstances, and these will be explored below. However, let us first indicate the solution in the general case in which simultaneity

may be present over the entire horizon. The solution can always be obtained in the following way. Define the stage s computation as the determination of the $s+1$ roots $z_n^{(s)}$ of the equations[1]

$$c'(z_n^{(s)}) + r'(I_o + \sum_1^n z_i^{(s)}) = c'(z_{n+1}^{(s)}) \quad (n = 1, \ldots, s)$$
$$c'(z_{s+1}^{(s)}) = p_{s+1}^e \qquad (6)$$

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[1] If there are any values of s for which no value $z_{s+1}^{(s)}$ of z_{s+1} satisfies this equation, then this set of $s+1$ equations can be ignored. They play no further part in the determination of the optimum program.

There are N sets of equations like (6), obtained by putting $s = o, 1, \ldots, N-1$. The s^{th} set yields a program for the first $s+1$ periods in which the firm sells nothing until the $(s+1)^{st}$ period. In particular, if $s = o$ then (6) reduces to (3a) when $n = 1$. In this case, provided the solution $z_1^{(o)}$ exists, it is the same as z_1^o, the fundamental solution. The firm's optimum first period output is

$$z_1^* = \max_s z_1^{(s)} \qquad \text{where } s = o, 1, \ldots, N-1 \qquad (7)$$

Essentially, what this calculation does is to tell the firm the farthest period in the future for which it is worthwhile to start accumulating inventory in

period one. $z_1^{(s)}$ is the amount it would pay the firm to produce in period one if it contemplated sales only during (and after) the $(s+1)^{st}$ period, and if its production obeyed the intertemporal efficiency condition specified by (4a). That z_1^* is, as indicated by (7), the largest of the $z_1^{(s)}$ can be seen as follows. Suppose $z_1^* = z_1^{(q)}$. In (6) put $s = q$. By successive substitution among the $s+1$ equations in (6) we get

$$
\begin{aligned}
c'(z_1^{(q)}) &= c'(z_2^{(q)}) - r'(I_o + z_1^{(q)}) \\
&= \\
&\vdots \\
&= p_{q+1}^e - \sum_{i=1}^{i=q} r'\left(I_o + \sum_{j=1}^{j=i} z_j^{(q)}\right)
\end{aligned}
$$

The right hand side of this equation can be called the 'corrected' value of p_{q+1}^e; that is, corrected for the cost of storing the marginal unit from period one until period $q+1$. Thus, in this form the equation determines the value of z_1 which equates marginal production cost in period one to corrected marginal revenue in period q when an optimum production program is planned in the intervening periods. Thus, the notion behind (7) is that the firm should sell period one's output in that (temporal) market whose (corrected) marginal revenue is greatest. Furthermore, the amount which should be produced for that market during period one is the amount which equates marginal production cost and (corrected) marginal revenue.

Having established z_1^* by (7), x_1^* and I_1^* follow easily. Optimum first period sales are

$$x_1^* = \begin{cases} x_1^o & \text{if } z_1^* = z_1^o \\ 0 & \text{if } z_1^* = z_1^{(q)} \text{ where } q > 0 \end{cases}$$

The optimum first period inventory is

$$I_1^* = \begin{cases} I_1^o & \text{if } z_1^* = z_1^o \\ z_1^* + I_o & \text{if } z_1^* = z_1^{(q)} \text{ where } q > 0 \end{cases}$$

Having established by (7) that $z_1^* = z_1^{(q)}$, the other roots of (6) when $s = q$ give us the optimum production quantities $z_n^* = z_n^{(q)}$ for the intervening periods $n = 2, \ldots, q + 1$, as well. The intuitive consideration here is that if it pays to hold period one's production until period q, then it certainly pays to hold the production of intervening periods until at least that time. Optimum sales and inventory levels for the intervening periods are

$$x_n^* = o,$$

$$I_n^* = I_o + \sum_1^n z_j^*$$

for any n in the interval $1 < n \leq q$, if such n exist.

This completes the determination of the optimum plan for periods one through $q+1$. The procedure for finding the optimum program for periods $q+2$ through N is as follows. Exactly the same procedure is followed as above except that we now start with period $q+2$ instead of with period one. Sets of equations ($N-q-1$ sets, the s^{th} of which contains s equations, where $s = 1, \ldots, N-q-1$) exactly like (6) must be solved simultaneously, except that they now start with period $q+2$ instead of with period one, as do (6) . The application of the criterion (7) to these new sets of equations will then lead to an optimum production z^*_{q+2} for period $q+2$. Proceeding in exactly the same way as before the entire optimum program over some new interval from $q+2$ to $q'+1$, say, can be determined. Then a third interval starting with $q'+2$ must be ascertained until, in this way, the entire N periods are covered. This procedure breaks the N periods into a series of not more than N subintervals. Each subinterval contains exactly one period in which positive sales are made. The normal case (as defined above) is that in which there are N such subintervals.

This procedure evidently involves extremely heavy computations if N is at all large. It is only feasible to compute provided some shortcuts can be found which will limit the degree of simultaneity involved. Several sets of considerations can be established which reduce drastically the computational burden implied by the procedure outlined above.

(i) The first relevant consideration is that the firm is normally interested in determining each period only the first period solution of the N-period program. Only period one's plan must actually be executed in period one and next period the firm will normally have a new N-period horizon, commencing

with that period, and it will again take only the first period decision of the new program. This calculation is of course much simpler than the determination of the entire N-period program.

Most of the attention in the next paragraphs and, indeed, in the rest of the chapter will thus be concentrated on the problem of finding the first period's optimum program. According to the procedure outlined above this still, however, involves an extremely burdensome computation. Each of N sets of nonlinear equations, the s^{th} set involving $s+1$ equations, must be solved simultaneously. Now the essential purpose of all this computation is to find the value of q, as defined above. Once this has been ascertained the most that is required is the solution of one set of $q+1$ simultaneous equations. Therefore the problem is to find ways of discovering q without solving all N sets of simultaneous equations. The following paragraphs are directed toward this goal.

(ii) Much the most important case is, of course, that in which the fundamental solution of the entire program is feasible, i.e., $x_n^o \geq 0$ for $n = 1, \ldots, N$. In this case $z_n^* = z_n^o$ for all n and no simultaneity is involved whatsoever. This is the case in which every period is a normal one. A sufficient, but by no means necessary condition for feasibility is $p_n^e \geq p_{n+1}^e$ for $n = 1, \ldots, N-1$. Roughly, if expected prices rise slowly relative to marginal storage cost, then the fundamental solution is likely to be feasible.

(iii) Suppose that $k_1 \leq N$ is the largest n for which $x_n^o < 0$, i.e., suppose $x_n^o \geq 0$ for $n \geq k_1$. Then $q \leq k_1$. In this case only k_1

rather than N sets of simultaneous equations must be solved. The maximum degree of simultaneity is the number of periods until the last nonfeasibility is observed in the fundamental solution.

(iv) Suppose k_2 is the smallest value of n for which $x_n^o < 0$, i.e., suppose that $x_n^o \geq 0$ for $n < k_2$. Then either $q = 0$ or $q \geq k_2$. This means that it is never worthwhile to save all of current production for sale in a period earlier than that in which the earliest nonfeasibility occurs. In particular, if nonfeasibility never occurs then the fundamental solution is optimum.

(v) Suppose $p_m^e = \max_n p_n^e$ and that $m < N$. This means that the highest price expected over the horizon occurs in period m which is before the final period. In this case $q \leq m$. This is so since it evidently does not pay to start accumulating inventory for sale in any period after m until m has passed. Any inventory accumulated up to period m will realize more revenue at that time than later.

(vi) Finally, the computational burden is reduced if special forms of the cost functions are introduced. Consider the following marginal cost functions, used extensively in the analysis of sections four and five:

$$c'(z) = cz,\ r'(I) = rI \quad \text{where} \quad c,r > 0 .$$

In this example the total production and carrying cost curves are quadratics and marginal costs are straight lines through the origin. In this case the fundamental solution is given by

$$z_n^o = p_n^e/c \quad \text{and} \quad I_n^o = \max. \left((p_{n+1}^e - p_n^e)/r \ , \ 0 \right) \tag{8}$$

The simultaneous calculations are now within bounds for even quite large N. (6), for example, reduces to the matrix equation.[1]

$$\begin{bmatrix} c+r & -c & 0 & \cdots & 0 \\ r & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & -c \\ r & \cdots & \cdots & r & c+r \end{bmatrix} \begin{bmatrix} z_1^{(s)} \\ \\ \\ \\ z_{s+1}^{(s)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ p_{s+1}^e \end{bmatrix}$$

---

[1] It may be useful to indicate briefly how the solution presented here differs from those of other writers who have included inventories in models of a perfectly competitive firm. The two things which appear to be new in this analysis are: (1) The precise conditions for the normal case in which the effective horizon is two periods, and (2) the theorem that, if expected prices beyond the second period affect the first period's production, then no first period sales are worthwhile.

The two classical papers on the subject of speculative inventory holding are by Shaw, [57], and Smithies, [59]. Shaw presents the correct solution when the horizon is two periods and, in the terminology used here, $x_1^o \geq 0$. His diagrams are restricted to the case of linear marginal production and storage costs, but this is not crucial to the argument. He considers the possibility $x_1^o < 0$ uninteresting "in view of the timorousness and lack of resources of the competitive producer." Smithies was the first to give a rigorous analysis of the case (with a continuous time parameter) of a firm with a downward sloping demand curve and proportional Storage costs. Essentially the same model (in discrete time) is considered by Lutz and Lutz in [37]. Both of these studies derive the usual price discriminator's condition that marginal revenue and marginal cost should be equal within and between all periods. They restrict consideration to rightward shifts in demand and this ensures the nonnegativity of I_n. Neither considers in detail the special case of perfect competition.

3. Dynamic stability without inventories

In this and the following sections the decision model developed in the previous section is used to investigate the dynamic stability of a single competitive market. No consideration is given to the question of stability in multiple markets. The main focus of the discussion will be the relation between stability and the accuracy of expectations. The usual approach in studies of stability is to assume that expactations (the p^e terms in the model in section 2) are generated in a particular way and to ask what time-path of actual prices will result. In this study the approach is to ask what expectations must be if the behavior they induce is to make those expectations turn out to be correct.[1]

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[1] For an interesting application of this technique to a different problem see [27].

Does this behavior lead to a stable or an unstable market adjustment? In particular, is it possible for a competitive market to be unstable even though expectations are correct?

In this section the rather simple case of a market in which there are no inventories is considered. Although this is essentially a trivial case, it may be useful to apply the analysis first to a transparent example. In section 4 inventories are introduced and it is shown that the assumption of numerically accurate expectations leads to a richer and more interesting analysis. In section 5, the results of section 4 are generalized to the case in which expectations

are correct in a statistical rather than in a numerical sense.

In this as in other studies of stability it is much easier to obtain results if linear approximations of certain relationships are employed. This means that the results obtained are valid only in the vicinity of the equilibrium position. This is true of all the results in the remainder of this Chapter. In particular, the firm's marginal production cost will be approximated by the linear form used in the example in the previous section: $c'(z) = cz$. In the next section, when inventories are introduced, the marginal carrying cost will also be approximated by a linear form, again as in the example. Both here and in the following sections the addition of a constant to these marginal costs leaves all the stability analysis unaffected. Since in this section the firm is assumed not to hold inventories, its program in the n^{th} period is completely specified by the first relation in equation (8):

$$z_n = p_n^e/c \qquad (8a)$$

One of the important assumptions employed in the rest of this Chapter is that all firms in the industry have the same price expectations. (This is already contained in the notion that expectations are correct.) On this assumption the aggregation of the firms' supply curves into an industry curve presents no problem. Assuming that all firms' marginal production costs are proportional to output then, even though the coefficient c be different for different firms, the industry output will be determined by an equation of the form (8a). Industry output will be $\Sigma z_n = p_n^e \Sigma 1/c$ where the sums are over all firms, not over

time. In this situation there is no point in changing notation and in the further analysis (8a) will refer to industry totals. Precisely similar remarks apply to the aggregation of inventories and the second relation in (8) will be used to represent total industry inventory in the following sections. In the absence of inventory holding, industry sales x_n^S are the same as industry production z_n. The industry demand curve is also assumed to be linear in the current actual price,

$$x_n^D = a-bp_n \tag{9}$$

The actual and expected prices which clear the market in period n are those which satisfy

$$x_n^D = x_n^S \tag{10}$$

Equations (8a), (9) and (10) do not yet close the model since (7a) contains expected price while (9) contains actual price. A determinate time-path of price would result if an additional relation relating expected price to actual past prices were introduced. For example, the simple textbook version of the cobweb model is the case in which $p_n^e = p_{n-1}$.

Express (8a), (9) and (10) in the form

$$a-bp_n = (1/c)p_n^e \tag{11}$$

It is obvious that, in this model, expectations can be accurate only if the system is in equilibrium. Putting $p_n = p_n^e$ in (11) gives us $p_n = \bar{p}$ where

$$\bar{p} = \frac{ac}{1 + bc} \tag{12}$$

is the equilibrium price. In fact, for the accuracy of expectations in any period it is both necessary and sufficient that price be equal to its equilibrium level.

Now let us close the model by assuming expected prices to be generated by the adaptive expectations formula[1] (see (1) in Chapter III).

$$p_n^e = (1-B)p_{n-1}^e + Bp_{n-1} \tag{13}$$

---

[1](11) and (13), except for minor notational changes, constitute the model considered by Nerlove in [51].

It is clear from the above discussion that when the expectations generated by (13) are inserted in (11) the resulting time-path of actual price must be such that expectations are in fact wrong except when the market is in equilibrium, i.e., when $p_n = \bar{p}$. This was proven above regardless of the way in which expectations are formed. In fact, in this case, the time-path of expectational errors has exactly the same form as the time-paths of both the actual and expected prices. This can be seen as follows. Between (11) and (13) either p_n or p_n^e can be eliminated resulting in a difference equation in the remaining variable. In this case, whichever variable is eliminated, the ensuing equation is of the same form, viz.,

$$v_n = (1-B-B/bc)v_{n-1} + aB/b \qquad (14)$$

where v_n equals p_n or p_n^e. (14) has the solution.

$$v_n = (1-B-B/bc)^n (v_0 - \bar{v}) + \bar{v}$$

where $\bar{v} = \bar{p}$ and v_0 is the initial condition (either actual or expected price in period zero). From this equation we get a similar equation for the expectational error in period n,

$$p_n^e - p_n = (1-B-B/bc)^n (p_0^e - p_0)$$

Here p_o^e is the expected price, and p_0 the actual price, in period zero. Thus, starting with an arbitrary initial expectational error, $p_0^e - p_0$, the time-path of the error term is of exactly the same form as that of both the actual and expected prices. In particular, if actual price coverges (the market is stable) then so does the error; if the initial error is positive $(p_o^e - p_o)$, then so are all succeeding errors. This is an example of the use of a simple autoregressive expectational equation in order to deduce the time path of the variable of which the expectation is an estimate (see Chapter III, section 3). Yet the mechanism generating the variable leads to a time-path such that the expectations cannot possibly be correct. Not only that, but the expectational errors follow a time-path as simple as that of the variable itself. Presumably, an intelligent decision maker would be able to observe at least that his

expectational errors were all biased in the same direction and would then refrain from using the simple routine which led to the situation.

In closing this section it should be emphasized that these remarks are directed at the use of virtually all mechanical expectations formulas in stability analysis. Exactly the same criticisms would apply if the textbook cobweb model, in which $p_n^e = p_{n-1}$, were used. The adaptive expectations formula has served merely as a peg on which to hang a much more general criticism.

4. Dynamic stability with inventories: nonstochastic case

In this section the possibility of inventory holding is introduced in the dynamic model considered in the previous section. This complicates the analysis considerably, but it is still possible to obtain definite results on the assumptions introduced above. Formal results in this section will be restricted to the normal case, defined in section 2 as that in which the firm behaves as if it looked only one period ahead in formulating its plans. This is the only case which is relevant in an investigation of stability in the vicinity of equilibrium since, as was shown in section 2, when the normal case breaks down the firm sells nothing. When this happens price rises to the level which cuts off all demand and this is evidently not a point in the vicinity of equilibrium. In any case the meaning of the analysis becomes doubtful when prices are reached at which no sales are made.

In this normal case the production and inventory plan which the industry follows in period n is completely specified by equations (8).

This plan can be expressed by

$$z_n = p_n^e/c \tag{15a}$$

$$I_n = \max \left\{(p_{n+1}^e - p_n^e)/r, 0\right\} \tag{15b}$$

Here the price terms are all expectations held when the production and inventory decisions in the period in question must be made. Employing the convention introduced in the previous section we shall reinterpret the symbols in (15) to be industry totals. Industry sales are

$$x_n^S = z_n + I_{n-1} - I_n \tag{16}$$

Substituting (15) in (16) we get

$$x_n^S = p_n^e/c + \max \left\{(p_n^e - p_{n-1}^e)/r, 0\right\} - \max \left\{(p_{n+1}^e - p_n^e)/r, 0\right\}$$

The linear demand curve (9) and the condition (10) that the price clear the market each period are again assumed. In this section the implications will be sought of the assumption that all price expectations are numerically correct, i.e., $p_n = p_n^e$ for all n. In the next section this will be generalized to include the case in which the distribution function, but not the numerical values, of future prices are correctly anticipated. Dropping the superscripts, using the definition (16), and substituting (15a), (15b) and (9) in (10) we get

$$a-bp_n = p_{n/c} + \max\left\{(p_n-p_{n-1})/r,0\right\} - \max\left\{(p_{n+1}-p_n)/r,0\right\} \qquad (17)$$

(17) is the relation which price must satisfy if expectations are to be fulfilled each period. It is the fundamental dynamic relation on which all subsequent analysis in this section is based. It is a second order nonlinear difference equation, but fortunately one whose stability properties are easily deduced. Its solution, the time-path of price (and of course of expected price) depends on two arbitrary initial prices p_o and p_1 whose values are given to the market exogenously.

The equilibrium price in the market is the stationary solution of (17) obtained by putting $p_n = \bar{p}$ for all n. The equilibrium price which this yields is the same as that found in the model of the previous section, equation (12), in which there were no inventories. This is because in equilibrium the firms hold no inventories and the price which clears the market in this case is unaffected by the cost of carrying inventory. If the initial prices p_0 and p_1 are both equal to $\bar{p}$, this price will continue until some exogenous force disturbs the market.

What happens if the equilibrium is disturbed? Superficially, the possibility of inventory holding against an expected price rise appears to be a destabilizing factor in the market. An expected increase in price induces the firm to withhold some current production for future sale and this is just the behavior which makes price rise and hence fulfills expectations. In fact, this connection is more than superficial and in any case in which correct expectations are possible, (17)

is explosive in an upward direction. More specifically, the following theorem will be proved:

> For any initial conditions neither of which is equal to $\bar{p}$, but for which a solution to (17) exists, this solution is unstable in an upward direction.

The mathematical proof of this theorem is tedious, though it involves only elementary algebra, and it is relegated to Appendix A. The following literary proof appears to be rigorous, and the Appendix proof follows it step by step.

It is convenient to consider two cases separately according to the relation between the initial prices p_o and p_1.

(i) Suppose $p_0 < p_1$. In the first place, $p_2 < p_1$ is inconsistent with (17), i.e., once price starts to rise expectations of a fall cannot be correct. If price in period one exceeds that in period zero firms will have some inventory on hand at the end of period zero. On the other hand, if price is expected to fall in period two below its period one level, firms will unload this inventory in period one. Not only this, but production will also be less in period two than it was in period one since the price is lower and therefore a smaller output equates price and marginal cost. Thus, even if no inventory is held at the end of period two, the amount put on the market in period two must be less than that put on the market in period one. Since the demand curve is downward sloping this will bring a higher price than ruled in period one and therefore expectations of a lower price cannot be correct. This argument can be summarized by saying that if $p_0 < p_1$, then expectation of a decline in price

in period two would lead firms to put less on the market than in period one and hence this would make the price higher rather than lower.

In the second place, a continuous rise in price is always possible, consistently with correct expectations. If $p_o < p_1$, and firms expect p_2 to be greater still, there is always some value of p_2 which, if expected, will induce firms to hold in inventory at the end of period one just that amount which makes p_1 clear the market. The same reasoning applied to the next period ensures that the price expected in period two actually materializes. Together, these results imply that there is a unique price p_2 such that, if expected, it will materialize, and that this price exceeds p_1. The same reasoning applied to period three shows that $p_3 > p_2$ and, inductively, that every price exceeds the preceding one.

(ii) Suppose $p_o > p_1$. Here there are two subcases.

(a) Suppose $p_o > p_1 > p_2$. This is impossible. If it were true there would be no inventory at the end of period zero and none at the end of period one. Hence, in period one production equals sales and this is possible only if $p_1 = \bar{p}$. But we cannot also have $p_2 < \bar{p}$ since the amount available for sale in period two would then be less than that available in period one and would therefore bring a higher price. If $p_o \geq p_1$ and $p_1 = \bar{p}$, then $p_2 = \bar{p}$ is also a possibility. In this case the market remains in equilibrium.

(b) Suppose $p_o > p_1 < p_2$. This is also impossible. In this case we would have no inventory at the end of period zero but a positive amount at the end of period one. But production in period one would be less than that in period zero. Hence the amount put on the market in period one cannot be as great as the amount put on the market in period zero, and it would therefore bring a higher price, contradicting the assumption that $p_o > p_1$.

(a) and (b) together prove that (ii) is impossible. This completes the proof of the theorem.

The force of the above theorem appears to be that if we wish to retain the essential characteristic of competition, that the price clears the market each period, then we must search for stabilizing factors in such markets among expectational hypotheses which, in one sense or another, permit incorrect expectations. If this argument is accepted, it becomes important to ask what kind of expectational errors should be introduced. In the spirit of the discussion in the previous Chapter, an obvious case to investigate is that in which expectational errors are statistically stable. This is undertaken in the next section.

5. Dynamic stability with inventories: stochastic case

Until now no account has been taken of the effect of uncertainty on the behavior of the firm or market. It is the purpose of this section to do so. As was pointed out in section 1, only uncertainty concerning price, and not quantity demanded, is relevant in a perfectly competitive market. It is assumed that this ascertainty can be represented by

$$p_n = p_n^e + u_n \qquad (18)$$

where u_n is a randum variable which is identically and independently distributed from period to period and has a zero expected value. p_n^e is thus the expected value of p_n, though it is convenient for the moment to leave unspecified the exact information on which this expectation is conditional. Since it is assumed that all firms have correct expectations, it is not necessary to distinguish between the actual and anticipated distributions of p_n: (18) represents both. It should, however, be noted that (18) involves a reinterpretation of the symbol p_n^e. In previous sections it has represented the numerical value which price was expected (with certainty) to assume in period n. In this section it represents the "expected value" of price in the usual sense of the average of all values which p_n might assume, each weighted by its probability.

(18) (and the cost functions introduced in section 2) still do not completely specify the firm's decision problem. We must also specify what information is available at the time each decision is made. The following classification will indicate some of the possibilities:

(a) The firm might be able to observe p_n and then decide on z_n and x_n;

(b) The firm might have to decide on z_n before observing p_n (but after observing p_{n-1}), but be able to make its sales decision x_n after observing p_n;

(c) The firm might have to decide on both z_n and x_n before observing p_n (but after observing p_{n-1});

(d) The firm might have to decide on all z_n and x_n for $n = 1, \ldots, N$ before observing p_1.

These are listed in order of increasing degree of "precommittal" required. The kinds of problems involved in programming the firm's operations are quite different in the four cases. Case (d) is essentially the same as the non-stochastic case considered in sections 2-4, and the analysis of those sections applies completely.[1]

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[1]This is because the firm must commit itself at time zero to all future sales and production levels within its horizon. For this purpose the expected price p_n^e is a "certainty equivalent" and the techniques of section 2 apply.

Cases (a) - (c) represent basically different decision problems from the non-stochastic case.[2]

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[2]Case (c) has, however, an important characteristic in common with the nonstochastic case. This arises when, as was true in the stability analysis of section 3 and is true in the stability analysis below in this section, the two-period approximation is used in which the firm makes its sales, output and inventory decisions each period only on the basis of prices anticipated in that and the next period. In this case, p_n^e and p_{n+1}^e are certainty equivalents for p_n and p_{n+1} respectively in the making of the n^{th} period's decisions and the results of section 4 apply completely.

In some ways case (b) -- in which the firm makes its production decision before knowing p_n but decides how much to sell and store after observing p_n --

is the most plausible and it is this case which is analyzed in the remainder of this section. This case takes into account the fact that in competitive markets production decisions normally entail more precommittal than do sales decisions. Unfortunately, the firm's programming problem is still a very complicated one which is largely unsolved.[1] Hence the following approach will be used.

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[1]Any finite-horizon programming problem such as this one is solvable in a sense. One can, in principle, always start at period N and work backward to period one. What is meant in the text is that it has not been possible to characterize the solution in a way that is useful for the purpose of stability analysis.

In sections 3 and 4, it was found that, in order to analyze the stability of the market, a two-period approximation had to be used in which it was assumed that each period's decisions were made only on the basis of prices anticipated in that period and in the next one. A precisely analogous assumption will be made here, namely that each decision is made on the basis of exactly one anticipated price. This means that z_n is chosen taking into account the possible values that p_n may take; x_n and I_n are chosen after p_n has been observed, taking into account the possible values that p_{n+1} may take.[2] p_n^e can thus be

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[2]Unfortunately, here we no longer have the comfort provided by the theorem proved in section 2 for the nonstochastic case to the effect that in any period in which the model with the truncated horizon does not provide the optimum policy, no sales would be made. In the present case the "correct" value of z_n would exceed that indicated by the two-period approximation whenever p_{n+1}^e was large. Whether sales would be made in any period would depend on whether the price which materialized was abnormally high or not (i.e., in the realized value of u_n for that period).

interpreted as the expected value of p_n conditional on the realized prices of all previous periods. The form of the dependence of p_n^e on earlier observed prices is deduced below. In this case p_n^e and p_{n+1}^e are certainty equivalents for the n^{th} period's decisions and the programming problem for case (b) -- and for the other cases as well -- becomes trivial. In addition to this assumption, the linear marginal production and inventory cost functions which were used in sections 3 and 4 will also be used here. With all these assumptions, the firm's behavior can be summarized as follows:

$$z_n = p_n^e/c \tag{19a}$$

$$I_n = \max \left\{(p_{n+1}^e - p_n)/r \ , \ 0\right\} \tag{19b}$$

$$p_n = p_n^e + n_n \tag{19c}$$

$$x_n^s = z_n + I_{n-1} - I_n \tag{19d}$$

The form of (19a) follows from the fact that z_n must be set before observing p_n. Hence, marginal production cost is equated to the expected value of p_n, i.e., p_n^e. The form of (19b) follows from the fact that the period n sales and inventory decisions are made after observing p_n. Hence the firm equates the expected value of the price change to marginal storage cost if p_{n+1}^e exceeds p_n and carries no inventory otherwise.

In defense of the two-period approximation the following argument should be made. It is clear that, were the firm to look ahead more than one period, this could not induce the firm to carry less inventory than is indicated by (19b). (19b) indicates the inventory that is profitable if the firm knows it must sell next period. If, instead, it took account of the fact that next period it will have the option of keeping the inventory longer, this could not make inventory carrying less profitable. Hence, the inventory level indicated by the "correct" programming problem must be at least as large as that indicated by (19b). Now it is clear that in the models considered in this chapter, the possibility of instability arises from the fact that, if firms expect price to rise, they take goods off the market to hold in inventory for future sale and this makes the price rise. Hence, if the market in which firms' actions are governed by (19) to be unstable, this creates the presumption that if the "true" model were used -- in which firms store even more -- the market would be even more unstable. This is a presumption rather than a proof, but it provides some confidence in the following results.

In order to complete the model, it remains only to introduce a demand equation. For this purpose the linear form (9) used in previous sections will be used again, except that a random term will be added to provide the "source" of the uncertainty represented by (19c). This can be written

$$x_n^D = a - b\, p_n + \epsilon_n \qquad (20)$$

where ϵ_n has the same properties as u_n, namely an identical and independent

distribution from period to period, and a zero mean.[1] ϵ_n and u_n are

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[1]There is no substantive difference between (20) and the alternative representation

$$x_n^D = a - b\, p_n^e + \epsilon_n' \,.$$

In view of (19c), (20) can be written

$$x_n^D = a - b\, p_n^e - b\, u_n + \epsilon_n \,.$$

Putting $\epsilon_n' = \epsilon_n - b\, u_n$, the two forms are seen to be equivalent.

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obviously related and this relationship is explored below. It should be pointed out that ϵ_n and p_n cannot be independent. ϵ_n affects the quantity demanded at each price and, given the supply equation, the price which clears the market.[2]

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[2]This is precisely the reason that in statistical models of simultaneous estimation of economic relationships all the endogenous variables are dependent on all the random terms.

The model is completed by introducing the identity

$$x_n^S = z_n + I_{n-1} - I_n \tag{21}$$

and the market-clearing condition

$$x_n^S = x_n^D \,. \tag{22}$$

Finally, recall the convention introduced in section 3 that the same variables will be reinterpreted as market totals for the purpose of stability analysis. We are now ready to investigate the stability of the market represented by equations (19) - (22).

The substitutions which led to (17) now give us the fundamental difference equation

$$a - bp_n + \epsilon_n = p_n^e/c + \max\left\{(p_n^e - p_{n-1})/r, 0\right\} - \max\left\{(p_{n+1}^e - p_n)/r, 0\right\} \qquad (23)$$

which, together with (19c) showing the relation between the p's and the p^e's, constitutes the system whose stability properties are to be investigated. This is a nonlinear stochastic difference equation -- a type whose behavior is generally very difficult to analyze. However, quite definite statements can easily be established about this particular system. The analysis is very similar to that undertaken in the previous section and the algebra is again relegated to an appendix.

The stationary solution of (23), defined by the equalities

$$p_n = p_{n-1} = p_n^e = p_{n+1}^e , \quad \epsilon_n = u_n = 0$$

is the same as (12),

$$\bar{p} = \frac{ac}{1+bc} \qquad (24)$$

Partly as an exercise, first consider the possibility that p_n is a purely random process so that lagged values of price are of no use in predicting current

price. In this case, $p_n^e \equiv \bar{p}$ for all n and $p_n = \bar{p} + u_n$. This represents the situation in which firms treat each price as a random deviation from the equilibrium price and observed prices have no effect on anticipations. If this were true, it follows from (19a) that z_n would assume the same value, $\bar{p}/c$, in each period. However, when p_n was less than $\bar{p}$, p_{n+1}^e $(= \bar{p})$ would exceed p_n and firms would store some of their current production for future sale. If the next period's price, p_{n+1}, exceeded $\bar{p}$, firms would put not only that period's production but also their accumulated inventory on the market. This means that at any $p_{n+1} > \bar{p}$ more will be put on the market in period $n+1$ if p_n was less than $\bar{p}$ than if p_n exceeded $\bar{p}$. Hence, the amount put on the market in period $n+1$ depends on p_n and, since the realized value of p_{n+1} obviously depends on the amount supplied in $n+1$, p_{n+1} also depends on p_n. Hence the distribution of price is not independent from period to period and this contradicts the anticipations on which this behavior was based. This argument shows that if current price is assumed to have no effect on future price, this will lead firms to behave in such a way that current price does affect future price, thus contradicting anticipations.

We must now consider what time paths of p_n might be solutions of the system. A complete analysis requires separate consideration of several possibilities classified according to the arrangement of the two initial prices on which the solution of (23) depends. This is tedious and is presented only in the appendix. Furthermore, the analysis follows that of the previous section step by step and requires mainly changes in interpretation. In any case, the

basic argument is very simple and is common to all the possibilities.

The behavior of the dynamic system (23) and (19c) can be summarized as follows. It is only when expected price rises consistently and explosively that anticipations lead to behavior which implies an actual time path of prices which is consistent with anticipations. Thus, with some reinterpretations to allow for the fact that the system is now stochastic, the conclusion is the same as that reached in the previous section, namely that in the vicinity of equilibrium, the market must be unstable in an upward direction. This conclusion is established as follows.

First, consider the possibility of consistently declining anticipated prices. When price is expected to decline on the average (i.e., $p^{e}_{n+1} < p_n$), it is not worthwhile for firms to hold inventories. Furthermore, the lower the anticipated price, the smaller the amount that firms find it worthwhile to produce (since marginal production cost rises with output). Therefore, the lower the price, the less, on the average, is supplied. But demand is, on the average, greater at lower prices. Therefore, a smaller supply cannot, again on the average, clear the market at a lower price.

Essentially the same argument applies to all possible sets of initial conditions and the more detailed analysis of the following case will serve as a typical example. (The other possibilities are considered in Appendix B .) Suppose $p_o < \bar{p} > p_1$, and that $p^{e}_{n+1} < p_n$. Clearly, $I_n = o$ since, in each period, firms expect next period's price to be lower than the current price. In this case, z_2 is the amount that will be placed in the market in period 2.

However, z_2 will clearly be less than the amount that would clear the market on the average if p_2^e were as high as $\bar{p}$. Hence this smaller amount cannot, on the average, clear the market at a price lower than $\bar{p}$ and p_2 cannot therefore, on the average, be as low as p_2^e. Thus, expectations are contradicted.

Thus argument applies step by step to other cases in which, e.g., the initial prices are both above the equilibrium level $\bar{p}$. In this case the argument shows that the amount that firms put on the market is more than can be sold, on the average, at $\bar{p}$ $(< p_2^e)$ and therefore the realized value of p_2 cannot, on the average, be as great as p_2^e.

Now consider the reverse situation in which price rises are anticipated, i.e., in which $p_{n+1}^e > p_n$. Here things are quite different. As price rises through time, so does production. However, the price rises also induce firms to put some of their production in inventory in anticipation of further rises. Provided prices rise, on the average, at the appropriate rate, the amount that firms find it worthwhile to take off the market for speculative purposes is just the amount that results in the price rises whose anticipation induced the inventory accumulation in the first place. Hence, the anticipations of rising prices are confirmed and firms have no reason to alter them.

Again, an exhaustive classification of possibilities is tedious and the following is presented as a typical case for more detailed analysis. Suppose $p_o > \bar{p} < p_1$ and that $p_{n+1}^e > p_n$. Clearly $I_n > 0$ since, in each period, the expected value of price in the next period is greater than the current price.

Now it is easy to show that there exists some relation between p_n and p^e_{n+1} which induces firms to put on the market just the amount that, on the average, will be demanded at the anticipated price. To see this, fix p^e_2 ($> \bar{p}$) and therefore z_2 . Now if p^e_3 were only slightly above p_2 , then firms would put on the market in period 2 nearly the total amount available to them, $z_2 + I_1$. But since $p^e_2 > \bar{p}$, z_2 is greater than the amount that would be demanded on the average at the lower price of $\bar{p}$. Hence, on the average, p_2 could not be as great as $\bar{p}$, and this contradicts the expectation $p^e_2 > p_1 > \bar{p}$. On the other hand, suppose p^e_3 were very large relative to p_2 . This would induce firms to make I_2 nearly as large as $z_2 + I_1$. Thus, the amount which firms put on the market in period 2 would be less than would be demanded, on the average, at p^e_2 and therefore this could not, on the average, be the market-clearing price. The foregoing shows that, for fixed p^e_2 , a p^e_3 too small in relation to p_2 will lead to a distribution of realized prices in period 2 with an average less than p^e_2 , while a p^e_3 too large in relation to p_2 will lead to a distribution of realized prices in period 2 with an average greater than p^e_2 . It follows that, for some intermediate p^e_3 ($> p_2$) , the distribution of realized prices in period 2 will have a mean equal to p^e_2 . Furthermore, in this example, the larger is p^e_2 the larger must be the difference $p^e_3 - p_2$ in order that expectations be realized. This is because, the larger is p^e_2 ,

the larger will z_2 be. However, a larger p_2^e means, on the average, a smaller demand. Hence, the larger is p_2^e the more the firm must put in inventory if p_2 is to clear the market on the average at a level p_2^e. But, in order to induce firms to add larger amounts to inventory, the difference $p_3^e - p_2$ must be larger. Roughly, this last point can be summarized by saying that the higher the price is the faster it must rise.

When the initial prices are below $\bar{p}$, the analysis is exactly the same except that in this case firms must be decumulating inventory (at a slower and slower pace) until p_n^e rises above $\bar{p}$.

We can summarize as follows what has been demonstrated, or asserted, above. Starting from any pair of arbitrary initial prices, only expectations of price rises lead to market behavior which, statistically speaking, fulfills the expectations. Thus, on the average, price must rise. Furthermore, the higher the price the larger the amount by which price must, on the average, rise. Thus, in this statistical sense, the market must be unstable in an upward direction.

In conclusion it might be worthwhile to make several brief comments of a more general nature on speculative markets.

First, as is true of all models of stability in the vicinity of equilibrium, this model contains assumptions which are clearly invalid when the departures from equilibrium are extreme. In the first place, the assumptions of linear demand and marginal cost curves are less accurate the larger the departure from the initial point. Perhaps, more important, are the assumptions that are

emphasized by recalling that profitability of inventory speculation in this model refers only to "paper profits." It is trivially obvious that the speculative inventories cannot be profitably decumulated by the market. When deviations from equilibrium are no longer "small" this may cause changes in market behavior. For example, the capital market may no longer supply funds at a constant interest rate for the purpose of carrying speculative inventories. Alternatively, firms may no longer be satisfied to plan their operations with a one-period horizon, or large inventories in the hands of firms may lead to pessimistic expectations. Even though no such expectations may exist which can be realized, the market may become "chaotic." These and other factors provide nonlinearities in the real world which become important when divergences from equilibrium are large and, presumably, prevent price from moving further away from equilibrium.

Second, it has often been realized in the literature that expectations concerning economic phenomena may be self-confirming. What does not seem to have been sufficiently appreciated is that not all expectations can be self-confirming. Depending on the characteristics of the market in question (e.g., on whether inventories can be carried) some kinds of expectations will be self-confirming and some kinds will not. In the model considered here, only very special kinds of expectations will induce firms to behave in a way which leads to the confirmation of expectations. In this model, though not necessarily in others, the only expectations which can lead to their confirmation result in a statistically unstable market.

Third, it is often said that expectations which are correct must stabilize the market because they induce speculators to reduce supply when price is low and to increase supply when price is high. This argument rests on a confusion between the systematic and stochastic parts of price movements. It is true in the model considered in this section that the more firms alter their inventory in response to deviations of p_n from p_n^e the smaller, on the average, will be such deviations. However, this notion considers only the effect of speculation in ironing out random fluctuations in price. The more fundamental aspect of speculation is that speculation is profitable whenever current price is low relative to anticipated future prices. This notion has to do not with the random but with the systematic component of price. In this sense, at least in the vicinity of equilibrium, speculation can clearly be destabilizing, even though it is based on correct expectations concerning the future trend of prices.

APPENDIX A

The purpose of this appendix is to prove the theorem in section 4, that price is unstable in an upwards direction when it is determined according to (17), i.e., when

$$a-bp_n = p_{n/c} + \max\left\{(p_n-p_{n-1})/r,0\right\} - \max\left\{(p_{n+1}-p_n)/r,0\right\} \qquad (17)$$

An elementary proof of this theorem is made possible by the fact that (17) is piecewise linear, i.e., for any given set of initial conditions p_o and p_1, p_2 is a linear function of p_o and p_1. The nonlinearity lies in the fact that different initial conditions will imply a different linear equation. The following proof parallels that in the text at each step.

(i) Suppose $p_o < p_1$. In the first place, $p_o < p_1 > p_2$ contradicts (17). From (17), this double inequality would imply

$$a-bp_1 = p_1/c + (p_1 - p_o)/r$$

or

$$p_1 = \frac{cp_o}{bcr+r+c} + \frac{acr}{bcr+r+c} \qquad (25)$$

A consideration of the properties of the difference equation

$$p_n = \frac{cp_{n-1}}{bcr+r+c} + \frac{acr}{bcr+r+c}\ , \qquad (25a)$$

of which (25) is the special case where $n = 1$, establishes the assertion (i). (25a) is stable and converges monotonically to the stationary solution $p = \bar{p}$.

Hence, there are three possibilities:

(a) If $p_o > \bar{p}$, then (25a) indicates that $p_1 < p_o$ which contradicts the assumption $p_o < p_1$;

(b) If $p_o < \bar{p} < p_1$, then again (25a) indicates that $p_o < p_1 < \bar{p}$ and we have a contradiction;

(c) If $p_o < p_1 < \bar{p}$, then in this case (25) is satisfied. However, it leads to the relation $p_2 < p_1 < \bar{p}$ and it is shown under (ii) that this set of prices, taken as initial conditions for the determination of p_3, is inconsistent with (17).

In the second place, a continuous price rise is possible. From (17), if p_2 is to exceed p_1, it must satisfy

$$a - bp_1 = p_1/c \quad -(p_2 - p_1)/r + (p_1 - p_o)/r$$

or

$$ep_2 - e_1 p_1 + ep_o + a = 0 \tag{26}$$

where

$$e = 1/r \quad \text{and} \quad e_1 = b + 1/c + 2/r$$

This is a special case of the second order equation whose solution is

$$p_n = \mu_1^n K_1 + \mu_2^n K_2 + \bar{p} \tag{27}$$

where

$$\mu_1 = \left[e_1 + (e_1^2 - 4e^2)^{\frac{1}{2}} \right] /2e \ , \quad \mu_2 = \left[e_1 - (e_1^2 - 4e)^{\frac{1}{2}} \right] /2e$$

Note that $e_1 > 2e > 0$. Hence $e_1^2 > 4e^2$. It follows that both μ_1 and μ_2 are real and positive and that $\mu_1 > 1$. K_1 and K_2 depend on p_o , p_1 , μ_1 and μ_2 . (27) is therefore an explosive system and furthermore some sets of initial prices will give K's resulting in a continuously rising price.[1]

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[1] It is easy to see that if (27) governs the behavior of the system for a sufficiently long time, it will cause the breakdown of the 'normal' case on which it is based.

(ii) Suppose $p_o > p_1$. There are two possibilities.

(a) $p_o > p_1 > p_2$; this is impossible. From (17) this would imply $p_1 = \bar{p}$. But, applying the same reasoning to the next period, we get $p_2 = \bar{p}$, which contradicts the assumption $p_1 > p_2$.

(b) $p_o > p_1 < p_2$; from (17), this requires

$$a - bp_1 = p_1/c - (p_2 - p_1)/r$$

or

$$p_2 = (1 + br + r/c)\, p_1 - ar \qquad (28)$$

This is a special case of the difference equation which has the solution

$$p_n = (1 + br + r/c)^n (p_o - \bar{p}) + \bar{p} \qquad (29)$$

This is explosive since $1 + br + r/c > 1$. If $p_o < \bar{p}$, it indicates $p_2 < p_1$, contradicting the premise. If $p_o > \bar{p}$, (29) indicates $p_2 > p_1$. Then, when the p_2 indicated by (29) materializes, the initial conditions for determining p_3 are p_1 and p_2. Since $p_2 > p_1$ we are back in the case discussed under (i) in which upwards instability must ensue.

APPENDIX B

The purpose of this appendix is to establish algebraically the conclusions reached in section 5 concerning the stability of the system

$$p_n = p_n^e + u_n \tag{19c}$$

and

$$a - bp_n + \epsilon_n = p_n^e/c + \max\left\{(p_n^e - p_{n-1})/r, 0\right\} - \max\left\{(p_{n+1}^c - p_n)/r, 0\right\} \tag{23}$$

The easiest way to analyze this system is as follows. Take p_o and p_1 as initial conditions and consider the following four possibilities:

(i) $p_1^e > p_o$, $p_2^c > p_1$. Eliminating p_n^e and p_{n+1}^e by (19c) , (23) tells us that, in this case, p_2 must satisfy

$$a - bp_1 + \epsilon_1 = (p_1 - u_1)/c - (p_2 - u_2 - p_1)/r + (p_1 - u_1 - p_o)/r$$

or

$$ep_2 - e_1 p_1 + ep_o + a + \epsilon_1 + u_1/c + u_1/r - u_2/r = 0 \tag{30}$$

Except for the random terms, this is the same as (26) in Appendix A, and e and e_1 have the meanings assigned to them there. (30) also tells us what the relationship between ϵ and u must be. In order that (19c) be satisfied by (30) it is necessary and sufficient that

$$\epsilon_1 = -(1/c + 1/r)\, u_1$$

We know from Appendix A that the characteristic equation of (30) has two real, positive roots, at least one of which is greater than one. Therefore, taking the expected value of p_2 from (30), we know that if anticipations satisfy condition (i), p_2^e will exceed p_1 and, statistically, an explosive upward movement of price can result. Thus, if anticipations satisfy (i), an unstable upward movement of price will result, and if an unstable upward movement of price results, anticipations will continue to satisfy (i). If p_2 is generated by (30), it and p_1 become initial conditions and (30) can be used to generate p_3. Therefore, anticipations specified by (i) are self-sustaining and lead to the market behavior specified in the text.

(ii) $p_1^e < p_o$, $p_2^e > p_1$. In this case, using (19c), (23) reduces to

$$a - bp_1 + \epsilon_1 = (p_1 - u_1)/c - (p_2 - u_2 - p_1)/r$$

or

$$p_2 = (1 + br + r/c)p_1 - ar - (\epsilon_1 + u_1/c)r + u_2 \qquad (31)$$

Except for the random terms, (31) is the same as (28) in Appendix A. In this case, in order that (19c) be satisfied, the random terms must be related by

$$\epsilon_1 = -u_1/c$$

There are now two possibilities. If $p_1 < \bar{p}$, (31) indicates that $p_2^e < p_1$ (since the coefficient of p_1 is greater than one) and this contradicts (ii), i.e., these expectations cannot be self-fulfilling. If $p_1 > \bar{p}$, (31) indicates

$p_2^e > p_1$ and p_2 can be generated by (31). If this occurs, p_1 and p_2 are initial conditions for the determination of p_3 and, provided $p_3^e > p_2$, (i) is again relevant and p_3 is determined by (30). If $p_3^e < p_2$ (given that $p_2^e > p_1$ and $p_1 > \bar{p}$), it is shown under (iii) below that expectations cannot be realized.

(iii) $p_1^e > p_o$, $p_2^e < p_1$. In this case, again using (19c), (23) reduces to

$$a - bp_1 + \epsilon_1 = (p_1 - u_1)/c + (p_1 - u_1 - p_o)/r$$

or

$$p_1 = \frac{cp_o}{bc+r+c} + \frac{acr}{bcr+r+c} + \frac{\epsilon_1+(1/c+1/r)u_1}{bcr+r+c} \qquad (32)$$

which, except for the random terms, is the same as (25) in Appendix A. Thus, if expectations were as specified in (iii), the system would need only one initial condition, say p_o, and (32) would generate p_1. However, it is easy to show that (32) cannot lead to the realization of expectations. First, recall that the coefficient of p_o in (32) is between zero and one. Then there are two possibilities.

(a) $p_o > \bar{p}$. In this case, (32) indicates $p_1^e < p_o$, contradicting (iii);

(b) $p_o < \bar{p}$. In this case, (32) indicates $p_o < p_1^e < \bar{p}$.

However, if we then use the same form as (32) to generate p_2 from p_1, we get $p_2^e > p_1$, contradicting (iii).

This case says that once price rises have been anticipated, firms cannot then anticipate that price will begin to fall. Roughly, this is because, in the last period before the first anticipated price decline, firms would unload all their inventories. Since production will also decline next period, supply will clearly be greater in the period preceding the anticipated price decline than it will in the period in which the decline takes place. But, this larger supply cannot, on the average, clear the market at a higher price and therefore the anticipated price decline cannot take place.

(iv) $p_1^e < p_o$, $p_2^e < p_1$. It is obvious that this is impossible. In this case we would have $I_o = I_1 = 0$ and, on the average, the market would be cleared in period 1 only if p_1^e were equal to $\bar{p}$. But if this were true, $p_2^e < p_1$ could not be realized.

We can now summarize these results as follows. Anticipations of rising prices always lead to behavior which fulfills the expectations in question (i). In some cases, (ii), an initial anticipation of a price decline, followed by anticipations of price rises in subsequent periods, can be sustained. No other anticipations can be sustained.

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