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Buffer Stocks, Sales Expectations, and Stability:

A Multi-Sector Theory of the Inventory Cycle

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Implications for the stability of the economy of certain alternative inventory practices on the part of individual firms will be appraised in this study. Production is time consuming. Entrepreneurs are not endowed with perfect foresight. Firms hold inventories of finished goods as buffer-stocks in order that unanticipated demand may be satisfied. They attempt to adjust inventories to an appropriate level in the face of incomplete knowledge of future demand.

Eric Lundberg [23] and Lloyd Metzler [24] have both formulated macro-economic models in order to analyze the inventory cycle. My approach departs from theirs in that I consider complications that arise in aggregating the behavioral patterns assumed for individual firms in deriving conclusions concerning the dynamic properties of the economy. As in Metzler's study, a simple servo-mechanism type of behavior is attributed to the individual firm. But in contrast to the approach adopted by Metzler, I consider the economic implications of a multitude of interacting firms all attempting to adjust inventories to a level deemed appropriate in the face of incomplete knowledge of future market conditions.**

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** Pigou has made a most interesting comment concerning Aftalion's furnace analogy. "The really important point is that other people, beside himself, are putting coal on the fire, and that he has no adequate means of knowing to what extent they are doing this." [27, p. 70].

Macro-economic model construction neglects complications that arise from the problem of aggregation. An analysis of stability conditions for various multi-sector buffer stock models will reveal that dynamic properties depend upon a multitude of parameters, some of which are suppressed in aggregative model construction. Conditions for stability under various types of buffer stock behavior are found to differ fundamentally from those developed by Metzler in his macro-analysis.

The challenge raised is similar to that faced by Goodwin [14] and Chipman [9] in the disaggregation of the multiplier process. But the task is more difficult in that a crucial role is explicitly assigned to errors of expectations and inventory stocks. In the development of the matrix and multi-sector multiplier theories, no explicit mention was made of the role of inventories. Although Chipman acknowledges in a more recent paper that inventories must indeed play a role in the process portrayed by his disaggregated multiplier model, the specification of stability conditions has been made in abstraction from this fact.**** It will be found that the

* "In the multiplier approach demand for outputs is regarded as preceding the production of inputs, the initial production of outputs being made possible by the temporary depletion of inventories of inputs." [10, p. 5].

multiplier models of Goodwin and Chipman are but a special, restricted case of the model presented here. Under more realistic assumptions, the conditions for stability are found to be quite different from those specified by Chipman and Goodwin.

In a discussion of the implications for stability of alternative inventory practices it is most appropriate to start, as in the theory of competition, with a discussion of the behavior of the individual firm. The second stage of the analysis is that of deriving the behavior of individual sectors or industries from the assumed firm behavior. Then, the interrelations between the various sectors must be considered in tying together the equations representing the behavior of the individual sectors in order to obtain a model of the whole economy. Finally, the behavior of the model for various values of the parameters may be explored.

This procedure will be followed for several alternative models of inventory behavior. The first model, which embodies the basic behavioral assumptions of Metzler for the individual firm, will be found to have disconcerting implications for the stability of the economy. The dynamic

properties resulting from the first set of assumptions suggest that revisions are worthy of investigation not only in order to incorporate more realistic behavioral assumptions for individual firms but also to determine what alternative assumptions may lead to a model with more reasonable dynamic properties. Various patterns of inventory and output adjustment behavior and the effects of alternative assumptions concerning expectations are explored. A final model is presented involving the assumption that firms attempt only a partial adjustment of inventories to the desired level in each time period; the assumption of a delayed adjustment form of inventory behavior under conditions of uncertainty is found to lead to a model which may be stable for certain realistic values of the parameters.

In order to make this difficult task tractable, numerous simplifying assumptions will be made. The inventory behavior pattern assumed for individual firms will be relatively simple. Capacity restraints on the level of output are to be ignored. Time will be treated as a discrete unit. It will be assumed that the production period is the same for all commodities and, in addition, that the production processes of all firms are synchronized so that output begins at the same point of time for all firms; essentially, the production process is assumed to be of the "point-input point-output" type so frequently encountered in capital theory. In addition, price phenomena will be neglected throughout the discussion. None of these assumptions is without precedent in either the literature of managerial economics on optimal inventory policy or in the multitude of aggregative economic models of the accelerator and related theories. No attempt will be made here to justify the assumption of price rigidity by reference to the literature on full cost pricing or the theory of oligopoly. This assumption, motivated by convenience, serves to suppress the roles of speculative inventory holdings and substitution. While a more general model would relax some of the assumptions found necessary here in order to facilitate the theoretical analysis, it must be emphasized that the introduction of such complications as price flexibility might well result in a structure which could not conceivably serve as a useful framework for empirical investigation.

Granted that simplifying assumptions must be made in any analysis, it may still be asked whether an appropriate set has been chosen for the problem at hand. Perhaps the assumption of price rigidity is most vulnerable to this criticism. But a statement by J. R. Hicks suggests that the approach adopted here may be worthwhile [19, p. 145].

Both the manufacturer and the retailer are, for the most part "price makers" rather than "price takers"; they fix their prices and let the quantities they sell be determined by demand... A model in which quantities bear the brunt of disequilibrium fits most of the facts distinctly better (than the model of Value and Capital).

My own empirical investigation of the behavior of manufacturers' inventories in the United States suggests that the buffer-stock model does provide an appropriate framework for examining inventory behavior [22]. Errors of expectations and delayed adjustment behavior serve to explain observed inventory phenomenon. These complications are introduced into the theoretical analysis. Contrariwise, I found no empirical support for the hypothesis that manufacturers speculate in inventories, adjusting their holdings of stocks in response to anticipated price changes; consequently, it was appropriate to neglect the complications of speculation in the theoretical study. Unfortunately, I have not succeeded in introducing unfilled orders into my theoretical analysis, although the empirical evidence suggests they are a factor of considerable importance in determining the desired level of inventories of purchased materials and goods in process.

Empirical investigation may suggest appropriate complications to be considered in theoretical analysis. A restraint upon the complexity of the theoretical constructs is imposed if the model is to serve as a useful framework for empirical research. But a theoretical investigation of the dynamic implications of alternative sets of assumptions concerning the behavior of the firm may also assist in empirical studies.

A major problem encountered in any econometric investigation is to limit the range of models to be studied. Although the assumption of profit maximization may serve to partially restrict the hypotheses to be tested, major simplification is almost inevitably involved in moving from theorems derived from the assumption of normative behavior to the equations of a completely specified model. It is suggested here that another source of a priori knowledge is provided by an examination of the dynamic properties that follow from a particular set of assumptions. The empirical analysis may be restricted to those models of firm behavior implying reasonable dynamic properties for the economy.

When the econometric investigation is completed, it is necessary to demonstrate that the conclusions are significant in more than a statistical sense. My empirical investigations suggest that the observed behavior of manufacturers' inventories may be explained in terms of errors made by firms in anticipating sales volume and delayed inventory adjustment behavior. But a theoretical investigation of the role of expectations within the framework of a purely competitive environment by Arrow and Nerlove suggests that the stability of a competitive economy may be independent of expectations [4]. Again, Metzler has shown in a brilliant article that the stability of competition does not depend upon the speed of adjustment if all commodities are gross substitutes [25]. Does this imply that these two aspects of actual inventory behavior revealed in our empirical investigation are of no importance in examining questions of stability? The theoretical argument that follows reveals that expectations and delayed adjustment behavior may play an important role in the determination of the dynamic properties of an inventory holding economy. In the concluding section, I show that they must be considered in evaluating certain policy measures advanced as means of stabilizing the economy.

PART I THE ELEMENTARY MODEL

A. The Behavior of the Firm

It seems appropriate to assume that under conditions of price rigidity entrepreneurs will carry inventories for the purpose of avoiding the unsatisfied market that would otherwise occur whenever demand exceeded anticipated sales. But when sales exceed expectations, inventory stocks are reduced below the desired level; conversely, when sales fall short of demand, unplanned inventories are accumulated. Consequently, production is planned so as to either exceed or fall short of anticipated sales in order to restore inventories to the desired level.* But it

* Entrepreneurs in fact have the option of eliminating excess stocks by price reductions and of raising prices in periods of shortage. This type of behavior has been excluded by the assumption of price rigidity. Paul A. Samuelson has derived stability conditions for a model in which price adjustments occur when existing stocks exceed an equilibrium level as a result of a divergence between current production and consumption. Foundations of Economic Analysis [30, p. 275].

must be mentioned that Mills has demonstrated that under appropriate conditions this type of firm behavior is consistent with the assumption of profit maximization [26].

Suppose that the firm's desired level of inventory is linearly related to sales. If $I^d(t)$ represents desired inventories for a firm at time t and $\bar{X}(t)$ anticipated sales, this assumption may be expressed by the equation:

$$(1.1) \quad I^d(t) = c + b \bar{X}(t) .$$

The parameter b , the marginal desired inventory coefficient, relates the desired level of inventories to sales. Now if an entrepreneur adjusts production in an attempt to maintain the desired level of inventories, the level of output, $Q(t)$, will be determined by the equation:

$$(1.2) \quad Q(t) = \bar{X}(t) + I^d(t) - I(t-1) ,$$

where $I(t-1)$ is the actual level of inventories of the firm at time $t-1$. Unless sales anticipations for time $t-1$ were correct, actual inventories at time $t-1$ will have either exceeded or fallen short of the desired level. If sales fell short of anticipations, surplus inventories will have been accumulated and conversely. Therefore, actual beginning inventories will have been determined by the equation:

$$(1.3) \quad I(t-1) = I^d(t-1) + \bar{X}(t-1) - X(t-1) ,$$

or more concisely:

$$(1.4) \quad I(t-1) = c + (1 + b)\bar{X}(t-1) - X(t-1) .$$

Substituting equation (1.1) and (1.4) into (1.2) one obtains:

$$Q(t) = \bar{X}(t) + c + b \bar{X}(t) - [c + (1+b) \bar{X}(t-1) - X(t-1)] .$$

This last relationship simplifies to:

$$(1.5) \quad Q(t) = (1+b) \bar{X}(t) + X(t-1) - (1+b) \bar{X}(t-1) .$$

It is interesting to note that if $b = -1$, if the marginal desired inventory coefficient is equal to minus one, the output of the firm is independent of anticipations and is equal to sales in the preceding period. Alternatively, if sales anticipations are at a constant level independent of current and past sales levels, output is again found to equal sales of the immediately preceding period. But anticipations can be neglected only in these special cases. They are most easily introduced if they are assumed to be static, if $\bar{X}(t) = X(t-1)$. Then equation (1.5) becomes:

$$(1.6) \quad Q(t) = (2+b) X(t-1) - (1+b) X(t-2) .$$

While it will be assumed that the output of each firm in the economy is determined in this fashion, it must be observed that wide fluctuations in sales might lead to conditions in which the equation could not be assumed to represent the determination of output. First, a rapid fall in sales might imply negative outputs in order that excess inventories could be eliminated within a single period; in reality, inventories may not be liquidated at a rate higher than actual sales. Second, inventories cannot be negative. At least for relatively small fluctuations, however, these complications may be neglected: the output of each firm in the economy may be assumed to be determined by equation (1.6).

B. The Industry and the Economy

It is tempting to assume that equation (1.6), a relation developed on the basis of considerations with regard to the firm, describes the determination of output for the whole economy. This is the essence of the procedure followed by Metzler in his study of the inventory cycle; it is a practice that characterizes macro-economic model building. It will be shown here that if a more realistic approach is followed, the problem is not only more complicated; the conditions for stability are altered in a most fundamental fashion.

In order to derive economic implications from the assumption that firms behave according to equation (1.6), it is convenient to assume that all firms producing a given commodity have the same marginal desired inventory coefficients. Since this is the only parameter entering into the linear equations determining the output of each firm of a particular industry, total output for the industry may be derived as a function of past industry sales by simply summing the quantities and sales figures for the individual firms in the industry. This uniformity assumption permits a reduction in the complexity of the system by replacing the multitude of equations in a given industry by a single equation explaining industry output on the basis of past sales. The behavior of interest, the output of each sector of the economy, may then be expressed in the matrix counterpart of (1.6).

$$(1.7) \quad Q_t = (2I + B) X_{t-1} - (I + B) X_{t-2} .$$

Here Q_t and X_t are vectors whose components represent industry output and sales respectively, and B is the diagonal matrix of marginal desired inventory coefficients.

It may be observed that the output of a whole industry, but not that of any individual firm, is determined on the basis of past sales of the industry. The output of the individual firm is indeterminate. But the same problem is encountered in the theory of pure competition under the assumption of constant costs. Samuelson suggests that "under the purest conditions of competition the boundaries of the (firm)...become vague and ill-defined, and also unimportant." [30, p. 79]. Here too we need not be concerned with the scale of operation of the individual firm; only the output of the industry is of interest.

Equation (1.7) is not sufficient, however. The assumption of uniform marginal desired inventory coefficients for each firm in a given industry does not provide a deterministic economic model. Additional equations connecting outputs and sales are required. A possible procedure would be to assume that only one commodity is produced in the economy; this assumption is implicit in Metzlers' inventory cycle models. But an alternative approach is available.

A system of equations relating the sales of each sector to the level of output of all sectors is required. Production conditions will be assumed to be the same for all firms in a given industry. Sales may be regarded as the sum of purchases by all other firms for production purposes and a final bill of goods representing government demand and sales that are not related to current and past levels of output. If we assume that no stocks of inputs are held, purchases of commodities at time t by each sector must be related to the quantity that the sector plans to produce in the next period. The inputs required for production purposes may be considered as specified by a matrix of technological coefficients $A = (a_{ij})$, where $a_{ij} \geq 0$ represents the quantity of

output of sector i required per unit of output of the j th sector. The n equations relating sales of each of the i sectors to the output of all other industries, because of the production lag, has the form:

$$x_{it} = \sum_j a_{ij} Q_{jt+1} + y_{it}, \quad (i=1, \dots, n)$$

or in matrix notation:

$$(1.8) \quad X_t = A Q_{t+1} + Y_t .$$

Sales are equal to the inputs required for output forthcoming in the next period plus Y_t , the final bill of goods vector whose components represent the sales of each commodity that are independent of output. The equation is meaningful, of course, only if $X_t \geq 0$ and $Q_t \geq 0$. The components of the matrix A of technological coefficients might be regarded as fixed if substitution were not a technological possibility and if constant returns to scale prevailed. But the technological assumption of fixed proportions need not be adopted. The assumption of price rigidity, already introduced in the development of the inventory behavior equation for individual firms, together with the assumption of constant returns to scale, serves to establish the same result. Under either set of assumptions, the empirical counterpart of the A matrix is provided by the Leontief input-output matrix of flow coefficients.

Alternative interpretations of the A matrix are possible. For example, we could restrict attention to three sectors only; manufacturing, wholesale, and retail trade. Another interpretation of the A matrix is provided by the Hayekian type of technology in which higher stages of production feed their output to the lower stages. Such special cases of the general problem are all subsumed within the analysis that follows.

This method of achieving closure abstracts from fixed investment in plant and equipment, buildings, and so forth. But Baumol has implied in a discussion of an aggregative model that the buffer stock type of behavior may be attributed to all investment, to explain a divergence between ex ante and ex post investment [5]. Only a rephrasing of the argument to follow rather than a substantive change would be required in order to include fixed investment in this way. Alternatively, non-inventory investment may be relegated to the final bill of goods.

The time lags in equation (1.8) are crucial. This type of formulation is essential if production is to be regarded as a time consuming process. The differential equation models discussed by Georgescu-Roegen [13], David Hawkins [17], and Leontief [20] assume that production is instantaneous. But the essence of the inventory problem may well lie in the fact that production does require time. Although an alternative time lag formulation may be appropriate for analyzing other problems, it seems clear that a model of inventory behavior must recognize that the inputs used in the production of a commodity must have been fabricated in an earlier period.

With the aid of equation (1.8), a simple process of substitution suffices to eliminate the expression for sales from equation (1.7) so as to have a relation involving past and present levels of output and the final bill of goods alone. This yields:

$$(1.9) \quad Q_t = (2I + B) A Q_t - (I + B) A Q_{t-1} + (2I + B) Y_{t-1} - (I + B) Y_{t-2}.$$

Alternatively, an equation describing the path of sales may be derived by premultiplying equation (1.7) by the A matrix and substituting the resulting expression for sales into equation (1.8) :

$$(1.10) \quad X_t = A(2I + B)X_t - A(I + B)X_{t-1} + Y_t .$$

The expression for quantity may be transformed so as to reveal that current levels of output are determined by past levels of output and the magnitude of the final bill of goods.

$$(1.11) \quad Q_t = T Q_{t-1} + K(2I + B)Y_{t-1} - K(I + B)Y_{t-2} ,$$

where $K = [I - (2I + B)A]^{-1}$ and

$$T = -K(I + B)A .$$

A similar expression may be derived for sales.

Consumption expenditure has not been mentioned in this formulation. It can be relegated to the final bill of goods if it may be assumed to be independent of current income. Alternatively, consumption expenditure may be made dependent upon the current level of output by simply changing the interpretation of certain coefficients. No fundamental change in the structure of the model is required. Let us regard the o 'th sector as the labor sector. Then a_{oj} must be the quantity of labor required per unit of the j th output and a_{io} the slope of the (linear) Engel's curve relating consumption of commodity i to income. Various alternative lags in consumption behavior may now be considered by appropriate specification of the parameter b_o . Labor income at time t is $x_{ot} = \sum_j a_{oj} q_{j,t+1}$ by equation (1.8). Now if $b_o = -1$ we have by equation (1.7) that $q_{o,t+1} = x_{ot}$; but since the goods required for "production" at time $t+1$ are purchased at time t , this is equivalent to unlagged consumption behavior! If, on the other hand, $b_o = -2$, we have $q_{ot} = x_{t-2}$ and consumption is lagged one period, as with the Robertsonian consumption function. Any appropriate setting of b_o between these two values makes consumption depend on both past and current income.

An alternative procedure may be utilized to remove consumption from the final bill of goods. This procedure has the advantage that it does not require the introduction of a separate equation for the consumption sector. In addition, it permits all the marginal desired inventory coefficients to be regarded as non-negative. The relation of consumption by individual commodities to labor income may be expressed by the equation:

$$(1.12) \quad C_t = C + A^c y_t .$$

Here A^c is a column vector whose components represent marginal propensities to consume by commodity type and C a vector of constant components of consumption expenditure. Income, y_t , may in turn be considered to be dependent upon the level of output, Q_{t+1} , according to the extent to which labor is utilized to produce a unit of output, and possibly the level of income itself, for labor may be consumed directly. This gives the equation:

$$(1.13) \quad y_t = A^r Q_{t+1} + a_{00} y_t ,$$

where $A^r = (a_{01}, a_{02}, \dots, a_{0n})$, a row vector, and a_{00} are defined in a similar fashion to the coefficients of A . Substituting one obtains:

$$(1.14) \quad C_t = C + \left(\frac{1}{1 - a_{00}} \right) A^c A^r Q_{t+1} .$$

Here $A^c A^r$ is an $n \times n$ matrix, of course. Equation (1.8) can be replaced by the relation:

$$(1.15) \quad X_t = \left[A + \left(\frac{1}{1 - a_{00}} \right) A^c A^r \right] Q_{t+1} + Y_t ,$$

where Y_t now includes only those elements of consumption that are independent of income, the vector C , as well as other autonomous expenditure. By substituting $A^* = A + \left(\frac{1}{1 - a_{00}} \right) A^c A^r$ into equation (1.15) it is made identical in form to (1.8); the other equations derived for the case in which all consumption was relegated to the final bill of goods are made immediately applicable to the more general case.

C. Feasibility and Stability

The model developed in the preceding section purports to describe the generation of output for each period in terms of past levels of output and sales. The model was derived by assuming that the output of each firm in the economy was determined by an elementary buffer stock adjustment equation under conditions of static expectations. Here certain properties of the model will be derived.

It is convenient to observe the behavior of the model represented by equation (1.11) under static conditions before analyzing its dynamic properties. If output and the final bill of goods are both unchanging, we have from equation (1.9) :

$$Q = (2I + B)AQ - (I + B)AQ + (2I + B)Y - (I + B)Y .$$

But this expression simplifies to the familiar equation:

$$(1.16) \quad Q = AQ + Y .$$

The equilibrium of the inventory model for an unchanging final bill of goods, a vector of constant outputs, is:

$$(1.17) \quad Q = (I - A)^{-1} Y .$$

This static solution is identical to that encountered in open input-output analysis. The model is called feasible in the static sense if $Y \geq 0$ implies that the static solution Q is non-negative. Clearly, a model can be valid only under conditions that imply $Q \geq 0$; a negative output for some sector could only be interpreted as a production process operating in reverse, as the steel industry, for example, consuming steel in order to produce coke and iron ore. A theorem established by Hawkins and Simons [18] states that the system is feasible unless it is inefficient in the sense that some productive process requires to produce a unit of output, both directly as an input and indirectly in the production of commodities required as inputs, one or more units of its own output. Efficiency is a most reasonable assumption for an economy in which production does not require time, for an inefficient production process would be unprofitable under any set of non-negative prices, not all zero. When production requires time, however, it is conceivable that an inefficient process would not be unprofitable if the rate of interest were negative. While this possibility has been considered by von Neumann [31] and Irving Fisher [12, pp. 191-2], it does seem reasonable to exclude from consideration technologies implying a negative rate of interest, to assume that the technology is that of an efficient economy so that the model is feasible in the static sense.

The task of analyzing the dynamic properties of the model represented by equation (1.11) is facilitated, just as with the single difference equations encountered in aggregative models, by working in terms of deviations from the static solution. This procedure will be most fully appreciated if a problem of prediction is considered. It may be observed

that (1.11) is the "reduced form" of the system of equations (1.10); the vectors Y_t and Y_{t-1} being regarded as exogenous and the vector Q_{t-1} representing the predetermined variables of the system; the vector of endogenous variables Q_t is to be determined. If one desired to predict future levels of production on the assumption that equation (1.11) portrays the behavior of the economy, the prediction will have to be conditional upon knowledge of the path of the unexplained exogenous variables, the final bill of goods Y_t . Suppose, in order to facilitate the argument, that the path of Y_t is not only known; suppose for the sake of simplicity that the final bill of goods is a constant vector Y .*

* The assumption that the final bill of goods is fixed is introduced only to simplify the discussion; it is not essential to the argument. The procedure for a fluctuating final bill of goods is analogous to that utilized in analyzing the elementary case where only a single difference equation is involved. The conditions for stability are independent of the time path of the final bill of goods. An account of the procedure for the special, single difference equation problem is given by Allen [3, Ch. 6]. For the more general case see Leontief [20, pp. 63-5].

Then the level of output of each sector is given for the next period by the matrix expression $Q_1 = TQ_0 + KY$; again substituting this result into equation (1.11) yields $Q_2 = T^2Q_0 + (T + I)KY$.

Clearly, the assumptions permit prediction any number of periods into the future. But by working in terms of deviations from equilibrium it is possible to circumvent this clumsy, iterative procedure. Subtracting the static solution (1.16) from (1.9) yields the homogeneous expression:

$$(1.18) \quad Q_t - Q = (2I+B)A (Q_t - Q) - (I+B)A(Q_{t-1} - Q) = T(Q_{t-1} - Q),$$

the last equality following from the definition of T presented in deriving (1.11). Consequently, one obtains by induction:

$$(1.19) \quad Q_t - Q = T^t (Q_0 - Q) .$$

But surely, all that is necessary to obtain Q_t from this last expression is to add Q to both sides of the equation:

$$(1.20) \quad Q_t = T^t (Q_t - Q) + Q ,$$

where Q is obtained by equation (1.17).

The stability of a difference equation system is most conveniently discussed with reference to the homogeneous equation (1.19). Will the system converge to the static solution for all initial deviations from equilibrium? Equation (1.19) reveals that this definition of stability requires:

$$(1.21) \quad \lim_{t \rightarrow \infty} T^t (Q_0 - Q) = 0, \text{ for all vectors } (Q_0 - Q) .$$

Whether or not a given transition matrix T is stable is a question that will be explored in a moment, but first it is necessary to discuss the significance of the question of stability.

Consideration of a special but unrealistic case of the buffer-stock inventory model will clarify aspects of this concept of stability. It will also emphasize that stability is but one, not necessarily the most important property of a dynamic system; stability, it will be seen, may not be a desideratum.

Suppose that all marginal desired inventory coefficients equal minus two, that $B = -2I$. Inspection of equation (1.10) reveals that under this assumption the model reduces to:

$$(1.22) \quad X_t = AX_{t-1} + Y_{t-1} \text{ or } Q_t = AQ_{t-1} + Y_{t-2} .$$

The buffer stock inventory model simplifies in this special case to the matrix or multi-sector multiplier. The special case has the same static solution as previously specified. The system is stable if the economy is efficient, for the condition of efficiency is equivalent to the requirement that all characteristic roots of A be within the unit circle on the complex plane.

There is another property of interest for this model. Not only is it feasible in the static sense; it will not degenerate into a phase in which negative sales are implied for certain sectors for any non-negative set of initial conditions. To show that the sequence X_0, X_1, X_2, \dots generated by the relation $X_t = AX_{t-1} + Y$ contains only non-negative vectors for $A \geq 0$, $Y \geq 0$, and initial conditions $X_0 \geq 0$ a simple indirect proof suffices. Suppose the contrary; then there must be some first element in the sequence, X_{t^*} , that is not non-negative. But since this is the first vector with a negative component, $X_{t^*-1} \geq 0$, so the equation gives us $AX_{t^*-1} + Y = X_{t^*}$. But this too must be non-negative, contrary to assumption, for A , X_{t^*-1} , and Y are all non-negative.

Since the matrix or multi-sector multiplier cannot degenerate into a state of negative outputs, it might be asserted that the question of stability is of little import. If the economy is unstable, if the largest characteristic root of A is greater than unity, the system is capable of balanced, endogenously explained growth.* This is true provided only

* Gerard Debreu and I. N. Herstein prove that $A \geq 0$ implies the existence of an eigen vector with non-negative components; Econometrica, vol. 21, (October, 1953), p. 600. This result of Frobenius also follows as an immediate corollary of the theorem demonstrated in the preceding paragraph.

that the final bill of goods is non-negative. If the final bill of goods is zero, the system is said to be closed; the static solution is null; output is completely determined by lagged endogenous variables. Regardless of whether the closed system is stable or unstable, it will approach a unique path of balanced output, provided that A is indecomposable.*

* If the matrix is indecomposable, it has but one eigen vector with all components of the same sign. Since this vector is associated with the largest root, it will eventually dominate the system if it is included in X_0 . It must be included, for if it were not included some other root would dominate, a root associated with an eigen vector with components of mixed signs; but this would imply that eventually the outputs of certain sectors would be negative, and it has been shown that this is an impossibility.

While the dynamic properties of the special, matrix multiplier case are of interest as an illustration of the nature of the concept of stability, the bizarre conclusion that inefficiency implies the possibility of endogenous growth does more than reveal that strange consequences follow from the peculiar assumption that the marginal desired inventory coefficients all equal minus two. Not only are the properties of the model derived under unreasonable assumptions concerning the type of inventory position that firms desire to maintain. There is also the danger that the system will degenerate into a contradictory condition implying negative inventories. As the system expands, either endogenously as a consequence of inefficiency or as a result of enlargements in the final bill of goods, the inventories held by each sector are inevitably reduced; the system necessarily breaks down when the level of output for some sector reaches a point corresponding to

negative inventories.* For this particular version of the buffer stock

* Even when the marginal desired inventory coefficients are non-negative, it is possible for the model to follow a path implying negative inventories. But this situation can only arise as a result of rapid changes in the level of output and is not dependent upon the level of output itself. Consequently, the initial inventory endowment of the system does not place a restriction on the extent to which the economy can expand.

inventory model the question of stability enters only by the back door, as it relates to the dynamic infeasibility of the system, the danger of negative inventories.

This review of the dynamic properties of the special matrix multiplier case of the buffer stock inventory model has served to reveal that the question of dynamic feasibility as well as stability may be of interest. In addition, it must also be emphasized that a closed linear system explaining the outputs of various sectors of the economy does not necessarily approach a uniquely determined path of final output. Even Harrod's concept of the warranted rate of growth has this property only for certain values of the parameters.**

** Cf. the discussion of Harrod's model below, p. 28.

David Hawkins has shown that the type of closed dynamic model he considers has but a single balanced path of growth; a slight disturbance of the system from this path is likely to send the system into a phase that eventually degenerates into a state of infeasible, negative outputs [17].***

*** For this type of differential equation model, interest centers on the question of whether the system has but one root with positive real part. Only then will the system approach a unique path of balanced growth for all feasible initial conditions.

The property of stability, in the sense in which it has been defined here, is of greater interest for models that are not closed, for models in which no attempt is made to explain certain exogenous variables, i.e., the final bill of goods. If an open system is unstable, even a minor disturbance may send the system into a path diverging further and further from the static solution. This would not be so serious, of course, if all the disturbances were in the positive direction, for this implies that the system would follow a path of endogenous growth. But if a system is capable of endogenous growth, if T is such that for a given static solution Q there exists some set of initial conditions Q_0^* for which the system explodes upwards, then the system will eventually collapse for initial conditions $Q_0^{**} = 2Q - Q_0^*$ into a contradictory condition implying negative outputs for all sectors. Furthermore, for certain transition matrices instability may be of a perverse form such that outputs of sectors diverge from the equilibrium in opposite directions, implying that although endogenous growth is an impossibility, a collapse of certain sectors into negative output is a certainty for some sets of initial conditions. It must be mentioned that certain problems of an empirical nature follow from instability. Not only does instability raise statistical difficulties in parameter estimation, in the determination of the coefficients of the matrix T ; even if the coefficients of T were precisely known, an error however small in the determination of initial conditions Q_0 could lead to errors in prediction of larger and larger magnitude the further ahead one attempted to predict by applying equation (1.20).

Examination of the special case in which all desired inventory coefficients are assumed to equal minus two has revealed that the question of stability is of importance for open models, such as the inventory model under review here. It is of much greater interest to explore the dynamic

properties of the inventory model for the more realistic situation in which all desired marginal inventory coefficients are assumed to be non-negative; entrepreneurs are assumed, in other words, to desire as large or larger inventories at higher levels of sales.* This task will be

* Although the problem of feasibility is also of interest, it will not be explored here. If the marginal desired inventory coefficients are non-negative, sufficiently small disturbances cannot cause a stable system to degenerate into a state of negative stocks or outputs. The complication of stability for larger disturbances may be dealt with by embedding it within a larger possibly piecewise linear system; it is now but one of several regimes; the nature of the alternative regimes and the rules for switching from one regime to another must be specified unambiguously. Leontief modified the Hawkins dynamic model in this way; see [20, pp. 68-76].

attacked in stages. As a first step, a theorem will be presented defining necessary conditions for the stability of the system when all marginal desired inventory coefficients are equal to zero. It will be shown that under these more realistic conditions a much stronger restriction upon the characteristic roots of A is required for stability.

The theorems are stated under the assumption that the economy is efficient. This is equivalent to the condition that the characteristic roots of A are all within the unit circle on the complex plane. As has already been mentioned, this guarantees the existence of a feasible static solution for any positive final bill of goods. In his discussion of the Correspondence Principle, Professor Samuelson has suggested that "known properties of a (comparative) statical system can be utilized to derive information concerning the dynamic properties of a system"[30, p. 284]. Here it will be shown that for the buffer-stock inventory model, the necessary and sufficient condition for feasibility under static conditions does not imply stability.

In the development of the stability conditions, frequent use is made of properties of the characteristic roots of square non-negative, Frobenius matrices presented by Debreu and Herstein [11]. The proofs of the theorems are simplified by restricting the argument to the case in which the matrix of flow coefficients A is equivalent under a similarity transformation to a diagonal matrix, but this is not truly restrictive.*

* The theorems are only slightly weakened when it is abandoned; furthermore, no empirical investigation could lead to its rejection for Bellman [6, p. 25] has shown that for any matrix $T = (t_{ij})$ and $\epsilon > 0$ there exists a matrix $T^* = (t_{ij}^*)$ which has the property and such that $|t_{ij} - t_{ij}^*| < \epsilon$.

NOTATION: If A is any square matrix and λ_i is a characteristic root of A , then $|\lambda_i|$ is the modulus of λ_i and $r(A) = \max_i |\lambda_i|$.

THEOREM 1: If $r(A) < 1$, and if $T = -(I-2A)^{-1}A$, then a necessary and sufficient condition for $r(T) < 1$ is $r(A) < 1/3$.

PROOF: The relation between λ_i , a characteristic root of A , and γ_i , the corresponding characteristic root of T will be established first.

If $PAP^{-1} = \Lambda = \text{diag}(\lambda_i)$, the diagonal matrix of characteristic roots of A , then

$$P(I-2A)P^{-1} = PIP^{-1} - 2PAP^{-1} = I - 2\Lambda.$$

$$\text{Consequently, } [P(I-2A)P^{-1}]^{-1} = (I - 2\Lambda)^{-1}.$$

$$\text{But } [P(I-2A)P^{-1}]^{-1} = P(I-2A)^{-1}P^{-1}, \text{ for } P(I-2A)^{-1}P^{-1} \cdot P(I-2A)P^{-1} = I.$$

$$\text{Therefore, } P(I-2A)^{-1}P^{-1} = (I - 2\Lambda)^{-1}.$$

Now $PAP^{-1} = \Lambda$, so it follows that

$$\begin{aligned} PTP^{-1} &= -P(I-2A)^{-1}AP^{-1} = -P(I-2A)^{-1}P^{-1}PAP^{-1} \\ &= -(I - 2\Lambda)^{-1}\Lambda. \end{aligned}$$

Because the difference, inverse and products of diagonal matrices are diagonal, it follows that the matrix to the right of the last equality is diagonal; consequently, PTP^{-1} is also diagonal. Since T is thus shown to be equivalent under a similarity transformation to a diagonal matrix, it follows that the components of the diagonal matrix are its characteristic roots (T also has the same characteristic vectors as A). Consequently, we may write:

$$PTP^{-1} = - (I-2A)^{-1}A = \text{diag}\left(\frac{-\lambda_i}{1-2\lambda_i}\right) = \text{diag}(\gamma_i) .$$

Considering any term of the last equality, we have:

$$\gamma_i = \frac{-\lambda_i}{1-2\lambda_i} ,$$

the relationship between the roots of A and T .

NECESSITY: Suppose $r(A) \geq 1/3$. Since $A \geq 0$, it follows by a theorem of Debreu-Herstein that $\lambda^+ = r(A) \geq 1/3$ is a characteristic root of A . Now $\lambda^+ \neq 1/2$, for then $I-2A$ would be singular and T would be undefined. Suppose that $1/3 \leq \lambda^+ < 1/2$; then,

$$\gamma^+ = \frac{-\lambda^+}{1-2\lambda^+} \leq -1 \text{ is a characteristic root of } T, \text{ and therefore}$$

$$r(T) \geq 1 .$$

If $1/2 < \lambda^+ < 1$, the only remaining possibility, then

$$\gamma^+ = \frac{-\lambda^+}{1-2\lambda^+} > 1 , \text{ and again } r(T) > 1 .$$

SUFFICIENCY: It will be shown that a contradiction follows from supposing that $r(A) < 1/3$ but $r(T) \geq 1$.

Since $r(A) < 1/3$ and $A \geq 0$, $(I-2A)^{-1}A \geq 0$ (Debreu-Herstein present a proof of this familiar theorem). Consequently, $T = - (I-2A)^{-1}A \leq 0$.

Therefore, the application of the Debreu-Herstein theorem to $-T$ establishes the existence of a root γ of T with the property:

$$\gamma = -r(T) < -1 .$$

Now suppose that $\lambda = a + bi$ is the corresponding root of A . Then, from the expression relating the roots of A and T :

$$\gamma = \frac{-a - bi}{1 - 2a - 2bi} , \text{ or } \gamma(1-2a) - \gamma 2bi = -a - bi .$$

Equating imaginary parts of the last expression yields:

$\gamma 2bi = bi$, so $b = 0$ for otherwise $\gamma = 1/2$, contrary to assumption. Consequently, λ must be real and we have:

$$\gamma = \frac{-\lambda}{1 - 2\lambda} < -1 .$$

But this too leads to a contradiction; it is not consistent with the assumption that $\lambda^+ = r(A) < 1/3$.

The Theorem applies only to the special case in which all marginal desired inventory coefficients equal zero. But it is, of course, more reasonable to assume that the marginal desired inventory coefficients are non-negative. Is it possible that the conditions for stability may be weakened if some of the coefficients are greater than zero but none are negative? The following Theorem shows that this is not the case.

THEOREM II: Let $T = -[I - (2I+B)A]^{-1}(I+B)A$ and

$$T^* = -[I - (2I+B^*)A]^{-1}(I+B^*)A ,$$

where $A \geq 0$ and $B \geq 0$ are square matrices and $r(A) < 1$.

Then $B^* \geq B$ implies either:

i/ $r(T^*) \geq r(T)$, or

ii/ $r(T^*) \geq 1$.

PROOF: T will first be expressed in terms of a non-negative matrix.

$$[I - (2I+B)A]T = -(I+B)A .$$

$$(I - A) T = - (I+B)A + (I+B)A T .$$

$$T = - (I-A)^{-1}(I+B)A + (I-A)^{-1}(I+B)AT .$$

If $G = (I-A)^{-1}(I+B)A$, this last equality yields:

$$T = -G + GT = -(I-G)^{-1}G .$$

Now $r(A) < 1$ implies $(I-A)^{-1} \geq 0$; furthermore $B \geq 0$ by assumption.

Therefore, G is the product of non-negative matrices and must itself

be non-negative. It is apparent that G^* may be derived in an

analogous fashion from T^* . Furthermore, $B^* \geq B$ implies:

$$G^* = (I-A)^{-1}(I+B^*)A \geq (I-A)^{-1}(I+B)A = G \geq 0 .$$

It follows by a theorem of Debreu-Herstein that $r(G^*) \geq r(G)$.

Now suppose that $1 > r(G^*) \geq r(G)$, then

$$(I-G)^{-1} = I + G + G^2 + \dots , \text{ and consequently,}$$

$$T = - (I-G)^{-1} G = -G - G^2 - \dots ; \text{ similarly,}$$

$$T^* = - (I-G^*)^{-1} G^* = -G^* - G^{*2} - \dots$$

Here T and T^* are expressed as sums of convergent geometric series of non-positive matrices. But every term in the series expression for T^* is no larger than the corresponding term for T ; consequently:

$$T^* \leq T \leq 0 .$$

Now applying a theorem proved by Debreu-Herstein for non-negative matrices to $-T^* \geq -T \geq 0$ yields the following inequality:

$$r(T^*) = r(-T^*) \geq r(-T) = r(T) , \text{ which is case 1/} .$$

This has been shown only for the case where $r(G^*) < 1$.

Suppose that $r(G^*) \geq 1$. Since $G^* \geq 0$, G^* has a characteristic root $g^* = r(G^*) \geq 1$. But then the relation between T^* and G^* implies that T^* has a corresponding root given by the equation:

$$t^* = \frac{-g^*}{1 - g^*}$$

(This may be established by an argument similar to that utilized in the proof of Theorem I.)

Inspection of this equation reveals that $g^* \geq 1$ implies $t^* \geq 1$. Therefore, $r(T^*) \geq 1$, which is case ii/.

It is now apparent that a rather strong restriction must be placed upon the matrix of flow coefficients in order to insure stability. Before summarizing these results it will be useful to prove still a third theorem that establishes that even if the stability conditions were satisfied, the model would behave in a most peculiar way.

THEOREM III: Let $T = -[I - (2I+B)A]^{-1}(I+B)A$, where $A \geq 0$ and $B \geq 0$ are square matrices. Then $r(A) < 1$ and $r(T) < 1$ imply $T \leq 0$.

PROOF: It was demonstrated in the proof of Theorem II that there exists a matrix $G \geq 0$ such that $T = -(I-G)^{-1}G$. Clearly, $(I-G)^{-1} \geq 0$ implies $T \leq 0$. Now since $G \geq 0$, $r(G) = g^+ \geq 0$ is a root of G and in addition, $(I-G)^{-1} \geq 0$ if and only if $g^+ < 1$. But $g^+ \geq 1$ implies $t^+ = -g^+/(1-g^+) \geq 1$ is a characteristic root of T , contradicting the assumption that $r(T) < 1$.

These theorems have rather startling implications with regard to the type of buffer-stock inventory behavior now under consideration. Suppose that the marginal desired inventory coefficients are non-negative.

Then a necessary condition for stability is that the largest characteristic root of A , the matrix of technological coefficients, be less than one-third. Unless all the marginal desired inventory coefficients are zero, even this stringent condition will not suffice for stability. If the system is stable, it will be prone to generate a cycle with most peculiar characteristics, for the transition matrix is negative. A simple example will serve to illustrate this strange cycle. Suppose that the economy is initially in equilibrium with some given final bill of goods Y . If the final bill of goods changes to Y^* , $0 \leq Y^* \leq Y$, outputs will fall in the next period below the new equilibrium level. Then in the subsequent period they will rise above the new equilibrium level and so forth. For this type of disturbance, the length of the inventory cycle is two time periods. Such a saw-tooth cycle necessarily develops if every sector is producing below the equilibrium level, as in a depression. All this follows from the fact that a necessary condition for stability, $T \geq 0$, together with $(Q_t - Q) \leq 0$, implies $T(Q_t - Q) = Q_{t+1} - Q \geq 0$, and conversely.* The

* Aggregative model builders should also be concerned with this problem. In Harrod's growth model, for example, the concept of the "Warrented Rate of Growth" is derived from a simple first order difference equation resulting from coupling a simple accelerator equation determining investment with a consumption function to obtain the expression $y_t = ay_t + b(y_t - y_{t-1})$, where y_t is national income at time t , a is the marginal propensity to consume, and $b \geq 0$ is the accelerator coefficient. But this equation may be solved so as to yield the alternative equation: $y_t = b/(a+b-1)y_{t-1}$. Clearly, if y_t is not to fluctuate in sign it is necessary for $b > 1-a$, for the acceleration coefficient to be larger than the marginal propensity to save. But the marginal propensity to consume is a relation between flows while the accelerator coefficient is a relation between a stock and a flow and has a time dimension. Consequently, the model is either feasible or infeasible depending upon the length arbitrarily chosen for the unit time period. The dimensional problem can be circumvented by working in terms of the differential equation version of the model originally presented by Lundberg [23, p. 185].
fn.

conclusion that the inventory cycle will be of two (production) time periods in length holds, of course, only for a particular type of disturbance. Nevertheless, the possibility of an inventory cycle of such curious form demonstrates that the model can give rise to cycles of an entirely different type from those that have plagued the American economy in the postwar period.

These theoretical results raise an empirical question concerning the actual values of the characteristic roots of T . Empirical estimates of the matrix of flow coefficients A have been published by the Harvard Economic Research Project [28] for a number of different levels of aggregation. Although the precise determination of the characteristic roots of a matrix is a difficult computational task, a lower bound for the largest characteristic root can be easily determined for A is non-negative.* Inspection of matrices of size six, eleven, and twenty-one

* Specifically, $r(A) \geq \min_j \sum_i a_{ij}$ and $r(A^x) \leq r(A)$ if $A^x \leq A$.

sectors reveals that in no case can the largest characteristic root be less than three-fifths. Clearly, these empirical results together with the theorems developed here demonstrate that under the assumptions of the model the reasonable condition of non-negative desired inventory coefficients implies that inventories are a destabilizing factor in the economy.

The stability conditions for the buffer stock inventory model presented here differ markedly from those prescribed by Lloyd Metzler for the static expectations case of his macro inventory cycle model [24, pp. 117-8]. This contrast is most apparent for the special case of the multi-sector inventory model in which the marginal desired inventory coefficients are

all assumed to equal zero. Metzler found that for a single commodity economy this implied that the system, while subject to oscillations, was stable, provided that the marginal propensity to consume is positive but less than unity. Here it has been shown that if the multi-commodity nature of the economy is recognized, the magnitude of the technological coefficients must be considered in analyzing questions of stability. Even if the marginal propensity to consume is zero, stability requires that the largest characteristic root of the matrix of technological coefficients be less than one-third.* A priori considerations suggest

* Because Metzler neglects production conditions entirely, the special, single commodity case of the disaggregated model is not identical to Metzler's.

that Metzler's conditions upon the marginal propensity to consume is satisfied; while a priori considerations also suggest that the characteristic roots are all less than unity in absolute value, nothing implies that they are less than a third. Empirical evidence suggests that at least one of the roots is above this critical value. Clearly, recognition of the fact that more than one commodity is produced in the economy alters in a most significant if disconcerting fashion the conditions for stability.

D. Perfect and Independent Expectations:

The discouraging conclusion that inventories play a destabilizing role was developed under the assumption of static expectations. The conditions for stability are dependent upon this assumption. Here the conditions for stability under alternative assumptions concerning the nature of expectations will be presented.

The case in which entrepreneurs are assumed to have perfect expectations, their anticipations of future sales being precisely fulfilled, will be considered first. It might be objected that this is a meaningless context in which to explore the consequences of buffer stock behavior. D.H. Robertson has remarked that in Keynesian analysis "the organ which secretes (the rate of interest) has been amputated, and yet it somehow still exists --- a grin without a cat." [29, p. 25]. By assuming perfect foresight, the very element of errors of judgement required to justify the existence of buffer stock inventories has been eliminated. But our analysis of perfect foresight is undertaken only to demonstrate that the unstable elements of the model are not the consequence of errors of expectations. It will be shown that errors of judgement are neither a necessary nor a sufficient cause of instability. Intuition might well suggest that perfect foresight would insure stability. This is not the case. Indeed, the type of buffer stock inventory model now being considered is necessarily unstable for all reasonable values of the parameters under the assumption of perfect expectations.

In order to demonstrate the unstable nature of the buffer stock inventory model for the case in which expectations are precisely fulfilled, it is necessary to return to equation (1.5). If actual sales at time t , X_t are substituted for anticipated sales in this equation one obtains the expression:

$$(1.23) \quad Q_t = (I+B)X_t - BX_{t-1}$$

Output in period t is just sufficient to satisfy sales requirements and to adjust inventories to a new level desired on the basis of the accelerator principle. But sales at time period t are equal to the input requirements for next periods output plus the final bill of goods,

as explained by equation (1.8). Consequently, the above expression may be premultiplied by A in order to obtain the relation:

$$(1.24) \quad X_{t-1} = AQ_t = A(I+B)X_t - ABX_{t-1} + Y .$$

This obviously has the same static solution as the models discussed previously.

In order to demonstrate that the system will be unstable for all matrices of inventory coefficients $B \geq 0$, provided that the economy's technology is efficient, it is helpful to rearrange the homogeneous formed equation (1.24) so as to obtain:

$$(I-A)X_t = (I+AB)X_t - (I+AB)X_{t-1} .$$

Premultiplication by the non-negative matrix $(I-A)^{-1}$ yields:

$$(1.26) \quad X_t = RX_t - RX_{t-1}, \text{ where } R = (I-A)^{-1}(I+BA), \text{ or} \\ = -(I-R)^{-1}RX_{t-1} = TX_{t-1} .$$

Since A may be assumed to represent the technology of an efficient economy, a simple manipulation of well known convergence properties of Leontief matrices reveals that $P(I-A)^{-1}P^{-1} = I^+ PAP^{-1} + PA^2P^{-1} \dots = I + \lambda + \lambda^2 \dots$. But A has a non-negative characteristic root λ^+ , for $A \geq 0$. Therefore, $R \geq (I-A)^{-1} \geq 0$ has a real root $r^+ \geq 1$. But now it is obvious from (1.26) that the transitions matrix for this model has a corresponding real root $t^+ = -r^+/(1-r^+) > 1$. This establishes that if manufacturers had perfect expectations, buffer-stock inventory behavior would imply instability. Accurate foresight, far from contributing to stability, insures that the system will be unstable even for matrices A and B that would lead to stability under conditions of static expectations.

The case of perfect foresight is not the only alternative to the assumption of static expectations. As a third case, suppose that the expectations of entrepreneurs are completely independent of current and past sales; for definiteness assume that $\bar{X}_t = \bar{X}$ for all t . Then, it follows from equation (1.5) that the expectations term drops completely out of the system. We now have $Q_t = X_{t-1}$. Premultiplying by A as before we obtain:

$$(1.27) \quad X_t = AQ_{t+1} + Y = AX_t + Y.$$

Clearly the system is not only stable. The system converges immediately to its equilibrium. The transition matrix is null.

Can any of the three types of expectations behavior that have been considered be regarded as realistic? The assumption that anticipations are completely independent of current developments insults the intelligence of the entrepreneur. The assumption of perfect foresight, on the other hand, attributes the power of the soothsayer to the business man. Perfect foresight, we have seen, implies instability. The assumption of static expectations is a not too unhappy compromise between these two extremes. While with static expectations stability is a possibility, it is a stability characterized by a most eccentric style of cyclical behavior. My own empirical investigations, based on observed manufacturers' sales and inventory behavior, suggest that manufacturers' expectations are considerably more accurate than is implied by the assumption of static expectations. On the other hand, they certainly are not perfect. I have not explored the theoretical implications of the more interesting case in which expectations are assumed to lie between the value obtained with a naive projection and actual developments. Rather, we turn to a modification of the basic buffer-stock principle which also has empirical

foundation. We shall see that when manufacturers do not attempt an immediate adjustment of actual inventories to the equilibrium level, the conditions for stability are relaxed.

PART II: DELAYED ADJUSTMENT INVENTORY BEHAVIOR

The possibility that entrepreneurs do not attempt a complete adjustment of inventories to the desired level within each production period will be considered here. Costs of changing production and stocks may lead a firm to attempt only a partial adjustment of actual inventories to the desired level in any one period. The basic behavioral pattern for each firm may be assumed to be similar to the distributed lag investment function suggested by Goodwin [15], Chenery [8], and others. But perfect foresight need not be assumed. It will be shown that this generalization of the buffer-stock model leads to a system with quite different and more promising dynamic properties.

A. Static Expectations

It will again be assumed that I_t^e , the equilibrium level of inventories that entrepreneurs desire to hold, is linearly related to sales:

$$(2.1) \quad I_t^e = c + b \bar{X}_t .$$

But only a partial adjustment of inventories to the equilibrium level is planned by each firm for the production period. Consequently, planned inventories, I_t^p , are determined by the equation:

$$(2.2) \quad I_t^p = d(I_t^e - I_{t-1}^a) + I_{t-1}^a = cd + bd\bar{X}_t + (1-d)I_{t-1}^a, \quad 0 \leq d \leq 1.$$

The planned change in inventories is thus assumed to be some constant proportion of the discrepancy between the actual and the equilibrium level of inventories.

The development of the model now proceeds much as before. The actual level of inventories at time t will exceed the planned level if sales anticipations are not fulfilled:

$$(2.3) \quad I_t^a = I_t^p + [\bar{X}_t - X_t] = cd + (1 + bd)\bar{X}_t - X_t + (1-d)I_{t-1}^a .$$

The level of output at time t is equal to anticipated sales plus the excess of planned inventories for time t over the level of inventories in the preceding period.

$$\begin{aligned} Q_t &= \bar{X}_t + I_t^p - I_{t-1}^a \\ &= \bar{X}_t + cd + bd\bar{X}_t + (1-d)I_{t-1}^a - cd - (1+bd)\bar{X}_{t-1} + X_t - (1-d)I_{t-2}^a . \\ (2.4) \quad &= (1+bd)\bar{X}_t - (1+bd)\bar{X}_{t-1} + (1-d) \left[I_{t-1}^a - I_{t-2}^a \right] + X_{t-1} . \end{aligned}$$

But this last equation may be simplified, for the actual change in inventories is equal to the discrepancy between output and actual sales; that is, $I_{t-1}^a - I_{t-2}^a = Q_{t-1} - X_{t-1}$. Consequently, a simple substitution eliminates inventories from equation (2.4) yielding:

$$(2.5) \quad Q_t = (1+bd)\bar{X}_t - (1+bd)\bar{X}_{t-1} + (1-d)Q_{t-1} + dX_{t-1} .$$

If static expectations are again assumed, a further simplification may be achieved:

$$(2.6) \quad Q_t = [1 + (b+1)d] X_{t-1} - (1+bd)X_{t-2} + (1-d)Q_{t-1} .$$

All this, of course, applies to the individual firm; but the same principles utilized earlier in connection with the simpler model may again be applied to justify the assumption that each industry behaves according to equation (2.6). Consequently, if the identity matrix is substituted for 1 and the scalars d and b are replaced by diagonal matrices D and B of reaction and desired inventory coefficients, respectively, the expression may be interpreted as a matrix equation explaining the determination of output for each industry on the basis of past sales and outputs.

The next step of the analysis is to eliminate the expression for sales by making use of knowledge of the technology of the economy. Equation (1.8) may again be applied in precisely the same way as before in order to achieve this crucial simplification. One obtains:

$$(2.7) \quad Q_t = (I+BD+D) (AQ_t + Y_t) - (I+BD) (AQ_{t-1} + Y_{t-1}) + (I-D)Q_{t-1} .$$

If the final bill of goods is assumed constant, the same static solution as before is obtained. Furthermore, the homogeneous form of the equation reduces to:

$$(2.8) \quad \begin{aligned} Q_t &= (I+BD+D)AQ_t + (I-D-A-BDA)Q_{t-1} \\ &= [I-(I+BD+D)A]^{-1} [I-D-(I+BD)A] Q_{t-1} , \text{ or in brief} \\ &= TQ_{t-1} , \text{ where} \\ T &= [I-(I+BD+D)A]^{-1} [I-D-(I+BD)A] \\ &= I + [I-(I+BD+D)A]^{-1}D(A-I) . \end{aligned}$$

Observe that if D , the matrix of reaction coefficients, is equal to the identity, if all firms desire to adjust their inventories completely in each time period, this more complicated model reduces to that analyzed in detail in Part I.

But suppose that this is not the case, suppose that entrepreneurs do not attempt a complete adjustment of inventories to the desired level. In order to demonstrate that the stringent conditions for stability on the characteristic roots of A are weakened when entrepreneurs attempt only a delayed adjustment of their inventories, it is most convenient to adopt a strong uniformity assumption, to assume that all firms in all industries have the same reaction and marginal desired inventory coefficients. If in addition, A may be assumed to be equivalent under a similarity transformation to a diagonal matrix, $PAP^{-1} = \Lambda = \text{diag}(\lambda_i)$, then:

$$(2.9) \quad PTP^{-1} = [I - (1+bd+d)\Lambda]^{-1} [(1-d)I - (1+bd)\Lambda]$$

This is a diagonal matrix; consequently, if λ_i is a characteristic root of A there exists a corresponding root of T given by the equation:

$$(2.10) \quad t_i = \frac{1-d-\lambda_i-bd\lambda_i}{1-(1+bd+d)\lambda_i}$$

The conditions on d and the roots of A such that all t_i are within the unit circle on the complex plane are desired.

In order to answer this question of stability, it is convenient to analyze the behavior of a function mapping conceivable, real characteristic roots of A into corresponding characteristic roots of T :

$$(2.11) \quad t(b, d, \lambda) = \frac{1-d-\lambda-bd\lambda}{1-(1+bd+d)\lambda}, \quad \begin{array}{l} -1 \leq \lambda \leq 1, \text{ and real} \\ 0 \leq d \leq 1 \\ 0 \leq b \end{array}$$

The following properties of the function are easily derived:

(a) If $t(b,d,\lambda) = 1$, either $d = 0$ or $\lambda = 1$.

(b) If $t(b,d,\lambda) = -1$, $\lambda = \frac{2-d}{2+2bd+d}$.

(c) If $1-(1+bd+d)\lambda \neq 0$, the function exists and has continuous partial derivatives.

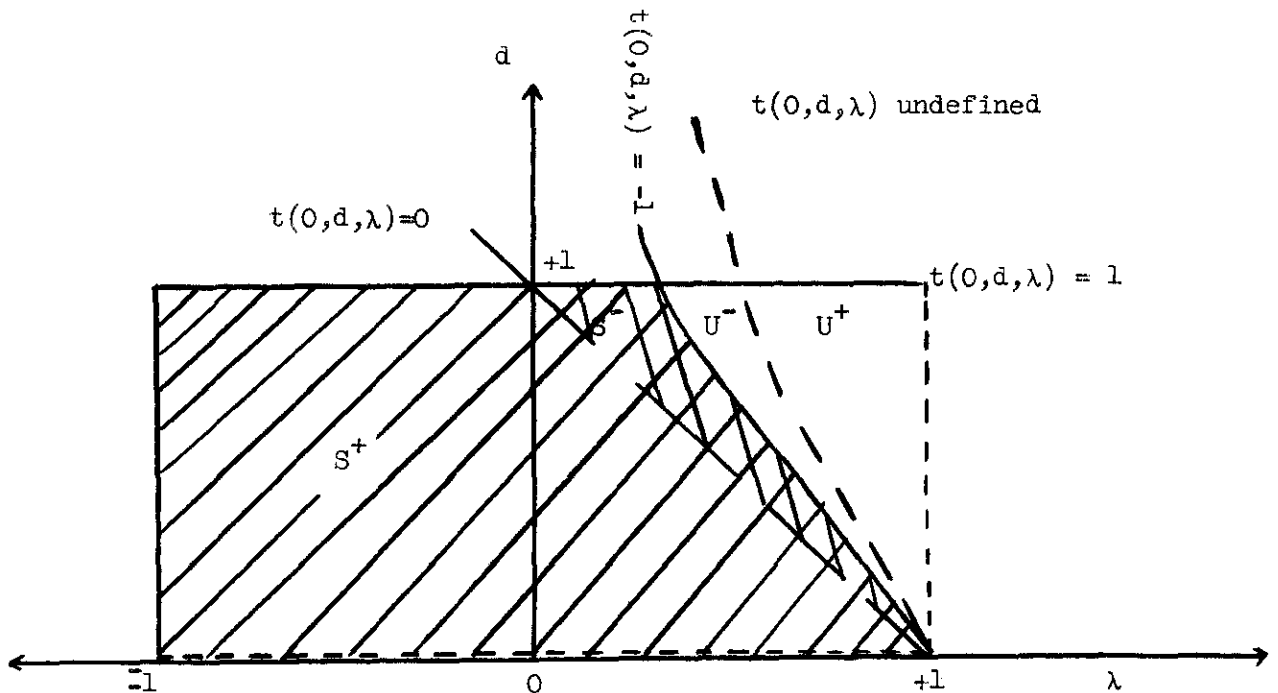
(d) $\frac{\partial t}{\partial \lambda} = \frac{-(1+b)d^2}{[1-(1+bd+d)\lambda]^2} \leq 0$, if defined.

(e) $\frac{\partial t}{\partial b} = \frac{d^2(\lambda-1)\lambda}{[1-(1+bd+d)\lambda]^2} \begin{cases} < 0 & \text{for } 0 < \lambda < 1 \\ > 0 & -1 < \lambda < 0 \end{cases}$

(f) If $t(b,d,\lambda) = 0$, $1-d/(1+bd) = \lambda$.

(g) $\frac{\partial t}{\partial d} < 0$ if $\lambda > 0$

For the special case in which $b=0$, these conditions suffice to partition the domain of definition of the function into the regions labeled on the following diagram:



The region S^+ on the graph corresponds to positive values of t that are less than unity; the region S^- to the negative values of t that are greater than minus one. The regions U^- and U^+ correspond to values of t that are either less than minus one or greater than plus one; the dotted line to the particular values of d and λ for which t is precisely equal to unity. Now $A \geq 0$ and $r(A) < 1$ by assumption; consequently, $\lambda^+ = r(A) < 1$ is a characteristic root of A . Clearly, if $b = 0$, a necessary condition for stability of T is that $-1 < t(b, d, \lambda^+) < 1$. This establishes the necessity half of the following theorem only for the special case in which all marginal desired inventory coefficients equal zero.

THEOREM: Let T be defined as in (2.8); assume that $b_i = b$ and $d_i = d$ for all i . Then $b \geq 0$, $0 < D \leq I$, $A \geq 0$, and $r(A) = \lambda^+ < 1$ imply that a necessary and sufficient condition for $r(T) < 1$ is

$$\lambda^+ < \frac{2-d}{2+2bd+d}.$$

NECESSITY: Let $\sigma = \frac{2-d}{2+2bd+d}$ and let $\gamma = 1/(1+bd+d)$. Now $\gamma \neq \lambda^+ = r(A)$, for then T would be undefined. Furthermore, it is apparent that $\sigma < \gamma$. Therefore, if $\lambda^+ > \sigma$, either $\sigma < \lambda^+ < \gamma$, or $\gamma < \lambda^+ < 1$. If $\sigma < \lambda^+ < \gamma$, then $t(b, d, \lambda^+) < t(b, d, \sigma) = -1$ by properties "a", "c", "d". If $\gamma < \lambda^+ < 1$, $t(b, d, \lambda^+) > 1$, by the same properties.

SUFFICIENCY: Suppose that $t = t(b, d, \lambda^+)$ is less than unity in absolute value. It will be demonstrated that if $\lambda = \alpha + \beta i$ is any other root of A , then $t(b, d, \lambda)$ is inside the unit circle on the complex plane. From the definition of $t(b, d, \lambda)$ it follows that:

$$|t(b, d, \lambda)| = \frac{|1-d-\lambda-bd\lambda|}{|1-(1+bd+d)\lambda|} = + \sqrt{\frac{[1-d-(1+bd)\alpha]^2 + (1+bd)^2\beta^2}{[1-(1+bd+d)\alpha]^2 + (1+bd+d)^2\beta^2}}$$

Now $r(A) = \lambda^+ < 1$; therefore, by properties "a", "c", and "d",
 $-1 < t(b,d,\alpha) < 1$, or $[1-d-(1+bd)\alpha]^2 < [1-(1+bd+d)\alpha]^2$. In addition,
 $d \geq 0$, so $(1+bd)^2 \geq (1+bd+d)^2$. This establishes that the ratio of
the terms under the radical in the equation is less than unity, for
both terms in the denominator of the ratio are larger than the
corresponding terms of the numerator. Consequently, $t(b,d,\lambda^+) < 1$
implies $t(b,d,\lambda) < 1$. QED.

This theorem demonstrates that under conditions of static expectations, the requirements for stability are weakened when entrepreneurs attempt only a delayed adjustment of their inventories to the equilibrium level.

It is also possible to say something concerning the effects of changes in certain parameters of the model upon the system's stability. Let us evaluate the effects of various adjustments under the assumption that initially the characteristic root of the transitions matrix of largest absolute value, t^+ , is negative. In the first place, suppose the economy becomes more efficient as a result of technological change that dominates in the sense that the new technology matrix A^* has the property $0 \leq A^* \leq A$. A theorem of Debreu-Herstein establishes that with the new technology the largest characteristic root of A will be smaller, provided that A is indecomposable. Under these circumstances, property (d) implies that the largest characteristic root of the new transition matrix will be smaller in absolute value than before the technological improvement, contributing to stability. Next, suppose that the marginal desired inventory coefficient decreases in magnitude, again under the supposition that $t^+ < 0$. Property (e) implies that this will decrease the absolute value of the largest characteristic root of T , contributing to stability. Again, a slower reaction speed, a smaller value of d , would

contribute to stability according to property (g) . Evaluation of an unstable system with $t^+ > 0$ is complicated, in that large changes in the parameters may involve the discontinuity. Inspection of the diagram suggests that if the changes are sufficiently small, the effects will be in exactly the reverse direction from those enumerated for the opposite case in which $t^+ < 0$.

B. Perfect Expectations:

The conditions for stability of the delayed adjustment, buffer-stock inventory model under the assumption of perfect foresight will now be explored. It has already been demonstrated that perfect foresight together with immediate adjustment implies instability. It will be shown that with perfect expectations the model is necessarily unstable for any efficient technology, at least under the strong uniformity assumption that all marginal desired inventory and all reaction coefficients are of the same magnitude. This suggests that errors in foresight and delayed adjustment are prerequisites for reasonable dynamic behavior.

If all industries have the same marginal desired inventory and reaction coefficients and if $\bar{X}_t = X_t = AQ_{t+1} + Y_t$, equation (2.5) reduces to:

$$(2.12) \quad Q_t = (1+bd) (AQ_{t+1} + Y_t) - (1+bd)(AQ_t + Y_{t-1}) + (1-d)Q_{t-1} \\ + d(AQ_t + Y_{t-1}) .$$

Here b and d are scalars representing the uniform marginal desired inventory and reaction coefficients respectively.

If $Y_t = Y$, the same static solution as before is obtained. Whether or not the final bill of goods is constant, the stability of the system may be analyzed in terms of deviations from equilibrium; only the homogeneous

form of equation (2.12) need be investigated:

$$(2.13) \quad Q_{t+1} = \left\{ I + [(1+bd)A]^{-1}(I-dA) \right\} Q_t + [(1+bd)A]^{-1}(d-1)Q_{t-1} .$$

Now suppose that the eigen vectors of A are linearly independent, so that $P^{-1}AP = \Lambda$, the diagonal matrix of characteristic roots of A . Let Z_t be an n component column vector of composite bundles of goods defined as $Z_t = PQ_t$. Then an expression for Z_{t+1} may be derived from equation (2.13) as follows.

$$Z_{t+1} = PQ_{t+1} = P \left\{ [(1+bd)A]^{-1}(I-dA) + I \right\} P^{-1}PQ_t + P[(1+bd)A]^{-1}(d-1)P^{-1}PQ_{t-1} ,$$

or more simply:

$$(2.14) \quad Z_{t+1} = \left\{ I + [(1+bd)\Lambda]^{-1}(I-d\Lambda) \right\} Z_t + [(1+bd)\Lambda]^{-1}(d-1)Z_{t-1} .$$

But each of the matrices in this last equation is diagonal. Consequently, the new definition of commodities in terms of composite bundles of goods Z_t effectively separates variables so that there are now n independent second order difference equations of the form:

$$(2.15) \quad z_i(t) + \gamma_{i1} z_i(t-1) + \gamma_{i2} z_i(t-2) = 0 \quad \begin{array}{l} i= 1,2,\dots,n \\ t= 1,2,\dots \end{array}$$

where:

$$\gamma_{i1} = -1 - \frac{(1-d\lambda_i)}{(1+bd)\lambda_i}$$

$$\gamma_{i2} = \frac{1-d}{(1+bd)\lambda_i}$$

Clearly, all n of these difference equations must be stable if (2.13) is to be stable. It will be shown that if the matrix of technological coefficients $A \geq 0$ is indecomposable and $r(A) < 1$, then at least one of these n difference equations is unstable. $\lambda_1 = r(A) > 0$ is a root of A . Consequently, the corresponding difference equation of the set (2.15) has real coefficients. Samuelson has specified that a necessary condition for stability of a second order difference equation with real coefficients is that $1 + \gamma_1^+ + \gamma_2^+ > 0$. [30, p. 436]. Applying this test to the equation corresponding to the largest root of A , however, reveals that $1 + \gamma_1^+ + \gamma_2^+ = d(\lambda^+ - 1) / [(1 + bd)\lambda^+] < 0$, the numerator being negative and the denominator positive for b and d are positive and λ^+ is positive but less than unity. Perfect expectations, at least under the strong uniformity assumption, imply instability regardless of the speed of adjustment.

PART III. SUMMARY AND CONCLUSIONS:

A class of disaggregated buffer stock inventory models has been revealed to possess surprising properties. Perfect expectations necessarily imply instability for the types of difference equation systems that have been considered. With static expectations, stability would require a much more efficient technology than the American economy possesses if entrepreneurs attempt an immediate adjustment of inventories to the desired level within each production period. My empirical investigation revealed that errors in anticipating future sales and delayed adjustment are facets of actual inventory behavior. The theoretical investigation suggests that expectational errors and delayed adjustment behavior are essential for stability. Reaction lags and imperfect foresight, far from contributing to economic instability, are important stabilizers of the economy.

The magnitude of fluctuations in the inventory component of GNP is widely recognized, thanks to the research of Moses Abramovitz.* As a

* In his analysis of inventory behavior in the inter-war period, Abramovitz revealed that the inventory component of GNP was subject to major fluctuations from peaks to troughs of the business cycle [, Ch. 21].

consequence, tax measures designed to stabilize inventory investment have been proposed. Abramovitz himself suggested that a tax on the average value of inventories, by inducing firms to operate with lower stocks, would contribute to economic stability [2, pp. 293-4]. Albert G. Hart argues that "a tax at a substantial rate (25 per cent, say) to be applied each quarter to the value of any increase or decrease in each firm's inventories, compared with the same date of the year previously," might better contribute to the same objective if the scheme were not administratively infeasible [16, pp. 452-3]. J. Keith Butters appraised the consequences for economic stability of alternative provisions for evaluating inventory for tax purposes [7].

Disaggregation of the Metzler type inventory model provides a theoretical framework suitable for evaluating such policy issues. The process of aggregation had suppressed the technological coefficients of the system, a set of additional parameters offering further degrees of freedom for consideration in the formulation of stabilization policy. Technological advance may be expected to contribute to stability, we found, under fairly general conditions. Disaggregation also reveals essential prerequisites for stability. A tax on the size of inventory holdings designed to reduce the average value of inventories might well miss its objective of dampening cycles in economic activity engendered by

fluctuations in inventories. Even when the desired inventory equation is assumed to be linear, a reduction in the size of inventories that entrepreneurs desire to hold at relevant levels of output does not insure increased stability, for the crucial marginal desired inventory coefficients might still be larger than before. While it might be argued that the adoption of an alternative proposal, a tax levied each quarter on the change in the value of inventories held by firms over the corresponding quarter of the preceding year, would necessarily lower the marginal desired inventory coefficient, this would by no means establish that such a tax would contribute to the stability of the economy. Ceteris paribus, the faster entrepreneurs attempt to adjust inventories to the desired level, the less stable the economy. A tax on inventory investment might well reduce the size of the marginal desired inventory coefficients; the possibility that it would induce firms to attempt a tighter inventory policy, a more rapid adjustment of inventories to the desired level, must also be admitted.

REFERENCES

- [1] Abramovitz, Moses, Inventories and Business Cycles, With Special Reference to Manufacturers' Inventories, New York: National Bureau of Economic Research, 1950.
- [2] Abramovitz, Moses, "Inventory Policy and Business Stability," Regularization of Business Investment; Conference of Universities-National Bureau Committee for Economic Research (Princeton: Princeton University Press, 1954), pp. 293-4.
- [3] Allen, R. G. D., Mathematical Economics (London: MacMillan, 1956).
- [4] Arrow, Kenneth J. and M. Nerlove, "A Note on Expectations and Stability." Econometrica, 26 (1958), p. 297.
- [5] Baumol, W. J. , "Notes on Some Dynamic Models," Economic Journal, LVIII (December, 1948).
- [6] Bellman, Richard; Stability Theory of Differential Equations, New York: McGraw-Hill, 1953.
- [7] Butters, J. Keith, Effects of Taxation: Inventory Accounting and Policies, Cambridge: Harvard University Press, 1949.
- [8] Chenery, Hollis E., "Overcapacity and the Acceleration Principle," Econometrica, 20 (January, 1952).
- [9] Chipman, John S., The Theory of Inter-sectoral Money Flows and Income Formation, Baltimore: Johns Hopkins, 1951.
- [10] Chipman, John S., "A Note on Stability Workability and Duality in Linear Economic Models," Metroeconomica VI, (April, 1954) pp. 1-10.
- [11] Debreu, Gerard and I. N. Herstein, "Nonnegative Square Matrices," Econometrica, 21 (October, 1953) pp. 597-607.
- [12] Fisher, Irving, The Theory of Interest, New York: MacMillan, 1930.
- [13] Georgescu-Roegen, Nicholas, "Relaxation Phenomena in Linear Dynamic Models," Activity Analysis of Production and Allocation, edited by T. C. Koopmans, New York: John Wiley, 1951.
- [14] Goodwin, R. M., "The Multiplier as a Matrix," Econ. Journal, LIX (December, 1949), pp. 537-556.
- [15] Goodwin, R. M., "Secular and Cyclical Aspects of the Multiplier and Accelerator," Income, Employment, and Public Policy: Essays in Honor of Alvin H. Hansen, New York: Norton and Co., 1948.
- [16] Hart, Albert G., "Government Measures Designed to Promote Regularization of Business Investment," Regularization of Business Investment.

- [17] Hawkins, David, "Some Conditions of Microeconomic Stability," Econometrica, 16 (October, 1948), pp. 309-23.
- [18] Hawkins, David and Herbert A. Simon, "Note: Some Conditions of Microeconomic Stability," Econometrica, 17 (July-October, 1949), pp.245-48.
- [19] Hicks, J. R., "Methods of Dynamic Analysis," Economic Essays in Honor of Eric Lindahl, Stockholm: Economics Tidskrift, 1956. pp. 139-
- [20] Leontief, Wassily, Studies in the Structure of the American Economy, New York: Oxford Univ. Press, 1953.
- [21] Lovell, Michael C., "Inventories and Stability: An Interindustry Analysis," Unpublished Ph.D. Dissertation, Harvard University, March, 1959.
- [22] Lovell, Michael C., "Manufacturers' Inventories, Sales Expectations, and the Acceleration Principle," Cowles Foundation Discussion Paper No. 86, December 11, 1959, (mimeographed).
- [23] Lundberg, Eric, Studies in the Theory of Economic Expansion, London: P. S. King and Son, 1937.
- [24] Metzler, Lloyd A., "The Nature and Stability of Inventory Cycles," Review of Economics and Statistics, XXIII, (August, 1941), pp. 113-129.
- [25] Metzler, Lloyd A., "Stability of Multiple Markets: The Hicks Conditions," Econometrica, 13 (October, 1945), pp. 277-292.
- [26] Mills, E. S., "Expectations, Uncertainty, and Inventory Fluctuations," Review of Economic Studies, XXII (1954-5), pp. 15-23.
- [27] Pigou, A. C., Industrial Fluctuations, London: MacMillan and Co., 1927.
- [28] Report on Research for 1953, Cambridge: Harvard Economic Research Project, 1954, (hctographed).
- [29] Robertson, D. H., "Mr. Keynes and the Rate of Interest," Economic Journal, XLVIII (June, 1938), reprinted in Essays in Monetary Theory, London: Stables Press, 1940.
- [30] Samuelson, Paul A., Foundations of Economic Analysis, Cambridge: Harvard University Press, 1947.
- [31] von Neumann, John, "A Model of General Economic Equilibrium," Review of Economic Studies, XIII (145-6), pp. 1-9.