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ANALYSIS OF A PRODUCTION PROBLEM BY DYNAMIC PROGRAMMING

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Analysis of a Production Problem by Dynamic Programming

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Summary

Let demand for a product be a random variable, which is independently and identically distributed in successive periods. Consider a firm producing this commodity subject to the following costs: the direct cost of production is proportional to output; the cost of changing the rate of production is proportional to the size of the change with possibly a different cost coefficient applying to upward and to downward shifts; the cost of storage is proportional to the size of the stock at the beginning of the period, and the cost of (penalty for) shortage is proportional to the size of the shortage. Let unfilled demand be carried over into the next period. The problem is to find the optimal rate of production conditional on the rate of production in the last period and on the current level of stock. This paper discusses the nature of the solution, but leaves open all questions concerning the most efficient methods of computation.

1. Introduction
2. Formulation
3. Simplification
4. Iteration
5. Characterization

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1. Introduction

The introduction of risk into the analysis of the classical single product firm leads to essentially new results only when the static framework is dropped. The problem of determining optimal output which appears so simple in static theory, reveals then its true difficulty: the output decision must be revised for each period in the light of information about the prevailing situation. Variables that might influence this decision are for instance the forecast level of demand and the accumulated inventory of product. In the simplest case the expected value of demand is itself stationary. More precisely: demand in each period is a random variable, independent of demand in other periods, and subject to a known, constant probability distribution. This will be the situation considered in this paper.*

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* Actually the introduction of forecasts of the mean of the probability distribution of demand into the model would raise no great difficulties, provided all other parameters (e.g., the variance) of the distribution remain unchanged.

It has been observed [Arrow, pp. 29, 30] that the costs of production that arise in a decision problem of this kind are threefold: costs attaching to the change of the rate of production, to the rate of production itself, and to the accumulated rate of production, namely inventories or shortages. A dynamic production problem involving risk and all three kinds of cost has been investigated for the special case that all cost functions involved are of the second degree. [Simon] It then turns out -- not surprisingly -- that the optimal decision rule is linear in terms of the observed variables and that therefore the form of the probability distribution of demand is immaterial since only the expected value of demand is relevant.

In the light of recent successes with linear production relationships in the economic analysis of the firm it is interesting to consider the case in which all the cost functions are linear or piecewise linear with kinks occuring where stock or the change in the rate of production are zero. Since the cost of production during a given period now depends on the rate in the preceding period, the information relevant to the production decision must include this past rate as well as the inventory level. This raises the mathematical difficulty of the problem above that of an ordinary inventory problem -- the classical sequential decision problem involving one state variable. As we intend to show, it is possible to discuss the nature of the solution, although its computation in practical applications encounters formidable difficulties, which must be expected in a bivariate dynamic programming problem. In this paper we shall concentrate on the qualititative aspects of the sultion which are of primary interest to economic analysis, and reserve for a later paper all questions of practical computation.

2. Formulation

Notation

ξ	optimal rate of production
x	past rate of production
y	level of inventory
t	demand during the current period -- a random variable
c	proportional cost of production
h	cost of storage per commodity unit per period
g	shortage penalty per commodity unit per period
k - i	cost per unit of an upward change in the rate of production
k + i	cost per unit of a downward change in the rate of production
a	discount rate
P(t)	probability distribution of demand

$$f(y) \begin{cases} hy & y > 0 \\ -gy & y < 0 \end{cases}$$

The problem requires that a decision about the rate of production be made in the face of the fact that this decision will be revised one period hence. It is the beauty of dynamic programming that it allows to formulate and solve such decision problems where present actions whose consequences depend on as yet unmade future decisions must be chosen in the best possible way. This is a step forward from the traditional dynamic theory of the firm in which a simultaneous decision about all future actions had to be assumed at one time.

The basic idea is that of introducing explicitly the expected value of present and discounted future cost under an optimal policy conditional on the present state of the system. This cost is usually called the "loss function." Since of all the problem data only the rate of production and the inventory of product are subject to change these will be the state variables. Hence the loss function depends on the current level of commodity stock and of the rate of production. To obtain the expression of the loss function consider the state of the system one period later. Then the current rate of production is the rate we now choose, say ξ and the level of stock is $\xi + y - t$ where t is the unknown demand during the present period. If this expression is negative it means a backlog of unfilled orders. Now a demand t occurs with probability $dP(t)$. The expected value of discounted future cost at the beginning of the next period is therefore, in terms of our unknown loss function, φ

$$\int \varphi(\xi, y + \xi - t)\, dP(t)$$

At the present time this cost must be discounted by a discount factor $\underline{a}$. In addition there are costs during the first period, namely, cost of production

$$c\xi$$

cost of inventory or shortage $f(y)$

and cost of changing rate of production $k|x - \xi| + i(x - \xi)$

This adds up to a present value of the total discounted stream of cost equal to

$$(1) \qquad k|x - \xi| + c\xi + f(y) + a \int \varphi(\xi, \xi + y - t)dP(t)$$

The aim of the decision is to minimize cost; this is introduced by requiring that the expression (1) be a minimum. We then have a definition of the loss function, a definition which contains the thing defined again on the right-hand side:

$$(2) \qquad \varphi(x, y) = f(y) + \underset{\xi}{\text{Min}}\; k|x - \xi| + i(x - \xi) + c\xi + a\int\varphi(\xi, \xi+y-t)dP(t)$$

This apparently circular definition is in fact a functional equation of the dynamic programming type which determines the loss function φ as its solution. Once φ is known the optimal policy $\xi = \prod(x, y)$ results from the prescription of choosing ξ so as to minimize (1).

Functional equations of the dynamic programming type are those which involve a maximum or minimum operator on the right-hand side,

$$\varphi(z) = \underset{\zeta}{\text{Min}}\; F(\varphi, z, \zeta)$$

In most practical applications F is a linear operator; in the present case F is a linear integral operator. We postpone our demonstration of the existence and uniqueness* of a solution until Chapter 4. Our first concern will be to

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* The present Dynamic Programming equation is of Bellman's type 3 -- i.e., everything that does not fall into the simple classes 1 and 2 -- for which general existence and uniqueness theorems have not been given in the literature.

simplify the equation by suitable transformations.

3. Simplification

Let μ, σ be the mean and variance of the demand distribution $P(t)$. How is the solution of the functional equation (2.2) affected, when the distribution is standardized: $\mu = 0$, $\sigma = 1$?

Write $\quad x = \mu + \sigma X \qquad y = \sigma Y$

$\qquad \xi = \mu + \sigma Z \qquad t = \mu + \sigma T$

Then $P(t) = P(\mu + \sigma T) = Q(T)$, say where Q is standardized.

$$(3.1) \qquad \varphi(\mu + \sigma X, \sigma Y) = f(\sigma Y) + \underset{Z}{\mathrm{Min}}\, k\sigma|X - Z| + i\sigma(X - Z) + c(\mu + \sigma X) + a\int \varphi(\mu + \sigma Z, \mu + \sigma Z + \sigma Y - \mu - \sigma T)\sigma dQ(T)$$

Observe that $f(\sigma Y) = \sigma f(Y)$. Now set

$$\varphi(\mu + \sigma X, \sigma Y) = \frac{c\mu}{1-a} + \sigma\Phi(X, Y) .$$

Substituting this in (3.1) one has after an easy calculation

$$(3.2) \qquad \Phi(X, Y) = f(Y) + \underset{Z}{\mathrm{Min}}\, k|X - Z| + i(X - Z) + cX + a\int\Phi(Z, Z + Y - T)dQ(T)$$

which is formally identical with (2.2). This proves

Lemma 1: Suppose that $\Phi(X, Y)$ is the solution of (3.2) for a standardized distribution Q with zero mean and unit variance. Then the solution of (2.2) having a distribution P of the same family with mean μ and variance σ is given by

$$\varphi(x, y) = \frac{c}{1 - a}\mu + \sigma\, \Phi\left(\frac{x-\mu}{\sigma} , \frac{y}{\sigma} \right)$$

If $Z = \prod(X, Y)$ is the optimal policy associated with (3.2), then

$$\xi = \mu + \sigma \cdot \prod\left(\frac{x-\mu}{\sigma} , \frac{y}{\sigma} \right)$$

is the optimal policy associated with (2.2).

We turn next to the elimination of $c\xi$ and of $i\cdot(x - \xi)$ from the equation (2.2).

Lemma 2: Equation (2.2) is equivalent to

$$(3.3) \qquad \psi(x, y) = f^*(y) + \underset{\xi}{\text{Min}}\, k|\xi - x| + a\int\psi(\xi, \xi + y - t)dP(t)$$

where
$$f^*(y) = f(y) + (c - i)\,\frac{(1 - a)^2}{a}\; y$$

and

$$\varphi(x, y) = (i - c)x + (i - c)\,\frac{1 - a}{a}\, y + \psi(x, y)$$

Proof: Define $\psi(x, y) = \frac{1-a}{a}\cdot(c - i)y + (c - i)x + \varphi(x, y)$

Substituting for φ in (2.2) one has

$$\psi(x, y) = f(y) + cx + \frac{1 - a}{a}(c - i)y + ix + \underset{\xi}{\text{Min}}\Big(k|\xi - x| - i\xi$$
$$+ a\int\psi(\xi, \xi + y - t)dP(t) - (1 - a)(c - i)(y + \xi - \mu) - a(c - i)\xi\Big)$$

Without loss of generality we may set $\mu = 0$ according to Lemma 1.

Ordering terms we obtain

$$\psi(x, y) = f(y) + \frac{(1-a)^2}{a}(c - i) + \underset{\xi}{\text{Min}}\Big(k|\xi - x| + a\int\psi(\xi,\xi + y - t)dP(t)\Big) \quad \text{QED.}$$

From now on let us denote again

$$\psi(x, y) \text{ by } \varphi(x, y)$$

$$\left.\begin{array}{ll} h\cdot y + \frac{c - i}{a}(1 - a)^2 y = h^* y & \text{if } y > 0 \\ -g\cdot y + \frac{c - i}{a}(1 - a)^2 y = g^*|y| & \text{if } y < 0 \end{array}\right\} \text{ by } f(y)\,.$$

The equation (2.2) assumes then the particularly simple form

$$(3.4)\qquad \varphi(x, y) = f(y) + \operatorname*{Min}_{\xi} k|\xi - x| + a\int\varphi(\xi, \xi + y - t)dP(t)$$

It accords well with economic intuition, that the proportional cost of production should affect the solution only through interest charges on cost that are incurred before the realization of a sale.

4. Iteration

Some important properties of the solution to (3.4) can be brought out only by a study of suitable approximation processes. A natural approach to the solution is that of the following iteration scheme:

$$\varphi_0 = f(y) + \operatorname*{Min}_{\xi} k|\xi - x| = f(y)$$

$$\varphi_{n+1}(x, y) = f(y) + \operatorname*{Min}_{\xi} k|\xi - x| + a\int\varphi_n(\xi, \xi + y - t)dP(t)$$

a. We show if $\int|t|dP(t) = \lambda$ exists, then the sequence of functions φ_n is bounded

$$(4.1)\qquad \varphi_n \leqq \frac{a}{(1-a)^2}\, g\lambda + \frac{ag}{(1-a)^2}\,|x| + \frac{g}{1-a}\,|y|$$

Proof: (by induction) $\varphi_0 = f(y) < \frac{g}{1-a}\,|y|$

Suppose that (4.1) is satisfied for φ_n.

$$\varphi_{n+1}(x, y) \leqq f(y) + a\int\varphi_n(x, x + y - t)dP(t)$$

$$\leqq g\cdot|y| + a\int \frac{a}{(1-a)^2}\, g\lambda + \frac{ag}{(1-a)^2}\,|x| + \frac{g}{1-a}\,|x + y - t|dP(t)$$

$$\leqq g\cdot|y|\left(1 + \frac{a}{1-a}\right) + g|x|\left(\frac{a^2}{(1-a)^2} + \frac{a}{1-a}\right) + g\lambda\left(\frac{a^2}{(1-a)^2} + \frac{a}{1-a}\right)$$

$$= \frac{g}{1-a}\,|y| + \frac{ga}{(1-a)^2}\,|x| + \frac{ga}{(1-a)^2}\,\lambda \qquad \text{QED}$$

b. The sequence $\varphi_n(x, y)$ is monotone:

$$\varphi_{n+1}(x, y) \geqq \varphi_n(x, y)$$

Proof: (induction)

$$\varphi_1(x, y) = f(y) + \underset{\xi}{\text{Min}}\, k|\xi - x| + a\int \varphi_0(\xi, \xi + y - t)dP(t)$$

$$\geqq f(y) = \varphi_0(x, y)$$

Suppose $\varphi_n(x, y) \geqq \varphi_{n-1}(x, y)$.

$$\varphi_{n+1}(x, y) = f(y) + \underset{\xi}{\text{Min}}\, k|\xi - x| + a\int \varphi_n(\xi, \xi + y - t)dP(t)$$

Now

$$k|\xi - x| + a\int \varphi_n(\xi, \xi + y - t)dP(t) \geqq k|\xi - x| + a\int \varphi_{n-1}(\xi, \xi + y - t)dP(t)$$

The inequality is preserved when we take the minimum on either side. Hence

$$\varphi_{n+1}(x, y) \geqq f(y) + \underset{\xi}{\text{Min}}\, k|\xi - x| + a\int \varphi_{n-1}(\xi, \xi + y - t)dP(t) = \varphi_n(x, y) \text{ QED.}$$

Since the sequence $\varphi_n(x, y)$ is monotone and bounded for every x,y it converges to a limit function. By construction of the sequence, this limit function must be a solution of (3.4).

The economic meaning of monotone convergence is this: every iteration extends the horizon of the firm by one period. The cost function naturally increases with the number of periods under consideration.

c. $\varphi(x, y)$ is a (jointly) convex function of x and y .

Proof (induction on $\varphi_n(x, y)$):

$\varphi_0(x, y) = f(y)$ is convex jointly in x and y .

Suppose $\varphi_n(x, y)$ is convex.

It is easily shown that the function

$$\Phi_n(x, y) = \int \varphi_n(x, x + y - t)dP(t)$$

is also convex jointly in x and y. Since $k|\xi - x|$ is convex jointly in ξ and x it follows that the minimand

$$k|\xi - x| + a\Phi_n(\xi, y) = \psi_n(\xi, x, y) \text{ (say)}$$

is convex jointly in x, ξ, and y.

Consider

$$\underset{\xi}{\text{Min}}\ \psi_n(\xi, x_0, y_0) + \underset{\eta}{\text{Min}}\ \psi_n(\eta, x_1, y_1) =$$

$$\underset{\xi,\eta}{\text{Min}}\ \psi_n(\xi, x_0, y_0) + \psi_n(\eta, x_1, y_1) \geqq$$

$$\underset{\xi,\eta}{\text{Min}}\ 2\ \psi_n\left(\frac{\xi + \eta}{2}, \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right) \quad \text{by convexity of } \psi_n$$

$$= \underset{\xi}{\text{Min}}\ 2\ \psi_n\left(\xi, \frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

If follows that

$$\varphi_{n+1}(x_0, y_0) + \varphi_{n+1}(x_1, y_1) \geqq 2\ \varphi_{n+1}\left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right)$$

i.e., φ_{n+1} is convex.

By induction $\varphi_n(x, y)$ is convex for any n. Since the limit of a sequence of convex functions is convex, $\varphi(x, y)$ is convex QED.

d. Since $\varphi(x, y)$ is convex, its right and left hand derivatives exist and agree except for an at most denumerable number of points where both have jumps. The value of the left hand derivative is the left hand limit, that of the right hand derivative the right hand limit. [Fenchel, p. 71]

e. <u>Uniqueness theorem</u>: Suppose that the absolute first moment of P exists

$$\int |x| dP(x) = \lambda < \infty$$

and that the set X of (admissible or optimal) ξ is bounded. Then the solution of (3.4) is unique.

Proof: Suppose there are two solutions φ_1, φ_2. Let ξ_1, ξ_2 be the respective optimal policies. Then

$$\varphi_1(x, y) = k|\xi_1 - x| + f(y) + a\int \varphi_1(\xi_1, \xi_1 + y - t)dP(t)$$

$$\varphi_2(x, y) \leqq k|\xi_1 - x| + f(y) + a\int \varphi_2(\xi_1, \xi_1 + y - t)dP(t)$$

by definition of ξ_1.

Subtracting the first from the second statement we have

$$\varphi_2(x, y) - \varphi_1(x, y) \leqq a\int \Big(\varphi_2(\xi_1, \xi_1 + y - t) - \varphi_1(\xi_1, \xi_1 + y - t)\Big)dP(t) .$$

In the same way one obtains with ξ_2

$$\varphi_2(x, y) - \varphi_1(x, y) \geqq a\int \varphi_2(\xi_2, \xi_2 + y - t) - \varphi_1(\xi_2, \xi_2 + y - t)dP(t).$$

The two inequalities imply

$$|\varphi_2(x, y) - \varphi_1(x, y)| \leqq \underset{\xi=\xi_1,\xi_2}{\text{Max}}\ a\int |\varphi_2(\xi, \xi + y - t) - \varphi_1(\xi, \xi + y - t)|\ dP(t)$$

Iterating this inequality k times(i.e., substituting for the integrad k times in succession the expression on the left) we have

$$|\varphi_2(x, y) - \varphi_1(x, y)| \leqq a^k \sup_{\xi \in X} \int |\varphi_2(\xi,\xi + y - t) - \varphi_1(\xi, \xi + y - t)|\ dP^{(k)}(t)$$

where $P^{(k)}(t)$ is the k^{th} convolution of the distribution function $P(t)$.

We shall overestimate φ_2 when replacing it by the bound (4.1)

$$\frac{a}{(1-a)^2} \; g\mu + \frac{ag}{(1-a)^2} \; |x| + \frac{g}{1-a} \; |y|$$

and underestimate φ_1 by setting it zero.

Thus

$$|\varphi_2(x, y) - \varphi_1(x, y)| \leqq a^k \sup_{\xi \in X} \int \frac{a}{(1-a)^2} \; g\mu + \frac{ag}{(1-a)^2} \; |\xi| + \frac{g}{1-a} \; |\xi + y - t| \; dP^{(k)}(t)$$

Substitute the largest element in X for ξ and evaluate the integral

$$|\varphi_2(x, y) - \varphi_1(x, y)| \leqq a^k \cdot [\frac{a}{(1-a)^2} \; g\mu + \frac{ag}{(1-a)^2} |\xi_{max}| + \frac{g}{1-a} (|\xi_{max}| + |y|) + \frac{g}{1-a} \int |t| \; dP^{(k)}(t)$$

Now $\int |t| \; dP^{(k)}(t) \leqq k \int |t| \; dP(t)$ and the right hand side exists.

Thus

$$|\varphi_2(x, y) - \varphi_1(x, y)| \leqq a^k (c_1 + c_2 |\xi_{max}| + c_3 |y| + c_4 k)$$

Since $0 < a < 1$, for any fixed y the right hand side is smaller than any $\epsilon > 0$ provided k is sufficiently large. This proves that $\varphi_2(x,y) \equiv \varphi_1(x,y)$,

QED.

5. Characterization

How is the minimum of the right hand side of (3.4) characterized? What can be said about the optimal policy without actually computing $\varphi(x, y)$?

The right hand side of (3.4) is convex and hence differentiable except for at most a denumerable number of values ξ, at which left and right hand derivatives exist continuous from the left or to the right, respectively.

Recall now the necessary conditions for a minimum of a function with one-sided derivatives: In order that $f(z)$ be at a minimum it is necessary that

$$f_{+z} \geqq 0$$

(5.1)

$$f_{-z} \leqq 0$$

where f_{+z}, f_{-z} denote right and left derivatives, respectively. At a point of differentiability this implies $f_z = 0$. At a point of non-differentiability not both

$$f_{+z} = 0 \qquad f_{-z} = 0 \ .$$

Assume first that the convex function

$$\Phi(\xi, y) = a\int\varphi(\xi, \xi + y - t)\, dP(t)$$

is differentiable for all ξ . When applying (5.1) to the right hand side of (3.4) three cases must be distinguished. Let $\xi = \hat{\xi}$ be the minimizer.

If $\hat{\xi} = x$ (case 1)

$$k + \Phi_\xi(x, y) \geqq 0$$

$$-k + \Phi_\xi(x, y) \leqq 0 \qquad \text{or}$$

(5.2) $$-k \leqq \Phi_\xi(x, y) \leqq k \ .$$

If $\hat{\xi} > x$ (case 2)

(5.3) $$\Phi_\xi(\xi, y) = -k$$

If $\xi < x$ (case 3)

(5.4) $$\Phi_\xi(\xi, y) = k \ .$$

Now Φ is a convex function of ξ. Hence the curve described by (5.4) in a ξ, y diagram lies to the right of the curve (5.3) [Figure 1]. The set of points satisfying (5.2) must lie between the two curves as boundaries.

The optimal policy can therefore be characterized as follows: if for a given stock level y the past rate of production falls between two preassigned limits

$$\alpha(y) < x < \beta(y)$$

then the rate of production should not be changed;

If $x > \beta(y)$ reduce the rate to $\beta(y)$

If $x < \alpha(y)$ increase the rate to $\alpha(y)$.

The two curves $x = \alpha(y)$, $x = \beta(y)$ delimit the set of production and stock level combinations for which the production process is "in control." If it is "out of control" the policy calls for an adjustment of the rate just sufficient to bring the process back into control.

What happens at $\xi = \hat{\xi}$ where the one-sided derivatives $\Phi_{+\xi}$, $\Phi_{-\xi}$ disagree?

In case 3, for example, the minimality condition is then

$$\text{(5.1a)} \qquad \begin{aligned} &\Phi_{+\xi} \geqq k \\ &\Phi_{-\xi} \leqq k \end{aligned} \qquad \text{"=" not in both.}$$

There are now two different points $(\hat{\xi}, \overset{+}{y})$ and $(\hat{\xi}, \bar{y})$, say where, respectively [fig. 2]

$$\Phi_{+\xi}(\hat{\xi}, \overset{+}{y}) = k$$

$$\Phi_{-\xi}(\hat{\xi}, \bar{y}) = k \ .$$

Convexity implies

$$\Phi_{-\xi} < \Phi_{+\xi}$$

and

$$\overset{+}{y} > \bar{y} \ .$$

Since $\Phi_{\xi y} \leq 0$ in the neighborhood* and $\Phi_{-\xi}$ is continuous from the left and $\Phi_{+\xi}$ is continuous from the right, we also have that both $\Phi_{+\xi}$ and $\Phi_{-\xi}$ are decreasing from $\overset{+}{y}$ to $\bar{y}$. Hence the points satisfying

$$\text{(5.4a)} \qquad \begin{matrix} \Phi_{-\xi}\ \hat{\xi},(y) \leqq k \\ \Phi_{+\xi}\ \hat{\xi},(y) \geqq k \end{matrix} \qquad \text{not both "="}$$

are all located on the vertical line segment $\xi = \hat{\xi} \qquad \bar{y} \leqq y \leqq \overset{+}{y}$, attached at both ends to continuous segments of the curve $x = \beta(y)$. Therefore even where $\Phi_{+\xi}$ and $\Phi_{-\xi}$ disagree condition (5.4a) describes a segment of the upper boundary of the control zone, namely a vertical segment. The same is true in case 2 with respect to $x = \alpha(y)$.

In case 1 we have

$$\text{(5.2a)} \qquad \Phi_{+\xi} \geqq -k \qquad \Phi_{-\xi} \leqq k$$

and not "=" in both.

Comparison with (5.4a) and a corresponding condition (5.3a) shows that the points satisfying (5.2a) must again lie in the interior or on the boundary of the control zone.

The problem has thus been reduced to one of determining the two boundary curves $\alpha(y)$ and $\beta(y)$. What are their general properties? How can they be computed?

Intuition suggests that they be downward sloping. This can be proved by induction (in the manner of section 4): By demonstrating that the sign of $\Phi_{\xi y}$ is non-negative wherever the second derivatives of Φ exist.

If the cost of changing production $k = 0$ then the two boundary curves agree, the loss function is a function of $y + \xi$ only and the solution is to

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* See below on this page.

set $y + \xi$ equal to an optimal value. The control line is therefore of the form $y + \xi$ = constant.

In order to compute the boundaries of the control zone one cannot avoid solving equation (3.4) for the loss function $\varphi(x, y)$. The straightforward method in dynamic programming problems of this type is iteration (section 4). Unfortunately the computations at each step are cumbersome even for elementary distributions $P(t)$. In the first iteration the boundary curves form a pair of straight lines with slope -45° . For large values of k they do not exist: the whole plane is then "in control." This is the case in the following example:

$$a = 1$$
$$k = 8$$
$$h = 3$$
$$g = 7$$
$$dP(t) = 1 \qquad 0 \leqq t \leqq 1$$

Figure 1 shows the control zone for the second iteration of the same problem. The boundary curves are given by

$$y = \alpha(x) = -2x - \frac{1}{2} + \sqrt{1.55 + x}$$

$$y = \beta(x) = \begin{cases} 1.99 - 2x & \text{for} \quad 0 \leqq x \leqq .99 \\ -2x + 2.5 + \sqrt{1.35 - x} & \text{for} \quad .99 \leqq x \leqq 1 \end{cases}$$

Further iteration calls for the use of an electronic computor.

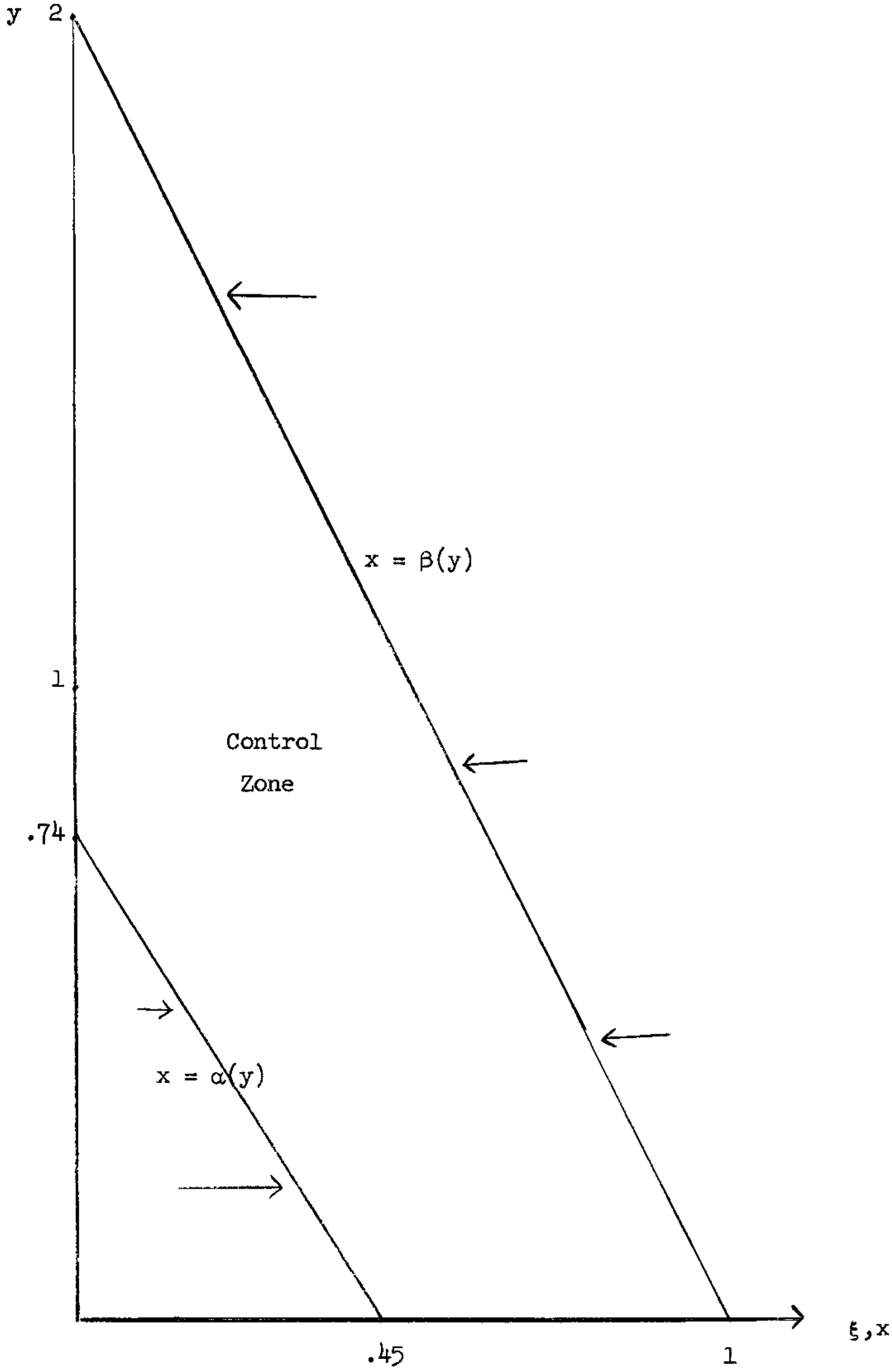

Figure 1

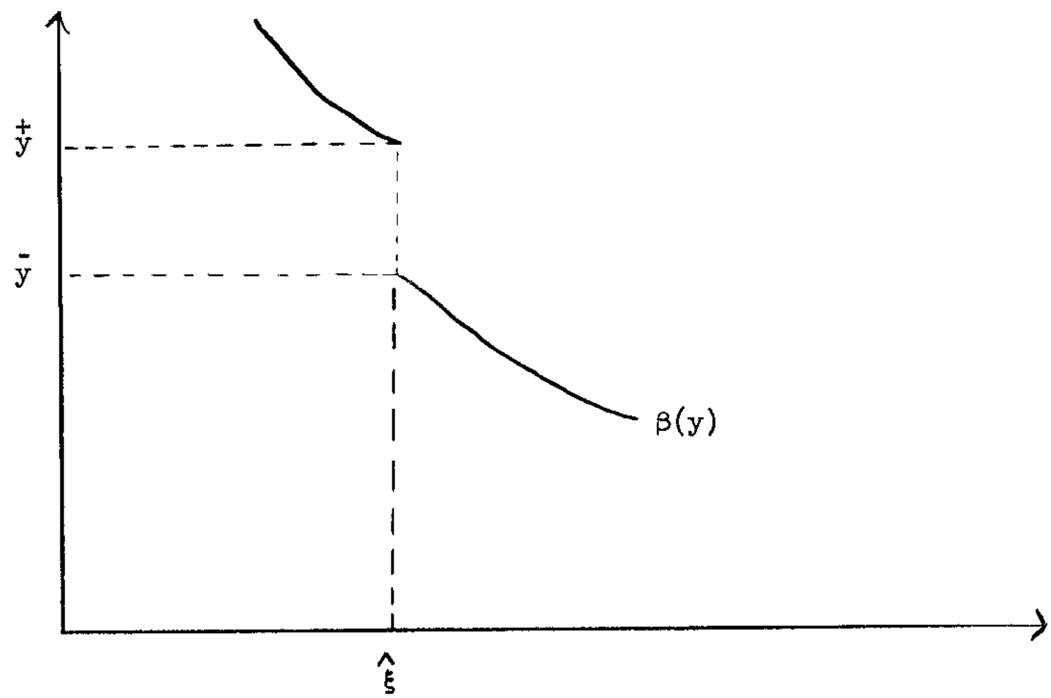

Figure 2

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