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A Model of Bank Portfolio Selection [*]

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I. Introduction

Over the course of the last century, the implications of the assumption of profit maximization for the behavior of the firm have been tracked down in ever greater detail. Curiously, however, this firm has almost always been a seller of non-financial goods; banking has been studiously exempted from the application of such theory. The exemption is curious because the commercial bank seems in many respects more likely to fit the conditions of such static theory than the product manufacturer. The "method" of production and the "product" itself do not change, and hence the unpleasant necessity of neglecting some of the most interesting features of markets in order to devise marginal conditions does not arise. Even the proverbial conservatism of bankers is a prop to such theory, for it may well make banking less prone to upsetting expectational factors than other markets.

The reason for this neglect of banking probably lies in the implication of straightforward profit maximization: that the bank should acquire a portfolio consisting entirely of the asset whose yield (less any costs of maintainence or acquisition) is greatest.* But this procedure

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* Diversification can be explained only if the bank is a monopsonist in the market of the highest-yield asset or if it is required by law to carry reserves of low-yield assets.

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misses the very essence of banking, which is to "borrow short and lend long." Thus, the "profit" which a bank derives from its portfolio must be interpreted in terms of not only the money return but also the liquidity

and capital certainty which the portfolio offers. There is no reason why the concepts of profit maximization cannot be applied to bank operations, provided that "profit" is conceived in this broader sense.

The crux of bank operations is uncertainty, and hence any reference to profits must be in a probabilistic sense. In this paper, it will be assumed that the bank considers the expected value of its profits (i.e., additions to surplus during the planning period) under various conditions of risk,* prin-

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* In deference to received literature, the word "risk" is used rather than "uncertainty" since the bank is assumed to know, with certainty, the parameters of the probability distributions.

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cipally that of change in size of deposits. The problem is somewhat analogous to recent demand-risk-inventory theory for the selling firm,** where cash (and

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** Cf. pp. 256-259 of K. J. Arrow, T. Harris, and J. Marschak, "Optimal Inventory Policy," Econometrica, Vol. 19, No. 3 (July 1951), pp. 250-272.

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other assets readily convertible into cash) represents inventories, the carrying cost of these "inventories" is the surrender of earning power, and various penalties are incurred for insufficient "inventories."

This approach to commercial bank operations is not new, having been first indicated by Edgeworth in 1888,*** although at that time bankers still con-

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*** F. Y. Edgeworth, "The Mathematical Theory of Banking," Journal of the Royal Statistical Society, Vol. 51, Part 1, (March 1888), pp. 113-127.

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sidered loans to be liquid (in the sense of self-liquidating) and securities

frozen, a view which lingered into the 1920's.* Edgeworth indicated the importance

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* For a review of this revolution in bankers' ordering of relative liquidities,
whereby securities became "secondary reserves" and loans frozen assets, cf.
B. Suviranta, "The Shiftability Theory of Bank Liquidity," Economic Essays in
Honor of Gustav Cassel (London: 1933), pp. 623-635.

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of probability to banking through the device of a simple game:

> I have imagined a new game of chance, which is played in this
> manner: each player receives a disposable fund of 100 counters,
> part of which he may invest in securities not immediately realizable,
> bearing say 5 per cent per ten minutes; another portion of the 100
> may be held at call, bearing interest at 2 per cent per ten minutes;
> the remainder is kept in the hands of the player as a reserve against
> certain liabilities. ... [22 digits are drawn at random every two
> minutes, and the difference between their sum and their expected sum,
> 99, is calculated.] The special object of the reserve above mentioned
> is to provide against demands which exceed that average. If the
> player can meet the excess of demand with his funds in hand, well;
> but if not he must call in part, or all, of the sum placed at call,
> incurring a forfeit of 10 per cent on the amount called in. But if
> the demand is so great that he cannot even thus meet it, then he
> incurs an enormous forfeit, 100£ or 1000£.**

** Edgeworth, "The Mathematical Theory of Banking," p. 120.

Unfortunately, Edgeworth then proceeded to solve for optimum portfolio by a

kind of enlightened common sense, claiming that "the calculus cannot indeed,

I think, by itself determine what chance of great disaster it might be prudent

to incur for the probability of a moderate gain."*** If however, values can

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*** Edgeworth, "The Mathematical Theory of Banking," p. 121.

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be placed upon the various aspects of this "great disaster," the calculus can

do just that.

II. Assumptions

It is uncertainty, in its various guises, far more than anything else which makes the banker's job a difficult one. The important areas of this uncertainty arise because the bank cannot know exactly:

1. How large will be its deposit liabilities at any moment of the future.

2. The market value of the non-matured securities in its portfolio at any moment of the future.

3. What proportion of its borrowers will be forced to default the loans which the bank has extended to them.[*]

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[*] There is also the possibility of default on securities, but Government obligations comprise so large a part of banks' portfolios that this area of uncertainty may be neglected.

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4. The degree of "frozen-ness" of the loan portfolio at any moment of the future, where this degree depends upon the ability (and, to a certain extent, willingness) of customers to accept refusal of loan renewals.[**]

[**] The nominal maturity distribution of the bank's loan portfolio may have little to do with the actual degree of "frozen-ness" of its loans. A study by the Federal Reserve Bank of Cleveland indicated that continuous borrowing through renewal of short-term loans was quite widespread. While only six per cent of the loans of banks in that district matured in five years or more, 25 per cent of the total dollar amount of loans had been made by borrowers who had been in debt continuously to the same banks for over five years. Federal Reserve Bank of Cleveland, Monthly Review, September 1956.

While the first element of uncertainty is particularly critical to the bank, the last three areas are clearly not unimportant. If bonds were always marketable at par and loans callable on demand, without possibility of default, the bank could never become illiquid no matter how erratic the behavior of its deposits. The greater the extent to which any or all of these latter three uncertainties

exist, the greater becomes the bank's first concern for the future course of deposits. Thus, no one of these four aspects may be properly neglected in a model of bank operations.

The assets which the bank can hold may be divided into three general categories: cash assets, securities, and loans. Since the problem of diversification within each of these portfolios (that is, what types of securities and loans are held) will not be of concern in this paper, each of these categories will be assumed internally homogeneous. "Cash" assets in fact consist of Federal Reserve Bank reserves, vault cash, net balances with other banks, and bills of very near maturity; here no such distinctions will be made, all "cash" being assumed 1) to provide no earnings and 2) to be completely free of risk of capital value change. The category "securities" will be assumed to include a homogeneous group of securities 1) without default risk, 2) readily saleable upon established markets, 3) with maturity date beyond the end of the bank's present planning horizon, and 4) with a fixed coupon per bond per planning period. The distinction between "cash" and "securities", is clearly one of degree and not of kind. The portfolio of an actual bank will invariably consist of a variety of assets in the range from cash to fairly long-term bonds; in this simplified representation, the choice of the bank is narrowed. "Loans" are assumed 1) to be not callable during the planning period, 2) to be not marketable, and 3) to be "shiftable" only to the extent that they are eligible as collateral for borrowing from the Federal Reserve Banks. Thus, the essential difference between "securities" and "loans" is that there is a market for the former so that securities may be readily converted into cash, although at an uncertain price, while loans can be so converted only through the Federal Reserve Bank. Since these are assumed to be the only assets which the bank can hold,* it must be true that cash plus securities plus

* Non-financial assets comprised less than one per cent of the assets of member banks of the Federal Reserve System in 1956.

loans equals deposits[*] plus total capital accounts; the bank is assumed to have no

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* Deposits are also assumed internally homogeneous, i.e., no distinctions are made between demand and time deposits. The question of the bank's optimal proportion of time to demand deposits is briefly treated in Appendix D.

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liabilities other than deposits, and, of course, total capital accounts (which will be called simply "net worth") cannot be withdrawn from the bank.

What has been called the "planning period" is that span of time upon which the bank concentrates all its attention and over which it sets, and does not plan to alter, its asset portfolio. This is obviously unrealistic, for every bank is always planning and re-planning its asset portfolio. Even if the fact of continually maturing securities—which forces the bank to re-plan by automatically replacing securities with cash assets—is removed, as it is in the model, actual banks would make continual changes in their portfolio plans. Nevertheless, it is equally true that portfolios are not planned with the intention of making frequent changes, and it seems more realistic to assume that the basic portfolio decisions with respect to the fundamental components, cash, securities, and loans, are made fairly seldom and with reference to a sizable span of time. Forcing this flexible procedure into a planning period of fixed length is very simplifying but, it is hoped, not badly distorting.

The choice of this portfolio is assumed to depend entirely upon the circumstances of the ensuing period and in no way upon past events, past portfolio selections, or expectations of events occurring after the end of the ensuing period. To expose the importance of the assumptions of this last sentence some further amplification is required. First, it is assumed that the bank is not influenced by past events except insofar as these affect its estimates of the future. Thus, if the bank has had an unusual proportion of its loans defaulted

in the previous period (the third element of uncertainty), this may induce it
to re-value its estimates of such risk but it does <u>not</u> reduce its loans solely
on the basis of a "once burnt, twice shy" code of behavior. More relevant
to the real world is the connection to the "pin-in" effect. Any such effect is
assumed away in the model; previous declines in the security price level
(as a result of the second element of uncertainty) cannot directly induce the
bank to carry a greater proportion of securities than it otherwise would
desire. Second, past portfolio selections do not influence the current choice
(except, of course, through their influence on the bank's appraisal of the
ensuing period); this requires that there be leeway in the portfolio at the
start of any period. Clearly, cash and securities can be converted into
other assets at any time* so this assumption really applies only to loans --

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* The fact that transaction costs of change give the existing portfolio some
inertia is assumed to be of little importance at the margin.

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if all the bank's borrowers "required" renewals of their loans (the fourth
element of uncertainty) for the ensuing period, the bank would not be able
to reduce its loans, whether it wished to or not. Thus, under this assumption
the model can only consider those banks for which the proportion of "required"
renewals, while perhaps large, is never so large that the bank cannot start
the new period with the exact quantity of loans it wishes to make. Third,
the bank knows, or estimates with complete confidence, all the parameters of
its environment that are relevant to its portfolio choice for the ensuing
period. While the bank recognizes the possibility of change in certain of
these parameters, it does not <u>expect</u>** any change and it knows, or estimates with

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** The words "expect" and "expected" are used throughout in their probability
sense.

complete confidence, all the relevant parameters of the frequency distribution

governing these changes. And fourth, the portfolio of the current period is

not affected by expectations of change in the parametric climate of the next

period. In short, the assumptions about the "period" are those required to keep

the model manageable and static -- "once burned, once shy," unitary elasticity

of expectations (in the Hicksian sense[*]), and limits to the degree of the

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[*] J. R. Hicks, Value and Capital, 2nd ed., Oxford, Oxford Press, 1946, p. 205.
This is not strictly correct since expectations are there assumed single-valued.
In this paper, any expectations that are not single-valued are assumed to have
a probability distribution which, at every moment of time has an arithmetic
mean equal to the then-existent value.

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fourth area of uncertainty.

The first area of uncertainty, that of the future course of the level

of deposits, implies that the bank must be prepared for the possibility

that withdrawals exceed additions to deposits over a particular time-span.

A net reduction of deposits will always occur over the moment of time during

which one depositor makes a withdrawal. Not infrequently, a bank will find

its deposit levels declining over a few days or weeks. And it is not im-

possible that seasonal, cyclical, or secular factors will cause a fall in

deposits over long periods. At the beginning of the planning period, the

bank recognizes that its level of deposits during the period may follow a

myriad of possible paths, of which continuous rises or continuous falls are

but two. In reality, the complete shapes of the possible paths are im-

plicitly considered in the specification of the bank's asset portfolio.

But one aspect of the shape of each of these possible paths is of such great

importance to the bank that it will here be assumed to be the only aspect

considered by the bank -- namely, the lowest point to which deposits fall in

each of the paths. For, it is at this "deposit-low" of the period that the

bank is forced to make the most radical adjustment of its asset portfolio

in order to meet the demand of its depositors.[*]

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[*] Of course, if deposits rise throughout the period, the "deposit-low"
is zero and the bank need make no adjustments as far as meeting withdrawals
is concerned. Thus, the "most radical" adjustment may well be no adjustment
at all.

This assumption of sole concern with "deposit-lows" is not in itself

sufficient to permit complete neglect of the time-shape of deposit changes

since the date of occurrence of any "deposit-low" may still be important

to the bank. In the interest of simplicity, this problem of the time-path

of deposits will be avoided in the following way. At some point toward

the end of the period, deposits will reach their low,[**] at which time the

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[**] If deposits, on the average, should rise during the period, the "deposit-
low" will probably not be much, if at all, below zero and will probably occur
toward the beginning of the period. Since little asset readjustment is
required in this case and since we neglect the net rise in deposits that
follows, the assumption that the "deposit-low" occurs at the end of the period
is innocuous.

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bank makes any asset adjustments required; this perhaps necessitates selling

some of its securities and/or borrowing on the collateral of some of its

securities or loans.

This is not the place for a full discussion of the complex manner

(use of Federal Funds, security sales, Federal Reserve Bank discounts, etc.)

in which banks in fact can and do meet the problem of insufficient reserves

(i.e., insufficient cash assets). Basically, the process may be simplified

into the following stages:

Stage 1. The bank meets net withdrawals from its cash assets as long as it can without drawing these assets down below their minimum required level.

Stage 2. Should the cash assets prove insufficient, the bank sells from its security portfolio, at the going market price, and continues to do so as long as it has securities to sell.[*]

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[*] For simplicity, it is assumed that the banks sell, rather than borrow from the Federal Reserve Banks on the collateral of securities. Given bankers' dislike of debt and the fact that interest charges would probably exceed transaction costs of selling and later repurchasing, sales rather than borrowing would occur in the world postulated by the model, that is, a world of no "pin-in" effects and unitary elasticity of expectations of bond prices.

Stage 3. Should the sale of all its securities also be inadequate to meet the deposit depletions, the bank borrows from the Federal Reserve Bank on the collateral of its outstanding loans.[**] This it continues to do as

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[**] Alternatively, one may think of this process as straightforward redis-counting in the traditional, if in fact little used, manner.

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long as necessary or until its stock of such collateral is exhausted.

To these three stages, a fourth might be added: should all its assets be converted, to the greatest extent possible, into means of payment and still be insufficient to cover deposit withdrawals, Edgeworth's "great disaster" would occur par excellence -- the bank would then be in the throes of a liquidity crisis beyond its ability to handle. At the very least, it would have to call for exceptional aid from the Federal Reserve System; it might be forced to close its doors, and it might find itself insolvent as well. However, it will be assumed that such a "Stage 4" is so costly to the bank that no bank's optimum portfolio permits any possibility of this occurrence.

Such a result may (and, in fact, does, if seldom) occur, but it could only

happen, by the assumptions here, through a misestimation by the bank of the

parameters of its operations.[*]

* It is possible that some bank may be forced to accept the possibility of
such a "Stage 4" if its net worth were low and its lowest possible "deposit-
low" very near zero. Of course, the bank could meet this situation by holding
very large cash assets, but this may be so unprofitable as to induce it to
accept the possibility of "Stage 4." Such a bank is not considered in this
paper.

The method of treatment of the four areas of uncertainty may now be

more accurately specified.

1. The size of deposits at any moment of the future. Although the

bank expects (in the probability sense) deposits to stay at their start-of-

period level, it recognizes that they may fall or rise. The relevant dis-

tribution function is that relating each of the various "deposit-lows" during

the ensuing period to the probability of its occurrence. If the random

variable, u, is defined to be the "deposit-low" as a fraction of initial

deposits, the distribution of u may be defined only over the range, zero

to unity. For simplicity, the frequency distribution of u, $f(u)$, is

assumed to be a linearly increasing function of the amount of which u exceeds

s, where s is the smallest "deposit-low" (as a fraction of initial deposits)

to which the bank assigns a non-zero probability (and clearly, $0 \leq s \leq 1$).[**]

** This "triangular" distribution is assumed because it is believed to be the
best simple approximation to the actual distribution of banks' "deposit-lows."
For a theoretical derivation of the distribution, see Appendix A. Alternatively,
a uniform distribution of "deposit-lows" is considered in Appendix E.

Since the cumulative of $f(u)$ must equal one, specification of s is suf-

ficient to determine:
$$f(u) = \frac{2(u - s)}{(1 - s)^2}$$

2. <u>The market value of its securities at any moment in the future</u>. A "unit" of securities is defined as a dollar's worth at the market prices prevailing at the start of the planning period; this "unit" carries a coupon paying g dollars per "unit" per period where it is assumed, without undue restriction, that $0 < g < 1$. The market price at the end of the period may be written as $(1 + w)$ where w , the change in security prices during the period (absolute and percentage), is assumed to be uniformly distributed over the range, -a to a $(0 < a < 1)$.[*] Since the "deposit-low" occurs toward the

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[*] Unfortunately, fixing the distribution of w so that its mean is zero implies that the mean of the equivalent distribution of interest rates is greater than zero, in contradiction of the assumption of static expectations. In the case of consols the mean of the distribution of changes in the interest rate is:

$$g \left[\frac{a^2}{3} + \frac{a^4}{5} + \frac{a^6}{7} + \ldots \right]$$

which is sufficiently near zero for small values of g and a that this contradiction is not serious. This results from the property of the number system, whereby the average of the reciprocals does <u>not</u> equal the reciprocal of the average.

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end of the period, the price of securities sold at the moment of the "deposit-low" may also be considered to be $(1 + w)$.

3. <u>The proportion of loans defaulted</u>. This aspect of bank uncertainty will be most summarily treated, not because it is felt to be unimportant to a complete theory of bank operations but because its basic effects upon the bank's portfolio can be seen in the present model without complex treatment. It is assumed that the bank charges a pure interest rate of e per dollar of its loans; it then adds to this rate some amount according to the default risk which just suffices to insure that the bank will not lose through defaults in the long run. The final "gross" rate is e' (where

$0 < e < e' < 1$), but we shall here concern ourselves only with the bank's
earnings net of default.[*]

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[*] It must always be remembered, however, that to the extent that the bank
is worried about the time-path of defaults or the default rate is positively
related to the quantity of loans, the present model will overstate the amount
of loans which the bank will desire to make.

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4. The degree of intra-period "frozen-ness" of the loan portfolio.

The fear on the part of the bank that it may not be able in an emergency

to reduce its loans sufficiently, even over several periods, means that

any debt incurred to help meet deposit depletions may well be long-term

debt, a position which bankers dislike.[**] It is because of this that Stage 2,

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[**] Much nebulous writing has appeared on this subject, but the bankers'
aversion seems real enough, probably basically deriving from their fear
that heavy indebtedness will have adverse effects upon their relations
with depositors, borrowers and correspondent banks. The view is not with-
out its dissenters, however; for example, see A. Murad, "The Ineffectiveness
of Monetary Policy," Southern Economic Journal, Vol. 22, No. 3 (Jan. 1956),
pp. 339-351. Of bankers' supposed aversion to steady borrowing, Murad says
(p. 346):

> It may be that bankers feel that way or say that they feel that
> way, but they certainly do not act that way. Whenever they have
> reserve deficiencies they borrow and if necessary remain for
> years in debt to the Federal Reserve banks.

As a general phenomenon, this last sentence is open to great doubt, for the
fact of increasing or large aggregate indebtedness is not proof of a de-
creasing or small antipathy toward permanent indebtedness.

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sales of securities, is assumed to precede Stage 3, borrowing on loan

collateral, in the process of meeting deposit depletions. For the same

reason, the cost of such borrowing in Stage 3 may be interpreted to include

not only the charge of the Federal Reserve Bank but also a subjective "cost"

of being in what may prove long-term debt.

While use of the "discount window" is a privilege and not a right, no Federal Reserve Bank would refuse to extend advances to a bank which found itself unable to cover exceptionally large deposit withdrawals without such aid. The only questions before the bank are, then, how much borrowing could they do on the basis of their total loan portfolio and how much would it cost. If the bank gets into Stage 3, it can take a typical dollar's worth of its loans to the Federal Reserve Bank and receive an advance of $(1 - m)$ dollars, where m , which might be labelled the "excess-collateral rate," is, of course, between zero and unity. On this advance, the borrowing bank is charged an interest cost which, it is here assumed, is different from the real "cost" because of bankers' dislike of such debt. There are many ways in which such a "cost" might be handled, but the one to be assumed in this paper is that the real "cost," q , of such borrowing is to some extent greater than the interest charge, where $0 < q < 1$.[*]

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[*] If there were no addition of a subjective "cost" and the bank knew that it could repay its debt in exactly one period, q would be equal to the Federal Reserve Bank discount rate (per period). To the extent that there is a subjective element or such last-resort borrowing is felt to be of longer duration, q may be well above the discount rate.

Cash assets are of two types, those required to be held as reserves in the Federal Reserve Bank and those which the bank holds (in various forms) in excess of these requirements. The amount of the former at any moment of time must be a specified fraction of the bank's deposit liabilities. While, in fact, a rise in reserve requirements usually results in the lowering of the amount of other cash assets which the bank feels it requires, it is here assumed that the amount of cash assets other than required reserves is also a specified fraction of its current deposit liabilities. Thus, the "required"

amount of cash assets can be written as a fraction, k , of the bank's deposit
liabilities, where k is somewhat larger than the reserve requirement ratio.
Most banks would meet these requirements almost continually, but as the model
is set up, it need only be prepared to meet them at the moment of the "deposit-
low" to be sure of having a sufficient amount at every other moment of the
period. Assuming a fixed fraction of cash assets in this fashion means that
Stage 1 is not possible. But this assumption is not as restrictive as it
might seem at first since the excess of cash assets over k would probably
be very small unless g were near zero and/or a extremely large.[*]

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[*] The real-world analog to the parameter g is the difference between long-
term security rates and the bill rate. Not infrequently, this difference
is very slight, but such times have little relevance here for the assumption
of static expectations concerning future interest rates is then almost cer-
tainly violated.

Each of the balance sheet items will be written as a fraction of start-
of-period deposits -- the fraction of cash assets being k , of securities,
B , of loans, L , and of net worth, N . B and L are variables under
the control of the bank, while N is assumed previously determined and
unalterable at least over the ensuing period.

It remains only to fix the criterion by which the bank balances its
portfolio between possible gains and losses. One often stated by bankers
themselves is that they minimize the probability of losses (or, in reverse,
maximize the probability of some gain); but this implies that the portfolio be
prepared to meet any possible deposit reduction out of cash assets, while
in fact banks do incur an unnecessary, if small and profitable, risk of
losses. A variant of the above is the minimization of the probability of
incurring losses within the constraint of a reasonable expected profit. Such

a criterion is rejected on the grounds that setting the definition of

"reasonable" is more important than the minimization process that follows.

A criterion advanced by recent portfolio theory[*] is that some point is

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* Cf. H. Markowitz, Portfolio Selection (New York: 1959) , Part IV;
J. Tobin, "Liquidity Preference as Behavior Towards Risk," Review of Eco-
nomic Studies, Vol. 25, No. 2 (Feb. 1958), pp. 65-86; and I. O. Scott,
"The Availability Doctrine: Theoretical Underpinnings," Review of Economic Studies,
Vol. 25, No. 1 (Oct. 1957), pp. 41-48.

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chosen, according to the selector's preferences, on the frontier (or locus)

of the maximum expected return for every possible variance of return.

This criterion was introduced because it was useful in explaining diversi-

fication; in the present model, the variable, variance of return, is not

needed to explain diversification and so, for simplicity, will not be in-

cluded in the text.[**] Here, the bank is assumed to choose its asset portfolio

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** The implications of a bank preference map which is a function of both
expected gain and variance of gain are drawn, for a simplified version of
the model, in Appendix B. Also, some inference about the change in results
which this would cause is made there.

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so as to maximize its expected additions to net worth during the period.

Thus, it maximizes its expected additions to net worth function with respect

to one of the two asset variables, B and L , the other being then determined

by the accounting identity:

(1) $1 + N = k + B + L$

In summary, the symbols to be used in the paper are:

Variables:

   1.   B , securities as a fraction of initial deposits.

2. L , loans as a fraction of initial deposits.

Random elements:

3. u , the "deposit-low" of the period, as a fraction of initial deposits. u is defined over the range, $0 < s < u < 1$, by the distribution,

$$f(u) = \frac{2(u - s)}{(1 - s)^2} \quad .$$

4. w , the change between the start of the period and the "deposit-low" (and the end) of the period in the market price of securities. w is defined over the range, $-a < w < a (0 < a < 1)$, by a uniform distribution,

$$f(w) = \frac{1}{2a} \quad .$$

Parameters:

5. N , net worth, unchanging, as a fraction of initial deposits.

6. g , the coupon per dollar's worth of securities (at initial market prices); $0 < g < 1$.

7. e , the earning rate on loans (net of default risk); $0 < e < 1$.

8. k , the amount of cash assets which the bank holds, as a fraction of current deposit liabilities; $0 < k < 1$.

9. q , the "cost", both actual and subjective, of borrowing a dollar during the period from the Federal Reserve Bank; $0 < q < 1$.

10. m , the "excess-collateral rate." A dollar of loans as collateral enables the bank to borrow $(1 - m)$ dollars from the Federal Reserve Bank (the latter acting in its capacity of "lender of last resort"); $0 < m < 1$.

III. Structure of the Model.

The amount of profit which the bank makes during the period will clearly depend upon the "stage" into which the "deposit-low" forces it, and, for Stage 3, upon how far into that stage it goes. The expected

addition to net worth (ΔN) for each stage is:

Stage 2. (Securities sales are required to handle the "deposit-low".)

$$(2) \qquad \Delta N = gB + wB + eL$$

The profit is composed of: gB, earnings on securities; wB, capital gains or losses on securities; and eL, earnings (net) on loans.

Stage 3. (Borrowing on loan collateral is required.)

$$(3) \qquad \Delta N = gB + wB + eL - q(1 - m)x$$

where x is the amount of loans put up as collateral (and $(1 - m)x$ the amount borrowed) and

$$(4) \qquad (1 - m)x = (1 - k)(1 - u) - (1 + w)B$$

The profit is composed of: gB, securities earnings; wB, capital gains or losses on the (completely) sold securities; eL, earnings on loans; and $q(1 - m)x$, the cost of the bank's borrowings.

According to the values assumed by the random variables, u and w, the bank finds itself in one of these two stages. The ranges of u and w which bring about each stage are:

Stage 2. This may occur in either of two ways:

i) $s < u < 1$; $-1 + \dfrac{(1-s)(1-k)}{B} < w < a$. It may be possible that the most extreme (conceivable) deposit depletion can be met through bond sales alone, provided that the price of securities rises sufficiently (or falls sufficiently little) during the period.

ii) $1 - \dfrac{(1+w)B}{1 - k} < u < 1$; $-a < w < -1 + \dfrac{(1-s)(1-k)}{B}$. If securities prices do not rise enough, only a certain degree of possible

deposit withdrawals can be handled by means of bond sales alone.

Stage 3. $s < u < 1 - \frac{(1 + w)B}{1 - k}$; $-a < w < -1 + \frac{(1 - s)(1 - k)}{B}$. Any

deposit depletion too extreme to be met by securities sales alone <u>can</u> be met

by bond sales <u>plus</u> borrowings on loan collateral.

These three cases are exhaustive since we have already excluded the

possibility of a Stage 1, the running down of cash assets to meet withdrawals,

and of a Stage 4, where even borrowings on all the bank's loan collateral are

insufficient to cope with the deposit losses. But the fact that the three

possibilities listed above cover the entirety of the ranges of w and u

does not automatically imply that every stage is relevant for every bank --

for example, the net worth of a bank may be so high that it is able to cover

any conceivable deposit depletion by bond sales alone, even if the price of

bonds drops to their lowest conceivable level. In technical terms, for a stage

to be possible of occurrence, the lower limits of both u and w (for that

stage) must indeed be lower than the upper limits. If all three of the

cases cited above -- Stage 2(i), Stage 2(ii), and Stage 3 -- are considered

relevant to the bank, then the following assumption about the size of the

bank's securities holdings is being made implicitly:

$$(5) \qquad \frac{(1 - s)(1 - k)}{1 + a} < B < \frac{(1 - s)(1 - k)}{1 - a}$$

for if the left-hand inequality does not hold, Stage 2(i) does not exist;

and if the right-hand inequality does not hold, Stage 2(ii) and Stage 3 do

not exist.

The bank for which Stage 3 is not even a remote possibility is not only

rare but uninteresting, so there should be few qualms about assuming the

right-hand inequality. But the left-hand inequality is not so easily handled;

a taxonomic approach would construct the model for both directions of the inequality sign, but here only that one which is felt most closely to describe reality will be extensively treated.[*] In a world where banks

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[*] Enough has been worked through for the other case to indicate that the results are similar. See Appendix C for a brief treatment of the bank which is able, for some conceivable level of bond prices, to meet the worst conceivable "deposit-low" without recourse to borrowing.

---

do not expect extremely large bond price fluctuations and do make a significant amount of loans relative to their net worth and lowest possible "deposit-lows," most banks are probably not able to cover their lowest conceivable "deposit-lows" by means of securities sales alone, even if bond prices rise to as high a level as is considered possible. Thus, in what follows, it is assumed that Stage 2(i) is not a possibility, or, in other words, that inequalities (5) may be replaced by:

$$(6) \qquad 0 < B < \frac{(1 - s)(1 - k)}{1 + a}$$

IV. Determination of the Optimum Portfolio.

The stage in which the bank finds itself is determined, as follows, by the value assumed by the random variable, u :

$$\text{Stage 2(ii).} \qquad 1 - \frac{(1 + w)B}{1 - k} < u < 1 ; \quad -a < w < a .$$

$$\text{Stage 3.} \qquad s < u < 1 - \frac{(1 + w)B}{1 - k} ; \quad -a < w < a .$$

The expression for the expected addition to net worth, $E(\Delta N)$, is:

$$(7) \qquad E(\Delta N) = \int_{w=-a}^{a} \int_{u=s}^{1} [gB + wB + eL] \left| \frac{u - s}{a(1-s)^2} \right| du\ dw$$

$$+ \int_{w=-a}^{a} \int_{u=s}^{1 - \frac{(1+w)B}{1 - k}} [-q(1-k) + q(1+w)B + q(1-k)u] \left| \frac{u - s}{a(1-s)^2} \right| du\ dw$$

which becomes, after integration,

$$(8) \qquad E(\Delta N) = gB + eL - \frac{q}{3(1-s)^2(1-k)^2} [(1-s)(1-k)-B]^3 - \frac{a^2 q B^2}{3(1-s)^2(1-k)^2} [(1-s)(1-k)-B]$$

L may be eliminated as a variable by means of the accounting identity (1);

then the derivative of $E(\Delta N)$ with respect to B is:

$$(9) \qquad dE(\Delta N)/dB = g - e + \frac{q}{(1-s)^2(1-k)^2} [(1-s)(1-k) - B]^2$$

$$- \frac{a^2 q B}{3(1-s)^2(1-k)^2} [2(1 - s)(1 - k) - 3B]$$

and the second derivative:

$$(10) \qquad d^2 E(\Delta N)/dB^2 = \frac{2q}{(1-s^2(1-k)^2} [-(1-s)(1-k)(1 + \frac{a^2}{3}) + (1+a^2)B]$$

If B is at its lowest permissible level (i.e., zero)[*] the first derivative,

---

* See inequalities (6).

$dE(\Delta N)/dB$, is positive, and, if B is at its highest value (i.e., $\frac{(1-s)(1-k)}{1 + a}$),

the derivative is negative, provided that the following inequalities hold:[**]

** See Appendix C. If the left-hand inequality does not hold, the present model
is inapplicable and that treated in Appendix C becomes the relevant one. Unless
very high values of a are considered, however, the expression on the left will be
very close to zero. If the right-hand inequality does not hold, there is a corner
maximum, with no securities entering into the portfolio.

(11)
$$\frac{2a^2(2-a)}{3(1+a)^2} < \frac{e-g}{q} < 1$$

If inequalities (11) hold, there exists a regular maximum and the optimum fraction of assets in securities can be found by setting the derivative (9) equal to zero. Solving this quadratic equation in B yields:

(12)
$$B = \frac{(1-s)(1-k)(1+\frac{a^2}{3})}{1+a^2} \left\{ 1 - \sqrt{1 - \frac{(1+a^2)(1-\frac{e-g}{q})}{(1+a^2/3)^2}} \right\}$$

Much of the complexity (or rather simplicity!) of the form of equation (12) results from the particular functional representation of the distribution of the random variables, u and w ; but there are three interesting properties of (12) which are not dependent upon the choice of distribution functions:

1. The fact that (1-s)(1-k) enters in linear fashion. In words, this quantity is the fraction which does not need to be held in cash assets (1-k) of the largest conceivable loss in deposits (1-s). The rest of equation (12), involving the parameters, a , e , g , and q , serves to fix the quantity of securities as a fraction of this term, (1-s)(1-k). Thus, the determination of the optimum portfolio can be divided into two problems, first, the cal-culation of the maximum amount of securities which the bank would ever need to sell to meet deposit losses (on the assumption that bond prices do not change), and second, the decision as to what fraction of this amount the bank will actually hold (which will depend upon the various earning and borrowing cost rates as well as the expected fluctuation in security values).

2. The manner in which the earning and borrowing cost rates enter the equation. By means of the single expression, (e-g)/q , the bank measures the relative advantage of loans vis-a-vis securities, the advantage being

greater the larger is the difference in earning rates and/or the lower is the cost of borrowing on loan collateral. It is interesting to note that, if the bank is to include any securities at all in its optimal portfolio, it is not necessary that the borrowing cost be greater than the loan earning rate (i.e., a "penalty" rate is not essential), but only that it be larger than the difference between the earning rate on loans and that on securities.

3. The fact that N , the net worth of the bank, plays no part in the determination of the optimum quantity of securities. This independence between optimum B and N implies that any change in N induces a change in the bank's loans of the same amount and direction. This conclusion is not surprising if one recognizes that net worth, from the viewpoint of the bank's liquidity problems, can be treated as a deposit liability with no possibility of withdrawal.

What is really interesting about equation (12) is not the level of B , and hence of L , but rather the way in which the optimum holdings of these assets vary as a result of changes in the different parameters. The first and second partial derivatives of B with respect to the various parameters are given in the table below.

$$\partial B/\partial x \qquad \partial^2 B/\partial x \partial y \quad \text{where}$$

	$\partial B/\partial x$	$\partial^2 B/\partial x \partial y$ where					
		$y = a$	$y = q$	$y = g$	$y = e$	$y = k$	$y = s$
where $x = s$	-	$\pm^1$	-	-	+	+	0
$x = k$	-	$\pm^1$	-	-	+	0	
$x = e$	-	$\pm^2$	+	-	+		
$x = g$	+	$\mp^2$	-	+			
$x = q$	+	$\mp^2$	-				
$x = a$	$\mp^1$	$\mp^3$					

1. upper or lower sign holds according as: $\dfrac{e - g}{q} \gtrless \dfrac{1}{9}$

2. upper or lower sign holds according as: $\dfrac{e - g}{q} \gtrless \dfrac{3 - 2a^2}{9}$

3. evaluated at $a = 0$; upper or lower sign holds according as:
$$\frac{e - g}{q} \gtrless \frac{1}{9}$$

(see Appendix E)

A similar table of partial derivatives could be constructed for the changes in optimum holdings of loans, but this is not necessary; since, by accounting identity (1), $L = 1 + N - k - B$, the first and second partial of L with respect to any parameter (except k) is simply the negative of the relevant partial of B . It can be shown that an increase in k decreases L as well as B , and that the second derivative $(\partial^2 L/\partial k^2)$ is zero.

One could find quantitative estimates of these derivatives by assuming particular parameter values, but more generally we can plot the value assumed

by $\dfrac{B}{(1-s)(1-k)}$ (written hereafter as B') for all possible values of a and

of the composite parameter, (e - g)/q. This is done, for various fixed levels

of (e - g)/q, in Figure 1. The dotted line is the border above and to the

right of which Stage 2(i) becomes a possibility.[*]

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[*] Although that region is neglected in Figure 1, numerical examples based
on Appendix C indicate that the curves could be smoothly extrapolated into
the Stage 2(i) area without much, if any, error.

Figure 1
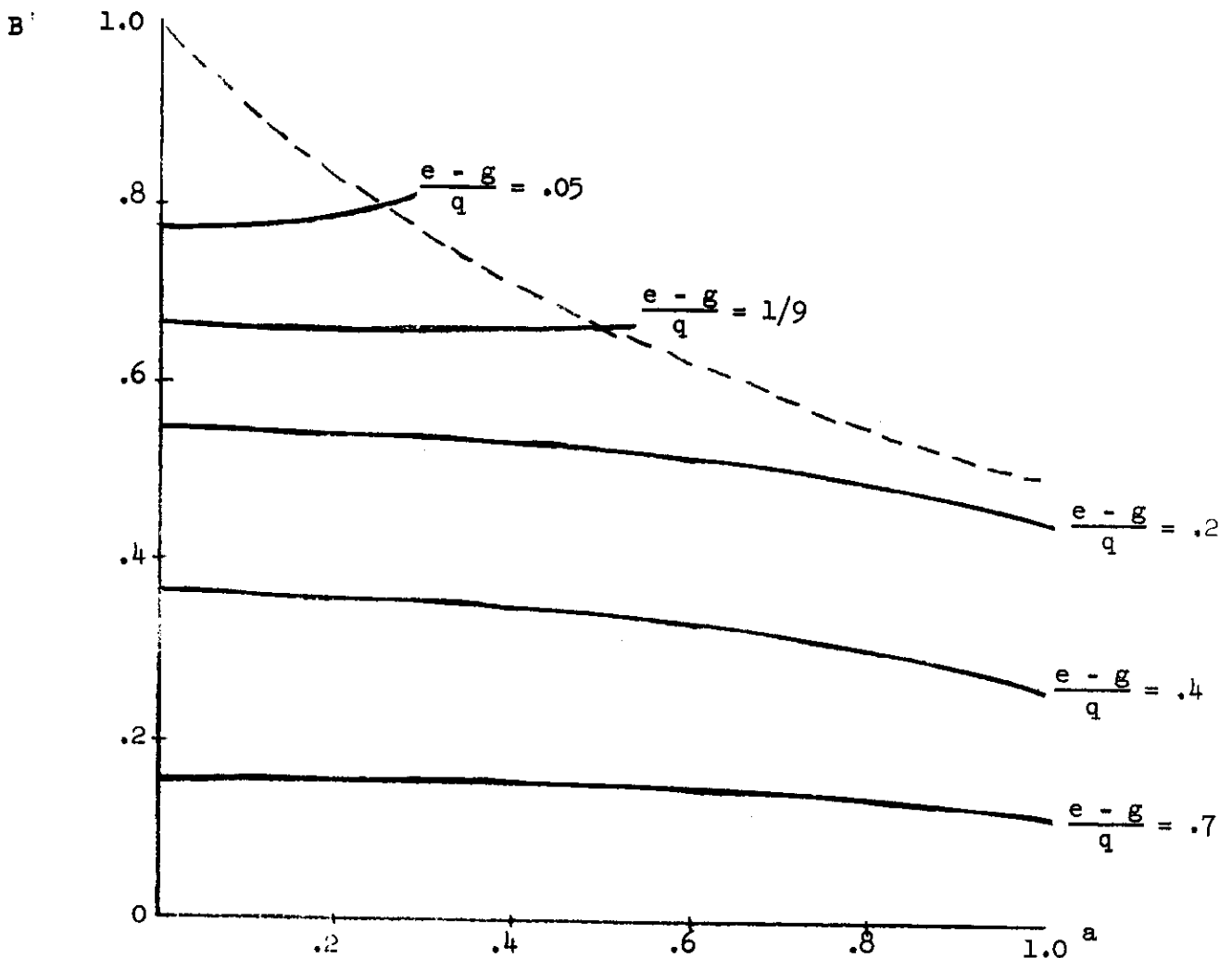
The most obvious lesson of Figure 1 is that the effect of changes in the anticipated fluctuation in security prices is uncertain, with respect to both direction and magnitude. When $(e - g)/q$ is in the neighborhood of 1/9 or of unity (i.e., when B' is in the neighborhood of .67 or zero), changes in the parameter, a , have almost no effect upon the composition of the optimum portfolio. The farther $(e - g)/q$ is from these critical values, the greater will be the effect on the portfolio of a . If $(e - g)/q$ is less than 1/9, greater certainty about the course of future bond prices will induce the bank to hold less securities; while if $(e - g)/q$ is between 1/9 and unity, greater bond-price certainty will induce larger holdings of securities. This unexpected result lends some, necessarily very qualified, support to the policy of maintainence of stable Government security markets for, in a recession when $(e - g)/q$ is probably very low, it will induce banks to make more loans and, in a boom when $(e - g)/q$ is probably high, it will induce them to restrict expansion of their loan portfolio.

Lest Figure 1 give the impression that the value of a is a critical determinant of the portfolio composition, another diagram, Figure 2, is included which relates B' to $(e - g)/q$ for two very different values of a. The solid line shows the relation at a = 0 and the dotted line at $a = \frac{1}{2}$.[*]

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[*] The dotted line is not continued to the point where $(e - g)/q$ is zero because for the very low values, Stage 2(i) becomes possible. For some values of a , as Figure 1 shows, the dotted line will cross the solid one for low values of $(e - g)/q$.

Figure 2
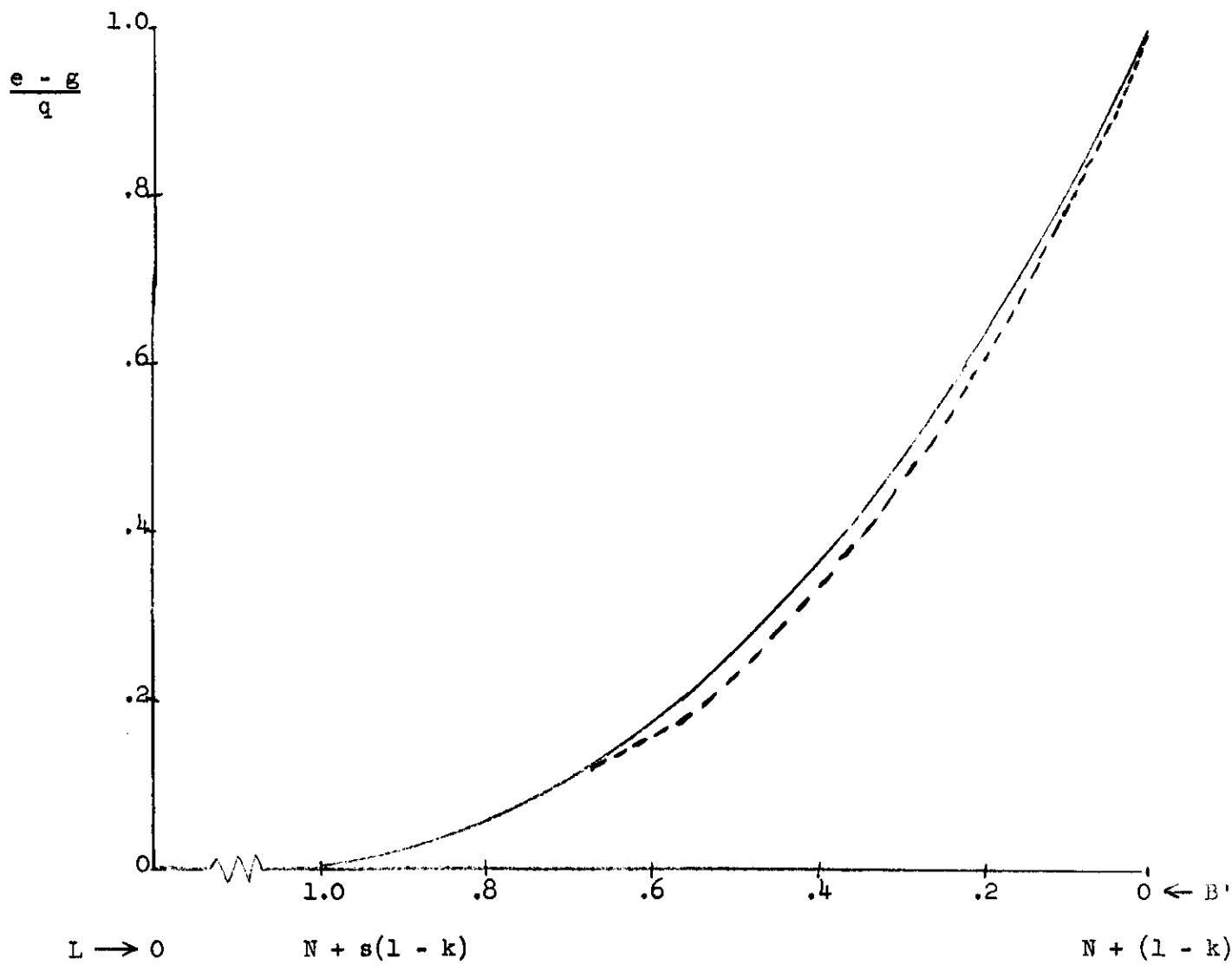
It will be seen from Figure 2 that a has no more than a very marginal effect
upon the portfolio,* and that the important parameters for the division of the

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* A rise in a from 0 to .50 never decreases the optimum security holdings
by more than .022(1 - s)(1 - k) (i.e., by more than 1/40 of deposits even if
both s and k are zero).

---

portfolio between loans and securities are, not surprisingly, the difference
in earning rate between loans and securities, (e - g), and the cost of bor-
rowing, q . It is the influence of these latter parameters (as well, of course,
as s and k) with which the rest of this paper is concerned; considerable
simplification will henceforth be achieved by the assumption that a is zero.
The first step will be to drop the unrealistic assumption that e is a constant,
unaffected by the quantity of loans which the bank makes.

V.   Imperfect Competition in the Loan Market

In order to conceive of the bank as, to some degree, a monopolist in
its loan market, the meaning of the loan demand curve must be analyzed. As
long as banks are a homogeneous group, each of which have available the same
information concerning the credit-worthiness of every potential borrower, the
rate of interest charged a customer for a loan is simply the going market rate
on riskless lending (the "pure" or "prime" rate, e) plus a certain risk
premium.** It would be a matter of indifference to both borrowers and banks to

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** This "certain risk premium" is here, it will be recalled, such as to insure
the bank against default losses in the long run.

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which bank a particular borrower went; each bank would get no business if
it charged more than the going rate and more business than it could handle
if less.

The actual banking mechanism differs, fundamentally, in two ways from this hypothetical competitive system.* First, banks do not all have the same

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* It differs as well in a third way, in that two banks (or the same bank at two different moments of time) may differ in their attitudes toward making risky loans. Certainly the assumption that all banks merely mark up the pure rate, e , so as to avoid default losses in the long run is no better than a very crude first approximation; but it is sufficient for present purposes.

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knowledge concerning the credit-worthiness of a potential borrower, and as a result, different banks do not add onto the pure rate the same risk premium for the same borrower. Other things being equal, the typical businessman is able to borrow at a lower gross rate in his own locale than elsewhere and at a still lower rate at his customary bank than at a new one. The more strange the borrower, the less sure is the bank of his ability and reliability,**

** And the more expensive it is to ascertain. If the bank has been dealing with the borrower for a long time, it does not need to incur the costs of careful credit investigation.

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and hence the higher will be the risk premium that is added to the prime rate. Second, the potential borrower knows all this and therefore tends not to shop around each time he seeks a loan -- he will accept the rate quoted by his traditional bank unless he is convinced that it is far out of line with the market situation.

Consequently, the bank is not faced with a horizontal demand curve for loans (in terms of the net rate) but has two degrees of freedom concerning the rate it charges. It can demand a rate higher than the prime rate plus its proper estimate of the risk premium and not lose all its customers because even this gross rate will be lower than many of its borrowers could get

elsewhere. Moreover, even those of its borrowers who could do better by taking a higher risk premium but a lower gross rate at another bank are not likely to realize this immediately.[*]

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[*] And having realized it, many may not wish and/or be able to take advantage of it immediately.

Just as the bank does not lose all its loan business by raising its pure rate above the going market rate, so also does it not gain an infinite amount of new loan demand by undercutting the going rate. For it would need to undervalue its risk premiums and overcome other bank-borrower inertias in order to gain the new business.

All the above assumes that the same pure rate must be charged to all borrowers. To the extent to which the bank can discriminate between borrowers,[**]

[**] The probability that this occurs is augmented by the fact that the gross rate will differ between borrowers anyway, and differential risk premiums help to disguise the existence of differential pure rates as well. Furthermore, the fraction (if any) of the loan which the bank insists (or strongly suggests) be retained in the borrower's deposit may vary among borrowers - and this practice is essentially nothing more than a rate increase.

---

it improves its situation (at least until its customers find out). At the extreme of perfect discrimination, the demand curve for the bank's loans in terms of the pure rate represents not its average but its marginal earning rate (net of default) per dollar of loans. While the following argument applies, with the requisite adjustments, equally well for this case, it will not be explicitly treated.

The marginal earning rate, or marginal revenus, of loans (net of default), e, should in the general case be written as a function of the amount of loans which the bank makes; however, since the fraction of initial deposits which the bank lends (L) is a linear transformation of the dollar amount of loans it

makes, e may equally well be considered a function of L . It is far beyond the scope of this paper to predicate the details of this functional relationship; the only property it is safe to assume is that e declines as L increases.

One possible procedure from this point would be to hypothesize a specific form for the function; for example, that e is linear in L :

(13) $e = h - jL$

where h and j are both positive; the <u>average</u> earning rate on loans would then be $(h - \frac{1}{2} jL)$. The technique of Section 4 can again be used to derive an explicit expression for the fraction of initial deposits which the bank optimally carries in securities. The monopolistic analog of equation (12), written for simplicity on the assumption that a = 0 , is:

$$(14) \quad B = \frac{j}{q} (1 - s)^2 (1 - k)^2 + (1 - s)(1 - k) \left\{ 1 - \frac{1}{\sqrt{\frac{h - g}{q} + \frac{j}{q} [(1 - s)(1 - k) - 2(1 + N - k)] + \frac{j^2}{q^2} (1 - s)^2 (1 - k)^2}} \right\}$$

which reduces to equation (12) with a = 0 whenever j = 0 and e = h . It is interesting to note three important differences between equations (12) and (14): first, the term (1 - s)(1 - k) no longer enters as a mere proportioning factor in the determination of optimum B ; second, j , slope of the marginal revenue from loans function, is an essential element in the determination of B ; and third, N now has an effect upon the size of B as well as L .

One could now proceed, as in the previous section, to derive the various properties of the derivatives of equation (14), but this will not be done

partly because of the mathematical complication, partly because the results would have validity only for the special case of a linear demand for loans function,[*] but mostly because it is unnecessary. Traditional economic

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[*] If e were made a function of powers of L higher than the first, the maximization equation would involve third (or higher) powers of B , and hence B could not be written as an explicit function of the parameters. Implicit differentiation would, of course, still be possible.

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theory suggests that, if we know the marginal revenue from loans function, we can deduce the relevant implications of the market if only we can discover the marginal cost function. And such a function we have already found - implicitly - in the derivation of equation (12).

VI. The Marginal Cost of Loans

Equation (12) is the expected profit maximizing relation between the optimum quantity of bonds and the parameters of the model. By means of the accounting identity (1), the following equation for the optimum amount of loans may also be found:[**]

[**] Throughout this section, too, the parameter, a , is assumed zero for simplicity.

$$(15) \qquad L = 1 + N - k - (1 - s)(1 - k) \left[1 - \sqrt{\frac{e - g}{q}} \right]$$

So long as all the parameters on the right-side of equation (15) are considered unalterable constants, L is simply a function of these parameters, and there is no need to go further. But in Section 4, the possibility was introduced

that the marginal earning rate of loans depends upon the quantity of loans made.
When this is so, it is better to view equation (15) as a relationship between
two variables, the optimum amount of loans, L, and the marginal loan earning
rate, e .* Then equation (15) may be put into a form more readily identifiable

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* The e which enters in equations (12) and (15) is the marginal and not
the average earning rate on loans. This can be seen by looking at equations
(8) and (9): the e appears as a result of the differentiation of the term
(eL), and this derivative of the total revenue from loans is the marginal
revenue from loans.

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as a marginal cost function merely by algebraic manipulation of e to the
left-side of the equation:

$$(16) \qquad e = q \left[\frac{L - N - s(1 - k)}{(1 - s)(1 - k)} \right]^2 + g$$

In effect, equation (16) is the bank's "marginal cost of loans" function
in that it shows what the marginal revenue (or marginal earning rate) of
loans must be if the bank is to make any given amount of loans.

The range of L over which equation (16) is relevant is, however,
limited. First, L must be less than (or equal to) the total of the non-
cash asset portfolio, (1 + N - k), since B has then taken its smallest
possible values, zero. Once L has attained this limit, the marginal cost
of loans, e , will equal (g + q) , and no further increases in e can
induce the bank to augment its loan portfolio. Hence, "capacity" limitations
imply that the marginal cost of loans curve becomes vertical (i.e., perfectly
inelastic with respect to e) at a value of L equal to (1 + N - k) . The
second limitation is less obvious. Equation (12) and hence equation (16),
has been derived on the assumption that there is a possibility of Stage 3

(where borrowing on the collateral of loans is required if the bank is to meet the withdrawals of its worst possible "deposit-low"). But this assumption is violated if L becomes less than $[N + s(1 - k)]$. Thus equation (16) is the relevant marginal cost function only over the range,

$$[N + s(1 - k)] < L < [1 + N - k] .$$

No elaborate theory is needed to understand the bank's actions in the range, $0 < L < [N + s(1 - k)]$, for then the bank faces no potential liquidity problems - sales of securities will always suffice to cover any conceivable amount of withdrawals. The bank's only concern is its earnings, and it will maximize these by making loans as long as the marginal earnings rate on loans exceeds the earning rate on securities; once e equals g , the bank will expand loans no further, holding the remainder of its non-cash asset portfolio in securities. The marginal cost of loans function is, therefore, horizontal (i.e., infinitely elastic with respect to e) at the level of g over the range of L less than $[N + s(1 - k)]$.

This marginal cost of loans curve is shown in Figure 3, where $L_0 = [N + s(1 - k)]$, $L_1 = [1 + N - k]$, and $e* = g + q$. The function has

Figure 3
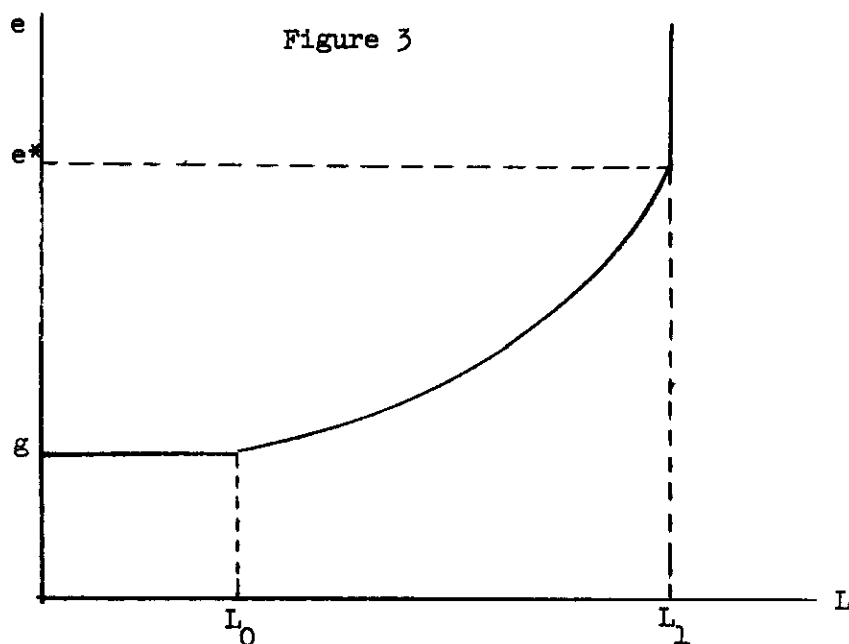
three principal regions: one of infinite elasticity for low values of L ;
one of zero elasticity once all possible loans have been made;   and one of
intermediate values of L where the elasticity declines continuously from
infinity to zero as L increases. It is true that the resemblance of this
to the traditional manufacturing cost curve is slight - both are equated to
marginal revenue to determine the optimum "output" - but this follows from
the fact that manufacturing cost curves are only indirectly aligned to
opportunity costs. As long as L is less than $[N + s(1 - k)]$, the bank's
loan costs are simply the opportunity costs of an alternative "output,"
i.e., holding securities. If L is large enough to make Stage 3 possible,
to these opportunity costs is added an illiquidity-incurring cost of in-
creasingly greater size as L rises.

Throughout this paper, we have neglected the possibility of a Stage 4
(where sale of all securities and borrowing on the collateral of all loans
are insufficient to cover the withdrawals of the worst conceivable "deposit-
low"). This neglect was justified on the grounds that no bank would ever choose
a portfolio that permitted any possibility of so fearful an occurrence. But
it is possible that Figure 3, as drawn, violates this assumption. Stage 4
emerges as a possibility if L is greater than $[N + s(1 - k)] / m$, where
m is the "excess-collateral rate" required by the Federal Reserve Bank for
the bank's borrowing on the collateral of its loans. For very small values
of m , it is clear that this consideration will be irrelevant,[*] but there are

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[*] In the extreme case, where m equals zero, Stage 4 can never occur, even
if N and/or s are zero.

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also values of m large enough to induce the Stage 4-avoiding bank to cease

its loan expansion, regardless of the marginal earning rate, before loans comprise the entire non-cash asset portfolio. This Stage 4 constraint becomes potentially operative if

$$(17) \qquad m > \frac{N + s(1 - k)}{1 + N - k}.$$

That bankers talk of being "loaned-up" while their portfolios still carry some securities is perhaps partial evidence that some such Stage 4 restriction generally does occur.[*] If so, the marginal cost curve becomes vertical not

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[*] However, as long as m is not one, there are always some values of N and s (large) that make Stage 4 impossible.

at L_1 but at L_2 (where loans equal $[N + s(1 - k)]/m$);[**] this is shown in

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[**] If there is uncertainty about the value of m (i.e., the bank is unsure to what extent the Federal Reserve System will support it in a liquidity crisis), the "loaned-up" limit will probably occur at a value of L less than that calculated by using the expected value of M . How much less we are not equipped to say, on the basis of our too simple assumption that the bank never incurs any possibility of Stage 4. Scott suggests the possibility that each bank's "loaned-up" limit of L is based upon other banks', on the grounds that it is certain that the Federal Reserve System will not permit a general liquidity crisis to occur under any conditions. See p. 219 (especially footnote 15) of I. O. Scott, "The Changing Significance of Treasury Obligations in Commercial Bank Portfolios," Journal of Finance, Vol. XII, No. 2 (May 1957), pp. 213-222.

Figure 4, where the dotted curve indicates the part of the Figure 3 marginal cost curve which becomes irrelevant as a result of Stage 4 considerations.

Figure 4
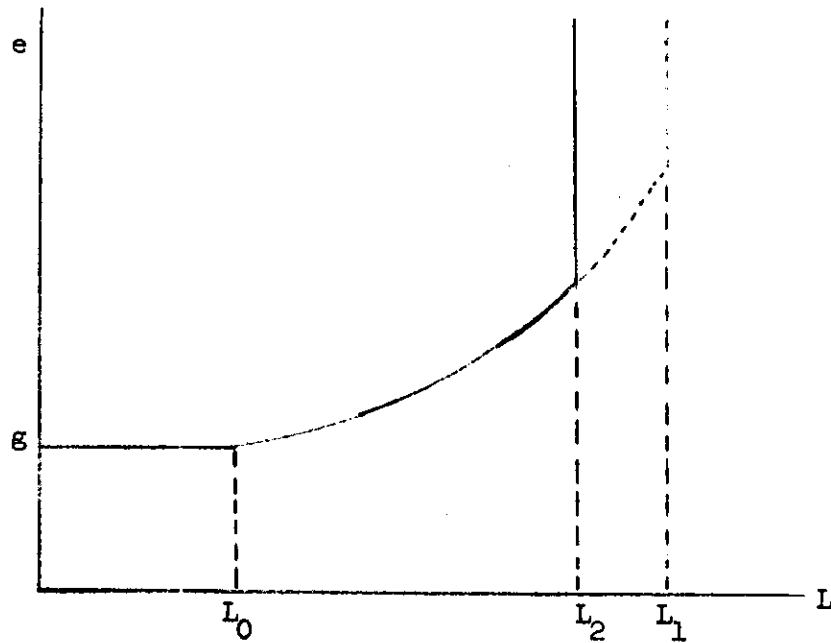
Perhaps more important than the shape of the bank's marginal cost of

loans function is the way in which this function changes as a result of

changes in the different parameters. Given the demand curve for the bank's

loans, any factor that causes the marginal cost function to rise (and/or

shift to the left) will tend·to bring about a reduction in the amount of

loans which the bank wishes to make (and vice versa). The changes in the

marginal cost function, for the four parameters, s , k , q , and g ,

are shown in Figures 5 - 8 respectively; the solid line is the marginal

cost for a lower value of the parameter and the dotted one for a higher,

and L_0' and L_2' represent the values of L_0 and L_2 pertinent to the
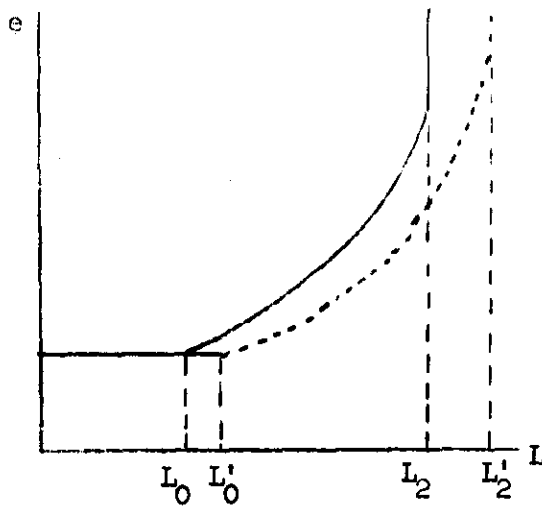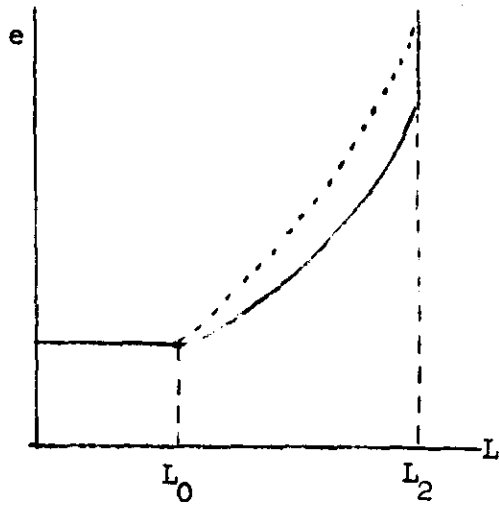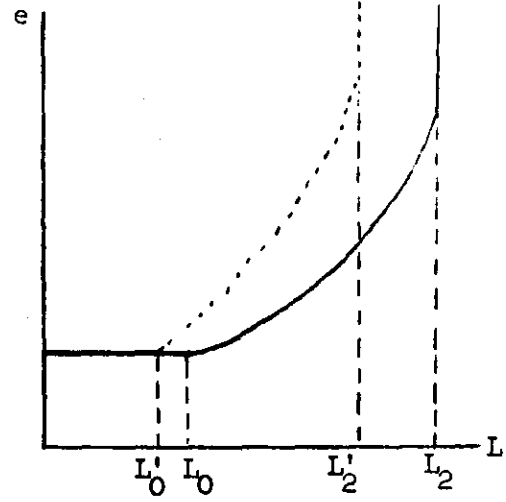
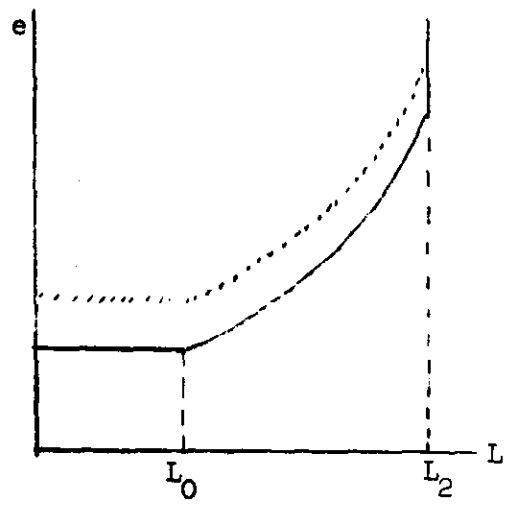higher parameter value.

Figure 5

Figure 6

Figure 7

Figure 8

A fall in s and a rise in k , q , and g all have the same general
effect of raising the marginal cost function, but the details of these
shifts differ. Only s and k are capable of affecting the "loaned-up"
limit (imposed by Stage 4 possibilities); and only g can affect the curve
in its perfectly elastic range, although changes in s and k can alter
the point at which the elasticity becomes finite. Although Figures 5-8
are illustrative and not empirically derived, they help to show how Federal
Reserve System actions may achieve an impact upon bank portfolios; the
traditional central bank policy weapons of reserve requirements, discount
rate, and open-market operations are seen to operate primarily, in the world
of the model, through the parameters, k , q , and g , respectively. And,
if these (or any other) central bank measures should alter the bank's un-
certainty about its deposit future, then there is an effect through s .

To derive implications for monetary policy directly from the bank's
marginal cost of loans function is a great temptation, but a dangerous one.
For the model here presented is no more than a theory of the "firm;" no
theoretical structure has been developed about the adjustment mechanism of the
banking "industry" nor of the other sectors of the economy, from whence
comes the banks' demand for loans. Knowledge about macro-economic behavior
requires, ultimately, macro-economic analysis. The hypotheses about the
individual bank's loan behavior here developed are only of value, from the
viewpoint of aggregative analysis, if they help to place it upon a more
firm micro-economic foundation.

VII. Broader Implications.

Not too long ago, it would have been considered presumptuous to claim
that knowledge of the bank's portfolio of earning assets could be useful to

analysis of monetary theory or policy. Once economists had become convinced

that commercial banks really could "create" money,[*] they became enamored

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* And this was not so long ago as we would like to think. As late as 1921,
Professor Cannan wrote: "If cloak-room attendants managed to lend out three-
quarters of the bags entrusted to them ... we should certainly not accuse
the cloak-room attendants of having 'created' the number of bags indicated
by the excess of bags on deposit over bags in the cloak-rooms." Page 31 of
"The Meaning of Bank Deposits," _Economica_, No. 1, January 1921, pp. 28-36.

---

with the fact that the amount of money thus created was limited by the

quantity of currency and reserves which the central bank issued. To give

precision to the formula relating currency and reserves to the money supply,

all that was needed was knowledge of the public's and the banks' propensities

to hold currency and the circumstances in which the banks keep excess reserves.

Behind such total concern for the money supply always lies the assumption,

explicit or implicit, that the velocity of money (or, in more acceptable

modern terminology, the relationship of aggregate demand to the money supply)

was constant, or at least fairly predictable.

In a world where money was used primarily for transactions purposes,

and where only a small and relatively unchanging fraction of the total was

used as a way of holding wealth, the quantity theory, at least in its more

sophisticated presentations, would be a good approximation of reality. And

neglect of bank portfolios, beyond the problem of changes in currency and

reserves holdings (either required or desired) would be thoroughly consistent

with the theory. What the bank's earning assets were technically labelled -

securities, advances, call loans, etc. - might, and obviously would, matter

to the banker, but to the monetary theorist they would be just different

ways of placing active money balances in the hands of the public.[*] Should

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[*] Rather curiously, through most of the nineteenth century, a great many
bankers and economists did believe that certain classes of bank assets were
inherently less inflationary than others. But the "real bills" doctrine
was founded not on a belief that the money thereby created was any less
"active," but on a mistaken notion concerning aggregate supply.

---

the bank decide to somewhat alter its portfolio from securities to loans,

the final result would be merely to reduce the spending potential of those

individuals who increased their holdings of securities while increasing,

to the same extent, the spending potential of the recipients of the new

loans.

Recognition that "idle" balances are neither an insignificant nor an

unchanging fraction of the total money supply and the fact that contemporary

governments provide a wide range of default-free forms of wealth-holding, from

currency to consols (varying each from the other only slightly, in liquidity

and yield), forces upon monetary theorists an entirely different mode of

analysis. While the theory of the determination of the velocity of money

was dramatically revised by Keynes in 1936, the relevance of this to monetary

policy has only gradually become apparent. It is only during the current

decade that monetary authorities have finally become as concerned with the

manner in which the banking system makes money available to the public as

with the total quantity. A movement by banks out of securities into loans

cannot be uninteresting to monetary policy: the bond-buying public may be

merely transferring a part of its wealth to a less liquid form, while the

recipients of new loans are almost certainly increasing their spending.

Thus the division of the bank's earning assets between securities and

loans is of relevance not only to the bank itself (concerned as it is with

liquidity and profits) but also to analysis of the inflationary impact of
different allocations of a given supply of money. Coincidentally - though
perhaps providentially so - the bank's choice between greater liquidity and
greater earnings is also society's choice between lesser and greater infla-
tionary forces. Most of the weapons of contemporary monetary policy can be
understood, and are in fact proposed, as an effort by the central bank to
"encourage" (where that word covers a spectrum of meanings from "suggest"
to "force") the commercial banks to hold assets which are relatively more
liquid (though the means of achieving this often involves making such assets
less liquid). It is toward an improved understanding of the ways in which
various aspects of monetary policy affect the bank's choice between different
earning assets, and hence aggregate demand, that this paper is aimed.

Appendix A : The "Deposit-Low."

Most banks have many depositors and the typical deposit transaction involves but a small fraction of the owner's account. For this reason, any attempt to derive theoretically the shape of the distribution of "deposit-lows" for a bank must recognize that any particular "deposit-low" is the result of a long series of individual deposit withdrawals and additions. Avoiding the question of what causes a depositor to alter the size of his account, we will deduce the "deposit-low" distribution on various assumptions about the probability of each deposit account transaction being an addition to or depletion of total deposits.

In line with the static nature of expectations in the text, let us assume that the bank "expects" no change in its total deposits during the ensuing period; this may be interpreted to mean that it believes that every dollar of transactions in its deposit accounts has a 50-50 chance of being a withdrawal of a dollar or a deposit of a dollar. It is certainly true, then, that the expected change of deposits, no matter how many transactions occur, will then be zero. But there will still be finite probabilities attached to "deposit-lows" less than zero. In general, if (2N) transactions occur, the probabilities of the "deposit-lows" are given by:

$$\Pr[0] = \frac{(2N)!}{2^{2N} \cdot (N!)^2}$$

$$\Pr[-2x] = \Pr[-2x + 1] = \frac{(2N)!}{2^{2N} \cdot (N - x)! \cdot (N + x)!}$$

where x is a positive integer, $\Pr[-2x]$ means the probability of a "deposit-low" of (2x) dollars less than initial deposits, and

$1 \leq x \leq N$.[*]

---

[*] In general, if the probability of a given deposit transaction being an accretion is p and the probability of a withdrawal $(1 - p)$, this formula becomes:

$$Pr[0] = (2N)! \sum_{i=0}^{N} \left(\frac{2N - 2i + 1}{i!(2N - i + 1)!}\right) p^{2N-i} (1 - p)^{i}$$

$$Pr[-2x] = Pr[-2x + 1] = (2N)! \sum_{i=0}^{N-x} \left(\frac{2N - 2i + 1}{i!(2N - i + 1)!}\right) p^{2N-2x-i}(1 - p)^{2x+i}$$

It would violate the assumption of expectations of no change in deposits, however, if any p other than .5 were chosen.

A smoothed picture of such a discrete probability function is given in Figure A-1,[**] where the solid curve represents the distribution for $(2N)$

[**] It should perhaps be noted that this distribution differs from Patinkin's receipts-expenditures distribution for individuals because there is here no assumption that withdrawals must equal accretions (i.e., both equal to N). Cf. D. Patinkin, Money, Interest and Prices, Evanston: 1956, Chapter 7 and Appendix to Chapter 7 (by Aryeh Dvoretzky). Note especially the difference between Figure A-1 below and Patinkin's Figure 9, p. 92.

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transactions and the dotted curve for a larger number of total transactions.[***]

[***] The slope of the frequency distribution is positive over its entire range, from $(-2N)$ to zero. It increases first at an increasing rate, but at a decreasing rate in the area to the immediate left of zero. The inflexion point occurs at $x^2 = \frac{1}{2} (N + 1)$.

Figure A-1
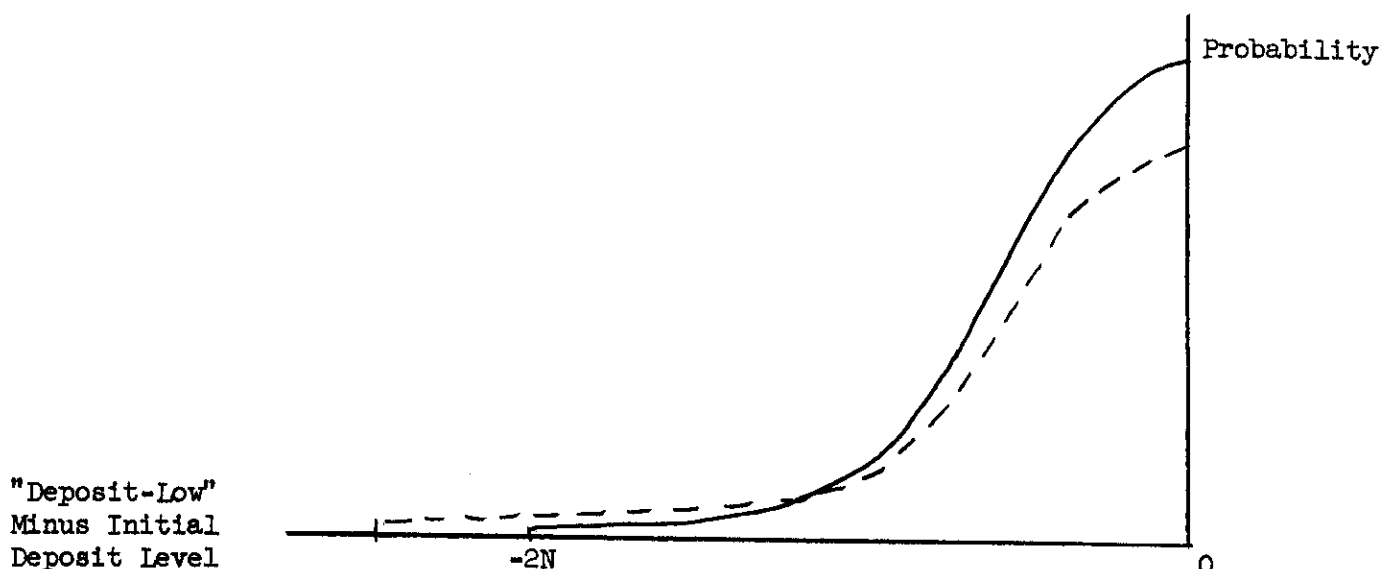
It is unlikely that any bank is in a position, however, to consider each transaction as totally independent of all or any previous transactions in the deposit accounts. At one extreme, if there is only one bank in the economy and there is no change in the public's desire to hold cash (as opposed to deposits), then any withdrawal must appear later as an addition; there is definitely an inverse relation between the probability that a given transaction will be an addition and the proportion of previous transactions which were additions. The more usual case, however, is that of a bank which experiences, or expects, or fears, positive correlation between the probability that a given transaction will be a deposit and the relative number of previous transactions which were deposits.

As an example of this, suppose that, whenever a majority of the previous transactions have been deposits, the probability that the next transaction will also be a deposit is somewhere between .5 and unity. In order to maintain

the assumption of static expectations, the probability of a withdrawal when a majority of the previous transactions were withdrawals must be equal to the probability of a deposit when a majority were deposits. The earlier formulae are now seen as the special case where these probabilities are .5. A rough picture of the effect of raising them above .5 is shown in Figure A-2, where the solid curve represents the same distribution as in Figure A-1, and the dotted curve the distribution when autocorrelation is introduced in the above manner. It is doubtful if the autocorrelation of deposit transactions is sufficiently large, for most banks, to cause a negative slope in the frequency distribution of "deposit-lows," but it may suffice to keep the probability of relatively low "deposit-lows" from being so small as to be negligible.

The "triangular" distribution has been chosen in the text as the closest simple approximation to this distribution. If a straight line were fitted through the distributions of Figure A-2, it could be seen that the "triangular" distribution understates the probability of occurrence of extremely low and zero-neighborhood "deposit-lows" and overstates the probability of the middle range of "deposit-lows."
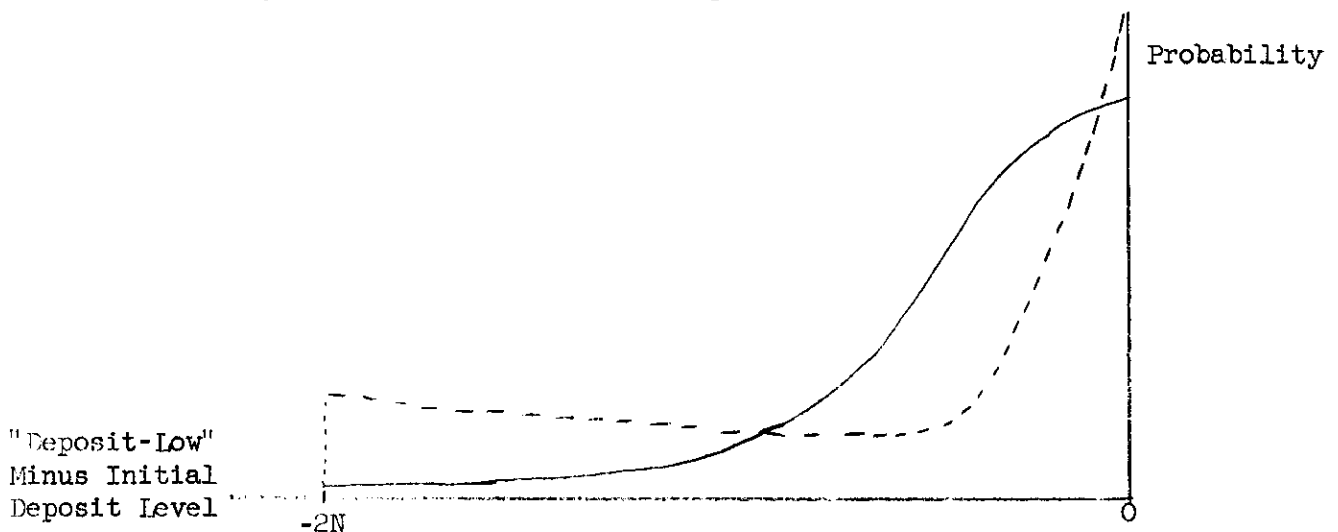

Figure A-2

Appendix B: Variance of Profits

Because the consideration of variance of earnings involves great com-
plication of the model, only a very simplified version of it will be discussed
here. In addition to the assumptions of the text, it is assumed that the bank
has no cash requirements $(k = 0)$, it has no net worth $(N = 0)$, there is no
"excess-collateral rate" $(m = 0)$, securities have no earnings $(g = 0)$ and
no possibility of price change $(a = 0)$, and it is considered possible that
all deposits be withdrawn during the period $(s = 0)$. Thus the accounting
identity (1) of the text becomes:

(B-1) $1 = B + L$

The distribution of "deposit-lows" is: $f(u) = 2u$; and the expected
additions to net worth function is simply:

(b-2) $E(\Delta N) = 2 \int_{u=L}^{1} (eL)u\,du - 2 \int_{u=0}^{L} (eL - qL - qu)u\,du = eL - \frac{qL^3}{3}$

Maximization of expected profit implies that:

(B-3) $L = \sqrt{e/q}$

A regular maximum will occur, with L between zero and one, as long as
$e < q$ (both are positive).

The variance of profit, written $s^2(\Delta N)$, is:

(B -4) $s^2(\Delta N) = 2 \int_{u=L}^{1} (eL)^2 u\,du - 2 \int_{u=0}^{L} (eL - qL - qu)^2 u\,du - [E(\Delta N)]^2$

$= q^2 L^4 [\frac{1}{6} - \frac{L^2}{9}]$

Minimization of S^2 (in the relevant range, $0 < L < 1$) clearly requires $L = 0$, at which point both E and S^2 are zero. No maximization process is needed to find the frontier of maximum E for each given S since, once one is specified, the other is uniquely determined. Although the equation of this frontier is complex, its slope at any point (determined by the value of L) is:

$$(\text{B-5}) \qquad dE/dS = \left| \frac{e}{q} - L^2 \right| \left| \frac{3 - 2L^2}{6L(1 - L^2)} \right|$$

which is positive in the range $0 < L < \sqrt{e/q}$, and negative beyond. Thus, the expected gain (E) can be increased only at the expense of increased variance of gain (S^2) up to the point of maximum E. These opportunity loci are plotted in Figure B-1 for three sets of values of e and q.

On the assumption that bankers are "risk-averters" and "diversifiers,"[*] it can be easily seen from Figure B-1 that they will choose L at least

---

[*] Cf. Tobin, "Liquidity Preference as Behavior Towards Risk," pp. 16-22. The labels "risk-averter" and "diversifier" are applied to those investors whose indifference curves between E and S are concave upward. For opportunity loci such as those of Figure B-1, "plungers" and "risk-lovers" might also diversify, i.e., choose L not equal to zero or one, but "diversifiers" necessarily will diversify.

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somewhat smaller than that value which maximizes E. However, the change in the portfolio induced by a change in e or q can only be guessed unless the bank's complete preference function between E and S is specified, a task which will not be attempted.

Figure B-1
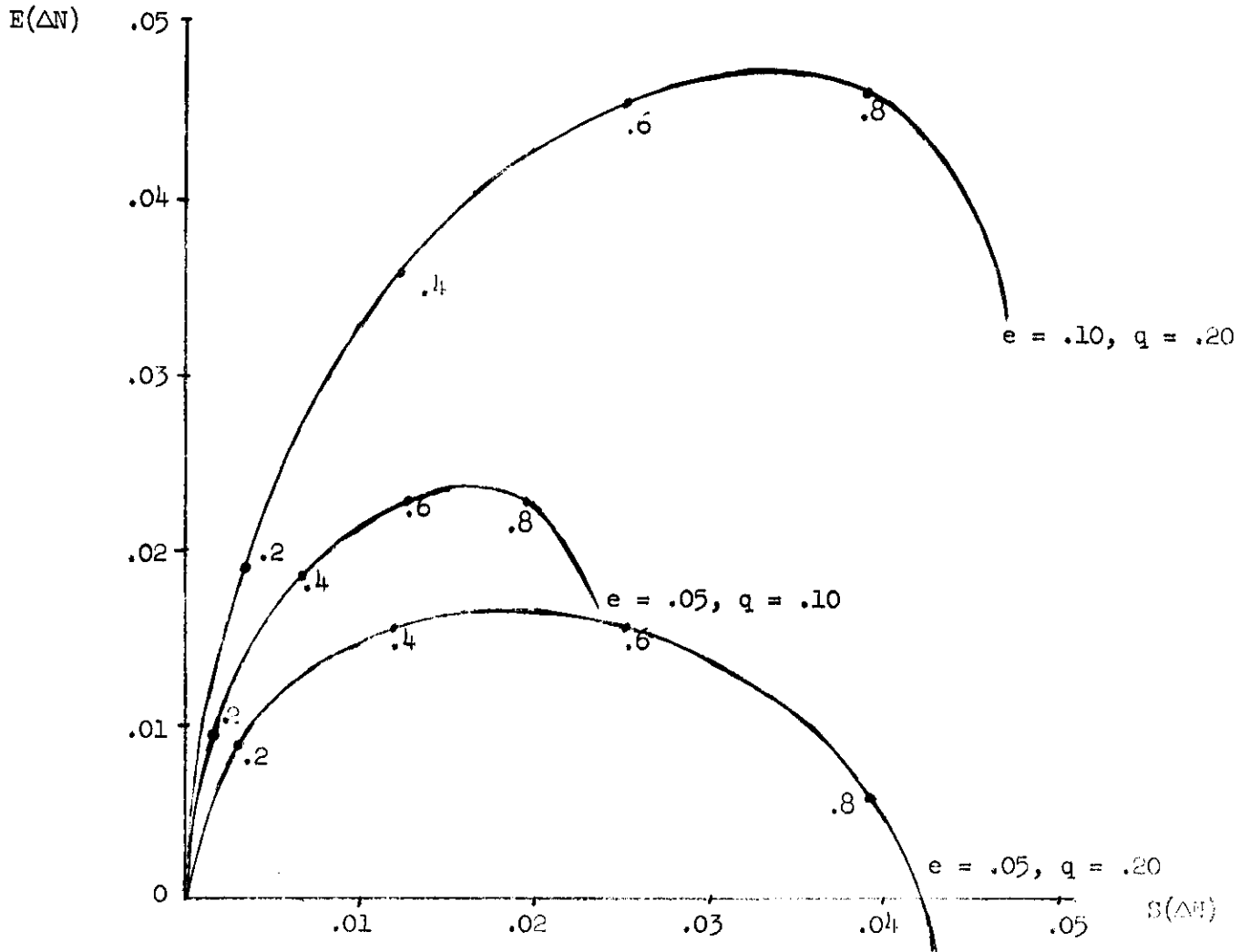
(The numbers beside the points are the fraction of total assets
held in loans at that point; the curves end at the point,
L = 1, except for the curve of e = .05, q = .20, which ends
at L = .86 where E becomes negative.)

Three conclusions may be drawn from this discussion:

1. The assumption that the bank maximizes E and neglects S entirely
is equivalent in the Tobin sense, to assuming that the bank is on the border
between risk-averting and risk-loving.[*] This is not as serious as the words

_____

[*] Tobin, "Liquidity Preference as Behavior Towards Risk," p. 19. In Tobin's
equation (3.7) maximizing expected return implies a marginal utility of
return (U'(R)) which is constant with respect to changes in R .

_____

imply since much of the bank's risk is reflected in the model of the text
in the expected profit function itself.

2. The size of the larger earning (and less liquid) portion of the
bank's portfolio (i.e., L) would generally be somewhat smaller than the
values derived in the text if variance of return were explicitly considered
and the bank assumed a "diversifier." While this might be more realistic
than to place all the risk elements in the expected return function, the
complications of such a procedure can be seen from the simple version of
the model presented here.

3. The effects of changes in parameters upon the various assets
become less determinate when variance is introduced. But inspection of
Figure B-1 indicates that the direction of changes is probably not altered
for plausible shapes of the indifference loci, and the magnitudes may not
be seriously different either.

Risk aversion on the part of the bank undoubtedly plays a critical
role in the determination of the composition within its loan and security
portfolios. As between two loans with the same expected earning rate, the
one whose returns have lower variance and/or lower correlation with the

returns of those loans already in the portfolio will certainly be preferred;
similarly, the bank will often accept a low-yield security into its bond
portfolio because its potential capital variation is small. While not denying
the importance of these considerations in the bank's choice between different
loans and between different securities, the model of the text maintains that
they are not crucial in the bank's prior choice as to the basic division of
the portfolio between loans and securities. The determinacy gained by as-
suming the bank an expected profit maximizer is felt to be worth the perhaps
slight loss of realism.

Appendix C: Stage 2(i)

If the left-hand inequality of equation (5) of the text is assumed,
then Stage 2(i) becomes a possibility, as well as Stage 2(ii) and Stage 3.
Then the expected addition to net worth $E(\Delta N)$, is as follows:

$$
E(\Delta N) = \int_{w = -1 + \frac{(1-s)(1-k)}{B}}^{a} \int_{u=s}^{1} [gB + wB + eL][\frac{u - s}{a(1-s)^2}]dudw
$$

$$
(C-1) \quad + \int_{w=-a}^{-1 + \frac{(1-s)(1-k)}{B}} \int_{u=1- \frac{(1+w)B}{1 - k}}^{1} [gB + wB + eL][\frac{u - s}{a(1-s)^2}]dudw
$$

$$
+ \int_{w=-a}^{-1+ \frac{(1-s)(1-k)}{B}} \int_{u=s}^{1- \frac{(1+w)B}{1 - k}} [gB + wB + eL - q(1 - k)
$$

$$
+ q(1 + w)B + q(1 - k)u][\frac{u - s}{a(1-s)^2}] dudw
$$

This becomes, once the integrations are performed,

$$(C-2) \qquad E(\Delta N) = gB + eL - \frac{q(1 - a)^4}{24Ba(1-s)^2(1-k)^2} \left[\frac{(1-s)(1-k)}{1 - a} - B \right]^4$$

After elimination of L by means of the accounting identity (1), the derivative of the expected profit with respect to B is:

$$(C-3) \qquad \frac{dE(\Delta N)}{dB} = -(e - g) + \frac{q(1-a)^4}{24B^2a(1-s)^2(1-k)^2} \left[\frac{(1-s)(1-k)}{1 - a} - B \right]^3 \left[\frac{(1-s)(1-k)}{1 - a} + 3B \right]$$

The second derivative (d^2E/dB^2) is negative throughout the range of B, the first derivative (equation (C-3)) is negative when B assumes its largest value (i.e., $B = \frac{(1-s)(1-k)}{1 - a}$), and the first derivative is positive when B assumes its smallest possible value (i.e., $B = \frac{(1-s)(1-k)}{1 + a}$), provided that the following inequality holds:

$$(C-4) \qquad 0 < \frac{e - g}{q} < \frac{2a^2(2 - a)}{3(1+a)^2}$$

Thus, if (C-4) holds, there is a value of B in the permissible range for which equation (C-3) is equal to zero; and it is that value of B which maximizes expected profits.

If the right-hand inequality of equation (C-4) does not hold, expected profits decline continually as B rises from its smallest to its largest permissible value. In that case, the maximizing value of B is smaller than $\frac{(1-s)(1-k)}{1 + a}$, and the model of the text is the appropriate one (i.e., Stage 2(i) is impossible). The common sense argument for choosing the assumption of the text that there is no Stage 2(i) is greatly strengthened by a consideration of the parameter values required if the right-hand

inequality of (C-4) is to hold: the table below gives the highest possible
values of (e - g) consistent with the existence of a Stage 2(i), for several
combinations of values of a and q :

	q = .05	q = .10	q = .30
a = 0	0	0	0
a = .1	.0005	.0011	.0033
a = .25	.0023	.0047	.0141
a = .50	.0055	.0111	.0333

Inasmuch as e generally does exceed g by at least one or two per cent,
surprisingly - if not implausibly - large values of a and q are required
if the bank is to choose to hold enough securities to be able to meet its
worst deposit depletions by securities sales alone - even if bond prices
rise to their highest conceivable value.

Nevertheless, the implications for the bank's optimum asset portfolio
of changes in various parameters can be determined by taking partial
derivatives of (C-3) while holding that equation equal to zero. The changes
in optimum B are as follows:[*]

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[*] The arithmetic is sufficiently messy that only the first derivatives
have been calculated.

$$\partial B/\partial e < 0$$

$$\partial B/\partial g > 0$$

$$\partial B/\partial q > 0$$

$$\partial B/\partial s < 0$$

$$\partial B/\partial k < 0$$

The sign of $\partial B/\partial a$ cannot be determined generally. That these signs are the same as those derived in the text, on the assumption that Stage 2(i) does not exist, is partial evidence that the influence of parameter changes upon optimum portfolios does not depend critically upon this choice of assumption.

Appendix D: Time Deposits

No distinction has been made, anywhere in the text, between different types of deposits, a neglect which will be here repaired by introducing time, as well as demand, deposits. The most obvious difference between the two classes of deposits lies in their cost, for time deposits are still permitted to, and do, earn interest for the depositor. On the other hand, the "earning power" of time deposits may exceed that of demand deposits in either or both of two ways. First, it is an institutional fact in most countries that the cash, or low-yield, reserve requirements on time deposits are lower than those on demand deposits. And second, to the extent that the lowest conceivable "deposit-low" is raised by the addition of less volatile time deposits, the bank may hold a larger portion of its portfolio in less liquid, but higher earning, assets.[*]

---

[*] Throughout this section, we neglect the possibility that greater time deposits may imply lesser private (or other financial intermediary) lending, and hence higher lending rates for the bank. Cf. J. Tobin, "Financial Intermediaries and the Effectiveness of Monetary Controls," Cowles Foundation Discussion Paper No. 63 (January 1959).

Banks attract new time deposits by offering to pay higher interest rates on them. Such increases in time deposits may "come" from several places: by a transfer from 1) the bank's own demand deposits; 2) the public's

deposits with other banks; and 3) deposits and shares of the non-bank

financial intermediaries.[*] From the point of view of the individual bank,

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* It is assumed that the public's currency needs are fixed.

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the relevant division is between the first case and the second and third

cases.[**] Here it will be assumed that the bank cannot, through its own

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** From the point of view of the banking system, the most interesting
division is between the first two cases and the third case. But even
then, one's conclusions depend upon whether one assumes the central bank
to fix the quantity of bank reserves or the price at which banks may
purchase additional reserves.

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volition, alter the total quantity of its deposits, but only, by changing

its time deposit interest rate, induce some of its depositors to hold

time instead of demand deposits. The fraction of total deposits held as

time deposits is assumed to grow from zero, as the time deposit interest

rate (i) increases from zero, to some maximum value at which point further

increases in i cannot induce further shifts from demand to time deposits.

Thus, time deposits as a fraction of total deposits (T) will lie between

zero and some maximum level (T*) depending upon i , where T* is less

than one.

This transfer of funds from demand to time deposits will affect the

values of k and s . On the assumption that vault cash requirements

are the same for both kinds of deposits, the overall cash requirements,

as a fraction of total deposits, will be:

$$(D-1) \qquad k = k_d - (k_d - k_t)T$$

where requirements on demand deposits (k_d) are assumed to exceed those on time deposits (k_t). The value of s , and the frequency distribution of the possible "deposit-lows," is not so easily calculated. It will depend upon the lowest possible "deposit-low" of time deposits (as a fraction of initial time deposits), s_t , and that of demand deposits (as a fraction of initial demand deposits), s_d ; but it will also depend upon the form of the distribution of each of these "deposit-lows" and the covariance of the changes of each of these two types of deposits. If the frequency distribution of each type's "deposit-low" is assumed "triangular" (with a lowest possible "deposit-low" of s_t and s_d for time and demand deposits, respectively), and the two changes are uncorrelated, the resulting distribution of "deposit-lows" for the total of deposits will not be "triangular" but will have a convex (from below) segment over very low "deposit-lows" and a concave segment for "deposit-lows" near unity.[*] Since the assumption

---

* If the fraction in time deposits is small enough that

(D-2)
$$T \leq \frac{1 - s_d}{(1 - s_d)+(1 - s_t)}$$

the frequency distribution of "deposit-lows" of total deposits, $f(u)$, will be:

$$f(u) = \frac{2}{3} \frac{[u-Ts_t-(1-T)s_d]^3}{T^2(1-T)^2(1-s_d)^2(1-s_t)^2} \quad \text{where: } Ts_t+(1-T)s_d < u < T +(1-T)s_d$$

$$= \frac{2}{3} \frac{3[u-Ts_t-(1-T)s_d]- 2T(1-s_t)}{(1 - T)^2 (1 - s_d)^2} \quad \text{where: } T +(1-T)s_d< u < (1-T)+Ts_t$$

(footnote continued)

$$= 2[u - Ts_t - (1-T)s_d] \left[\frac{1}{(1-T)^2(1-s_d)^2} + \frac{1}{T^2(1-s_t)^2} \right] - \frac{4(1-T)(1-s_d)}{3T^2(1-s_t)^2}$$

$$- \frac{4T(1-s_t)}{3(1-T)^2(1-s_d)^2} - \frac{2[u - Ts_t - (1-T)s_d]^3}{3T^2(1-T)^2(1-s_d)^2(1-s_t)^2} \quad \text{where:} \quad (1-T)+Ts_t < u < 1$$

The distribution, $f(u)$, will be 'triangular" only if s_t is one, i.e., time deposits never decline. If the direction of the inequality of (D-2) is reversed, the exact form of the distribution is altered, but the general shape is the same(and the distribution is "triangular" only if s_d is one).

---

of zero covariance between "deposit-lows" of time and demand deposits leads

to an understatement of frequencies in the low "deposit-low" segment, and

an overstatement in the high range, it will be convenient, and probably

less unrealistic, to assume that the aggregate "deposit-low" distribution

is also "triangular" with the point of zero probability, s , occurring at: [*]

(D-4) $\qquad s = Ts_t + (1 - T)s_d$

---

[*] This value of s is the lowest conceivable "deposit-low" of the distribution (D-3).

The costs of time deposits, iT , are subtracted from equation (8)

of the text to give the expected addition to net worth when the bank

considers a time deposit policy:

(D-5) $\qquad E(\Delta N) = gB + e(1 + N - k - B) - \frac{q}{3x^2} (x - B)^3 - iT$

where, for simplicity, bond prices are assumed not to fluctuate (i.e., a

equal to zero) and, for brevity, x is written for $(1 - s)(1 - k)$.
This expression, (D-5), is maximized by the bank not only with respect
to B , which yields equation (12) of the text (with a equal to zero),
but also with respect to i , for the interest rate paid on time deposits
is now a variable under the control of the bank. Differentiation of
(D-5) with respect to i yields, after substitution for the profit-maxi-
mizing value of B by means of equation (12):

$$(D-6) \quad \frac{\partial E(\Delta N)}{\partial i} = e(k_d - k_t) \frac{dT}{di} - (e - g)\left[1 - \frac{2}{3}\sqrt{\frac{e - g}{q}}\right] \frac{dx}{dT} \cdot \frac{dT}{di}$$

$$- \frac{d(iT)}{di}$$

where

$$(D-7) \quad \frac{dx}{dT} = \frac{d(1-s)(1-k)}{dT} = (1 - s_d)(k_d - k_t) - (1 - k_d)(s_t - s_d)$$

$$- 2T(s_t - s_d)(k_d - k_t)$$

Clearly, if i becomes sufficiently large that dT/di is zero
(i.e., time deposits are at their maximum level, T*), equation (D-6)
will be negative; for some high values of i , expected profits will be
reduced if time deposit rates are raised. At the other extreme, when
i , and hence T , is zero, equation (D-6) is positive; costs are not
increased as much as expected profits when i is increased slightly from
zero. Thus, expected profits at first increase and later decrease as i ,
and hence T , is raised. We can conclude, on the above assumptions, that
the optimum proportion of total deposits in time deposit accounts will

always be greater than zero.[*] Moreover, this conclusion still follows if

---

* W. L. Smith uses a simplified version of equation (D-6), on the assumption that e equals g (i.e., neglecting the liquidity differences between assets), to show that it is profitable for the bank to induce switches from demand to time deposits, by raising time deposit rate, if:

$$e > \frac{i}{k_d - k_t} \left[1 + \frac{1}{\eta} \right]$$

where η is the time deposit interest elasticity of time deposits. Of course, the above conclusions follow in this special case as well. Cf. footnote 7, page 544 of W. L. Smith, "Financial Intermediaries and Monetary Controls," Quarterly Journal of Economics, Vol. 73, No. 4 (Nov. 1959), pp. 533-553.

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time deposits are assumed to be reserve-saving though not deposit-stabilizing

(i.e., $k_d > k_t$, $s_d = s_t$), or vice versa (i.e., $k_d = k_t$, $s_d < s_t$).

Of course, there are many reasons beyond the scope of the model why the

optimum amount of time deposits might be zero for a particular bank. Most

obvious of these are that administrative considerations place a minimum on

the amount of such deposits which the bank will wish to attract and/or

that a time deposit rate markedly above zero is required to induce any

time deposits at all. On the other hand, there are reasons other than ex-

pected profits (as defined here) for desiring such relatively stable deposits

even if not profitable - for example, concern by the bank for the variability

of its expected profits (see Appendix B).

Appendix E: Uniform Distribution of "Deposit-Lows"

In order to indicate that the results of the model of the text do not depend too critically upon the type of distribution assumed for the bank's "deposit-lows," the same model is here worked through briefly on the alternative assumption that the "deposit-lows" are uniformly distributed over the range from s to unity; that is,

$$(E-1) \qquad f(u) = 1/(1 - s)$$

The expected addition to net worth is then exactly as given by equation (7) of the text with the appropriate alteration in the distribution of u . The analog to equation (8), expected profit after the integrations have been performed, is:

$$(E-2) \qquad E(\Delta N) = gB + eL - \frac{q}{2(1-s)(1-k)} [(1-s)(1-k) - B]^2 - \frac{a^2 q B^2}{6(1-s)(1-k)}$$

And the optimum proportion of securities (see equation (12)) is:

$$(E-3) \qquad B = \frac{(1-s)(1-k)}{(1 + a^2/3)} [1 - \frac{e-g}{q}]$$

It should be noted that the parameters enter in a less complex, but similar, manner (see pages 22-23).

The first and second partial derivatives of B have the following signs:

where		$\partial B/\partial x$	$\partial^2 B/\partial x \partial y$ where					
			$y = a$	$y = q$	$y = g$	$y = e$	$y = k$	$y = s$
where	x = s	-	+	-	-	+	+	0
	x = k	-	+	-	-	+	0	
	x = e	-	+	+	0	0		
	x = g	+	-	-	0			
	x = q	+	-	-				
	x = a	-	-					

Again, there is no basic difference between this table and that on page 24 although here more second partials are zero and there is no possibility of different signs in different regions of the parameters.