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INVESTMENT, REPLACEMENT, AND AN INDEX OF TECHNOLOGY

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Investment, Replacement, and an Index of Technology

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An earlier paper** presented a method of apportioning historical in-

** B. F. Massell, "Capital Formation and Technological Change in U.S. Manufacturing," Review of Economics and Statistics, (forthcoming). See also Robert Solow, "Technical Change and the Aggregate Production Function," Ibid. XXXIX, No. 3, pp. 312-320.

creases in output per manhour between increases in capital per manhour and technical change, and the conclusion was drawn that 90 percent of the increase in average labor productivity in U.S. manufacturing between 1919 and 1955 was attributable to improvements in the technology. However, a qualification was noted: Improvements in the technology do not descend from Heaven but are for the most part embodied in new capital goods (although some changes may be of purely organizational nature), so that a positive rate of gross investment is a necessary condition for technical change. Another consideration is that if the service life of capital goods is primarily determined by economic rather than technological factors,***

*** For support of this view, see George Terborgh, Dynamic Equipment Policy.

and if this variable in turn is a determinant of the level of technology, then the latter is partly a resultant of economic decisions. It is the

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purpose of the present paper to explore these issues, under a set of simplifying assumptions, so as to establish the relationships among gross investment, capital replacement, and the level of technological development. In terms of the model presented here, the importance of investment as a contributory agent for economic growth will be re-examined.

The following will be assumed:

(a) All technical change is embodied in new capital goods; thus it is not possible to improve the technology merely by rearranging existing inputs.*

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* This is just as much a polar case as the other model, which assumes technical change all to be organizational; a more general model would permit both types of progress.

(b) All additions to the capital stock embody the latest known techniques.

(c) Capital goods retain their full efficiency indefinitely; they are withdrawn from service for economic rather than technological reasons.

(d) The usual assumptions will be made regarding homogeneity of output and of input, so that index number problems can be ignored.

(e) There may be substitution between capital and labor before the capital is in place, but none afterward. In other words, once a machine is built, the number of men needed to operate it is technologically determined.

(f) Output from a given capital goods is a function solely of the level of technology associated with that asset.

From (f) we can write

$$(1) \qquad Q(t) = \int_{t-\Theta(t)}^{t} I(x)H(x)dx ,$$

where Q = output, $I(x)dx$ = gross investment, $H(x)$ = an index of the level of technology associated with capital produced at time x, and Θ = the age of the oldest capital in use. The value of Θ is chosen in the following way: The (assumed homogeneous) labor force is spread out over the capital stock, beginning with the newer vintages, according to the technologically determined labor requirement associated with each vintage. Assuming the capital stock to be larger than the labor force (Remember that capital never wears out!), eventually all laborers will be employed. All units of capital which are not manned are withdrawn from service (though they may be brought back subsequently). This can be expressed

$$(2) \qquad L(t) = \int_{t-\Theta(t)}^{t} I(x)r(x)dx ,$$

where L = the labor force and $r(x)$ = the labor-capital ratio associated with capital inputs built in year x.

If (1) is differentiated totally with respect to time, we obtain

$$(3) \qquad \dot{Q}(t) = I(t)H(t) - H(t-\Theta)I(t-\Theta)[1-\dot{\Theta}] .$$

Next, taking the time derivative of (2) and solving for $\dot{\Theta}$, we get

$$(4) \qquad \dot{\Theta} = \frac{\dot{L} - I(t)r(t)}{I(t-\Theta)r(t-\Theta)} + 1 .$$

An expression for the marginal productivity of new capital, $\pi(t)$, is

$$(5) \qquad \pi(t) = \frac{dQ(t)}{I(t)dt} = \frac{\dot{Q}(t)}{I(t)} .$$

Substituting (3) and (4) into (5), we can write (assuming $\dot{L}(t) = 0$),

$$(5a) \qquad \pi(t) = H(t) - H(t-\Theta)\frac{r(t)}{r(t-\Theta)} .$$

It can be seen that

$$\pi(t) > 0 \Longleftrightarrow \frac{H(t)}{r(t)} > \frac{H(t-\Theta)}{r(t-\Theta)}$$

i.e., the marginal productivity of investment is positive if and only if output per manhour associated with new capital exceeds that associated with marginal capital, a condition which must be satisfied for the capital to be built.

Ordinarily in a fixed proportions system, the marginal productivity of an input is indeterminate. Yet here we have shown capital's marginal product to exist, and we will presently demonstrate that there is an equilibrium wage rate for labor. The explanation of these results is quite simple: by introducing the possibility of a continuous trade-off at the margin between new and old machines, a marginal technological rate of substitution is implied. Fixing the size of the labor force results in determining also the quantity of capital; but there will always be substitution possibilities between the new and old models. Hence associated with each vintage of capital is a technological marginal product, expressing output per manhour obtainable from that vintage as compared with that which can be got from marginal capital.

Recognizing that the marginal productivity of new capital is a function of the replacement period of capital, as well as of time, we can write

$$\pi = \pi(\Theta\ ;\ t)$$

Let us assume that the technology variable can be expressed as an exponential function of time,

$$H(t) = H(0)e^{\lambda t}$$

Then, equation (5a) can be re-written

$$(8) \qquad \pi(t) = H(0)e^{\lambda t}\left\{1 - \left[\frac{r(t)}{r(t-\Theta)}\right] e^{-\lambda\Theta}\right\}$$

Assume further that $r(t) = \bar{r} =$ constant. Then (8) reduces to

$$(8a) \qquad \pi(t) = H(0)e^{\lambda t}[1 - e^{-\lambda\Theta}]$$

Partially differentiating (8a) with respect to Θ , we get

$$\pi_{\Theta} = H(0)\lambda e^{\lambda(t-\Theta)} > 0$$

Taking the second partial derivative with respect to Θ ,

$$\pi_{\Theta\Theta} = -H(0)\lambda^{2}e^{\lambda(t-\Theta)} < 0$$

Thus, as the age of the oldest capital in use increases, the marginal product of new capital will also rise, but at a declining rate. Moreover, in the limit, we have

$$\lim_{\Theta \to \infty} \pi = H(t)$$

From equation (4) it can be seen that (for $\dot{L}(t) = 0$)

$$\frac{d\Theta(t)}{I(t)dt} = - \frac{r(t)}{I(t-\Theta)r(t-\Theta)} < 0$$

Hence there is diminishing marginal productivity of new capital.

Next, differentiating (8a) partially with respect to t , we obtain

$$\pi_{t} = H(0)e^{\lambda t}(1 - e^{\lambda\Theta}) > 0$$

It can be seen from inspection that the second partial derivative is positive as well, indicating that the marginal product of capital rises over time at an increasing rate.

Let us now relax the restrictive assumptions regarding $H(t)$ and $r(t)$, and consider an economy which has a given labor force and a given inventory of capital goods of various vintages, extending back indefinitely in time. The central planners have one decision variable at their disposal: the level of investment in the present year. Output is determined by the existing labor force and the inventory of capital goods, assuming the allocation of labor over capital to be as described above.* All of this

* To avoid the possibility of output being a function of the same year's investment, which in turn forms part of output, we shall assume a one-year gestation period. Thus in effect this year's output will depend on last year's capital stock. This requires a minor change in notation.

year's output which is not invested by the planners will be consumed by the society in the present year. On the other hand, next year's output is an increasing function of this year's investment (and hence a decreasing function of this year's consumption). If we assume a two-year horizon, i.e., that the planners are concerned only with consumption this year and next, then clearly all of the next year's output will be consumed. The question that the planners wish to answer is: What is the trade-off between this year's consumption and next year's? This is given by the following expression:

$$(9) \qquad C(t+1) = \int_{t-\Theta(t)}^{t-1} H(x)I(x)dx + \int_{t-1}^{t} H(x)[Q(x)-C(x)]dx$$

Where C = consumption. (9) gives us a consumption locus, shown in figure 1; the curve does not extend below the 45 degree line, for if all of this year's output is consumed, the same quantity of goods will again be available the following year, assuming L to be constant during the interval. It can be seen that were L increasing, Θ would also increase, raising the profitability of investment and consequently the trade-off between $C(t+1)$ and $C(t)$.*

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* See equation (4).

The marginal trade-off between consumption in the two years (the slope of the locus) is given by

$$(10) \qquad \frac{dC(t+1)}{dC(t)} = - \left\{ H(t) - H(t - \Theta) \frac{r(t)}{r(t-\Theta)} \right\}$$

Again assuming that $H(t) = H(0)e^{\lambda t}$, and $r(t) = \bar{r}$,

$$(11) \qquad \frac{dC(t+1)}{dC(t)} = - H(0)e^{\lambda t}[1 - e^{-\lambda\Theta}]$$

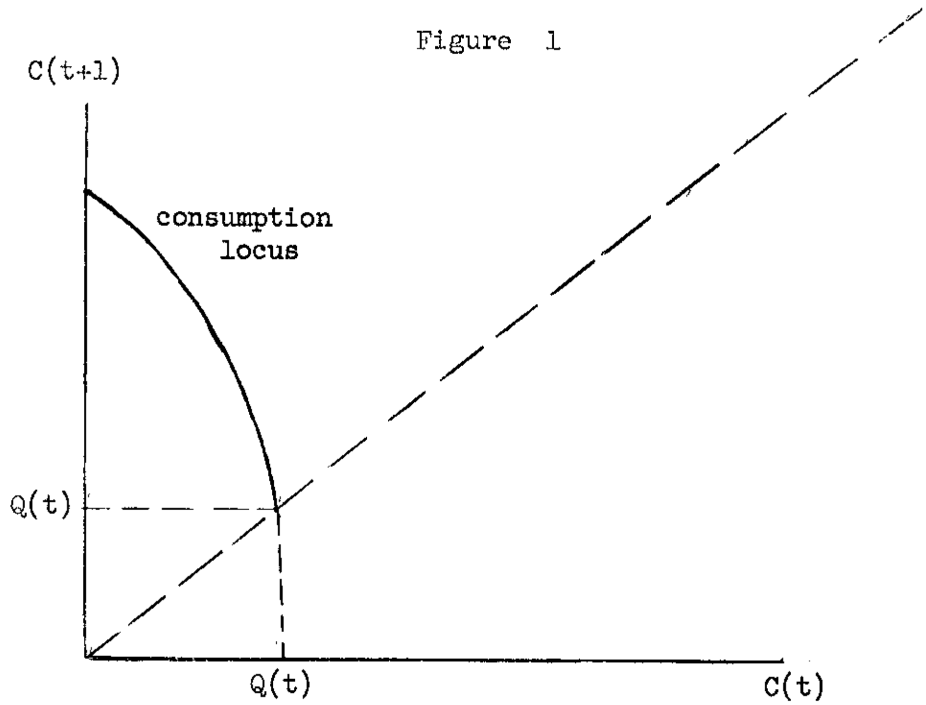
This reaches a maximum at the lowest point on the locus, at which $I(t) = 0$.

The opposite end of the consumption locus, where it intersects the vertical axis, is the case where $C(t) = 0$, and hence $I(t) = Q(t)$. In this case $C(t+1)$ is given by

$$(12) \qquad C(t+1) = Q(t) + \int_{t-1}^{t} H(x)Q(x)dx - \int_{t-\Theta(t-1)}^{t-\Theta(t)} I(x)H(x)dx$$

The slope of the locus is numerically least at this point; this slope is

Figure 1



a function of investment in year t , i.e., of the position on the curve. A large amount of investment leads to a small value of Θ , so that additional investment is relatively ineffective. We can express the marginal trade-off for the two limiting cases as follows:

$$\lim_{\Theta \to 0} \frac{dC(t+1)}{dC(t)} = 0 \qquad \text{and} \quad \lim_{\Theta \to \infty} \frac{dC(t+1)}{dC(t)} = H(t) \ .$$

The optimal policy within the framework of the two year horizon will depend on the interest rate. If B(t) = the present value of the two year consumption stream, and i is the interest rate, then

$$(13) \qquad B(t) = C(t) + \frac{C(t+1)}{1 + i}$$

Maximizing B with respect to I(t) , and simplifying, we obtain

$$(14) \qquad \frac{H(t)}{r(t)} - \frac{H(t - \Theta)}{r(t - \Theta)} = \frac{1 + i}{r(t)}$$

If H(t) is once more assumed to increase exponentially, then

$$(15) \qquad \hat{\Theta} = - \frac{1}{\lambda} \ln \left\{ [1 - \frac{1 + i}{H(t)}] \frac{r(t - \Theta)}{r(t)} \right\}$$

$$\frac{\partial \hat{\Theta}}{\partial t} < 0 \qquad \text{and} \qquad \lim_{T \longrightarrow \infty} \hat{\Theta} = 0$$

If competitive pricing is the order of the day, if the rate of technical change is anticipated, and if the supply of labor is perfectly inelastic, then we should expect the oldest capital in use to yield no rent; the entire output obtainable from the marginal machines is paid to the workers who operate them. Thus, it follows that the wage rate is given by

$$(16) \qquad W(t) = \frac{H(t - \Theta)}{r(t - \Theta)}$$

The rent on any asset must be equal to the difference between the output which can be produced with the asset and the payment which must be made to labor. Thus the rent on an asset of vintage x at time t is

$$(17) \qquad R^x(t) = H(x) - \frac{r(x)}{r(t - \Theta)} \cdot H(t - \Theta)$$

If the asset is new, then $x = t$, and

$$(18) \qquad R^t(t) = \pi(t)$$

It is now straightforward to determine the value V, of an asset of any vintage, x, at any time, t, with a continuous rate of interest, ρ

$$(19) \qquad V^x(t) = \int_t^{t^x} R^x(h)e^{-\rho h}dh$$

where

$$t^x - \Theta(t^x) = x$$

If $r(h) = \bar{r}$, $H(h) = H(0)e^{\lambda h}$, and $\rho = 0$, (19) becomes

$$(20) \qquad V^x(t) = H(0)e^{\lambda x}\left\{t^x - t - \frac{1}{\lambda}[1 - e^{-\lambda(t^x - t)}]\right\} .$$

In the case of a new asset, this can be simplified,

$$(20a) \qquad V^t(t) = H(0)e^{\lambda t}[\Theta - \frac{1 - e^{-\lambda\Theta}}{\lambda}] .$$

Let us now consider an economy experiencing steady exponential growth, with a labor-capital ratio that is technologically determined and constant over time. Assume further that the service life of capital is chosen initially and must subsequently remain unchanged. Labor input is given by

$$(21) \qquad L(t) = L(0)e^{gt}$$

and capital input can be expressed

$$(22) \qquad K(t) = rL(0)e^{gt}$$

Alternatively, the capital stock can be written

$$(23) \qquad K(t) = \int_{t-\Theta}^{t} I(x)dx$$

Equating (22) and (23), solving for $I(t)$, and simplifying,

$$(24) \qquad I(t) = \frac{grL(0)e^{gt}}{1 - e^{-g\Theta}}$$

Equation (1) can now be re-written

$$(25)\qquad Q(t) = \int_{t-\Theta}^{t} [H(0)e^{\lambda x}] \left[\frac{grL(0)e^{gx}}{1 - e^{-g\Theta}} \right] dx$$

which, when simplified, becomes

$$(26)\qquad Q(t) = \frac{grH(0)L(0)e^{(\lambda + g)t}}{g + \lambda} \left[\frac{1 - e^{-(\lambda+g)\Theta}}{1 - e^{-g\Theta}} \right]$$

An alternative expression for output is

$$(27)\qquad Q(t) = A(t)K(t)$$

where $A(t)$ is an index of the level of technology of the capital stock as a whole. Substituting (26) in (27) and solving for $A(t)$,

$$(28)\qquad A(t) = \frac{gH(0)e^{\lambda t}}{g + \lambda} \left[\frac{1 - e^{-(\lambda+g)\Theta}}{1 - e^{-g\Theta}} \right]$$

$A(t)$ can be regarded as an index of technology associated with the average production function, whereas $H(t)$ is the index associated with the "best practice" function.* Considered as a function of time,

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* The distinction between best practice and average production functions appears in a paper by Anne Gross, "The Technological Structure of the Cotton Textile Industry," in Leontieff, Studies in the Structure of the American Economy. See also K. Maywald, "The Best and the Average in Productivity Studies and in Long-Term Forecasting," Productivity Measurement Review, 9.

$H(t)$ represents the flow of inventions to the society and thus denotes an ideal technology path, the path that would prevail if all new inventions were instantaneously adopted. The actual level of technology, $A(t)$, is a weighted average of the $H(t)$, the weights equalling the proportion of the capital stock associated with each.

Equation (28) indicates the relationship between $A(t)$ and the capital replacement period, Θ. In appendix 1 it is shown that a lengthening of the service life of capital has an adverse effect on the technology. While Θ is often regarded as a technological fact of life, it seems likely that the primary motive for replacement is obsolescence rather than deterioration,* in which case Θ can largely be considered as

* Terborgh, op. cit. See also R. W. Goldsmith, "A Perpetual Inventory of National Wealth, Studies in Income and Wealth, Vol. 14 (NBER Conference on Income and Wealth).

a decision variable, chosen with regard to its impact on output.

While technology is a decreasing function of the replacement period, it is clear that, given the rate of growth of the labor force, gross investment also varies inversely with Θ, as given by

$$(29) \qquad \frac{\partial I}{\partial \Theta} = - \frac{rL(0)e^{g\Theta}}{(1-e^{-g\Theta})^2} < 0$$

In appendix 2, it is shown that, for any level of net investment, output is an increasing function of gross investment (and at a declining rate, for any value of gross investment greater than that required by the given increase in labor). If however, we define consumption as the difference between output and investment, we see that there is an optimal value of gross investment, and hence a corresponding optimal Θ, which will maximize consumption subject to a labor constraint. This is true because up to a point gross investment adds more to output than it subtracts from consumption, due to its effect on the technology; after this point, however, diminishing returns set in. In appendix 3, it is shown that the optimal service life can be found from the following equation:

$$(30) \qquad (g + \lambda)e^{-\lambda\Theta} - \lambda e^{-(g + \lambda)\Theta} = g\left[1 - \frac{g + \lambda}{H(0)r}\right]$$

The meaning of an optimal Θ , in a model with fixed proportions, is that there is an investment path which is optimal in the sense that it balances the marginal gain from an improved technology (resulting from a higher ratio of gross to net investment) against the marginal cost of this larger investment ratio. Beyond a certain point the cost of "buying-in" to the invention function outweighs the gain; in other words, there are diminishing returns to innovation. Technical change, is not, after all, a costless shift from one state knowledge to another, as neo-classical theory seems to suggest, but has associated with it a definite cost in terms of resources foregone.

APPENDIX 1.

$$\frac{\partial A}{\partial \Theta} = \frac{c}{[1 - e^{-g\Theta}]^2} \left\{ [1 - e^{-g\Theta}] \; [g + \lambda] \; e^{-(g+\lambda)\Theta} - g e^{-g\Theta} \; [1 - e^{-(g+\lambda)\Theta} \;] \right\}$$

where $c = T(0) \; e^{\lambda t} \left[\frac{g}{g + \lambda} \right]$; $g > 0$, $\lambda > 0$, $\Theta > 0$, $T(0) > 0$

$$\frac{\partial A}{\partial \Theta} = \frac{c}{[1 - e^{-g\Theta}]^2} \left\{ (\lambda + g)\Theta^{-(g + \lambda)\Theta} - (\lambda + g) \; e^{-(2g+\lambda)\Theta} - g e^{-g\Theta} + g e^{-(2g+\lambda)\Theta} \right\}$$

$$= \frac{c}{[1 - e^{-g\Theta}]^2} \left\{ (\lambda + g) \; e^{-(g+\lambda)\Theta} - \lambda e^{-(\partial g+\lambda)\Theta} - g e^{-g\Theta} \right\}$$

$$= \frac{c e^{-g\Theta}}{[1 - e^{-g\Theta}]^2} \left\{ (\lambda + g) \; e^{-\lambda\Theta} - \lambda e^{-(g+\lambda)\Theta} - g \right\}$$

< 0 if and only if $\lambda e^{-(g+\lambda)\Theta} + g > (\lambda + g) \; e^{-\lambda\Theta}$

if and only if $\lambda e^{-g\Theta} + g e^{\lambda\Theta} > \lambda + g$

Clearly, for Θ sufficiently large, this condition is satisfied. Moreover,

$$\lim_{\Theta \to 0} \; [\; \lambda e^{-g\Theta} + g e^{\lambda\Theta} \;] = \lambda + g$$

therefore if $[\; \lambda e^{-g\Theta} + g e^{\lambda\Theta} \;]$ is a monotone increasing function of Θ ,

for all $\Theta > 0$, then $\frac{\partial A}{\partial \Theta} < 0$ for all $\Theta > 0$.

$$\frac{\partial}{\partial \Theta} [\; \lambda e^{-g\Theta} + g e^{\lambda\Theta} \;] = - g\lambda e^{-g\Theta} + g\lambda e^{\lambda\Theta} = g\lambda(e^{\lambda\Theta} - \bar{e}^{g\Theta}) > 0 \; .$$

Q.E.D.

Appendix 2

Equation (24) can be solved for Θ, giving the following:

$$\Theta = -\frac{1}{g} \ln \left[1 - \frac{grL(t)}{I(t)}\right]$$

Substituting this in (26), and simplifying,

$$Q(t) = \frac{H(t)I(t)}{g + \lambda} \left\{ 1 - \left[1 - \frac{grL(t)}{I(t)}\right]^{1 + \frac{\lambda}{g}} \right\}$$

Differentiating partially with respect to $I(t)$, and simplifying,

$$\frac{\partial Q(t)}{\partial I(t)} = \frac{H(t)}{g + \lambda} \left\{ 1 - \left[1 - \frac{grL(t)}{I(t)}\right]^{\frac{\lambda}{g}} \left[1 + \frac{\lambda}{g} \cdot \frac{grL(t)}{I(t)}\right] \right\}$$

$$\frac{\partial Q}{\partial I} > 0 \Longleftrightarrow [1 - \xi]^{\gamma} [1 + \gamma\xi] = M < 1$$

where $\gamma = \frac{\lambda}{g}$, $\xi = \frac{grL(t)}{I(t)}$,

and where $\gamma > 0$ and $0 < \xi < 1$.

$\xi = 1 \Longrightarrow M = 1$.

$$\frac{\partial M}{\partial \xi} = -\gamma^2 (1 - \xi)^{\gamma - 1} < 0$$

Thus for $\xi > 0$, $M < 1$.

Hence $\frac{\partial Q}{\partial I} > 0$.

The second partial derivative can be expressed

$$\frac{\partial^2 \xi(t)}{\partial I^2(t)} = - \frac{H(t)\gamma\xi^2}{(\varepsilon+\lambda)I(t)} [1 - \xi]^{\gamma-1} [1 + \gamma] < 0 \quad .$$

Appendix 3

Consumption can be written as the difference between (26) and (24), where the time period is irrelevant (with respect to maximization) due to the assumption of exponential growth:

$$C = \frac{HgrL}{\lambda + g} \left\{ \frac{1 - e^{-(g+\lambda)\Theta}}{1 - e^{-g\Theta}} \right\} - \frac{grL}{1 - e^{-g\Theta}}$$

Differentiating partially with respect to Θ, and setting this expression equal to zero, and simplifying,

$$[g + \lambda][e^{-\lambda\Theta} - e^{-(g+\lambda)\Theta} - g(1 - e^{-(g+\lambda)\Theta}) = - \frac{g(\lambda + g)}{H} ,$$

from which equation (30) easily follows.