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A Keynesian Model Extended by Explicit Demand and Supply Functions for Investment-Goods

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1. Introduction

In static models of the Keynesian type the investment side of the economy is usually described by means of a single equation, which relates the level of real investment to the rate of interest (or to the rate of interest and real income). The form of this investment function implicitly reflects the characteristics of both the demand and the supply side of the market for investment-goods. The purpose of this paper is to develop a model which brings out these characteristics more explicitly. Our extended Keynesian model below is a formal exposition of the central model of Keynes' "General Theory" with an extended description of the investment side. For that reason we operate separately with functions describing the supply of, and the demand for, investment goods. We will compare the results of this extended model with those of the central model of Keynes' "General Theory" regarding such questions as: How will an increase in public investment, or a shift in the propensity to consume, affect total employment, or the price-level? The answers of the extended model are more complex expressions, which are shown to contain the answers of Keynes' central model as special cases. We find, also, that the extended model formalizes some qualifications often suggested verbally in discussions of Keynesian models.

Besides it can be argued that the extended Keynesian model below avoids a certain deficiency, or at least insufficiency, of most explanations of the Keynesian investment function, (and of most expositions of the acceleration-principle as well). The deficiency is, briefly, that the explanations concentrate upon the question of how much the investors want to increase their stock of capital equipment, and neglect to make inquiry about how soon they get their new capital goods, although the task is to explain the increase in capital equipment <u>per unit of time</u>. (Cf. p. 6 below.)

We will proceed so as to bring in our complicating assumptions in steps. We will mainly work from the simplifying assumption that the time of delivery (or construction) of investment-goods is fixed. Also to start with, we will assume that only the producers of consumer goods demand investment goods. Later it will be suggested how we might relax these two simplifying assumptions.

Our models below are static; and we apply the technique of comparative statics. But it is possible, I think, that the treatment of the investment side in the way suggested here, may prove useful also when we turn to concentrate on the question of how rapidly total employment, price level, etc. will respond to certain changes of economic policy.

2. The Central Model of Keynes' "General Theory"*

We start off with a brief analytical exposition of the central model of Keynes' "General Theory," which will serve as reference point for discussion of our extended Keynesian model. By the "central" model of Keynes' "General Theory" I mean the model which Keynes summarizes in his Chapter 18 "The General Theory of Employment Re-stated." The chapter where he says he will gather together the threads of his arguments.

We will denote this model "Mod. K."--In our description of "Mod. K." we use the following symbols:

Y ..... real national income (per unit of time)

C ..... nominal value of consumption (per unit of time)

I ..... nominal value of investment (per unit of time)

p ..... price level (the same for I- and C-goods)

w ..... money wage rate level

r ..... rate of interest level

N ..... total employment

M ..... nominal money supply

We shall differ from Keynes in the respect that we do not measure our variables in wage-units, i.e., we do not deflate nominal values by w in order to obtain expressions of real magnitudes. Instead we adopt the ordinary practice of using the price level as a factor of deflation.**

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* The General Theory of Employment, Interest and Money. We refer to this book as the "General Theory" or the "G. T."

** It can be argued that the usage of w as a factor of deflation is erroneous,(because Keynes often assumes that w is constant while p changes). Incidentally, if one uses the "wage unit approach" one will find that the Keynesian model contains a determined subset, (which means a simplification). In this subset real income is explained (expressed in wage units), but not the price level. (I have tried to develop this point in detail in Ekonomisk Tidskrift No. 4, 1958, pp. 258-261.)

The assumptions of "Mod. K." are, analytically expressed:

(2.1) $pY = C + I$

(2.2) $C = pF(Y)\ ,\ 0 < F' < 1$

(2.3) $M = pL(Y, r)\ ,\ L_1' > 0\ ,\ L_2' < 0$

(2.4) $I = pf(r)\ ,\ f' < 0$

(2.5) $Y = \Phi(N)\ ,\ \Phi'' < 0$

(2.6) $\Phi'(N) = w/p$

(2.7) w is fixed

(2.8a) M is fixed

(2.8b) r is fixed

(2.9) $N \leq N^S = g(\frac{w}{p})$

(2.8a) and (2.8b) are alternatives. If we apply (2.8a), the model (2.1) - (2.6) furnishes 6 equations to explain the 6 variables Y , C , I , p , r , and N . If we operate with (2.8b), the 5 equations (2.1), (2.2), (2.4), (2.5) and (2.6) form a determined model to explain the 5 variables Y , C , I , p and N . The former model, (2.1) - (2.8a), is Keynes' main alternative, the one he states in his Chapter 18.

(2.9) is a restraint which does not affect the degrees of freedom of the model. It says that the level of employment is less than, or equal to, the supply of labour at the actual real wage rate. (Cf. below.)

We will explain very briefly each of the above equations. Because of the controversies as to what Keynes actually said, or meant to say, in his

"General Theory," I will also try to point out carefully where each equation is stated in his Chapter 18.*

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* In my opinion there should not be much disagreement about the formal exposition of "Mod. K." A careful study of "The G. T." must - in case we use p instead of w as a factor of deflating - lead to the above representation. However, the formulation of the right hand side of (2.3) is open to question, as Keynes did not express himself clearly at this point. Don Patinkin suggests that the formulation: $M = pL_1(Y) + L_2(r)$, is a more correct interpretation of Keynes because he never permits the speculative demand to absorb an increased supply of money except at a lower rate of interest. See Don Patinkin, Money, Interest and Prices, p. 263 and p. 465.

(2.1) is a definitional relationship. (It is not explicitly mentioned in Chapter 18, but it is carefully stated in the Chapter 6 of "G. T.")

(2.2) is the familiar Keynesian consumption function.**

** To simplify our analysis we disregard the possibility that also r , or real value of cash balances, should enter into the consumption function.

(2.3) expresses that the real demand for money, $L(Y, r)$, shall equal the real money supply, M/p .

(2.4) expresses Keynes' investment function. It looks simple, but it is of a compound nature.*** It reflects the "marginal efficiency of capital,"

*** This function is sometimes interpreted as solely expressing the current demand for I-goods. (As an example see Don Patinkin, Money, Interest and Prices, p. 130.) This is, I think, very questionable, and it is certainly not in accordance with Keynes' own exposition, cf. "G. T." p. 136.

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e , to be a declining function of the amount of new capital goods, ΔK , which the investors wish to add to their stock of capital equipment; i.e., $e = h(\Delta K)$, $h' < 0$. If we particularly assume that the time of delivery (or construction) of the ΔK units of capital is fixed and equal to one time unit,

we can write $I = \Delta K$.* Assuming further that the investors limit their contracting for new capital goods at the point where e , which $= h(\Delta K) = h(I)$, is equal to r , we get: $I = h^{-1}(r)$ or $f(r)$, where $f' < 0$. The schedule $e = h(\Delta K)$,

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* The exposition will be more complicated if we allow the construction time of the demanded capital units to be a variable, which the investors and the producers of I-goods have to decide upon. It can be criticized that many authors just operate with $I = \Delta K$ without mentioning that this involves a specific and simplifying assumption. Also the "G. T." is short and insufficient on this point (see "G. T." p. 136).

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and thereby $f(r)$, depend upon the price expectations of the investors. Furthermore these functions also implicitly reflect the conditions of the supply of I-goods, as e also depends upon the marginal "supply price" of I-goods. (Cf. "G. T." pp. 135, 136).**

** Note that Keynes does not explicitly assume that the level of investment in the current period is partly predetermined by the volume of contracts the I-goods industry concluded in previous periods. If we want to bring this argument into "Mod. K." we may, however, simply assume that the volume of previously concluded contracts affects the form of the f-function, especially the constant term.

We may conceive of several important factors which contribute to determine the numerical value of f' , and to secure that $f' \leq 0$. Kalecki's"principle of increasing risk" can help us to explain why the single investor limits his demand of new I-goods, even if he reckons that the price of his own products, and the price of I-goods, are fixed (independently of his own adjustment).***

*** M. Kalecki, Essays in the Theory of Economic Fluctuations," p. 95.

Considering all actual and potential investors, the number of profitable investment projects (i.e., projects which are expected to yield an interest $\geq r$) is ordinarily supposed to increase when we (cet. par.) imagine a downward shift in

r . This increase may, however, be slight during crises of confidence. Hence, a low value of $|f'|$ is often supposed to express a state ofpessimistic price expectations. Further, if there is little or no excess capacity in the I-goods producing industry, an increase in ΔK would increase considerably the pressure on the facilities for producing capital, which will cause its supply price to increase, (as Keynes writes "G. T." p. 136). This, again, will cet. par. tend toward a low value of $|f'|$.

One thus can argue that the investment function, $I/p = f(r)$, takes several important factors into account. It does it, however, not explicitly. By our extended model below we will try to a certain degree to dissect Keynes' compact investment function.

In Chapter 18 Keynes refers to the equations (2.2), (2.3) and (2.4) as "the three fundamental psychological factors, namely, the psychological propensity to consume, the psychological attitude to liquidity and the psychological expectation of future yield from capital assets."

(2.5) states a production function. Keynes starts off his Chapter 18 (p. 245) by assuming as given "the existing quality and quantity of available equipment, the existing technique," etc. The given factors allow us, he says further (p. 246), to assume that "the national income depends on the volume of employment.... in the sense that there is a unique correlation between the two."

The assumption that $\Phi'' \leq 0$ is quite central in Keynes' analysis. Keynes assumes that when we imagine that N increases from a point where there is much excess capacity, $\Phi'(N)$ will to start with decrease very slowly, but later, when we get close to the full capacity point, $\Phi'(N)$ will fall strongly with N . (Cf. "G. T." p. 42, p. 296, and pp. 299-301.)--According to (2.6), employers will employ so many workers that the marginal product of labour equals the real wage rate. Implicitly, (2.6) expresses the demand for labour as a function of the real

wage rate. In Chapter 18 the relationship (2.6) is mentioned on p. 249: "Increase in output will be accompanied by a rise of prices (in terms of the wage-unit) owing to increasing cost in the short period." (Cf. also "G. T." Chapter 2, especially p. 17.)

(2.7) assumes the money wage rate to be fixed. Keynes' justification for this assumption is twofold. Firstly he argues that w actually tends to be rigid, at least in the downward direction. Secondly he holds that the assumption of fixed w gives analytical advantages.*

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* As to this second point cf. "G. T." Chapter 19, especially p. 257.

(2.8a) assumes that the nominal supply of money is fixed.

The assumptions (2.7) and (2.8a) are clearly stated in Chapter 18, p. 247. Here Keynes says: "Thus we can sometimes regard our ultimate independent variables as consisting of (1) the three fundamental psychological factors ..., (2) the wage-unit as determined by the bargains reached between employers and employed, and (3) the quantity of money as determined by the action of the central bank."

As mentioned, the equations (2.1)-(2.8a) form a determined model. On some occasions Keynes does not apply (2.3) and (2.8a) but assumes instead that r is directly fixed.** This gives a simpler, and somewhat different, model.

** When (2.8a) is replaced by (2.8b) the equations (2.1), (2.2), (2.4), (2.5) and (2.6) form a determined system. If we still operate with (2.3) this equation will give us the level of M which is necessary to keep the rate of interest on its fixed level.

This alternative and simpler model is indicated in Chapter 18, p. 245. "Our independent variables are, in the first instance, the propensity to consume, the schedule of the marginal efficiency of capital and the rate of interest, though,

as we have already seen, these are capable of further analysis."*

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* For an example where Keynes applied this simpler model, see "G. T." p. 260.

(2.9) expresses, as mentioned, the restraint that the level of employment cannot exceed the supply of labour, which is (in the classical way)supposed to be a function of the real wage.**

** Conversely it is, according to "Mod. K." possible that N is less than $N^T(= g(w/p))$. The workers will of course, especially the unemployed of them, not like such a situation. But as Keynes says (p. 291), that whilst labour is "always in a position to refuse to work" on a scale involving a real wage which is less than the marginal disutility of employment (i.e., $N > g(w/p)$), it is "not in a position to insist on being offered work" on a scale involving that $N = g(w/p)$.

There should be no doubt that Keynes attached the restraint (2.9) to his model. This is clearly stated in "G. T." p. 29, p. 30 and p. 292. In Chapter 18 it is mentioned on p. 246. Keynes says here that the factors which he considered as given also "furnish us with the supply function of labour (or effort); so that they tell us inter alia at which point the employment function for labour as a whole will cease to be elastic."

We may ask what would happen if the "effective demand" increases in a situation where $N = N^T = g(w/p)$. In this case Keynes will apply a different model. In particular, he then no longer assumes that the money wage is fixed and constant, (cf. "G. T." p. 301). Instead he suggests ("G. T." Chapter 21) that we may assume N and Y to be constants. Under these assumptions, an increase in the "effective demand" will raise p and w, but otherwise leave all real magnitudes unaffected.

3. An Extended Keynesian Model

As a point of deviation from "Mod. K." our extended model below describes separately (though very briefly) the demand and the supply side of the market for I-goods, and the demand and the supply side of the market for C-goods. It thus may become possible to express more explicitly relevant characteristics of inter alia the demand and the supply of I-goods.

We shall make use of the theoretical construction of the "week," which, e.g., Don Patinkin applies in his "Money, Interest, and Prices."* I.e., we

*Cf. "Money, Interest and Prices," pp. 8-9 and p. 37.

assume that the acts of consumption, production and delivery are going on continuously. However, buyers and sellers meet only once a week, e.g., on Monday morning. On this "meeting" the prices and quantities---for that "week" ---are supposed to be determined in accordance with the demand and supply functions and with the market clearance conditions. We describe thus a short term equilibrium.

We consider first the demand for investment-goods. We shall, in this section, assume that the producers of I-goods do not demand I-goods themselves. Only the producers of consumer goods demand capital goods. We imagine homogeneous units of capital. For the sake of simplicity we further assume that it takes a constant and given span of time, Θ "weeks,"** to construct, deliver

** To secure that the production will be constant throughout each single "week," we will assume that Θ is an integer, and that new capital is put to use at the beginning of the "week."

and install any quantity of I-goods. Thus, imagine a contract concluded this week, i.e., week no. t_o, about the construction of Q capital units. This contract gives rise to an investment of Q/Θ capital units per "week" from the beginning of week no. t_o to the beginning of week no. $(t_o + \Theta)$.— The production per week of the C-goods producers depend upon their inputs of labor and capital. At the beginning of the week t_o the size of their capital stock is a historically given datum. So is also the number of capital units they have already ordered, but not got. Let K_o denote the given stock of capital, including what is ordered, but not yet delivered. If the C-goods producers contract at t_o * for a number of Q capital units, their

*I.e., at Monday morning in "week" no. t_o.

stock of capital in week $(t_o + \Theta)$ will be $(K_o + Q)$.** Their planned input

**To abstract from the factor of depreciation, we assume that capital goods retain full technological efficiency,and constant maintenance costs, indefinitely.

of labor at $(t_o + \Theta)$ we denote n. Their planned output per week from $(t_o + \Theta)$ is consequently $H(K_o + Q, n)$, or $G(Q, n)$, as K_o is a constant. We assume that both G''_{11} and G''_{22} are negative,*** (diminishing returns to a single

*** G''_{11} denotes the second order derivative of $G(Q,n)$ with respect to Q, and G''_{22} the second order derivative with respect to n.

factor). We assume that the producers of C-goods, when deciding how much new capital they will contract for this week, act as if they maximize:

$$V = \int_{\Theta}^{\infty} [(p\ G(Q,\ n) - wn]\ e^{-r\tau} d\tau - qQe^{-r\Theta} .$$

V expresses the present value of expected ~~increase in their calculated~~ income. p denotes expected price level of C-goods, and w denotes the expected nominal wage level. We operate with the simple assumption that the producers of C-goods expect p and w to remain constant (on their existing levels this week). r denotes the existing level of the rate of interest, (we thus assume that the producers of C-goods will want to get at least this rate of interest on their capital outlay). q denotes the price level of I-goods. It is assumed that the purchasers of I-goods pay at the time of delivery. (The expression $qQe^{-r\Theta}$ thus expresses the present value of their payment for the Q capital units.)

A necessary condition for maximum of V (at some $Q > 0$) is that its partial derivative with respect to Q be zero. This condition gives:

$$(3.1) \qquad p\ G_1'(Q,n) = r\ q\ .$$

(3.1) expresses that the expected income of the marginal unit of capital shall equal its price times the rate of interest.

Another necessary condition for maximum of V is that its partial derivative with respect to n be zero, which gives:

$$(3.2) \qquad p\ G_2'(Q,n) = w\ .$$

(3.2) expresses that the planned input of labor at $(t_o + \Theta)$, when the Q capital units are installed, shall be so high that the marginal product of labor equals the real wage rate.

Actually, we are not interested in the variable n per se. As mentioned, n denotes the planned input of labor in the week $(t_o + \Theta)$. Thus, n does not directly affect the demand for labor <u>this</u> week. But, n affects the

marginal product of capital (at $t_o + \Theta$), $G_1'(Q,n)$, which in turn affects the ordering of capital goods this week, and thereby this week's employment in the I-goods industry. Therefore, in order to explain Q, p, q, etc., we have also to explain n . Only if we assume that G_{12}'' is zero for all actual values of Q and n , need we not include n , and the equation (3.2) in our model.

We shall assume that at the point of adjustment $G_{11}'' \; G_{22}'' > (G_{12}'')^2$. In that case (3.1) and (3.2) give a maximum of V . If this condition is not met, no certain values of Q and n give maximum V , given the values of p, w, q and r . The assumption is realistic, I think. When investors consider large increases in their stock of capital within a limited period of Θ "weeks," they must consider the problem of getting labor with the same degree of skill, and other difficulties.

(3.1) and (3.2) describe equilibrium conditions of a familiar type in analysis of the firm assuming perfect competition. We have, however, dealt with <u>total</u> production, etc., without going into aggregation problems. It may briefly be mentioned that the function G(Q,n) is, of course, supposed to possess characteristics which are typical to the individual production functions. Furthermore, when we consider an equilibrium situation, and consider equilibrium prices as given parameters, we may say that the single firms act as if they jointly and directly maximize the profit of the whole industry.

We now turn to the producers of I-goods. When at the beginning of week t_o , they are meeting the (potential) buyers of I-goods, they are already obliged to fulfill a certain number of previously concluded contracts. The work on those contracts has been started (before t_o), but is not yet completed. For their completion the producers of I-goods will employ, we assume, a fixed number of workers, L (throughout the week no. t_o).

Secondly, we assume that the I-goods producing sector also carries out public investment programs. For this purpose we suppose that a number of L_1 workers is employed during the week t_o. We will look upon L_1 as an exogenously given variable. An upward shift in L_1 expresses an increase in the level of public investment.

The sum $(L + L_1)$ we denote N_o. As L is supposed to be a constant, an upward shift in N_o is supposed to express an increase in the level of public investment.*

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*If, particularly, $\Theta = 1$, L is zero and $N_o = L_1$.

Thirdly, we assume that the producers of I-goods employ a number of N_1 workers, from the beginning of week t_o, to accomplish the orders they contract now (at t_o). They expect, we suppose, that this employment will render a production of $\Phi(N_1;N_o)$ capital units per week, (this does not include output of public investment goods) on an average in the period of time from t_o to $t_o + \Theta$. Why N_o should enter into this function needs an explanation. As mentioned an alternatively higher value of N_o means that more workers are engaged in public investment programs. This increase in N_o tends, cet. par., to diminish the capital labor ratio in the I-goods producing industry as a whole. $\partial\Phi/\partial N_o$ (which we denote Φ_2') may, therefore, be negative. We can imagine situations where the value of Φ_2' is negative and numerically quite high. That may be the case if the capacity of the I-goods industry is pressed and the marginal rate of substitution of capital for labor in the public investment activity is low. Conversely, if there is plenty of excess capacity and/or this marginal rate of substitution is very high, Φ_2' is probably zero or approximately zero. Furthermore, when N_o

increases it probably becomes more difficult to hire skilled workers (in case we consider an increase in N_1). Consequently, Φ'_1 may decrease when N_o increases, i.e., $\Phi''_{12} < 0$.*

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* An alternative and somewhat simpler way of proceeding is: We assume that the producers of I-goods, when they at t_o are meeting the (potential) buyers, have already obligations which bound them to produce Φ^o capital units per week (in week t_o). (We let Φ^o include also public investment.) Their total production per week (during week t_o) is $\Phi(N_I)$, where N_I denotes their total labor input. The size of the production which goes to accomplish the orders they conclude today (i.e., at the beginning of week t_o) is: $\Phi(N_I) - \Phi^o$.

In this case an increase in the level of public investment can be expressed by an upward shift in Φ^o.

By this alternative set up we omit, however, some possible ways that an increase in public investment may affect private investment. (Cf. our discussion in Section 4.A below.)

The producers of I-goods calculate, we assume, that the orders they conclude at t_o will give them an average net income, in the period t_o to $t_o + \Theta$, equal to:

$$\pi = q\Phi(N_1;\ N_o) - N_1 w \ .$$

It is here, among other things, assumed that the money wage rate is expected to remain constant during the period t_o to $t_o + \Theta$.

A necessary condition for maximum of π , (at some $N_1 > 0$), is that its derivative with respect to N_1 is zero; which gives:

$$q\Phi'_1(N_1;\ N_o) = w \ . \tag{3.3}$$

(3.3) expresses the usual condition that the marginal product of labor times the price of the product shall equal marginal cost.

The equations (3.1) and (3.2) describe the quantity demanded of I-goods, Q, as a function of p, q, w and r. (3.3) expresses the variable N_1, and thereby the supply of I-goods, as a function of q and w (and also of N_o).

We have, as a market clearing condition, that the supply of I-goods shall equal the demand:

$$\Phi(N_1;\ N_o)\ \Theta = Q\ . \tag{3.4}$$

We now turn to the market for consumption goods. The production of the C-goods producing industry, X_c, is supposed to depend upon its labor input, N_c:*

$$X_c = \psi\ (N_c)\ . \tag{3.5}$$

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* This applies to the production in the week no. t_o. The size of the capital equipment is constant throughout the week. (Cf. our assumption that Θ is an integer.)

We assume that ψ' is positive, and ψ'' negative, i.e., we assume decreasing returns (in the single factor N_c). There is a certain relationship between $G(Q, n)$ and $\psi(N_c)$. The latter function we can derive from the former by putting Q equal to zero and N_c equal to n. (To be, inter alia, able to write some formulas below in a shorter form, we have chosen to use two different symbols.)

The calculated net income per week of the producers of C-goods is $(p\ \psi(N_c) - w\ N_c)$. A necessary condition for maximum net income is that the derivative with respect to N_c be zero, which gives:

$$p\ \psi'\ (N_c) = w\ . \tag{3.6}$$

(3.6) assumes that the producers of C-goods will employ so many workers that the marginal product of labor equals the real wage, w/p .

The real demand for C-goods, X_c^D , depends, we assume, upon the size of total employment $(N_o + N_1 + N_c)$. This is as close as we can come to the Keynesian consumption function, which assumes the real consumption demand to depend upon real income, when we---in order to simplify---want to exclude real income from our list of variables.*

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* We allow the price levels of C-goods and I-goods to vary in different proportions, which complicates the definition of real income.

As an equilibrium condition, we have that the supply of C-goods shall equal the demand, i.e., $X_c = X_c^D$.

(3.7) $$X_c = g(N_o + N_1 + N_c) .$$

The nominal supply of money we denote by M . We assume that M is autonomously given, i.e., $M = \bar{M}$. As an expression for the <u>real</u> money supply we write: $\bar{M}/p$. The real demand for money is supposed to be a funcion of total employment and the rate of interest, i.e., $L[(N_o + N_1 + N_c), r]$. Here the argument $(N_o + N_1 + N_c)$ is supposed to express the demand for money caused by the transactions motive, while the argument r is connected with the speculative motive.

As an equilibrium condition, we have that the supply of money shall equal the demand:

(3.8a) $$\bar{M}/p = L[(N_o + N_1 + N_c), r] .$$

As an alternative to (3.8a) we may operate with the assumption that the rate of interest is directly fixed, determined autonomously by the monetary authority.

(3.8b) r is given.

Such an assumption may, as compared with (3.8a), often simplify our analysis considerably.

We finally assume that the level of the money wage rate is given, determined by bargains previously reached between employers and employed.

If we assume that r is fixed, the model (3.1)-(3.7) above provides seven equations to explain the seven variables: Q, n, p, q, N_1, N_c and X_c. If we alternatively operate with (3.8a), also r will be a variable, and the model (3.1)-(3.8a) provides eight equations to explain eight variables.

In "Mod. K." (Section 2 above) we operated with one aggregated production function for both I-and C-goods and with a single variable describing the general price level (as prices of the two mentioned types of goods are supposed to vary in the same proportions).* In our extended model we operate with

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* We may say that there is a certain asymmetry in the way Keynes treats investment and consumption. In "Mod. K." (see p.14 above) (2.5) describes the supply of both I- and C-goods, (2.2) describes the demand for C-goods, but (2.4) does not describe just the demand for I-goods. (2.4) states the level of HI/p, given r. As mentioned both characteristics of the demand and the supply side affect the form of the function $f(r)$.

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separate markets for I-goods and C-goods. As equilibrium conditions the demand equals supply in each market. By this deviation from "Mod. K." we are, to some extent, able to elaborate Keynes' compact investment function. Otherwise, however, the assumptions of the extended model are essentially quite similar

to those of the "Mod. K." * Comparing the equations: (2.4) of "Mod. K." is

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* When describing our extended model, we applied the theoretical construction the "week." This should not, I think, hinder a comparison of the two models. The construction of the "week" is helpful to interpret and formulate a short term equilibrium model, (where we concentrate on the income generating effects of investment decisions). It may also be useful to interpret "Mod. K." on the base of this construction.

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related to the equations (3.1)-(3.4) of the extended model. (2.6) is related to (3.3) and (3.6). Furthermore, (2.2) is related to (3.7), and (2.3) and (2.8a) are in the extended model expressed by (.8a). In both models the money wage is supposed to be a given constant. The condition (2.9) of "Mod. K." also applies to the extended model, though it is hitherto not mentioned in Section 3.

Both "Mod. K." and the extended model can describe situations of equilibrium with unemployment. Our assumptions do not remove the possibility that total employment is less than total labor supply N^S [$= g\,(w/p)$].**

** Indeed we may imagine situations where the potential investors are satisfied with the amount of capital they possess (including what is ordered but not yet delivered), or even where they find they have too much capital. In these situations N_1 will tend to be zero. However, such a tendency may be partly counterbalanced by a tendency towards a low value of q .

In the next section we proceed to compare the answers of the two models to the questions how employment and prices will respond to various changes (e.g., of economic policy).

4. A Comparison of the Conclusions of the Two Models

By means of comparative statics we can, within our two models, discuss how total employment and other endogeneous variables will respond to different shifts in parameters. In this section we intend to discuss quite thoroughly the effects within the two models of a change in the level of public investment. We will also, more briefly, look into the effects of shifts in the propensity to consume, in the rate of interest, in the money wage level and in the nominal money supply. Finally, we shall briefly try to generalize how the answers of the two models compare.

A. A Change in the Level of Public Investment

We consider first"Mod. K." The investment function (2.4) we will now write: $I/p = f(r) + I_o$. A shift in the level of the constant term I_o can be interpreted as an autonomous shift in the level of public investment. The investment function, together with (2.1) and (2.2) give: $Y = F(Y) + f(r) + I_o$. We assume that the rate of interest is fixed and constant (cf. Section 2). Differentiating with respect to I_o then gives: *

$$\frac{dY}{dI_o} = \frac{1}{1-F'} \quad . \tag{4.1}$$

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* An alternative way of proceeding is: The consumption function may be written: $C/p = F(Y) + C_o$. A shift in C_o can be interpreted as a shift in the level of public investment. In this case we get: $Y = F(Y) + C_o + f(r)$. Differentiating with respect to C_o leads obviously to the same results.

Using the equations (2.5) and (2.6) we further get:

$$\frac{dN}{dI_o} = \frac{1}{1-F'} \quad \frac{1}{\Phi'} \quad . \tag{4.2}$$

$$(4.3) \qquad \frac{dp}{dI_o} = - \frac{1}{(1-F')} \quad \frac{\Phi''}{\Phi'} \quad \frac{p^2}{w} \ .$$

(4.1) and (4.2) express the familiar Keynesian multiplier effect. The lower the marginal propensity to save $(1-F')$, the stronger this effect is. (4.3) shows the effect of an increase in I_o on the price level. An increase in I_o is seen to increase p. If $|\Phi''|$ is high, i.e., if we are close to the "full capacity point," p may change considerably even if the multiplier $1/(1-F')$, and thereby the effect on Y, is quite small.

In the above analysis we have disregarded the possibility that the level of public investment may directly and/or indirectly affect the level of private (real) investment. [When r has a fixed value, the level of private real investment was supposed to be given [equal to $f(r)$] independently of the value of I_o.]

We now turn to our "extended Keynesian model." This model takes the possibility just mentioned into account. Here an increase in real public investment is supposed to be associated with an increase in the number of workers carrying out public investment, which we express by an increase in N_o. The variable N_o enters into the function $\Phi(N_1;N_o)$. Thus a change in N_o can directly affect the supply of investment goods to the private sector. (Indirectly N_1 may be affected through the effect an increase in N_o has on p and q.)

We assume that the rate of interest is autonomously fixed. Our extended model thus consists of the equations (3.1) to (3.7) described in Section 3 above.

According to (3.4), $Q = \Theta\Phi(N_1; N_o)$. Further, according to (3.6) and (3.3), $p = w/\psi'(N_c)$ and $q = w/\Phi_1'(N_1; N_o)$. Inserting these expressions

for Q, p and q into (3.1) and (3.2) we get:

a) $$\Phi_1'(N_1; N_o)\ G_1'[\Theta\Phi(N_1; N_o), n] = r\,\psi'(N_c) .$$

b) $$G_2'[\Theta\Phi(N_1; N_o), n] = \psi'(N_c) .$$

(3.5) and (3.7) furnish a third equation:

c) $$\psi(N_c) = g(N_o + N_1 + N_c) .$$

By a), b) and c) we have a determined model to explain the three variables N_1, n and N_c.

We will discuss how, according to a), b) and c), a change in N_o will affect N_1 and N_c. Thereby we also learn how total employment $(N_o + N_1 + N_c)$ is affected.

Let us first assume that $G_{12}''(Q, n)$ is zero.* In this case N_1 and N_c

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* We then need not take into consideration the effects of a planned change in input of labor, together with the Q units increase in the input of capital, at the point of time $t_o + \Theta$. Intuitively, a planned increase (or a planned decrease) in labor input will, when $G_{12}'' \neq 0$, tend to increase G_1', and thereby raise the demand for capital goods. If, therefore, an increase in N_o is found to increase N_1 (and N_c) when $G_{12}'' = 0$, it does so to an even stronger degree when $G_{12}'' \neq 0$.

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are explained by the equations a) and c). Differentiating a) and c) with respect to N_o we get:

$$\frac{dN_1}{dN_o} = -\frac{(\psi' - g')\left\{G_1'\Phi_{12}'' + \Phi_1'\Phi_2'\,G_{11}''\Theta\right\} - r\,g'\psi''}{(\psi' - g')\left\{G_1'\Phi_{11}'' + (\Phi_1')^2\,G_{11}''\Theta\right\} - r\,g'\psi''} \qquad (4.4)$$

$$\frac{dN_c}{dN_o} = \frac{g'\left\{G_1'(\Phi_{11}'' - \Phi_{12}'') + \Phi_1'G_{11}''\Theta(\Phi_1' - \Phi_2')\right\}}{\text{same denominator}} \qquad (4.5)$$

To comfort the interpretation, let us at first look at the cases where Φ_2' and Φ_{12}'' are zero. A shift in N_o will then not directly affect the supply of I-goods to private firms. This assumption may be realistic when there is plenty of excess capacity in the production of I-goods (i.e., in a depressed economy). In this case (4.4) simplifies to:

$$(4.6) \qquad \frac{dN_1}{dN_o} = \frac{r\, g'\, \psi''}{(\psi' - g')\left\{G_1'\Phi_{11}'' + (\Phi_1')^2\, G_{11}''\Theta\right\} - r\, g'\, \psi''} \quad .$$

As to the sign of the denominator of (4.6) we shall first assume that $(\psi' - g') > 0$. I.e., when N_c increases the consequent increase in the output of C-goods will--cet. par.--exceed the consequent increase in the demand for C-goods. This assumption resembles the familiar assumption of "Mod. K." that the marginal propensity to consume is less than unity. Actually, $(\psi' - g')$ expresses something quite similar to the marginal propensity to save.

The first term of the denominator of (4.6) is thus negative, since Φ_{11}'', ψ'' and G_{11}'' are all supposed to be negative. However, $(-r\, g'\, \psi'')$ is positive. We shall assume that the numerical value of the first term exceeds that of the second term. This is, I think, (dynamically seen) necessary for stability; in a similar way as in "Mod. K." it is a necessary stability condition that the marginal propensity to consume, plus the possible marginal propensity to invest, be less than unity.*

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* Imagine a case where both $|\Phi_{11}''|$ and $|G_{11}''|$ are very small and where $|\psi''|$ is quite high, such that the denominator of (4.6) is positive. Imagine that, in a possible equilibrium situation, N_o shifts upward. Because of equation c) p.22, this will produce a tendency for a rise in N_c . A rise in N_c necessitates a sharp rise in p , [cf. (3.6)]. A rise in p will, if N_1 remains constant, disturb the equality (3.1): $p\, G_1' = r\, q$. Can a rise (or a fall) in N_1 restore this equality? Not in this case where $|\Phi_{11}''|$ and $|G_{11}''|$ are very small; [cf. also (3.3)].

Thus, according to our assumptions, the denominator must be negative.

The numerator of (4.6) is negative (as $\psi'' < 0$) . (4.6) thus says that N_1 will rise when N_o rises. Intuitively, when N_o increases, the demand for C-goods will tend to increase, whereby p rises; while w is assumed to be constant. The increase in p, therefore, gives private firms inducement to increase their demand for investment goods. Thus q and N_1 will rise.

If $|\psi''|$ is approximately zero, which is probably the case during a depression (with much excess capacity also in the production of C-goods), (4.6) shows that $dN_1/dN_o = \sim 0$. (In this case p will increase but slightly, and the mentioned inducement to the investors will not occur.)

In the case Φ_2' and Φ_{12}'' are zero, (4.5) reduces to:

$$(4.7) \qquad \frac{dN_c}{dN_o} = \frac{g' \left\{ G_1' \Phi_{11}'' + (\Phi_1')^2 G_{11}'' \Theta \right\}}{(\psi' - g') \left\{ G_1' \Phi_{11}'' + (\Phi_1')^2 G_{11}'' \Theta \right\} - r\, g' \psi''}$$

As the denominator is supposed to be negative, and the numerator is negative (because Φ_{11}'' and G_{11}'' are negative), this expression must be positive. We see that the effect on N_c of an increase is N_o is the stronger, the lower---cet. par.--- is $|\Phi_{11}''|$ or $|G_{11}''|$, or the higher is $|\psi''|$. If $\psi'' \sim 0$, i.e., if there is much excess capacity also in the C-goods industry, dN_c/dN_o approaches $g'/(\psi' - g')$. *

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* Cf. that we may interpret $(\psi' - g')$ as expressing the marginal propensity to save.

This last conclusion is quite similar to the conclusion "Mod. K." gives. The consumption function of "Mod. K." is: $C/p = F(Y)$. Thus, in this model we have: $d(C/p)/dI_o = F' dY/dI_o$, which because of (4.1) is $F'/(1-F')$.

When Φ_2' and Φ_{12}'' are zero, the effect of an upward shift in N_o on total employment is:

$$(4.8) \qquad \frac{d(N_o + N_1 + N_c)}{dN_o} = \frac{\psi' \left\{ G_1' \Phi_{11}'' + (\Phi_1')^2 G_{11}'' \Theta \right\}}{(\psi' - g') \left\{ G_1' \Phi_{11}'' + (\Phi_1')^2 G_{11}'' \Theta \right\} - rg' \psi''}$$

(4.8) says that $d(N_o + N_1 + N_c)/dN_o$ is, according to our assumptions, positive and > 1. (This also follows from our findings that dN_1/dN_o and dN_c/dN_o are both positive.)

If $\psi'' \sim 0$, (4.8) approaches $\psi'/(\psi' - g')$. Comparing this result to (4.2), we see that under specific assumptions, which may be realistic in a depression, our extended model gives results quite similar to those of the "Mod. K." In this case an upward shift in N_o will not, when r is fixed, affect the price levels of p and q, [cf. (3.3) and (3.6)]. The level of private investment is thus not affected either directly or indirectly.*

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* Not directly, because we assumed Φ_2' and Φ_{12}'' to be zero. Not indirectly, because the price levels p and q remain constant.

N_1 is then constant. But N_c has, in accordance with the simple logic of the multiplier, to increase so much that the demand for C-goods equals the supply (on a higher level of employment and income).

Imagine next a case where there is still excess capacity in the I-goods--, but not in the C-goods, producing industry, so that $\Phi_2' = \Phi_{12}'' = 0$, $\psi'' < 0$. According to (4.6), an increase in public investment will in this case stimulate private investment. The increase in N_1 may be considerable, but it will be small in the case $|G_{11}''|$ is very high. Furthermore, according to (4.8), the effect on total employment exceeds $\psi'/\psi' - g'$. The multiplier

effect is now strengthened,* we may say, because a rise in N_o rises p and thereby N_1 .

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* If we in this case imagine that an upward shift in N_o will, cet. par., increase G_1' , (for example because the increase in N_o is expected to facilitate transport possibilities), the effect may be still stronger.

Let us look into the case where $\Phi_2' < 0$ and $\Phi_{12}'' < 0$, i.e., where an increase in public investment affects unfavorably the facilities for the supply of I-goods to private firms. In this case (4.4) shows that dN_1/dN_o may be negative, e.g., if Φ_2' and ψ'' are approximately zero.

It is intuitively understandable that a low value of $|\psi''|$ tends towards a low value of dN_1/dN_o . In this case the private demand for I-goods is not much stimulated through a rise in p ; as p will only rise slightly, (3.6), p. 16.] As to the value of $|\Phi_2'|$, the higher this value is the higher is---according to (4.4)---dN_1/dN_o . To explain this we may stress that if $|\Phi_2'|$ is high the I-goods industry will have to increase N_1 considerably to maintain the same supply quantity to private firms (as before the increase in N_o). In this case N_1 may increase even if Q decreases.

Is it, according to (4.4), possible to conceive of a fairly realistic case where dN_1/dN_o is as low as -1 or lower? We can hardly imagine that the value of $|\Phi_{12}''|$ exceeds $|\Phi_{11}''|$. (Cf. p. 15.) If $\Phi_{12}'' = \Phi_{11}''$, and if $\Phi_2' = 0$, dN_1/dN_o may be negative, but is > -1 . If, however, at the same time $G_{11}'' \sim 0$, dN_1/dN_o will be close to -1 .

From (4.5) we see that a low value of $|\Phi_{12}''|$, and a high value of $|\Phi_2'|$ and of $|\psi''|$ tend towards a high value of dN_c/dN_o . Thus, the same forces which tend towards a rise in N_1 (when N_o increases), tend towards a strong

rise in N_c . If $\Phi''_{12} = \Phi''_{11}$, and $G''_{11} \sim 0$, dN_c/dN_o is approximately zero, while $dN_1/dN_o \sim -1$. In this case an increase in N_o will not increase total employment. (The level of employment must be limited for other reasons than lack of demand for I-goods.)

If dN_c/dN_o is positive, dp/dN_o is positive. This appears from the equation (3.6). Even if the rise in N_c is small, p may rise considerably if $|\psi''|$ is high. By equation (3.3) and (4.4), we find that the sign of dq/dN_o is uncertain. But dq/dN_o may be positive even when dN_1/dN_o is negative.

We will briefly look into the case where $G''_{12} \neq 0$. * Differentiating

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* Intuitively, we should expect that by and large our above conclusions about the sign of dN_1/dN_o , etc., will hold also, a fortiori, in this case. Cf. the footnote on p. 22 above.

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the three equations a), b) and c) (p. 22 above) gives:

(4.9) $$dN_1/dN_o =$$

$$-\frac{(\psi' - g')\Phi'_1\Phi'_2\Theta\left\{G''_{22}G''_{11} - (G''_{12})^2\right\} + \psi'' g'\left\{G''_{12}\Phi'_1 - G''_{22}r\right\} + G''_{22}G'_1\Phi'_{12}(\psi' - g')}{(\psi' - g')(\Phi'_1)^2\Theta\left\{G''_{11}G''_{22} - (G''_{12})^2\right\} + \psi'' g'(G''_{12}\Phi'_1 - G''_{22}r) + G''_{22}G'_1\Phi''_{11}(\psi' - g')}$$

(4.10) $$dN_c/dN_o =$$

$$\frac{g'\Theta\Phi'_1(\Phi'_1 - \Phi'_2)\left\{G''_{11}G''_{22} - (G''_{12})^2\right\} + g'G''_{22}G'_1(\Phi''_{11} - \Phi''_{12})}{\text{same denominator}}$$

Let us first look at the denominator. The first term must be positive. [Cf. p. 13, where we assumed that $G''_{11}G''_{22} - (G''_{12})^2 > 0$.] The third term is also positive, as both G''_{22} and $\Phi''_{11} < 0$. The second term may, however, be

negative, (it will certainly be negative if $G''_{12} \geq 0$). We shall assume that the numerical value of the second term does not exceed the sum of the first and third term.* Thus, the denominator is positive.

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* This is, I believe, a necessary condition for stability. Cf. the argument in the footnote of p. 23.

The numerator of (4.9) can be either positive or negative. As in the case of (4.4), we see that the numerator is probably positive, and accordingly dN_1/dN_o negative, if Φ'_2 and ψ'' are approximately zero. (4.9) also confirms other results we found when we assumed G''_{12} to be zero. We see, in accordance with (4.6), that if $\Phi'_2 = \Phi''_{12} = 0$,** dN_1/dN_o must be positive, and

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** I.e., a shift in N_o is not supposed to affect the supply of I-goods to private firms directly.

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the numerical value is higher the higher (cet. par.) $|\psi''|$ is. If, in this case, $\psi'' \sim 0$, also $dN_1/dN_o \sim 0$. This is the case of the simple "multiplier," described on pp. 24-26 above.

Turning to (4.10) we see that the numerator must be positive. (We assume, inter alia, that $|\Phi''_{12}| \leq |\Phi''_{11}|$.) Thus, an upward shift in N_o will rise N_c. (4.10) shows, as (4.5) does, that a low value of $|\Phi''_{12}|$, and a high value of $|\Phi'_2|$ and of $|\psi''|$ tend towards a high value of dN_c/dN_o . If $\Phi''_{12} = \Phi''_{11}$, and G''_{11} and $G''_{22} \sim 0$, $dN_c/dN_o \sim 0$, while according to (4.9) $dN_1/dN_o \sim -1$. I.e., in this particular case a shift in N_o will not affect total employment $(N_o + N_1 + N_c)$.

B. Shifts in the Propensity to Consume

We consider first "Mod. K." Let us use a linear representation of the consumption function: $C/p = \alpha Y + C_o$. We distinguish between two kinds of shift in the propensity to consume. First, a shift in C_o , which, cet. par., will change the average, but not the marginal, propensity to consume. Second, a shift in α , which will, cet. par., change both the marginal and the average propensity to consume. Actually, the effects of a change in C_o we have already discussed. (Cf. the footnote on p. 20.) The results are given in (4.1)-(4.3) above.

A linear consumption function together with the equations (2.1) and (2.4) give: $Y = \alpha Y + C_o + f(r)$. Assuming that r is fixed, and differentiating with respect to α , we get:

$$\frac{dY}{d\alpha} = \frac{1}{1-\alpha} Y \ . \tag{4.11}$$

Using (2.5) and (2.6) we further get:

$$\frac{dN}{d\alpha} = \frac{1}{1-\alpha} \frac{Y}{\Phi'} \tag{4.12}$$

$$\frac{dp}{d\alpha} = - \frac{1}{1-\alpha} \frac{Y\Phi''}{\Phi'} \frac{p^2}{w} \ . \tag{4.13}$$

(4.11) and (4.12) express a familiar Keynesian effect. The lower the marginal propensity to save, the stronger this effect is. Comparing (4.11) to (4.1) we see that $\frac{dY}{d\alpha} \frac{1}{Y} = \frac{dY}{dC_o}$. [Comparing (4.12) to (4.2) and (4.13) to (4.3) we find similar results for the variables N and p .] (4.13) shows that an increase in α will, cet. par., raise also the price level (as $\Phi'' < 0$). If $|\Phi''|$ is high, the price level may increase considerably even if the multiplier, $1/(1-\alpha)$, is quite low.

We turn to the extended model. We shall operate with a linear demand function for C-goods. E.G., we write: $g(N_o + N_1 + N_c) = d(N_o + N_1 + N_c) + C_o$. We shall first consider a shift in C_o, which means a change in the average, but not in the marginal propensity to consume.

This case of an upward shift in C_o is, analytically seen, quite similar to a case we discussed in Section 4A, namely, the case of a shift in N_o when Φ_2' and Φ_{12}'' both are zero; i.e., when changes in the level of public investment do not directly affect the supply of I-goods to private firms. When $\Phi_2' = \Phi_{12}'' = 0$, in effect neither N_o nor C_o enter into the equations a) and b) on page 22. They enter only on the right hand side of c), which is $g(N_o + N_1 + N_c)$, or $\alpha(N_o + N_1 + N_c) + C_o$, alternatively. The partial derivative of the right hand side of c) with respect to N_o or to C_o respectively is g' and 1, respectively. This shows us how dN_1/dC_o differs from dN_1/dN_o, which is given by (4.9). The expression for dN_1/dC_o has the same denominator as (4.9)*, and the numerator is also the same except that we

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* Notice that the denominator remains the same whatever parameter we differentiate with respect to.

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put Φ_2' and Φ_{12}'' equal to zero and substitute 1 for g'. By changing the numerator of (4.10) in a similar way and keeping the denominator we get the expression for dN_c/dC_o. We thus find that N_c increases, when C_o increases. Compare our more detailed conclusions above in connection with (4.6), (4.7), (4.9) and (4.10).

We then consider a shift in α, i.e., a shift in the marginal propensity to consume. Also in the extended model this case does not, analytically, differ much from the case of a shift in C_o. The partial derivative of the right hand side of c) with respect to α is $(N_o + N_1 + N_c)$. This shows us how

$dN_1/d\alpha$ differs from dN_1/dC_o . $dN_1/d\alpha$ is derived from (4.9) in a similar way as dN_1/dC_o, but in this case we substitute $(N_o + N_1 + N_c)$ for g' . (Consequently, we find that $dN_1/d\alpha$ times $1/(N_o + N_1 + N_c)$ equals dN_1/dC_o .)

C. An Autonomous Change in the Rate of Interest

We consider first "Mod. K." The equations (2.1), (2.2) and (2.4) give: $Y = F(Y) + f(r)$. Differentiating with respect to r gives:

$$\frac{dY}{dr} = \frac{f'}{1-F'} . \tag{4.14}$$

We see that (4.14) differs from (4.1), page 20, only by the factor f' in the numerator. Similarly, dN/dr equals (4.2) times f' , and dp/dr equals (4.3) times f' .

f' is supposed to be negative. Y, N and p will consequently increase when r shifts downward. The numerical value of f' is crucial. If $f' \sim 0$, a lowering of r will fail to stimulate economic activity even when the multiplier $1/(1-F')$ is high. A low value of f' may reflect a state of pessimistic expectations. It can also reflect that there is no excess capacity in the I-goods producing industry.

We turn to the extended model. Differentiating in the equations a), b) and c) on page 22 with respect to r , we get:

$$\frac{dN_1}{dr} = \frac{G_{22}'' \psi' (\psi' - g')}{\text{denominator as in (4.9)}} \tag{4.15}$$

The denominator is supposed to be positive (cf p. 28 above). The numerator is negative, as $G_{22}'' < 0$ and $(\psi' - g')$, which we interpreted as expressing the marginal propensity to save, is positive. (4.15) is consequently negative, i.e., a lowering of the rate of interest will stimulate employment in the I-goods industry.

Of equation c) page 22 we get:

$$(4.16) \qquad \frac{dN_c}{dr} = \frac{g'}{\psi' - g'} \frac{dN_1}{dr} .$$

We see that N_c will move in the same direction as N_1 does. The effect of a shift in r on total employment $(N_o + N_1 + N_c)$ is given by the sum of (4.15) and (4.16) as N_o is supposed to be constant.

$$(4.17) \qquad \frac{d(N_o + N_1 + N_c)}{dr} = \frac{\psi'}{\psi' - g'} \frac{dN_1}{dr} .$$

This result seems quite familiar. It resembles (4.14) which says that the effect on Y (and thereby on total employment) of a change in r equals the multiplier, $1/(1-F')$, times the effect on real investment. In our extended model we may interpret $\psi'/(\psi' - g')$ as expressing the multiplier,*

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* Cf. (4.8) in the case $\psi'' \sim 0$.

(while the change in N_1 expresses the change in real investment).

According to (4.15) N_1 will, as mentioned, rise when r shifts downward. But how much? We see that if Φ_{11}'' is ~ 0, and if $|G_{11}''|$ is low compared to $|G_{22}''|$, the numerical value of (4.15) is comparatively high. The assumption that Φ_{11}'' may be realistic during a depression. A low value of $|G_{11}''|$ compared to $|G_{22}''|$ means that there is diminishing returns to a greater extent for labor than for capital. If, conversely, $|G_{22}''|$ is low compared to $|G_{11}''|$, and if at the same time $[G_{11}''G_{22}'' - (G_{12}'')^2]$ has a high value, employment in the I-goods industry does not respond much to a downward shift in r .

According to (4.15) $|dN_1/dr|$ will be very low when $|\Phi''_{11}|$ is extremely high; i.e., when there is "no excess capacity" in the I-goods producing industry. Further we find that the higher $|\psi''|$ is , the more N_1 may increase when r is lowered.* $|dN_1/dr|$ may thus be very high in a case where $|\Phi''_{11}|$ is very low, $|G''_{22}|$ high, $[G''_{11}G''_{22} - (G''_{12})^2]$ low, and $|\psi''|$

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* This will be the case at least when $G''_{12} \geq 0$.

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very high. In this case a downward shift in r will raise p considerably, thus stimulating the demand for the goods of the capital producing sector, which was working far below capacity.

D. A Change in the Level of the Money Wage Rate

In both models the money wage w is supposed to be exogenously given. What are the effects of a shift in w ? We consider first "Mod. K." We operate with the alternative (2.8b), i.e., we assume that the rate of interest is (autonomously) fixed. In this case a change in w has no effects on N and Y . The fixed value of r determines I/p , which again through the multiplier determines N and Y (independently of the value of w). Therefore, by the way, w does not enter into the formulae (4.1), (4.2), (4.11), (4.12) and (4.14).

From the equation (2.6) we then see that p will change in the same direction and proportion as w .

Also our extended model possesses the specific result that a shift in w does not---when r is fixed---affect real magnitudes (as, e.g., employment), but changes the price level in the same proportion. That a shift in w does not change real magnitudes appears from the fact that w does not enter into

the set of equations a), b) and c) on page 22, which determines N_1 and N_c. It then follows from (3.3) and (3.6) that q and p must change in the same proportion as w.

Let us look at the case where we, instead of assuming that r is fixed, operate with:

$$M/p = L(Y,r),\ L_1' > 0,\ L_2' < 0\ ;$$

where M ---the nominal money supply---is autonomously given. Differentiating "Mod. K.", i.e., the equations (2.1)-(2.8a), with respect to w, gives:

$$\frac{dY}{dw} = \frac{Mf'\Phi'}{p\Phi''f'M - p\Phi'w[L_1'f' + L_2'(1-F')]} \quad . \tag{4.18}$$

We see that in this case a shift in w does affect Y (and consequently N). The denominator of (4.18) is positive, according to our assumptions; while the numerator is negative. Thus, a lowering of w will increase Y. (Intuitively: p decreases, whereby M/p increases and r tends to fall.) There will, however, not be any noticeable result when $|f'| \sim 0$, or if $|L_2'|$ approaches infinity (the case of "liquidity trap").

The expression for dp/dw has the same denominator as (4.18), the numerator being: $-\Phi' p^2[L_1'f' + L_2'(1-F')]$. Accordingly, p changes in the same direction as w, but now proportionally less.

Also, in our extended model, when we apply the alternative (3.8a) instead of (3.8b) (cf.pp. 17-18), total employment and other real magnitudes will depend on w. We shall, however, not go more closely into this matter.

E. A Change in the Nominal Money Supply

To discuss the effects of a change in M, we cannot any longer assume that the rate of interest is (directly) fixed. We apply the Keynesian liquidity function together with the assumption that the nominal money supply is

autonomously given.*

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* We then must expect somewhat more complicated formulae than (4.1)-(4.17) above, where we used the more simplifying assumption of a fixed r.

Differentiating (2.1), (2.2) and (2.4) with respect to M, we get:

$$\frac{dY}{dM} = \frac{f'}{1-F'} \quad \frac{dr}{dM} \ .$$

Thus, if an increase in M lowers r appreciably, it also will raise Y appreciably, except in the case where $|f'| \sim 0$.

Differentiating the equations (2.1)-(2.8a) with respect to M, we get:

$$\frac{dr}{dM} = \frac{-w\,\Phi'\,(1-F')}{p\,\Phi''\,f'\,M - p\,\Phi'\,w\,[L_1'f' + L_2'\,(1-F')]} \ . \tag{4.19}$$

(4.19) is, according to our assumptions, negative. An increase in M may lower r appreciably, except when $|L_2'| \to \infty$, (the case of the liquidity trap), or when $|\Phi''| \to \infty$, (in which case only the price level will increase).

Differentiating in our extended model, i.e., the equations (3.1)-(3.8a) with respect to M yields somewhat complicated formulae, which we shall not display here. We find, however, e.g., that dN_1/dM and dr/dM have opposite signs, and that assumptions which make the value of dN_1/dr small, also tend to make dN_1/dM small.

F. Summary

Comparing the results of the two models concerning our above questions, we observe firstly that formally seen the answers of "Mod. K." are much more simple. The answers of "Mod. K." are especially simple when r is supposed to be fixed. However, the answers of the extended model are not so much more complicated that we are unable to draw many conclusions without involving numerical examples.

We find that the comparatively simple answers of "Mod. K." are in essence implicit in the more compound answers of the extended model. The former ones can be deduced from the latter ones when we make some particular assumptions about the forms of the functions. These particular assumptions seem realistic in a depressed economy, where there is plenty of excess capacity both in the production of C-goods and of I-goods. We can thus say that the extended model justifies the use of "Mod. K.," where the investment side of the economy is described so very summarily, under certain circumstances. However, when we expect, for example, that an increase in the level of public investment will directly affect private investment, or that there is decreasing returns (to a considerable extent) with respect to labor in the producing of I-goods or C-goods, "Mod. K." may prove to be too simple.

5. Changes in the "State of Confidence"

The reader finds perhaps that the extended model to a certain extent "unlocks" Keynes' compact investment function because it separates the demand and supply side for I-goods. He may also consider this as a first and necessary step towards a more comprehensive and satisfactory treatment of the investment side. However, he probably objects that the extended model is very primitive with respect to the treatment of the risk/factor, price expectations, etc.

Our assumption concerning price expectations was that the levels of prices and of wages are expected to remain constant forever (on their current levels this "week"). (Cf. p. 12 above.) Even though this assumption is very specific, a few remarks may be offered to support it.

That the single producers consider future prices as constant parameters independently of how much they plan to produce is related to our assumption about a "free competitive market" ("automistic" market). - What matters is the expectedly constant relationship between p and w, and between q and w.

We use our extended model to discuss shift in certain parameters. If a shift in a parameter increases, e.g., the price level of C-goods p, and thereby p/w, we may say that our assumption about price expectations implies that the producers of C-goods expect this increase to be lasting. It could be more realistic to reckon with still more optimistic expectations, (or still more pessimistic expectation when p decreases), but this would usually not change our results as to the direction of the changes in our endogenous variables.

We can further mention that Keynes in "G. T." in the chapter about Long-Term Expectations, p. 148, claims: ..."the facts of the existing situation enter, in a sense disproportionally, into the formation of our long-term expectations; our usual practice being to take the existing situation and to project it into the

future, modified only to the extent that we have more or less definite reasons for expecting a change."*

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* Keynes investment function is, however, very generally formulated. It does not necessarily involve this particular assumption about "horizontal" price expectation.

Applying a construction of Shackle** we may, however, be able to include in

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** Cf. G. Shackle, "Interest-rates and the Pace of Investment," The Economic Journal, March 1946, p. 2 and p. 11.

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our extended model an expression which roughly describes "the state of confidence." Referring to the expression V on p. 12, we shall now not assume that the producers of C-goods calculate with a constant net income of $(pG(Q, n) - wn)$. They calculate, we suppose, with a net income of $(pG(Q, n) - wn)e^{-h\tau}$, where $\tau > t_o + \Theta$, and where, ordinarily, $h > 0$. One reason for such a more cautious calculation is, we assume, anxiety that the capital goods may become obsolete at some unpredictable date because of new inventions. Thus h tends to be positive. The value of h may further also depend upon price expectations. If we, e.g., are in a boom which is expected to last but a short time, h will tend to have a high value. If, conversely, the producers of C-goods believe in an inflationary movement (in p and w) h will tend to have a low value, and it might even be negative. Thus, a negative shift in h we will interpret as expressing a better "state of confidence."

Our expression V on p. 12 now becomes:

$$V' = \int_{\Theta}^{\infty} (pG(Q, n) - wn)e^{-(r+h)\tau}d\tau - qQe^{-r\Theta}$$

Instead of (3.1) we now get the following condition for maximum of V' :

$$(5.1) \qquad pG_1'(Q, n) = (r + h)q \ .$$

As a second condition we get, as before, (3.2).

The introduction of Shackle's parameter h thus alter only one equation in our model. The alteration in this one equation is solely that $(r + h)$ substitutes for r . Furthermore, we notice that the variable r enters only in this equation.

Consequently, to discuss the effects of a shift in h will be quite similar to our discussion p. 31 above, where we discussed the effects of a shift in r . A negative shift in h , i.e., a better "state of confidence," has the same effects as a lowering of the rate of interest. Both N_1 and N_c will increase, and increase more the lower is $|\Phi_{11}''|$ and $(G_{11}''G_{22}'' - (G_{12}''^2))$, and the higher is $|\psi''|$ and $|G_{22}''|$.

6. The "Extended Model" in the Case Both the Producers of C-Goods and of I-Goods Demand I-Goods

We shall suggest very briefly how the extended model appears in this case.

The equation (3.4), which expresses that the supply of I-goods equals the demand, we now write:

$$(6.1) \qquad \Phi(N_1;\ N_o, \Theta) = Q_1 + Q_c\ ;$$

where Q_1 denotes the number of capital units demanded by the producers of capital goods themselves, and Q_c the number demanded by the producers of C-goods.

The equations (3.1) and (3.2) remain the same, with the exception that the notation Q is changed to Q_c. (3.1) and (3.2) give us Q_c as a function of r, p, q and w. In a quite similar way we may express Q_1 as a function of r, q and w.

We assume that the producers of I-goods, when they decide how much new capital they will demand this "week," act as if the maximize:

$$V_1 = \int_{\Theta}^{\infty} (qH(Q_1, n_1) - wn_1)e^{-r\tau}d\tau - qQ_1e^{-r\Theta}$$

V_1 expresses the present value of expected increase in their calculated income. (Cf. the expression V on p. 12. Our assumptions here as to price expectations, etc., are similar. Cf. also p. 37 above.)

Necessary conditions for maximum of V_1 (at some $Q_1 > 0$) is that its partial derivative with respect to Q_1, and with respect to n_1, be zero. Which give:

$$(6.2) \qquad H_1'(Q_1, n_1) = r\ .$$

$$(6.3) \qquad H_2'(Q_1, n_1) = w/q\ .$$

We assume that H_{11}'' and H_{22}'' are negative (diminishing returns to a single factor). Further we assume that, in the point of adjustment, $(H_{11}''H_{22}'' - (H_{12}'')^2)$ is positive.

The equations (3.3) and (3.5)-(3.8) of section 3 we will use as before.*

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* There is a certain relationship between $\Phi_1(N_1; N_o)$ and $H(Q_1, n_1)$. The former we can derive from the latter by putting $Q_1 = 0$ and $n_1 = N_1$. (If considering shifts in N_o we actually should write: $H(Q_1, n_1; N_o)$.)

Our extended model now consists of the equations (6.1)-(6.3) of this section, and the equations (3.1)-(3.3), (3.5)-(3.8) of section 3.

If we assume that G_{12}'' and $H_{12}'' \equiv 0$ we can disregard the equations (3.2) and (6.3). (Cf. the footnote on p. 22.) In this case our model is comparatively simple. Q_1, the demand for I-goods of the producers of I-goods themselves, depends solely on r. Thus, Q_1 is constant whenever r is assumed to be constant. This means that some of our results above remain the same. The effects on N_1, N_c etc. of shifts in the level of public investment,** and in the propensity

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** If H_1' is supposed to be independent of N_o.

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to consume, and in w, will be the same as above (when we supposed G_{12}'' to be zero).

According to (6.2) and (6.3), Q_1 will increase when, cet. par., q rises, if $H_{12}'' > 0$. (Intuitively, planned input of labour n_1 will increase when q/w increases. This will raise H_1' when $H_{12}'' > 0$, i.e., when we have complementary factors.) Q_1 may tend to rise strongly, but not indefinitely, as a result of a shift in a parameter which raises q, when H_{12}'' is high compared to $(H_{11}''H_{12}'' - (H_{12})^2)$.

Our above model, where we assumed that only the producers of C-goods demand I-goods, gave in many cases conclusions which differed from those of the "Mod. K." Thus we found, p. , that in case $\Phi'_2 = 0$, $\Phi''_{12} = 0$, $G''_{12} > 0$, and the numerical value of ψ'' is very high, our extended model described a <u>strengthened</u> multiplier effect. The reason was that an increase in p induce the private investors to increase their demand for I-goods.* If $H''_{12} > 0$, we can, intuitively, conclude that the inclusion of the demand for I-goods of the I-goods producer themselves will probably mean that the multiplier effect is strengthened still more. The reason being that q tends to increase, (cf. (6.2) and (6.3)).

* In order to take this into account we would, in "Mod. K.," have to imagine an upward shift in the marginal efficiency of capital schedule. In the extended model such shifts are "built in" and explained by the model.

Conversely, we found, p. 26, that if $\Phi'_2 = 0$, $\Phi''_{12} = \Phi''_{11}$, $G''_{11} = 0$, $G''_{12} = 0$, dN_1/dN_o is approximately = -1 . In that case an increase in public investment would not increase total employment. Intuitively, the inclusion of the demand for I-goods of the I-goods producer themselves, would not change this conslusion. (In this case q would tend to remain constant.)

7. The Time of Delivery as a Variable

In Section 3 we considered Θ, the time of delivery, including installment, of all kinds of I-goods, to be a constant. We will now very briefly suggest how we could operate with Θ as a variable. We now will assume that q, the price per unit of capital, is a decreasing function of the time of delivery Θ, i.e., that investors will have to pay a premium for short deliveries.

In Section 3 both the investors and the producers of I-goods are supposed to consider q as a given parameter. Our model determines the value of q which secures equality between the demand for and the supply of I-goods. When we are going to explain both q and Θ we may first notice one problem concerning the construction of our model. We cannot possibly now assume that both the investors and the producers of I-goods consider q and Θ as given parameters. Such a way of proceeding will tend to leave us with one degree of freedom left.

A possible way of proceeding, which yields a determined model is: The investors have to reckon with a certain price schedule $q(\Theta)$, when they order capital goods. They can choose the time of delivery they want, but as $q' < 0$, it is expensive to choose a low value of Θ. The form of the schedule $q(\Theta)$ is supposed to depend upon the production function of the I-goods sector.

Referring to the expression V on page 12, we now shall assume that q is a given function of Θ, and that the investors act as if the maximize V with respect to Q, n, and Θ. Necessary conditions for maximum of V gives us the two equations (3.1) and (3.2) as before. In addition we now get a third equation:

$$pG(Q,n) = -Qq'(\Theta) + Qq(\Theta)r \quad . \tag{7.1}$$

(7.1) expresses that the expected income the investors lose by getting their capital goods a time unit later [the left hand side of (7.1)], shall equal the sum of their simultaneous gain by getting it somewhat cheaper, and their interest-gain by later payment.

We assumed (cf. p. 15) that the calculated average net income of the producers of I-goods is:

$$\pi = q(\Theta)\ \Phi(N_1, N_o) - wN_1 \ .$$

π depends, when w and $q(\Theta)$ are given, on N_1 and Θ. We now ask the question: What form of the function $q(\Theta)$ will make the producers of I-goods indifferent as to which value of Θ they contract, i.e., will make π only a function of N_1? This, under some assumptions, depends upon the form of the production function. We find, in general that $q' < 0$. Furtermore, we may, on the basis of a competition among the producers to get orders, argue that it is realistic to assume such a function $q(\Theta)$ confronting the buyers of capital goods.*

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* This point I have tried to develop more in detail in the Review of Economic Studies, February, 1960, p. 103.

The equations (3.1)-(3.8) together with (7.1) form a model which include the time of delivery as a variable.