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Random Orderings and Stochastic Theories of Responses*

H. D. Block, Cornell University
and J. Marschak, Yale University

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H. D. Block, Cornell University
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CONTENTS: I The Problem.--II Stochastic Concepts of Constant Utilities.--
III Random Orderings and Random Utilities.--IV Further Directly Testable Conditions for
the Existence of Constant Utilities.--V Further Directly Testable Conditions for the
Existence of Random Orderings.--VI A Survey.--VII Varying the Domain of
Testability.--VIII The Boundary Case: Perfect Discrimination or Preference.--
IX Remarks on Choices under Uncertainty.--X Statistical Tests when Replicas
are Few.--References.

I. THE PROBLEM

In interpreting human behavior there is a need to substitute "stochastic consistency of choices" for "absolute consistency of choices." The latter is usually assumed in economic theory but is not well supported by experience. It is, in fact, not assumed in empirical econometrics and psychology.

The stochastic approach brings out the affinity between the phenomenon of choice and the more general psychological phenomenon of response to physical stimuli or, for that matter, to questionnaires, or of the action of "judges" who compare the performances of individuals.**

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** While the stochastic approach uses measurable probabilities, another approach, also promising to give realism and unity to the understanding of choices and responses, makes use of the measurable time-delay of a choice or response. The shorter the delay the larger is the difference between certain values said to be attached by the subject to two alternative responses. A very long delay reveals a state of (almost) indifference, the "conflict-situation": Hamlet took a very long time to decide whether to kill his uncle. The logic of this approach and its combination with stochastic theories was treated by Cartwright and Festinger [1].

Let A be the set of all alternatives (or actions) ever to be considered by a given subject. Let F be a subset of A . Let $(a:F)$ denote a "multiple choice" (also called "first choice" out of a set), i.e., the following observed fact: when forced to choose one element of F the subject chose a . In the theory of choice, F is called the feasible set. When F consists of 2,3,... elements, the multiple choice is called binary, ternary,...

In another language, the set F is associated with a stimulus, and consists of all possible responses of the subject to that stimulus; A is then interpreted as the set of all possible responses to all stimuli. In experiments on so called preferences and attitudes, the stimulus consists in offering a list of alternative menus, alternative political candidates, alternative answers to an item in the questionnaire. This list, is then, the feasible set F . When a "judge" makes paired comparisons the set F is a pair of individuals. In experiments on perceptual discrimination, the stimulus consists of letting n (usually but not necessarily two) physical objects - sounds or light sources or weights - impinge upon the subject; his response consists in stating that one of those objects is louder or brighter or heavier than the other(s). F is then the set of the n possible responses (or $n+1$ if "I don't know" statements are permitted.)

Another example of F is the set of all family budgets open to a family with a fixed amount of money, given the prices of goods: see Hotelling [1]. Each element of F is a different way of allocating the money among different goods.

In this paper - subject only to some attempted generalizations in Section VII - we shall call the observed choice, or response, $(a:F)$ the basic or direct observation. These observations can be counted, and the constraints on (i.e., relations among) the resulting numbers (which may be possibly only 1's

or 0's) we shall call conditions. Some of these conditions we shall call directly testable. Only those conditions directly are/testable which involve the basic observations only. We shall encounter conditions that are not directly testable because they involve certain entities (constructs) that are not directly observable. An important class of these constructs will be called utilities (identical with sensations in psychophysics, values in anthropology, ethics, and older economics). These will be defined in a variety of ways but will always denote certain real numbers, constant or random, associated with the elements of the set A .

However, to involve basic observations only is a necessary but not a directly sufficient property of a/testable condition. For example, if the set A of alternatives is finite but of a size unknown, or too large to be exhausted by experiment, a constraint involving all alternatives simultaneously - such as Condition (o) in Section IV and (P_n) in Section V - cannot be tested.

The purpose of this paper is to study the logical relations between various conditions describing consistency of responses; and in particular the logical relations between certain conditions that are not directly testable and others that are. This will tell whether a given class of observational results would or would not justify the acceptance (or the rejection) of a theory.

Our particular way of defining the class of basic observations and, correspondingly, of the directly testable conditions is to some extent arbitrary. Depending on the range of possible experiments and other observations, it may be preferable to define the class more narrowly, e.g., by including binary choices only. Or we might define this class more broadly.

Following the practice of psychologists, we might admit the ranking, by the subject, of three or more objects as an observable fact, although the subject's observed action consists in this case of a verbal statement. (In the case of two objects, ranking and choosing is the same thing.) We might even admit as observable the subject's verbal statements of the relative "intensity" of his preferences.

Such variations of the domain of testability will be tentatively discussed in Section VII. In the rest of the paper, by using a particular demarcation of the class of ^{directly}testable conditions (the one most closely corresponding to the nature of economic observations), we are able to carry out a reasonably complete analysis of the relevant logical relations. The study may thus serve as a start when similar attempts are carried out under another definition of basic observations.

In particular, our operational approach seems to be unable to handle the following distinction that appears natural on grounds of common sense and may be important for predictions. If out of the pair $F = (a, b)$ of desirable objects a man chooses sometimes a and sometimes b , our introspection tells us that we may ascribe this to either or both of two different "causes":

1) he may have a difficulty in perceiving all the relevant characteristics of the objects, as when a and b are two 10-ton carloads of the same merchandise but the exact quantity (in pounds, net of package) or, for that matter, all the differences in quality, cannot be ascertained; 2) even if he knew exactly the differences in the characteristics of the two objects he might find them almost equally "desirable" or containing the promise of equal "satisfaction"; and he will vacillate as a result. To disentangle the two

"causes" - call them "perceptibility" and "desirability" (anticipated "satisfaction") - may be important if one wants to predict how people will act if perceptibility is kept constant while desirability varies; or vice versa. Economists in particular have mostly considered men's decisions under the assumption of perfect perceptibility, also called perfect information. In the more recent developments of economic decision theory, imperfect information is introduced, but a clear, and probably fruitful, distinction is kept between varying the nature of (so called) information and varying the satisfaction (the "payoff") attainable through the choice of action. Economists* use the term "utility" as interchangeable with desirabil-

* As pointed out by James Tobin in a private communication.

ity (satisfaction), and thus independent of perceptibility. Our concept is coarser. All the various definitions of utility given in this paper will be related to the empirical entities, called "alternatives." Each of these is identified precisely but combines the information and the desirability aspect in some unknown though presumably not too changeable fashion. Thus when two carloads are offered again, and, say, the firm's quality control facilities (or, in a similar example, a housewife's abilities to discriminate between cuts of meat) have not changed, predictions from previous behavior observations can be made. If discrimination is learned, through experience, in some more or less determined manner, the model has to be modified, of course; but we have not attempted to bring in learning.

Stochastic theory introduces the probability $p(a;F)$ of the basic observation $(a;F)$; and it is assumed that inferences about constraints on these probabilities can be made from a finite number of basic observations. These constraints may be described with the help of parameters of certain distributions, and some of these parameters have been called - by Fechner [1], a century ago - sensations. We shall call them constant utilities. Various types of constant utilities have been proposed. They will be studied in Section II. In addition, there emerge naturally the concepts of random utilities and random orderings, to be treated in Section III. Sections IV and V complete the study of directly testable conditions necessary and/or sufficient for the existence of constant and random utilities (and random orderings), respectively. Section VI sums up the main logical relations between the random and the various forms of constant utilities; and includes a very incomplete attempt to relate our various models to the work of other authors. As already mentioned, a generalized experimental situation - the combination of ranking and choosing - will be formulated in Section VII. Section VIII discusses the memory effect and other difficulties of experiments on choice and ranking, perhaps not present to the same extent in psychophysics proper. We shall postpone the description of Section IX until the end of the present introductory section. Section X deals tentatively with a special statistical problem: How to test statistically our directly testable conditions - i.e., the properties of the probabilities $p(a;F)$ - when (to avoid the memory effect and to avoid assuming that all subjects have identical distribution properties) - each set F can be offered only a few times, possibly only once.

Clearly, non-stochastic theory is a special, strong form of stochastic theory. And whenever a theory of choices or responses is being submitted to a statistical test there is, explicitly or not, an underlying model of stochastic behavior of the subject (or possibly of the error-making observer).

We shall conclude this Section with a quick look at the non-stochastic, or absolute, theory of choice. We shall formulate it in terms of our "basic observations," and not, as usual, in terms of the subject's verbal preference or indifference statements. Define* for all a, b in A :

* Preferences thus defined are "independent of irrelevant alternatives," a principle pointed out by Hotelling's pupil Arrow [1] and much used ever since. A good name for Arrow's principle is: "irrelevance of added alternatives."

" $a > b$ " (a preferred to b) means "never ($b;F$) if F contains a "

(1.1) " $a \succeq b$ " means "not : $b > a$ "

" $a \sim b$ " (indifference) means " $a \succeq b$ and $b \succeq a$."

Thus if the subject forced to choose between a and b chooses b nine (but not ten) times out of ten, the non-stochastic theory calls him indifferent. We have always $a > b$ or $b \succeq a$ since the subject is forced to choose. Therefore the relation " \succeq " is said to induce quasi-ordering on A provided the following condition (testable by basic observations) is satisfied:

(1.2) if $a \succeq b$ and $b \succeq c$ then $a \succeq c$. (Transitivity)

If, in addition, the set A is finite, $A = (a_1, \dots, a_n)$, we can associate with each of its elements a_i an integer $r_i (1 \leq r_i \leq n)$, called rank, such that

$$(1.3) \quad r_i \leq r_j \quad \text{if} \quad a_i \succcurlyeq a_j .$$

The vector $r = (r_1, \dots, r_n)$ which can also be regarded as an integer-valued function on the set of integers $N = (1, \dots, n)$ is called a ranking on N . If by some arbitrary convention ties are excluded, all r_i are different integers and r is a permutation. Clearly any function ω that is a strictly decreasing monotone function of the integers r_1, \dots, r_n induces a real-valued function on A and is order-preserving in the sense that

$$(1.4) \quad \omega(a_i) \geq \omega(a_j) \quad \text{if and only if} \quad a_i \succcurlyeq a_j .$$

ω , called the ("ordinal") utility function, is unique up to decreasing monotone transformations.

If the set A of alternatives is not finite an order-preserving function ω on A need not exist. However, Debreu [1] proved that an order-preserving function exists if A and the ordering relation " \succcurlyeq " satisfy a certain rather weak condition* that may justify the assumption of ordinal utility functions

* The condition is:

$$(1.5) \quad A \text{ is perfectly separable and is ordered by the relation } \succcurlyeq; \text{ and} \\ \text{for every } a' \text{ in } A \text{ the sets } (a|a' \succcurlyeq a) \text{ and } (a|a \succcurlyeq a') \text{ are closed.}$$

The condition is trivially fulfilled by every finite set; for it is perfectly separable and all its subsets are closed. Also, every subset of a finite-dimensional Euclidian space is perfectly separable; and it is reasonable to assume that the sets $(a|a' \succcurlyeq a)$ and $(a|a \succcurlyeq a')$ are closed when " \succcurlyeq " means a consumer's preference over the space of a finite number of commodities.

over the space of commodity-bundles.*

* Thus Abraham Wald [1] has suggested a method to evaluate the consumer's ordinal utility function ω from the basic observations $(a;F)$ - purchases made as F varies with incomes and prices - assuming a non-stochastic model and not attempting a statistical estimate. More recently, H. Theil and H. Neudecker [1] gave a stochastic generalization of the same model, specifying, as did Wald, ω to be quadratic in the quantities of the goods consumed.

In the stochastic models that will follow, the (testable) transitivity condition (1.2) and the "ordinal" utility function ω made possible by it will be suitably generalized. But in addition, some stronger testable conditions and, correspondingly, more strictly measurable utility functions will arise naturally.

A final remark: the case when the set A includes wagers so that choices are, in general, made under uncertainty, is more general than that of choices among sure alternatives. This case has been often treated, ever since Daniel Bernoulli [1] and, for that matter, Marshall [1], by ordering the wagers according to their "expected utility." This leads to a non-stochastic utility that is more strictly measurable than ω : It is unique up to increasing linear transformations. This model, too, lends itself to stochastic generalizations, as will be briefly discussed in our Section IX.

II. STOCHASTIC CONCEPTS OF CONSTANT UTILITIES

In general, $A = (a, b, \dots)$ will continue to denote the set of alternatives, and F the feasible subset. For mathematical ease, we shall assume A finite, unless otherwise stated, and identify it with $N = (1, \dots, n)$. A feasible subset will be $M \subseteq N$. The probability that the subject forced to choose an element of M chooses i , denoted previously by $p(i; M)$, can be written more briefly thus: $i(M)$. Clearly

$$(2.1) \quad i(M) \geq 0; \quad \sum_{i \in M} i(M) = 1.$$

When $M = (i, j, k, \dots)$, $i(M) = i((i, j, k, \dots))$ will be written simply $i(i, j, k, \dots)$. $i(M)$ will be called binary, ternary, ... probability when M consists of 2, 3, ... distinct elements. The binary probability will be sometimes written in still shorter forms: $i(i, j) = ij$, $i \neq j$. It will prove convenient to define $ii = 1/2$ so that always

$$(2.2) \quad ij + ji = 1,$$

whether i and j are distinct or not.

For easier reference, the various conditions will be labeled by (more or less suggestive) letters, thus: (x) . A theorem is an implication relation between conditions (the "hypothesis" and the "conclusion"); by using arrows several theorems can be combined into one. In addition to the usual signs \longrightarrow ("implies") and \longleftrightarrow ("implies and is implied by") we shall also need \dashrightarrow ("implies but is not implied by"). When $(x) \dashrightarrow (y)$, (x) is said to be stronger than (y) ; rejection of (y) forces rejection of (x) but acceptance of (y) is inconclusive. We shall also use the sign \dashv for "does not imply nor is implied by."

Each of the following three conditions (w), (v), (u), arranged in a sequence of increasing strength, constrains the set of probabilities $i(M)$ by postulating the existence of some real vector of order n (a real-valued function on the set N), called utility vector (utility function), and denoted by w, v, u , respectively. The stronger the constraint the more strict is the sense in which the utility vector is measurable, that is: the smaller is the group of transformations under which the vector remains indeterminate.

Condition (w): There is a constant real vector $w = (w_1, \dots, w_n)$ such that

$$(2.3) \quad w_i \geq w_j \quad \text{if and only if} \quad ij \geq 1/2.$$

w_i may be called weak utility of i ; and w , the weak utility function on N .

Condition (v): There is a constant real vector $v = (v_1, \dots, v_n)$ and, associated with it, a distribution function ϕ_v , strictly increasing [except when its value is 0 or 1],* such that

* Regarding the proviso in brackets, see Section VIII.

$$(2.4) \quad \phi_v(v_i - v_j) = ij; \quad \phi_v(0) = ii = 1/2.$$

v_i may be called strong utility of i ; and v , the strong utility function of N .

Condition (u): There is a constant positive vector $u = (u_1, \dots, u_n)$ such that for any i, j in $M \subseteq N$,

$$(2.5) \quad u_i/u_j = i(M)/j(M)$$

u_i may be called strict utility of i ; and u , strict utility function on N .

Clearly the weak utility function w is unique up to an increasing monotone transformation. It is analogous to the function ω of the non-stochastic model (Section I), with " $a_i \succ a_j$ " interpreted as " $ij \geq 1/2$." We can call the cases $ij >$ and $= 1/2$, stochastic preference and indifference, respectively. These concepts are implicitly used in experimental work as when, e.g., Mosteller and Nogee [1] define indifference as the case when the subject chooses one of the two offered alternatives half of the time. As in the non-stochastic model, (w) implies a testable

Condition (t) (transitivity):

(2.6) If $ij \geq 1/2$ and $jk \geq 1/2$ then $ik \geq 1/2$.

Since always $ij \geq$ or $\leq 1/2$, transitivity guarantees a (weak) ordering on N : the alternative is ranked above, below, or on the level of j according as $ij >$, $<$, or $= 1/2$. Since N is finite we have (as in Section I)

Theorem II.1: $(w) \longleftrightarrow (t)$ provided the set of alternatives is finite.

If we admit infinite sets of alternatives and redefine the condition (w) accordingly, we have $(w) \longrightarrow (t)$. But if we denote by (D) Debreu's condition (1.5) [which implies condition (t)] we can write

Theorem II.2: $(D) \longrightarrow (w) \longrightarrow (t)$.

Condition (v), the existence of "strong" utilities may be made plausible by a physical analogy. Because of random variations in the properties of the air, the metals, etc., the lowest voltage needed to produce a spark in a given direction is random; assume then that the probability of

its occurrence of the spark is the larger, the larger the voltage. Condition (v) has been used in psychophysics (and more recently in the scaling of attitudes) ever since Fechner [1]. It is stated in the psychologists' adage (Guilford, [1]): "equally often noticed differences (on the "sensation" scale) are equal [unless noticed always or never]." A similar thought might unify the recent attempts to measure the "power" of person X over Y : on the one hand, by the probability that Y obeys X ; and on the other hand, by the difference in the payoff (utility) to Y in the case of his obeying, as compared with disobeying, X . *

* These two points of view, the empirical and the game-theoretical, are represented in somewhat modified form by Dahl [1] and by Shapley and Shubik [1], respectively. If one accepts John Harsanyi's proposal to measure the power of X over Y by "how much difference it makes to Y between being X's friend or enemy," this difference between maxmax and maxmin payoffs (see, e.g., Luce and Raiffa [1]) might be related to the probability of Y's decision to be a friend.

If (v) is satisfied by some v , ϕ_v , then clearly for any increasing linear transform $v' = \alpha + \beta v$ there exists a $\phi_{v'}$ satisfying that condition. (If the set of alternatives is continuous, and the distribution ϕ_v strictly monotone, these are the only admissible transformations.**) The scale unit

** For suppose that $\phi_{v'}(v'(x) - v'(y)) \equiv \phi_v(v(x) - v(y))$. Then with $\psi = \phi_{v'}^{-1} \phi_v$ we have (i) $v'(x) - v'(y) = \psi(v(x) - v(y))$, (ii) $v'(y) - v'(z) = \psi(v(y) - v(z))$ and (iii) $v'(x) - v'(z) = \psi(v(x) - v(z))$. Adding (i) and (ii) and using (iii) we find, letting $a = v(x) - v(y)$, $b = v(y) - v(z)$, that $\psi(a+b) = \psi(a) + \psi(b)$. The only solution of this last equation which is not unbounded in every interval is, as is well known, $\psi(a) = \beta a$. Then from (i), with y fixed, we have $v'(x) = \beta v(x) + \alpha$.

for v may be chosen by setting, for example, the quartile $\phi_v^{-1}(.25) = 1$; this or a similarly chosen unit is sometimes called the "just noticeable difference," we believe. Moreover, because of (2.4) ϕ_v may be (and, in the continuous case, must be) so chosen that:

$$(2.7) \quad \phi(\xi) + \phi(-\xi) = 1$$

for all real ξ . Then ϕ_v is anti-symmetrical about the point $(0, 1/2)$; the median and the mean (if it exists) are zero. This is satisfied if ϕ_v is normal, as assumed by Fechner whose test of normality was, however, rather crude. Normality of ϕ_v is also assumed in most textbooks; we don't know whether a formal test of this assumption has been developed. The "logistic" distribution function $\phi_v(\xi) = 1/(1 + e^{-\xi})$ which, as we shall presently see, is required by the next stronger condition (u), also has the property (2.7).

Theorem II.3. $(v) \rightarrow (w)$. Proof. Sufficiency: Assume (v) and let $v_1 \geq v_j$; then $ij = \phi_v(v_1 - v_j) \geq \phi_v(0) = 1/2$; hence (w) is satisfied, with $w = v$. No necessity: Let $N = (1, 2, 3)$ and assume $12 > 13 > 23 > 1/2$. Then (w) is satisfied, with $w_1 > w_2 > w_3$; but no linear function of w will satisfy (v); for, by (v), if $1/2 < 23$ then $0 = v_1 - v_1 < v_2 - v_3$, $v_1 - v_2 < v_1 - v_3$; hence, by (v), $12 < 23$, contradicting the assumption, see also Theorem IV.1.

As to the "strict" constant utility condition (u), it has been postulated, and developed most fully, by Duncan Luce [2].* We shall now give some of his results. Because of (2.1), Condition (u) can be rewritten in the form

* Occasionally (in his Section II.D.3 entitled "A generalization to two or more alternatives") Luce seems to combine Condition (u) with another condition which we shall call (u') (existence of strict disutilities of last choices) and which is not generally consistent with (u): see our Theorem III.8 below.

$$(2.8) \quad i(M) = u_i / \sum_{j \in M} u_j, \text{ all } M \subseteq N.$$

It follows that the vector u is unique up to a positive factor $\lambda = u_i / i(N)$. With N fixed, a convenient normalization is $\lambda=1 = \sum_{i \in N} i(N)$; $u_i = i(N)$.

Putting $M = (i,j)$, $i(M) = ij$ we see that the probability of a binary choice is related to the ratio u_i / u_j by

$$(2.9) \quad \psi(u_i / u_j) = ij, \text{ where } \psi(\zeta) = \zeta / (1 + \zeta), \zeta > 0 \\ = 0, \zeta \leq 0.$$

Clearly ψ is a distribution function; for it is increasing, and $\psi(-\infty) = 0$, $\psi(\infty) = 1$. It is analogous to ϕ_v of condition (v), with ratios replacing differences, but is restricted to a particular form. Moreover, putting $v_k = \log u_k$ we recover condition (v) itself, with ϕ_v a "logistic curve":

$$(2.10) \quad \phi_v(\zeta) = 1 / (1 + e^{-\zeta}).$$

Theorem II.4: $(u) \dashv\vdash (v)$,

for (v) imposes constraints on binary choices only.

We shall now introduce two directly testable conditions equivalent to (u).

First consider

Condition (p.r) (constancy of probability-ratios): $i(M) > 0$ for every i in M ; and for every i, j in N , the ratio $i(M)/j(M)$ is constant over all sets $M \subseteq N$ that contain i and j .

(p.r) follows directly from (u); conversely, rewriting (p.r) as (2.11) $i(M)/i(N) = j(M)/j(N) = \dots = \frac{\sum_{k \in M} k(M)}{\sum_{k \in M} k(N)}$, putting $k(N) = u_k$ and using (2.1), we obtain (u). Now consider

Condition (c.p) (probability of choice as a conditional probability):

$$(2.12) \quad 0 < i(M) = i(N) / \sum_{k \in M} k(N) \quad \text{all } i \text{ in } M \subseteq N.$$

The fraction on the right side, is, of course, the conditional probability that the element i of N is chosen when it is known that the element chosen belongs to the subset M of N . On the other hand, the quantity on the left side is the probability that the element i of N is chosen when the subject is forced to make his choice from the subset M of N . Clearly these two quantities are not identical; (c.p) is a verifiable, empirical proposition which may or may not have intuitive appeal.* Condition (c.p)

* Luce tries to convey this appeal by an analogy with Arrow's (non-stochastic) principle which we called (above) the "irrelevance of added alternatives."

follows directly from (u) written in the form (2.8); on the other hand, replacing i by j in (2.12) and comparing the two equations one obtains (p.r) which we have seen to be equivalent to (u). We summarize this and the previous results of this Section in the following theorem, placing the directly testable conditions in the lower line.

Theorem II.5

$$\begin{array}{ccccc}
 (u) & \longmapsto & (v) & \longmapsto & (w) \\
 \updownarrow & & & & \updownarrow \\
 (p.r) & \longmapsto & (c.p) & \longmapsto & (t)
 \end{array}$$

III. RANDOM ORDERINGS AND RANDOM UTILITIES

In the non-stochastic theory briefly reviewed in Section I, the preference statement " $a_i > a_j$ " assigns probability 1 to a certain class of choices. In the stochastic models of Section II, the corresponding statement " $i_j > 1/2$ " merely sets a lower bound on that probability and is therefore weaker; however, some of those models are weaker than others.

A different way of weakening the non-stochastic theory consists in making the ("ordinal") utility function ω a random one; i.e., by defining a probability measure on the space of all real valued functions on A . To take up the economic example of Section I: A. Wald's problem of evaluating the (non-stochastic) utility function ω of consumers from their observed choices $(a;F)$, is replaced by the following more general one: estimate the probability distribution P on the space of utility functions ω , using the probabilities $p(a;F)$ already estimated from the observations $(a;F)$; or using these observations directly.

However, the probabilities $p(a;F)$ of the choices may be such as to exclude the existence of the probability distribution P on the ω -space. The present Section, and also Section V, deals with conditions on the $p(a;F)$ necessary and/or sufficient for the existence of P . Only the case of finite sets will be studied, so that ω can be regarded as a vector $(\omega_1, \dots, \omega_n)$. It will not change matters if we associate P with one particular monotone transform of ω , viz., the ranking $r = (r_1, \dots, r_n)$. For simplicity, no ties will be admitted. Then R , the set of all rankings on the set of alternatives $N = (1, \dots, n)$ consists of $n!$ elements. Denote by R_{iM} the set

of all rankings on N in which i is the first among all elements of $M \subseteq N$:

$$(3.1) \quad R_{iM} = \{r \mid r_i \leq r_j, \text{ all } j \in M\}, \quad i \in M;$$

Then clearly, for every $M \subseteq N$,

(3.2) R_{iM} and R_{jM} are disjoint for all $i \neq j$ in M ; and if we denote by R^M the set of all permutations on M , we have

$$(3.3) \quad \bigcup_{i \in M} R_{iM} = R^M; \quad \bigcup_{i \in N} R_{iN} = R^N = R.$$

We shall denote by i_r the element of N that has rank i when the ranking r is applied; i.e., $r_{i_r} = i$. Using this notation, a given ranking $r = (r_1, \dots, r_n)$ will be sometimes identified as $1_r 2_r \dots n_r$ (without commas and parentheses). Thus if $n = 3$ the ranking $r = 312$ means that $r_1 = 2$, $r_2 = 3$, $r_3 = 1$, and is therefore identical with $r = (2, 3, 1)$. We now state two conditions that will be shown to be equivalent. (We use the notation for finite sets introduced in Section II).

Condition (P) (existence of a probability distribution of rankings consistent with probabilities $i(M)$): There are $n!$ numbers $P(r)$ such that

$$(3.4) \quad P(r) \geq 0, \quad \sum_{R_{iM}} P(r) = i(M), \quad i \in M \subseteq N.$$

It follows, using (3.3), (2.1), that $\sum_R P(r) = 1$. We can call $P(r)$ the probability of the ranking r .

Condition (U): (existence of random utilities): There is a random vector
 $U = (U_1, \dots, U_n)$, unique up to an increasing monotone transformation, and
such that, for any i in M ,

$$(3.5) \quad \text{Prob}(U_i \geq U_j, \text{ all } j \in M) = i(M) .$$

U_i may be called random utility of i .

Theorem III.1: $(U) \longleftrightarrow (P)$. Proof. If we assume Condition (U) and define

$P(r) = \text{Prob}(U_{1_r} \geq U_{2_r} \geq \dots \geq U_{n_r})$ we obtain Condition (P). Conversely,

assume (P) and define the random vector U thus: for any non-random real vector $s = (s_1, \dots, s_n)$ let

$$(3.6) \quad \text{Prob}(U = -s) = \begin{matrix} P(s), & s \in R, \\ 0 & , s \notin R, \end{matrix}$$

where R is the set of permutations on N . Then Condition (U) is satisfied.

Theorem III.2: The existence of random utilities U does not imply the
existence of weak constant utilities w , let alone the existence of strong
or strict utilities, v or u . Proof. Let $N = (1,2,3)$ and $0 < \alpha < 1/6$.

Let the probabilities of rankings, $P(r)$ be

$$P(123) = P(231) = P(312) = 1/6 + \alpha > 0$$

$$P(321) = P(213) = P(132) = 1/6 - \alpha > 0 .$$

(Then $\sum P(r) = 1$). Let the probabilities of choices $i(M)$, $M \subseteq N$, be

$$1(N) = 2(N) = 3(N) = 1/3 ;$$

$$(3.7) \quad 1(1,2) = 2(2,3) = 3(1,3) = \alpha + 1/2 ;$$

$$2(1,2) = 3(2,3) = 1(1,3) = -\alpha + 1/2 .$$

Then $i(N) = P(123) + P(132)$, $i(1,2) = P(123) + P(132) + P(312)$, etc., and Condition (P) -- or its equivalent, (U) -- is satisfied. But Condition (t) is not:

$$i(1,2) > 1/2, i(2,3) > 1/2 \text{ but } i(1,3) < 1/2 .$$

this proves the present Theorem, using Theorem II.5.*

* This counterexample is due to C. Winsten. Another interesting proof is due to P. Halmos. Three dice A,B,C, are loaded so as to turn up with the following probabilities:

Face No	1	2	3	4	5	6
Die: A	0	0	.5	.5	0	0
B	0	.6	0	0	0	.4
C	.4	0	0	0	.6	0

Let a, b, c (independent chance variables) be the number of spots turning up on A,B,C, respectively. Then (P) is satisfied: the probabilities of the six possible rankings of the numbers a, b, c according to magnitude, must be consistent with the probabilities that one of the numbers is the largest in a given pair or triple. But (t) is not satisfied: $\text{Prob}(a > b) = .6 > 1/2$; $\text{Prob}(b > c) = .64 > 1/2$; $\text{Prob}(a > c) = .4 < 1/2$.

We shall show that the following conditions are necessary for (P), and therefore for its equivalent, (U):

Condition (e) (effect of enlarging the feasible set): If $L \subseteq M \subseteq N$

then

$$(3.8) \quad i(M) \leq i(L).$$

Condition (e₃): For any three elements i, j, k of N ,

$$(3.9) \quad i(i, j) \geq i(i, j, k); \text{ or, equivalently,}$$

$$(3.9') \quad i(i, j, k) \leq \min(ij, ik).$$

Clearly (e_3) is a special case of (e) ; it was exemplified numerically in (3.7), for $n = 3$. Applying to (P) the fact that

$$(3.10) \quad R_{iM} = (r | r_i \leq r_j, j \in L) \cap (r | r_i \leq r_j, j \in M - L) \subseteq R_{iL},$$

we obtain

Theorem III.3: $(P) \longrightarrow (e) \longmapsto (e_3)$.*

* Whether or not (e) implies (P) will be left open until Section V.

Theorem III.4: The existence of random utilities U is not implied by the existence of strong constant utilities v , let alone by the existence of weak constant utilities w . Proof. Denote by x, y, z the distinct generic elements of $N = (1, 2, 3)$. There exist three numbers (for example, $v_1 = .2, v_2 = .1, v_3 = 0$) such that

$$0 < v_x - v_y + 1/2 = \beta_{xy} < 1$$

$$0 < v_1 - v_2 = \gamma_1 < 1$$

$$0 < v_2 - v_3 = \gamma_2 < 1$$

$$0 < v_3 - v_1 + 1 = \gamma_3 < 1.$$

Since $\beta_{xy} + \beta_{yx} = 1 = \sum_x \gamma_x$, we can put $x(x, y) = \beta_{xy}$, $x(x, y, z) = \gamma_x$.

Then Condition (v) is satisfied, with $\phi_v(\xi) = \xi + 1/2$. But since

$\gamma_3 - \beta_{31} = 1/2 > 0$, $3(1, 2, 3) > 3(1, 3)$, Condition (e) , and therefore also

the stronger Condition (P) , and its equivalent, (U) , is contradicted.

In the work of Thurstone [1] the following condition seems to be used:

Condition (s.n.) (symmetrical normal): There is a normal random vector

$U = (U_1, \dots, U_n)$ that satisfies Condition (U) and has variances equal
 ($\sigma_{ii} = \sigma^2$, say) and all covariances equal ($\sigma_{ij} = \rho \sigma^2$, say). Clearly this
 implies both (U) and (v), with $v_i = U_i$ and ϕ_v normal with mean zero and
 variance $2\sigma^2(1 - \rho)$. Even the following, weaker condition is sufficient
 for the conjunction [(U), (v)]:

Condition (s) (symmetry of adjusted random utilities): There is a random
vector U satisfying condition (U), and a constant vector $v = (v_1, \dots, v_n)$
such that $X = U - v$ has a distribution function symmetric in its arguments.

Any increasing linear transform $X' = \alpha + \beta X$ will have the same symmetry
 property; and if the set of alternatives is continuous these are the only
 possible transformations. Clearly (n.s.) \rightarrow (s); moreover:

Theorem III.5: (n.s.) \rightarrow (s) \rightarrow [(U), (v)]. Proof. 1) Sufficiency of (s):

If (s) is true, so is (U); and $i(i,j) = \text{Prob}(U_i \geq U_j) = \text{Prob}(X_j - X_i < v_i - v_j)$
 = the value of a distribution function at $v_i - v_j$; because of symmetry this
 function is the same for all i, j and can be written as ϕ_v , satisfying (v).
 of [(U), (v)]:

2) No sufficiency/it suffices to find a non-symmetrical joint normal distribu-
 tion with zero mean and a constant variance of differences:

$$(3.11) \quad \sigma_{ii} + \sigma_{jj} - 2\sigma_{ij} = \tau, \text{ all } i \neq j.$$

Let, $n = 3$, $\sigma_{11} = \sigma_{22} = 1$, $\sigma_{33} = 1/2$, $\sigma_{23} = \sigma_{13} = 0$, $\sigma_{12} = 1/4$.

If U satisfies (U) then (v) is also satisfied, with ϕ_v normal with
 variance $\tau = 3/2$. But no translation can make the distribution function
 of U symmetric in its arguments.

Mosteller [1] dropped Thurstone's symmetry assumption but maintained normality. We don't know whether tests for the joint condition [(U), (v)] in general, or for the special case of normality with constraint (3.11), have been developed.

We shall now discuss another condition that is also stronger than the conjunction [(U), (v)] and is of particular interest, being equivalent to Condition (u) of Section II (existence of strict constant utilities).

For a given permutation r of the set N define

$$(3.12) \quad \pi(r) = \prod_{j=1}^{n-1} j_r(j_r, (j+1)_r, \dots, n_r).$$

For example $\pi(312) = 3(1,2,3) \cdot 1(1,2)$. Suppose this were the probability of the ranking 312. This would exemplify

Condition (π) (probability of ranking as the product of probabilities of successive first choices): Condition (P) is true and, for every ranking r of N , $P(r) = \pi(r)$ as defined in (3.12).

Substituting $\pi(r)$ for $P(r)$ in (3.4), condition (π) can be put into a form involving the "basic" probabilities (those of "first choices") only:

$$(3.13) \quad i(M) = \sum_{r \in R_{iM}} \pi(r) \text{ for all } M \text{ and } i \in M.$$

Theorem III.6. $(u) \longleftrightarrow (\pi)$. Proof outline. Sufficiency of (u). Define

$$(3.14) \quad \Omega(r) = \prod_{j=1}^{n-1} u_{j_k} / \sum_{k=j}^n u_{k_r}.$$

To prove that (u) implies (π), or (2.8) implies (3.13), is to prove the identity (an urn model discussed in Block and Marschak [2] makes it intuitively plausible)

$$(3.15) \quad \sum_{r \in R_{iM}} \Omega(r) = u_i / \sum_{h \in M} u_h,$$

for all $i \in M$, all $M \subseteq N$, and any positive numbers u_1, \dots, u_n . Without loss of generality let $M = (1, \dots, m)$, $m \leq n$, and, as in (3.3), denote by R^M the set of all permutations of M . Then $r^M = (r_1^M, \dots, r_m^M) \in R^M$. The set R^N is partitioned into m subsets of the form

$$R_{iM}^N = (r^N | r_i^N \leq r_j^N; /j \in M), \quad i \in M; \quad \text{all}$$

the set R^N is also partitioned into $m!$ subsets of the form

$$R^N(r^M) = (r^N | r_i^N \leq r_j^N \text{ if } r_i^M \leq r_j^M; /i, j \in M). \quad \text{all}$$

One obtains, by induction on m , the identities

$$\sum_{r^M \in R^M} \Omega(r^M) = 1; \quad \sum_{r^M \in R^M(r^L)} \Omega(r^M) = \Omega(r^L),$$

and these lead to (3.15). Necessity of (u). To show that (3.13) implies (2.8) let $u_i = i(N)$, all $i \in N$. Because of the identity (3.15), all numbers $i(M)$ that satisfy (3.13) also satisfy the system of equations

$$(3.16) \quad \begin{cases} i(M) = \sum_{r \in R_{iM}} u_{1_r} \prod_{j=2}^{n-1} j_r(j_r, \dots, n_r), \\ \sum_{i \in M} i(M) = 1. \end{cases}$$

One then proves the uniqueness of this solution: starting with $M=N$, then

taking all sets M with $n-1$ elements, then with $n-2$ elements and so on, one can solve (3.16) for all $i(M)$ and finds that they are all given by (2.8).*

* In more detail: If $M=N$ or $M=N-(n)$, the system (3.16) is readily solved for the $i(M)$. Now suppose that $i(M)$ has a unique value satisfying (3.16) for all M with more than h elements, $h \leq n-1$; and let $H = (1, \dots, h)$, $J = N - H$. Then (3.16) would imply that

$$\begin{aligned}
 (3.17) \quad i(H) &= u_1(N) + \sum_{j \in J} u_j \cdot i(N-(j)) \\
 &+ \sum_{j_1 \in J} \sum_{j_2 \in J-(j_1)} u_{j_1} \cdot j_1^{(N-j_1)} \cdot i(N-(j_1, j_2)) \\
 &+ \dots + \sum_{j_1 \in J} \sum_{j_2 \in J-(j_1)} \dots \sum_{j_{n-h} \in J-(j_1, \dots, j_{n-h-1})} \\
 &[u_{j_1} \cdot j_2^{(N-(j_1))} \dots j_{n-h}^{(H+(j_{n-h}))}] i(H) .
 \end{aligned}$$

Except for the two $i(H)$'s every probability appearing here is the probability of a first choice from a set of more than h elements and by the induction hypothesis is uniquely determined in terms of the u 's. Furthermore the quantity in brackets is equal to

$$\sum_{s \in R^H} j_1^{(N)} \cdot j_2^{(N-(j_1))} \dots j_{n-h}^{(H+(j_{n-h}))} \cdot 1_s(H) \cdot 2_s^{(H-(1_s))} \dots (h-1)_s^{((h-1)_s, h_s)};$$

thus the multiplier of $i(H)$ on the right side of (3.17) is equal to the probability, p , that the set J precedes the set H . Since $1(N) > 0$ and $1 \in H$ it follows that $p < 1$ and (3.17) can be solved for $i(H)$.

Theorem III.7. If (u) is true then (P) is true, with $P(r) = \pi(r) = \Omega(r)$.

This is clearly a corollary to III.6.

Some interest attaches to a condition that, in effect, appears in the work of Luce [2]. In the same way as Condition (u) treats the probabilities

of first choices, $p(i;M) = i(M)$, this new condition treats the probabilities of last choices. We may denote these by $p'(i;M) = i'(M)$ and remark that last choices are not "basic observations" in our sense. (However, see Section VII on considering the rankings themselves, and hence also the last choices, "observable").

Condition (u'). There exists a constant positive vector $u' = (u'_1, \dots, u'_n)$ such that for any i, j in M ,

$$u'_i/u'_j = i'(M)/j'(M) .$$

It is possible, however, to prove

Theorem III.8. If $n > 2$, (u') is inconsistent with (u) unless $i(N) = i'(N) = 1/n$ for all $i \in N$. Proof outline. Define, for each ranking $r = 1_r 2_r \dots n_r$ the inverted ranking $r^* = n_r(n-1)_r \dots 2_r 1_r$ (hence $r^{**} = r$); and define

$$\Omega'(r) = \prod_{j=1}^{n-1} u'_{j_r} / \sum_{k=j}^n u'_{k_r} .$$

By Theorem III.7, if (u) is true then $P(r) = \Omega(r)$. By the same theorem, if (u') is true then $P(r^*) = \Omega'(r)$, and hence $P(r) = P(r^{**}) = \Omega'(r^*)$. Therefore if both (u) and (u') are true,

$$(3.18) \quad \Omega(r) = \Omega'(r^*) .$$

Define the ranking $s = 2_{r^1} 3_{r^2} \dots (n-1)_{r^{n-1}}$. Since $\sum u_i = 1 = \sum u'_i$ without loss of generality, we have

$$\Omega(r)/\Omega(s) = (1-u_{2_r})/(1-u_{1_r}); \quad \Omega'(r^*)/\Omega'(s^*) = u'_{2_r}/u'_{1_r}.$$

Since $1_r, 2_r$ are arbitrary we have by (3.18) $u'_i/u'_j = (1-u_i)/(1-u_j)$; and summing on i , we find $u_j = 1-(n-1)u'_j$. Now define $t = 1_r 2_r \dots (n-2)_r n_r (n-1)_r$; then by (3.18)

$$\Omega(r)/\Omega(t) = u_{(n-1)_r}/u_{n_r} = \Omega'(r^*)/\Omega'(t^*) = (1-u'_{(n-1)_r})/(1-u'_{n_r});$$

and in general $u_i/u_j = (1-u'_i)/(1-u'_j) = [1-(n-1)u'_i]/[1-(n-1)u'_j]$; or

$(n-2)u'_i = (n-2)u'_j$. Hence, for $n > 2$, $u'_i = u'_j$ and $u_i = u_j$; so that

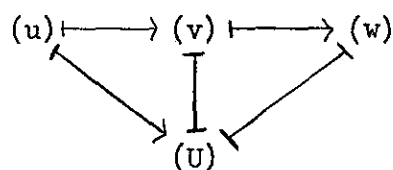
$i(N) = i'(N) = 1/n$ for all i .

Theorem III.9 $(u) \longmapsto (u, v) \longleftarrow (s)$

Proof. $(u) \longmapsto (U)$ by III.6 and III.1 since (π) is strictly stronger than (P) . Moreover $u \longmapsto (v)$ by II.4 and $s \longmapsto (U, v)$ by III.5. It remains to prove that (U, v) does not imply (u) . Consider any normal distribution symmetric in its arguments. Then by III.5, (u) and (v) are both satisfied; and the corresponding ϕ_v is normal, contrary to the implication (2.10) of (u) .

Some other results of this and the preceding sections are worth summarizing in

Theorem III.10



DIRECTLY

IV. FURTHER/TESTABLE CONDITIONS FOR THE EXISTENCE OF CONSTANT UTILITIES

We shall deal here with "strong" utilities \underline{v} only; ^{directly} testable conditions for the existence of "strict" utilities \underline{u} and "weak" utilities \underline{w} were given in Section II. ^{Directly} /Testable conditions for (v) - like those for (w) and unlike those for (u) - can, of course, involve binary choices only.

Unless otherwise stated the set of alternatives will be assumed finite: $N = (1, \dots, n)$. It is easy to see that we can rewrite

Condition (v): There is a real vector $v = (v_1, \dots, v_n)$ such that

$$(4.1) \quad \underline{\text{if}} \quad hi \geq jk \quad \underline{\text{then}} \quad v_h - v_i \geq v_j - v_k,$$

provided hi and jk are not both 0 or 1.

This implies that $hi \geq 1/2$ if, and only if, $v_h - v_i \geq 0$. It is seen immediately that (v) implies each of the following conditions:

Condition (t): (transitivity). If $ij \geq 1/2$ and $jk \geq 1/2$ then $ik \geq 1/2$.

Condition (t_s): (strong transitivity). If $ij \geq 1/2$ and $jk \geq 1/2$

then $ik \geq \max(ij, jk)$

Condition (t_s^{*}): If $ij \geq 1/2$ then $ix \geq jx$, all x in N ,

Condition (t_s^{**}): If there exists k such that $ik \geq jk$ then $ij \geq 1/2$.

Condition (6_w): (weak condition on 6-tuples). If $i_1 i_2 \geq j_1 j_2$ and

$$i_2 i_3 \geq j_2 j_3 \quad \underline{\text{then}} \quad i_1 i_3 \geq j_1 j_3.$$

Condition (m_w): (weak condition on m-tuples). For any $m \leq n$, if

$$i_h i_{h+1} \geq j_h j_{h+1}, \quad h = 1, \dots, m-1, \quad \underline{\text{then}} \quad i_1 i_m \geq j_1 j_m.$$

Condition (q): (quadruple condition). If $hi \geq jk$ then $hj \geq ik$.

Condition (6_s): (strong condition on 6-tuples). If $i_1 i_2 \geq j_2 j_3$ and

$i_2 i_3 \geq j_1 j_2$ then $i_1 i_3 \geq j_1 j_3$.

Theorem IV.1:

$$\begin{array}{ccccccccc}
 (v) & \longrightarrow & (6_s) & \longrightarrow & (q) & \longrightarrow & (6_w) & \longrightarrow & (t_s) & \longrightarrow & (t) \\
 & & & & & & \updownarrow & & \updownarrow & & \\
 & & & & & & (m_w) & & (t_s^*) & \longleftrightarrow & (t_s^{**})
 \end{array}$$

Proof: Obviously (v) implies all the other conditions; and obviously $(t_s) \longrightarrow (t)$. - To prove $(t_s) \longrightarrow (t_s^*)$ let $ij \geq 1/2$ and apply (t_s) to 3 cases: 1) $jx \geq 1/2$; 2) $jx < 1/2, xi \geq 1/2$; 3) $jx < 1/2, xi < 1/2$. To prove the converse let $ij \geq 1/2, jk \geq 1/2$; by (t_s^*) $ix \geq jx, jy \geq ky$; put $x = k, y = i$. - To prove $(t_s^*) \longleftrightarrow (t_s^{**})$, apply a contradiction argument using (2.2). - To prove $(6_w) \longrightarrow (t_s)$, let $ij \geq 1/2, jk \geq 1/2$ and apply (6_w) with $i = i_1, j = i_2 = j_2, k = i_3$, and 1) $j_1 = j, j_3 = k$; 2) $j_1 = i, j_3 = j$. - (m_w) clearly implies, and is itself obtained by successive application of, (6_w) . - To prove $(q) \longrightarrow (6_w)$ is straightforward. - To prove $(6_s) \longrightarrow (q)$ let $hi \geq jk, i_1 = h, j_1 = i_2 = i, i_3 = j_2 = j, j_3 = k$.

The absence of implications is proved by counterexamples: ^{*} (t) does

* For methods to construct these counterexamples see Block and Marschak [1]. Georgescu-Roegen [2] and Chipman [1] obtained a condition intermediate in strength between (t_s) and (t) by substituting Min for Max in (t_s) .

not imply (t_s) since, with $n = 3$ and $12 > 13 > 1/2$ and $23 > 1/2$,

(t) but not (t_s) is satisfied. - To prove that (t_s) does not imply (6_w)

let $n = 4$ and $1/2 < 43 < 32 < 21 < 31 < 42 < 41$. - To prove that (6_w) does

not imply (q) let $n = 4$ and $1/2 < 32 < 43 < 21 < 31 < 42 < 41$. - To prove that (q) does not imply (6_s) let $n = 5$ and $1/2 < 21 < 54 < 32 < 43 < 53 < 31 < 42 < 41 < 52 < 51$. To prove that (6_s) does not imply (v) a counterexample (with $n = 9$) was constructed, but must be omitted here.

All the conditions given so far are necessary but not sufficient for the existence of \underline{v} . With the exception of (m_w) they all involve binary choices among elements of subsets of fixed size m ; $m = 3$ in (t) and (t_s) ; 4 in (q); 6 in (6_w) and (6_s) . The exception (m_w) is only apparent as (m_w) is equivalent to (6_w) . Therefore all those conditions can be applied even when the size of \underline{N} (the number \underline{n} of all alternatives) is unknown. The following two conditions can be applied only if the size of \underline{N} is known.

Condition (m_s) (strong condition on \underline{m} -tuples). For any $m \leq n$,

$$i_1 i_m \geq j_1 j_m \quad \text{if} \quad i_h i_{1+h} \geq j_{r_h} j_{1+r_h} \quad h = 1, \dots, m-1,$$

where $r = (r_1, \dots, r_{m-1})$ is any permutation of the set $(1, \dots, m-1)$.

Condition (o) (ordering of the probabilities of binary choices). Arrange all the $i_j \geq 1/2$ in a sequence of $n(n-1)/2$ inequalities (involving only n distinct alternatives):

$$i_1 i_2 \geq i_3 i_4 \geq \dots \geq 1/2;$$

then the corresponding system of $n(n-1)/2$, inequalities in n distinct real numbers

$$v_{i_1} - v_{i_2} \geq v_{i_3} - v_{i_4} \geq \dots \geq 0$$

must have a solution.

The following needs no proof:

Theorem IV.2: $(v) \leftarrow (o) \rightarrow (m_s) \rightarrow (G_s)$

We have not investigated whether (m_s) implies (o) .

Thus, for a finite set N of alternatives we have the condition (o) [perhaps also (m_s) ?] sufficient and necessary for the existence of a strong utility vector v , and testable if N has known size. For finite N of unknown size our testable conditions are necessary only.

A summary of the main results of this section, assuming the set of alternatives finite, is given by

Theorem IV.3:

$$\begin{array}{ccccccc} & (v) & & & & & (w) \\ & \updownarrow & & & & & \updownarrow \\ (o) & \rightarrow & (m_s) & \rightarrow & (q) & \rightarrow & (G_w) & \rightarrow & (t_s) & \rightarrow & (t) \end{array}$$

If the set of alternatives is ascribed some continuity properties (to be defined presently), some of the necessary testable conditions become also sufficient. We give two examples. Suppose one postulates

Condition (s.c.) (stochastic continuity): For any elements a, b, c of the set A of alternatives such that $ab < \lambda < ac$, there is an element d in A with $ad = \lambda$.

Debreu [2] has proved

Theorem IV.4: $[(s.c.), (q)] \rightarrow (v)$.

Or one may postulate a stronger

Condition (d.b.p.) (continuous differentiability of binary probabilities):

The set a of alternatives is representable by a real interval, and

$p(a, b) = p(a; (a, b))$ is continuously differentiable with $\partial p / \partial b < 0$, $\partial p / \partial a > 0$.

In Block-Marschak [1] proof was given of

Theorem IV.5: $[(c.d), (6_w)] \rightarrow (v)$. To show this, one first proves a

$x, y \in [0, 1]$ and let

Lemma: Let $f(x, y)$ be continuously differentiable with $\partial f / \partial y < 0$,

$\partial f / \partial x > 0$. Then, in order that f admit the representation $f(x, y) = h(u(x) - v(y))$,

where h, u, v are monotone increasing functions of one variable, it is necessary

and sufficient that there exist functions $\alpha(x), \beta(y)$ such that $(\partial f / \partial x) / (\partial f / \partial y) =$

$\alpha(x) / \beta(y)$.

DIRECTLY

V. FURTHER /TESTABLE CONDITIONS FOR THE EXISTENCE OF RANDOM ORDERINGS

Directly /testable sufficient conditions for (U), the existence of random utilities (or, equivalently, for (P), the existence of random orderings) were given in Section III. It was shown, for example, that the probabilities $p(r)$ exist if the ratios $i(M)/j(M)$ do not depend on M . We have also shown that this condition is not necessary. On the other hand, the directly /testable condition (e) is necessary: if the $P(r)$ exist, then $i(M) \leq i(L)$ for any $i \in L \subseteq M \subseteq N$. We shall presently show that this condition is not sufficient. We have not directly found /testable conditions that are both sufficient and necessary for the existence of $P(r)$ on N . The following partial results may, however, prove useful. They still use finite sets of alternatives only, of known or unknown size. The case of continuous sets has not been explored.

Condition (e_3) defined in Section III, (3.9) as a special case of Condition (e) can also be conveniently denoted by (P_3) being the first link in a sequence of Conditions (P_m) , $m = 3, \dots, n$ which we shall state presently. Let us first state

Theorem V.1: If $n = 3$ then $(P_3) \longleftrightarrow (P)$. Proof. Sufficiency of (P_3) .

Denote by x, y, z the generic distinct elements of $N = (1, 2, 3)$, define $Q(zxy) = x(x, y) - x(x, y, z)$, and assume (P_3) or its equivalent, (e_3) . Then by (3.9), $Q(zxy) \geq 0$; and by (2.2), (2.1) the sum of the six numbers $Q(zxy)$ is equal to $3 - 2 = 1$; hence (P) is satisfied, with $P(xyz) = Q(xyz)$. Necessity of (P_3) : by Theorem III.3.

Condition (P_4) . For any four distinct elements w, x, y, z of N ,

$$(5.1) \quad w(w, x, y) \geq w(w, x, y, z);$$

$$(5.2) \quad w(w, x) - w(w, x, y) \geq w(w, x, z) - w(w, x, y, z) .$$

Theorem V.2: $(P) \longrightarrow (P_4) \longleftarrow (P_3)$. Proof. Add (5.1) and (5.2) to obtain (P_3) ($=e_3$) from (P_4) . The converse is clearly not true. Moreover, if (P) is true, then

$$(5.3) \quad w(w,x,y) - w(w,x,y,z) = P(zwxy) + P(zwyx),$$

$$(5.4) \quad w(w,x) - [w(w,x,y) + w(w,x,z)] - w(w,x,y,z) = P(zywx) + P(yzwx);$$

since the expressions on the right side are non-negative, (P_4) is satisfied. The converse is not true since (P) may involve subsets with more than 4 elements.

Theorem V.3: If $n = 4$, $(P_4) \longleftarrow (P)$. Proof. Sufficiency of (P_4) : If x,y,z,w are generic distinct elements of $N = (1,2,3,4)$, then (5.3), (5.4) is a system of 24 equations in as many unknowns, $P(r)$, non-negative and adding up to 1; the knowns being the $x(M)$, $M \subseteq N$. Each permutation r occurs exactly once in the set (5.3) and exactly once in the set (5.4). Thus the system breaks into 6 sets of 4 equations; each set involves exactly 4 permutations r (none of which appears in the other 5 sets) and is of the form

$$(5.5) \quad \begin{aligned} \alpha &= P(r^{(1)}) + P(r^{(2)}) \\ \beta &= P(r^{(2)}) + P(r^{(3)}) \\ \gamma &= P(r^{(3)}) + P(r^{(4)}) \\ \delta &= P(r^{(4)}) + P(r^{(1)}) \end{aligned} ,$$

where the knowns $\alpha, \beta, \gamma, \delta$ are non-negative. They obey the restriction $\alpha + \gamma = \beta + \delta$ so that the 4 equations are linearly dependent. A non-negative solution for the four $P(r)$ is obtained by putting $0 = P(r^{(4)})$ or $0 = P(r^{(2)})$ according as $\beta > \gamma$ or not. One then verifies (P) for all $x(M)$ and obtains

$\sum_R P(r) = 1$. Necessity of (P_4) : by Theorem V.2.

Theorem V.4: When $n = 4$, $(P_4) \longrightarrow (e)$. Proof. Sufficiency: by Theorems

V.3 and III.3. No necessity: (P_4) implies

$$(5.6) \quad w(w,x) - w(w,x,y) \geq w(w,x,z) - w(w,x,y,z) \geq 0;$$

when $n = 4$, (e) is the pair of sets of inequalities

$$(5.7) \quad w(w,x) - w(w,x,y) \geq 0$$

$$(5.8) \quad w(w,x,z) - w(w,x,y,z) \geq 0,$$

which are implied by, but do not imply, (5.6).

Conditions (P_4) and (P_3) are special cases of

Condition (P_m) : Let $y, z, x_1, \dots, x_{m-2}$ denote m generic distinct elements of a set M consisting of m elements, $3 \leq m \leq n$. Then for any h , $3 \leq h \leq m$,

$$(5.9) \quad S_h = y(y,z) - [y(y,z,x_1) + y(y,z,x_2) + \dots + y(y,z,x_{h-2})] \\ + [y(y,z,x_1,x_2) + y(y,z,x_1,x_3) + \dots + y(y,z,x_{h-3},x_{h-2})] \\ \dots \pm y(y,z,x_1, \dots, x_{h-2}) \geq 0.$$

Theorem V.5: $(P) \longrightarrow (P_n)$, with

$$(5.10) \quad S_m = \sum_r P(1_r 2_r \dots (m-2)_r yz), \quad m = 3, \dots, n;$$

the sum being taken over all permutations r of the m -element-set M which end in yz . Proof. Assume (P) and count the number of times, with sign, that any particular permutation is included in S_m . A term of the type $\dots y \dots z \dots$, where $k \geq 1$ elements besides z follow y , has the coefficient

$$(5.11) \quad 1 - \binom{k}{1} + \binom{k}{2} - \dots + \binom{k}{k} = (1 - 1)^k = 0,$$

and so the only remaining terms will be as specified on the right side of (5.10).

We have seen that, in addition, $(P_n) \longrightarrow (P)$ for $n \leq 4$. We do not know whether this is true for $n > 4$. And we note that (P_n) is not testable if n is unknown.

An interesting condition involving binary choices only was formulated independently by several authors (perhaps first by Guilbaud [1]):

Condition (c_3) (a condition on cycles of 3 elements): For any three distinct elements i, j, k of N ,

$$(5.12) \quad 1 \leq ij + jk + ki \leq 2 .$$

Theorem V.6: $(e_3) \iff (c_3)$. Proof. Sufficiency: Let $M = (i, j, k) \subseteq N$.

By (e_3) , $ij \geq i(M)$, $jk \geq j(M)$, $ki \geq k(M)$; add; apply (2.1), (2.2) to get the left inequality in (5.12); then replace i, j, k by k, j, i . /No necessity: use the example of Theorem III.4 where $ij + jk + ki = 3/2$ so that the binary probabilities obey (c_3) ; but one of the ternary probabilities contradicts (e_3) . We shall now prove

Theorem V.7: Condition (c_3) is sufficient and necessary for the binary probabilities to be consistent with Condition (e_3) . Proof. Sufficiency. Denote by x, y, z the generic distinct elements of $M = (i, j, k) \subseteq N$. We have to show that if the six binary probabilities xy satisfy (c_3) then there exist three numbers γ_x that have the properties required by (e_3) of the ternary probabilities, viz., $\sum_x \gamma_x = 1$, $0 \leq \gamma_x \leq \min(xy, xz)$ for all x . Clearly such numbers exist if

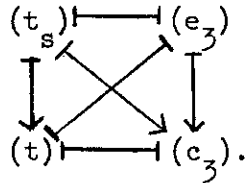
$$(5.13) \quad s = \min(ij, jk) + \min(ji, jk) + \min(ki, kj) \geq 1.$$

Let ij be the minimal, and hence - by (2.2) - ji maximal, among the six xy . If $kj < ki$, then, by (2.2), $s = ij + 1 \geq 1$; if $kj \geq ki$ then (c_3) implies $s \geq 1$. Hence (c_3) guarantees the existence of the desired numbers γ_x (in general not unique). Necessity: by Theorem V.6.

Theorem V.7 may be of importance if binary probabilities are the only observable ones (see Section VII). For the same reason the following Theorems,

V.8 and V.9, will be of interest. We shall first state the relations between (e_3) , (c_3) and the two transitivity conditions (t_s) and (t) of Section IV:

Theorem V.8



Proof. $(t_s) \implies (t)$, $(e_3) \implies (c_3)$ by IV.1, V.6. To show that (t_s) implies (c_3) assume, for some i, j, k in N , $ij + jk + ki < 1$, contradicting (c_3) ; then $ij + jk < 1$ and by (2.2) $ij < kj$; so that, if (t_s) [and therefore, by IV.1, (t_s^{**})] holds, then $ik < 1/2$, $ki > 1/2$, and by symmetry $ij > 1/2$, $jk > 1/2$, hence $ij + jk + ki > 3/2$, contradicting the assumption. To show that (c_3) does not imply (t) , let alone (t_s) , consider $N = (i, j, k)$ and make $ij = jk = ki = .4$, thus satisfying (c_3) but not (t) . On the other hand, the case $N = (i, j, k)$, $ij = jk = .1$, $kj = .6$ shows that (t) does not imply (c_3) . Finally to see that $(t_s) \implies (e_3)$ and $(t) \implies (e_3)$ note that (t_s) and (t) constrain only binary probabilities while (e_3) constrains ternary as well as binary ones, and its constraint on the latter is identical with (c_3) which was just shown not to imply (t_s) or (t) .

The following condition on binary probabilities due to Georgescu-Roegen [2] generalizes (c_3) :

Condition (c_m) : For any x_1, \dots, x_m in N ,

$$(5.14) \quad 1 \leq x_1 x_2 + x_2 x_3 + \dots + x_{m-1} x_m + x_m x_1 \leq m-1$$

A proposition stronger than (c_m) and - as will be proved presently - equivalent to (c_3) is

Condition (c^*) : (c_m) holds for all $m \leq n$.

Theorem V.9: $(c_3) \longleftrightarrow (c^*) \dashrightarrow (c_m)$. Proof. We have only to prove that (c_3) implies (c^*) and is not implied by (c_m) , $m > 3$. First show that the conjunction $[(c_3), (c_{m-1})] \dashrightarrow (c_m)$: by (c_{m-1}) ,

$$(5.15) \quad 1 \leq x_1x_2 + x_2x_3 + \dots + x_{m-1}x_1 \leq m - 2; \text{ and by } (c_3)$$

$$1 \leq x_{m-1}x_m + x_mx_1 + x_1x_{m-1} \leq 2, \text{ or by (2.2)}$$

$$(5.16) \quad 0 \leq x_{m-1}x_m + x_mx_1 - x_{m-1}x_1 \leq 1;$$

adding (5.15) and (5.16) we obtain (c_m) ; now put $m = 4, 5, \dots$ - To prove that (c_m) does not imply (c_3) let $m = n = 4$; if the matrix $[x_i x_j]$ is

$$\begin{pmatrix} .5 & .8 & .2 & .5 \\ .2 & .5 & .8 & .6 \\ .8 & .2 & .5 & .6 \\ .5 & .4 & .4 & .5 \end{pmatrix},$$

then (c_4) is satisfied but (c_3) is not: for $x_1x_2 + x_2x_3 + x_3x_1 = 2.4 > 2$.

The main results of this Section can be summarized in

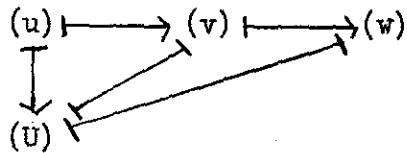
Theorem V.10: If $n \geq 4$ then

$$\begin{array}{ccccccccccc} (P) & \dashrightarrow & (P_n) & \dashrightarrow & (P_{n-1}) & \dashrightarrow & (P_4) & \dashrightarrow & (P_3) & \longleftrightarrow & (e_3) \\ \downarrow & & & & & & & & & & \downarrow \\ (e) & \dashrightarrow & (c_n) & \longleftarrow & \dots & \longleftarrow & (c_4) & \longleftarrow & (c^*) & \longleftrightarrow & (c_3) & \longleftarrow & (t_s) & \dashrightarrow & (t) \end{array}$$

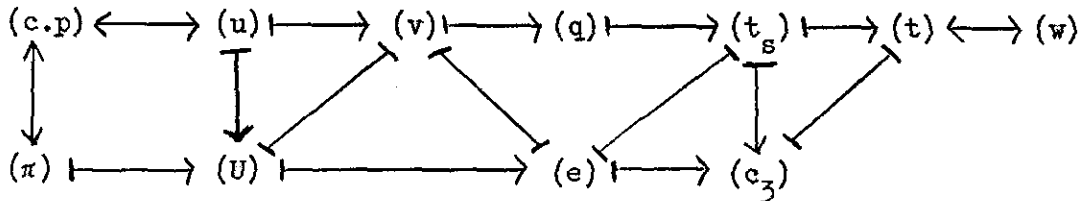
If $n = 3$ or 4 then $P \longleftrightarrow P_n$.

VI. A SURVEY

Using the results obtained in Sections II - III, we can summarize the relations between conditions postulating the existence of random orderings, and of constant utilities of various degrees of strength, in

Theorem VI.1

It is also useful to summarize the relations between Conditions (U), (u), (v) and (w) and some of the testable conditions, mainly from Sections IV - V:

Theorem VI.2

The following attempt to relate our various conditions to the hypotheses stated by earlier authors is very incomplete. Not only could we not do justice to the work of many psychologists; we have also failed to study thoroughly some of the statisticians' work.

In Section II, the ideas of Duncan Luce, imposing a very strong constraint on the probabilities of multiple choices of any order, were identified with condition (u); it can also be found in early unpublished papers of Reichenbach [1] and of Tornqvist [1]; and was stated orally (but taken back as unrealistic) by Debreu. Bradley and Terry [1] and Ford [1] applied this condition to binary choices only. A weaker constraint on the probabilities of binary choices is that of Fechner [1], identified with condition (v). This

condition, the still weaker condition (v), and an intermediate case (the "strong transitivity", (t_s)), was also studied by Clyde Coombs [1]; H. Gullikson [1]; the late S. Valavanis-Vail [1], to whom we owe the important Condition (t_s) ; Ward Edwards [1]; A. Papandreou et al [1]; Kenneth May [1]; J. Davis [1]; Donald Davidson (in collaboration with one of the present authors) [1].

In Section III, we identified an idea of Thurstone with our Condition (s.n.) (somewhat weakened in the subsequent work of Mosteller): a special form of combining the model (v) of strong constant utilities with the model (U) of random orderings. Conditions which can be identified with (U) were studied by G. Th. Gilbaud [1], N. Georgescu-Roegen ([1], [2]), David Rosenblatt [1], Duncan Luce [2], J. Marschak [1], and others.

M. G. Kendall [1], G. L. Mallows [1], I. R. Savage [1], E. Lehman [1], H.D. Brunk [1] and other statisticians (we owe these references to Lehmann) have studied rank-order statistics, "ranking models," and paired comparisons, starting with problems of statistical inference. For example, given a sample of binary choices, estimate the underlying (constant) ranking (perhaps a vector-parameter of the distribution of our \underline{U} ?); or test the hypothesis that a certain parameter is higher with one than with another of a pair of parent populations. To our regret, we have not had the opportunity to do the important job of stating the relations (if any) between our network of conditions and the stochastic models that ^{may} underlie the work of these authors. We shall briefly deal with statistical inference in our Section X .

VII. VARYING THE DOMAIN OF TESTABILITY

As remarked in Section I, the definition of the class of basic observations and therefore also of the class of testable conditions depends on the range of possible experiments and other observations. Moreover, condition (o) of Section IV, exemplifies the case of a constraint that involves all elements of the set of alternatives simultaneously; if this set is finite and of unknown size, such a condition is not testable, whatever the definition of the basic observation.

In this paper the basic observations have been defined as $(a;F)$: the actual choice of a single element a out of a feasible subset F of alternatives. It was suspected that the subject's verbal statements about how he ranks the elements of a given subset of three or more alternatives were less reliable, for the characterization and prediction of actual behavior, than his actual choices. But this suspicion may have been unnecessary. Psychologists do ask their subjects to rank three or more objects, according to the subject's intensity of perception, or his preferences, etc. They also ask "judges" to rank individuals according to some characteristics. Presumably some consistency is assumed to prevail among all the rankings of any offered subsets of alternatives, and also between these rankings and the actual choices from any offered subset. In a stochastic model involving a finite set of alternatives $N = (1,2,3,4)$ this would mean that, for example, a subject who, at a given time, ranks N in the order 1234, would, if asked to rank the subset $(2,3,4)$, produce the ranking 234; and if asked to choose from the subset $(2,3,4)$ would choose 2. The experimenter's faith in such consistency may be increased if the subject is told that the verbal statement about ranking will commit his actual choice when he is later presented

with a pair of alternatives (a procedure suggested in a non-stochastic context by W. A. Wallis as quoted by Savage [1]), or, for that matter, with any subset of alternatives.* With the rankings (on the set N of all

* On the other hand, the following procedure will not do for our present purposes (it was suggested in an earlier paper of Marschak [1]): to construct the subject's ranking of a set by asking him to make a choice from it; then removing the alternative chosen, and asking him to make a choice from the remaining subset; and so on. In the stochastic case, the subject may shift during the procedure from one ranking to another ranking of the same set.

alternatives) admitted as basic observations, our condition (P) can be reformulated as a testable one, with $P(r)$ indicating the probability of an observable ranking r ; that is, P is a distribution about which inferences can be made from experiments on rankings.

Testable Condition (P): The numbers $P(r)$ have the property (3.4).

Since ranking experiments can be made on any subset $K \subseteq N$, we may, denoting by $P(r^K)$ the probability of a ranking r^K of K , generalize (P) into the following

Testable Condition (P^K): The numbers $P(r^K)$ have the property (3.4) with K replacing N , and with $M \subseteq K \subseteq N$.

Note that when the subset K consists of two elements, the choosing from the subset and its ranking are the same thing. But the observation of ranking within triples and larger subsets does provide additional testable constraints not covered in the previous Sections.

Testability is extended still further if the following procedure is permitted. The subject is given a set $M \subseteq N$, where M consists of m objects,

and is told to select ℓ of them ($\ell \leq m$) and to rank these ℓ objects.

The ranking is denoted by $r^{L,M} = (r_{a_1}, \dots, r_{a_\ell})$ where $L = \{a_1, \dots, a_\ell\}$.

The condition $(P_{\ell,m})$ is said to hold if there exist $n!$ non-negative numbers $P(r^{N,N}) = P(r)$ whose sum is unity such that, for each set M having at least ℓ , but not more than m elements, and each subset L of M with ℓ elements

$$(*) \quad P(r^{L,M}) = \sum_r P(r) \begin{cases} |r_i \leq r_j & \text{if } i \in L, j \in M-L, \text{ and} \\ |r_i \leq r_j & \text{if } i \in L, j \in L \text{ and } r_i^{L,M} \leq r_j^{L,M} \end{cases}$$

Note: with $\ell = 1; (P_{1,n})$ becomes the condition (P) of Section III, since

$$(*) \text{ then becomes } \quad i(M) = \sum_r P(r) \quad r | r_i \leq r_j, j \in M$$

Thus $(P) \longleftrightarrow (P_{1,n})$

Clearly: $(P_{\ell,m}) \longmapsto (P_{\ell,m-1}) \quad (\ell < m)$

$(P_{m-1,m}) \longleftrightarrow (P_{m,m})$. Moreover

1. If $L = (i)$, then $P(r^{L,M}) = i(M)$
2. If $L = (i)$, $M = (i, j)$, then $P(r^{L,M}) = P(ij)$
3. If $L = (i, j)$, $M = (i, j)$, then $P(r^{L,M}) = P(ij)$
4. If $L = N$ or $N-1$, then $P(r^{L,N}) = P(r) = P(1_r 2_r \dots n_r)$.

So far, we have discussed the possibility of extending the domain of testable conditions beyond the definition given in Section I. On the other hand, the observational possibilities may be such as to make even that definition too wide.

For example they may be such as to make binary but not ternary, etc. choices observable. As remarked by H. Simon, to replace the experiment in which the

subject names the heaviest of two objects placed in his two hands, by a similar experiment with three objects (taken in successive pairs, or weighed on three fingers, etc.) may change experimental conditions so drastically as to invite inconsistencies. On the other hand, no such inconsistencies need be expected on experiments where the subject tells which of the 2, or 3, or possibly 4 lights is brighter. In the economics of the actual market, binary choices are often not observable where ternary or higher ones are: e.g., a speculator will 1) buy, 2) sell, or 3) do neither of these.

When the testability domain is extended some sufficient existence conditions become irrelevant. As a simple example let N consist of 3 elements and first assume that both binary and ternary choices are observable but rankings are not. Theorem V.1 gives a necessary and sufficient condition for the existence of probabilities of rankings as defined in Condition (P) of Section III. But it does not say that these probabilities will be equal to the numbers $P(r)$ of the Testable Condition (P) of the present Section, numbers about which inferences could be made if rankings were observable.

On the other hand, suppose that only binary choices among the 3 alternatives in N are observable, but neither ternary choices nor rankings are. Then Condition (c_3) ($1 \leq xy + yz + zx \leq 2$) becomes important: for, by Theorem V.2 and V.7, this condition is sufficient for the existence of the three ternary choice probabilities $x(x,y,z)$ and of the six probabilities of rankings $P(r)$; and nothing stronger can be stated with the observations at hand.

VIII. THE BOUNDARY CASE: PERFECT DISCRIMINATION OR PREFERENCE

The case when the binary probability $p(a;F)$, with $F = (a,b)$, takes the boundary values 1 or 0 is called, in psychology of perceptions, perfect (as distinct from imperfect) discrimination. In the language of the theory of choice there is then perfect (as distinct from imperfect, or stochastic) preference. The term can be extended to the case when the feasible set F contains any number of elements.

To take account of the boundary case, special provisos are needed to make conditions such as (t_s) , (v) , (u) consistent with empirical observations. To take the weakest of those conditions, (t_s) or its equivalent (t_s^{**}) : it implies that if there exists an element k of set N of alternatives, such that $ik = jk$ then $ij = 1/2$. Now, if three bodies i,j,k weigh, respectively, 1000, 10, and 6 grams, and xy is the probability that the subject says that x is heavier than y , then the observations will very likely, show $ij = ik$; but they will very unlikely show $jk = 1/2$. The conditions (t) and (w) , both weaker than (t_s) , are not endangered by this fact. But the stronger conditions (v) and (u) are. Without the proviso about boundary cases, made in Section II and again in Section IV, (v) would imply, in our case of three bodies, $v_j = v_k$ for each of the lighter ones, because of the existence of the very heavy body i . Similarly, in Condition (u) all the (strict) utilities u_i had to be assumed positive to prevent any of the probabilities $i(M)$ from being 0 or 1.

Perfect discrimination is circumvented experimentally by presenting to the subject, at each trial, objects not too strongly differing from each other

in the relevant physical dimension; and by keeping the subject uninformed of the physical dimension - for example, through blindfolding.

This approach is not sufficient in the case of experiments on choice to the extent to which these are intended to reveal "desirabilities," with "perceptibility" kept more or less constant (see Section I). Ideally, one would like to give the subject all the physical information, making the choice dependent only on his "desires." Suppose, then, that two alternatives differ from each other, however slightly, with respect to a single characteristic which is easily identified by the subject and is completely ordered with respect to the subject's choices. He will then presumably show perfect preference. Thus, in particular when a and b are quantities of an economic "good" and a exceeds b he will prefer a with probability one, even if the difference in quantities is very small. (In fact, this may be regarded as the definition of a "homogenous" economic good.) Also, if a and b are two vectors of quantities of the same m goods, $a = (a_1, \dots, a_m)$, $b = (b_1, \dots, b_m)$, and if a dominates b , i.e., if $a_i \geq b_i$ for all i and $a_j > b_j$ for some j , then a will be perfectly preferred to b . The same is true if the vectors a and b are interpreted as two money wagers, (where a_i and b_i are, respectively, the monetary gains awarded when event E_i happens) and a dominates b . On the other hand, if a and b are two vectors neither of which dominates the other, stochastic instead of perfect preference may occur. Accordingly, the pairs of commodity-bundles used in the experiments by Papandreou and others [1] and the pairs of wagers used in the experiments of Davidson and Marschak [1] were such as to exclude domination.

The vacillation of the chooser between the alternatives has thus been ascribed by Simon [1] and by Quandt [1] to the presence, in each alternative, of several relevant characteristics that themselves are completely ordered but to which the subject pays attention only one at a time. In general, we need not assume that every alternative is so reduced (albeit "unconsciously") to a bundle of characteristics, each of them an ordered one; so that the businessman decomposes, as it were, the job-applicant into his "personality factors," or considers a prospective plant site as a vector of numerical attributes. Yet in some important cases such vectorial representation is indeed manifest. This gives rise to disturbing discontinuities in applying our stochastic models. For example, denote by $a = (a_1, a_2)$, $b = (b_1, b_2)$, and $b' = (b'_1, b'_2)$, three pairs of "homogenous commodities" -- apples and oranges, not factories and vice-presidents; let ϵ_1 and ϵ_2 be two small positive numbers, and let $b_1 = b'_1 = a_1 + \epsilon_1$, $b_2 = a_2$, $b'_2 = a_2 - \epsilon_2$. Then, very likely, the binary probability $ba = 1$; while $b'a$ may be quite close to $1/2$; this big jump in probabilities would result from a very small physical change in the alternatives offered.

For such reasons, the psychophysist's proviso that the "differences" should not be "noticed always or never" is doubly important in designing and interpreting experiments on economic choices,* if one wants such experiments to throw light on desirabilities, as separated from the perceptibility aspect of the choice phenomenon.

* The case of absolute preferences is discussed by Georgescu-Roegen [2] in the context of the economic theory of demand (for homogenous commodities). Chipman [1] drew attention to important anomalies in the case of statistical decisions.

IX. REMARKS ON CHOICE UNDER UNCERTAINTY

As visualized at the end of Section I we shall now outline possible stochastic reformulations of the theory of choice under uncertainty, weakening in particular the "expected utility" condition.

A wager associates each alternative (in the sense defined previously) $i = 1, \dots, n$ with an event. Denote by $q = (q_1, \dots, q_n)$ any wager in which the event giving rise to alternative i has probability q_i (possibly zero). In its non-stochastic form the expected utility condition says that there is a constant vector $\bar{\omega} = (\bar{\omega}_1, \dots, \bar{\omega}_n)$, unique up to an increasing linear transformation, and such that

$$(9.1) \quad \sum \bar{\omega}_i q_i \geq \sum \bar{\omega}_i q'_i \quad \text{if } q \succsim q',$$

where the second " \succsim " means "preferred or indifferent." Since the observer cannot be sure that two subjects behave as if they ascribed the same probability to any given event, not only the "utility-numbers" $\bar{\omega}_i$ but also the probabilities q_i have to be interpreted in a subjective sense, thus: for a given subject, there exist a vector $\bar{\omega}$ and, associated with each wager, a vector q such that (9.1) is satisfied.

It has been shown* that this condition is implied by other conditions

* By von Neumann and Morgenstern [1], with q interpreted "objectively"; their "axioms" and proof were subsequently simplified by various writers, most concisely by Herstein and Milnor [1]. The "subjective" interpretation of q was introduced by Ramsey [1] and made precise by Savage [1]. See also Ward Edwards [1], D. Luce [1] generalized the stochastic theory of choices under uncertainty by introducing random subjective probabilities.

(called "axioms"), that apparently appeal more immediately to what our intuition

recommends as rational, or consistent. As we remarked at the beginning, consistency in an absolute sense is likely to be rejected by experience.*

* See the examples of actual behavior found by Allais [1] and discussed also by Savage [1], pp. 101-104.

But it can be replaced by more general conditions, of stochastic consistency; these imply and, or are implied by, statistically testable hypotheses. In particular, each of the conditions (w), (v), (u), (U) of our Sections II and III can be combined with the expected utility principle to yield a stronger condition to be denoted by (\bar{w}) , (\bar{v}) , (\bar{u}) , (\bar{U}) , respectively. In formulating these conditions, we shall denote by Q the set of all wagers (i.e., the set of all vectors q of order n , with $\sum q_i = 1$, $q_i \geq 0$, all i); the feasible set of wagers will be denoted by $F \subseteq Q$; the probabilities of choices will be denoted as before, e.g., $q(F) = p(q; F)$; $qq' = q((q, q'))$.

Condition (\bar{w}) : There is a constant vector $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n)$ such that

$$(9.2) \quad \sum w_i \cdot q_i \geq \bar{w}_i q'_i \quad \text{if } qq' \geq 1/2,$$

a generalization of (9.1).

Condition (\bar{v}) : There is a constant vector $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n)$ and, associated with it, a distribution function $\varphi_{\bar{v}}$, strictly increasing except at its value 1 and 0, and such that

$$(9.3) \quad \varphi_{\bar{v}}(\sum \bar{v}_i q_i - \sum \bar{v}_i q'_i) = qq' .$$

Condition (\bar{u}): There is a constant positive vector $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)$ such that, for every $F \subseteq Q$ and every q, q' in F ,

$$(9.4) \quad \frac{\sum q_i \bar{u}_i}{\sum q'_i \bar{u}_i} = \frac{q(F)}{q'(F)} .$$

Condition (\bar{U}): There is a random vector $U = (U_1, \dots, U_n)$ such that for every $F \subseteq Q$ and every q in F ,

$$(9.5) \quad q(F) = \text{Prob} (\sum (q_i - q'_i) \bar{U}_i \geq 0, \text{ all } q' \text{ in } F).$$

Clearly \bar{w} and \bar{U} (unlike w and U) are unique up to an increasing linear transformation. So is \bar{v} (by virtue of the continuity of the set Q : see footnote on p. 2.5). And \bar{u} is unique up to a positive multiplier (as is u).

The ^{directly} testable conditions of the previous Sections - e.g. (t), (t_s), (q), etc. - can be appropriately reformulated, and some new ones added. A case most easily accessible to observations is that of even-chance wagers: i.e. the case when the subjective probabilities are = 1/2 for each of a pair of alternatives. Let iEj be the wager promising i if event E happens, and j otherwise; then the subjective probability of E (and therefore also each of the subjective probabilities q_i and q_j) is said to be equal 1/2 if the subject is indifferent between iEj and jEi . Or, using the stochastic interpretation and denoting by $(iEj \cdot jEi)$ the probability of choosing the first out of that pair of wagers: $q_i = q_j = 1/2$ if $(iEj \cdot jEi) = 1/2$. In principle this property is operationally ascertainable.* If in an even chance wager, $q_1 = q_2 = 1/2$,

* For still greater experimental ease, Davidson and Marschak [1], following the device used by Davidson, Suppes and Siegel [1], and guided by considerations of continuity, assumed that when the alternatives i, j, i', j' are money amounts then the even-chance event E has also the following testable property.

$$(iEj \cdot j' Ei) = 1 = (jEi \cdot i' Ej)$$

provided $i - i' = j - j' = \epsilon$, a small positive amount.

and in another, $q'_3 = q'_4 = 1/2$; the expressions $\sum q_i \bar{v}_i$ and $\sum q'_i \bar{v}_i$ in Condition (v) become, respectively, $(\bar{v}_1 + \bar{v}_2)/2$ and $(\bar{v}_3 + \bar{v}_4)/2$; and so with similar expressions in (\bar{u}) , (\bar{w}) and (\bar{U}) . We can denote the even-chance wager aEb simply by \bar{ab} (hence $\bar{aa} = a$), and the probability of choice between two such wagers by $(\bar{ab} \cdot \bar{cd})$. Then Condition (\bar{v}) can be rewritten - analogous to (4.1) and with the proviso that the considered probabilities of choice are not 1 or 0 - thus:

$$(9.6) \quad \underline{\text{If}} \quad (\overline{a_1 a_2} \cdot \overline{b_1 b_2}) \geq (\overline{b_3 b_4} \cdot \overline{a_3 a_4}) \quad \underline{\text{then}}$$

$$v_{a_1} + v_{a_2} + v_{a_3} + v_{a_4} \geq v_{b_1} + v_{b_2} + v_{b_3} + v_{b_4}$$

This clearly implies the following directly testable set of conditions:

Condition (8_E) (octuple condition on even-chance wagers): If $(\overline{a_1 a_2} \cdot \overline{b_1 b_2}) \geq (\overline{b_3 b_4} \cdot \overline{a_3 a_4})$ then $(\overline{a_{r_1} a_{r_2}} \cdot \overline{b_{s_1} b_{s_2}}) \geq (\overline{b_{s_3} b_{s_4}} \cdot \overline{a_{r_3} a_{r_4}})$, where (r_1, r_2, r_3, r_4) and (s_1, s_2, s_3, s_4) are arbitrary permutations of $(1, 2, 3, 4)$.

Theorem IX.1. $(\bar{v}) \longrightarrow (8_E)$.

It is easily shown that the set (8_E) of conditions is not only necessary but also sufficient for (9.6) provided the stochastic continuity condition (s.c) of Section IV is satisfied with respect to the considered set of even-chance wagers.

If some of the eight alternatives in Condition (8_E) are made identical, directly weaker testable conditions arise, involving a smaller number of distinct alternatives, and necessary for (9.6) and Conditions (\bar{v}) . In certain conjunctions, and assuming stochastic continuity, they are also sufficient for (\bar{v}) , over the considered set of even-chance wagers. They were studied by Davidson and Marschak [1]; Debreu's paper [3], though dealing with even-chance wagers in a non-stochastic context, is also relevant.

X. TESTING STOCHASTIC CONSTRAINTS WHEN REPLICATIONS ARE FEW

All of our "stochastic" conditions were constraints on probabilities of certain responses of the subject to varying external conditions. More precisely these probabilities were of the form $p(a;F)$ although in Section VII more general forms were briefly discussed. These probabilities define the coordinates of the parameter space, and / each directly testable condition ("hypothesis") defines a certain region of this space. The test consists in deciding, on the basis of observations, whether the parameter point - the set of probabilities $p(a;F)$ for all F - falls into that region.

A trial is easily replicated if the same feasible set F - the same set of stimuli - can be presented to a random sample of subjects provided it is reasonable to assume that each member of the parent population is characterized by the same relevant probability distribution of responses. This assumption has been made, in effect (with respect to choices of foods, birthday gifts, etc.) by Jones, Peryan and Thurstone [1], Thurstone and Jones [1], and H. Gulliksen [2].

If this assumption of identical distributions is not justified the hypothesis must be tested separately for each subject to find the proportion of the observed subjects that fail to satisfy certain strong conditions - such as (q) and therefore (v); the proportion of subjects failing to satisfy weaker conditions - such as (t) and, therefore (w); etc. One can thus estimate the frequency distribution of the various "degrees of stochastic consistency" among the parent population even though the probabilities $(a;F)$ are not the same for two persons with the same degree of consistency. Two persons may both obey (w) and both fail to satisfy (v) even though one of them tends to vegetarianism

more than the other. In psychophysics, two persons may obey the Fechnerian condition (v) and yet have different functions ϕ_v ; e.g., have different quartiles or "just noticeable" differences (see Section II). A frequency (stochastic) distribution of degrees of consistency, in a given culture, is a useful thing to know. One may be interested in training people towards greater consistency, because of its importance for decision-making and leadership. This is not the same thing as inculcating identical tastes.

The number of replications, for the same subject, of the same set F of stimuli is limited by the fact that, in the course of successive replications, the subject's characteristic probabilities $p(a;F)$ for a given F may change. Inasmuch as this may be due to fatigue psychophysicists do not seem to have found it necessary to curtail replications too much; they regard relative frequencies with which the same response a is elicited by a fixed F (e.g., a fixed pair of sounds) as workable estimates of the probabilities $p(a;F)$.

True, in psychophysical experiments the subject must be prevented from identifying the stimuli and remembering them. For example, he must be blindfolded. If he were asked to choose the heavier of two objects (or the string producing the higher pitch) but were not blindfolded, the visual stimulus, added to that of the weight (or sound), might help him to identify and remember the objects and thus possibly influence him when the same object is presented again.

For a similar reason, successive replications may lack independence when subjects have to respond to verbal stimuli, as in questionnaires on attitudes or on voting preferences. Students of attitudes try therefore to repeat a question in a different form, preferably after a lapse of a little time. The

same lack of serial independence is likely to occur in the case of choices between verbally stated economic alternatives such as commodities and wagers. It might not be true of "blindfolded" choices between wines. But, as pointed out in Section I, the theorist of choice wants to separate preferences from difficulties of perception; his ideal is to observe the decisions of people giving them as complete information as possible.

Accordingly, experimenters on choices have used only a small number of repetitions of the same pair of alternatives (Papandreou et al., [1]) or have avoided such repetitions altogether (Davidson-Marschak, [1]). This creates special problems in devising statistical tests of significance, and at any rate makes such tests relatively weak. To bring out the logical nature of the problem, we shall concentrate on the mathematically easiest (though empirically weakest) case when each set F is presented once only; and we shall limit ourselves to the simplest of our hypotheses, viz., condition (t): see (2.6).

During the course of experiment the subject is asked to choose one from each of the sets (a,b) , (b,c) , (a,c) . Denote by p_a, p_b, p_c the binary probabilities ab, bc, ca . Then $p = (p_a, p_b, p_c)$ is a point in $[0,1]^3$. Assume that the subject has a certain probability distribution \mathcal{P} of choosing points in $[0,1]^3$; the random choice of triples (a,b,c) leads to a random choice from $[0,1]^3$ with the distribution \mathcal{P} on it. Let $q_i = p_i - 1/2$, $i = a,b,c$. The region of $[0,1]^3$ where condition (t) is not satisfied (call it \underline{T}) is characterized by the fact that all three q_i are ≥ 0 or ≤ 0 provided they are not all = 0. That is, apart from a set of measure 0, all three q_i have the same sign. It is desired to estimate $\mathcal{P}(\underline{T}) = 0$.

It may be useful to think of an analogous one-dimensional problem. A coin-making machine is characterized by an unknown probability distribution of the chance variable p (probability of a coin falling heads); one is permitted to toss coins, each only a few times (perhaps only once), in order to get evidence about the distribution of p ; for example, one wants to test the hypothesis that the proportion of coins with $p > 1/2$ (i.e., biased in favor of heads) is not more than a preassigned number. The region " $p > 1/2$ " clearly corresponds to our region T , and can be so denoted; the preassigned number is an upper bound θ' on $P(T) = \theta$. Now, an infinite number of coins of which exactly 10% are biased for heads, would, if each is tossed once, produce more than 5% (and less than 55%) heads; hence if out of a very large number of coins tossed, exactly $\bar{z} = 5\%$ have fallen heads, all we can say is that, the proportion of coins biased for heads is less than $\theta' = 10\% = 2\bar{z}$. Thus, regardless of the number of coins we are able to toss (each once), we cannot in general make statements involving a pre-assigned upper bound θ' on the proportion of coins belonging to a given region; if θ' had been 15% then we could assert $\theta < \theta'$, but if θ' had been 7% then we could not assert $\theta < \theta'$. Thus, if we want to be sure that we can make an assertion, we can choose θ' only after the tossings have been observed; if we are willing to accept the possibility of making no assertion we can preassign θ' . Similarly, in testing the stochastic transitivity condition (t) we shall be able to post-assign better than we can preassign an upper bound θ' on θ ; and this regardless of the number of triples (a,b,c) we can present - each once - to the subject. The situation there is further complicated by the fact that we have only a finite sample and we introduce another parameter t' to analyze the confidence level.

Furthermore if each coin could be tossed twice we would still be unable to guarantee an assertion with a pre-assigned upper bound to Θ , but the ratio Θ'/\bar{Z} of the post-assigned bound to the observed proportion of heads would become smaller. And a similar statement can be made about the consequences of duplicating the presentation of each triple (a,b,c) we can present - each once - to the subject.

We shall now develop a test of Condition (t). Let $q_i = p_i - \frac{1}{2}$; then

$$(10.1) \quad f(p) = p_a p_b p_c + (1 - p_a)(1 - p_b)(1 - p_c) = 1/4 + q_a q_b + q_b q_c + q_c q_a .$$

If $p \in T$ then $f(p) \geq 1/4$. If $p \notin T$ then $f(p) \leq 1/2$ for if $q_a = -\alpha$, $q_b = \beta$, $q_c = \gamma$ (α, β, γ non-negative), $f(p) = 1/4 + \beta\gamma - \alpha(\beta + \gamma) \leq 1/4 + \beta\gamma \leq 1/2$, and the other possibilities are covered by symmetry. Define three random variables X_a, X_b, X_c : $X_a = 1$ if a is chosen from (a,b) and $X_a = 0$ otherwise; $X_b = 1$ if b is chosen from (b,c) and $X_b = 0$ otherwise; $X_c = 1$ if c is chosen from (a,c) and $X_c = 0$ otherwise. Define

$$(10.2) \quad Z = X_a X_b X_c + (1 - X_a)(1 - X_b)(1 - X_c) .$$

For a given vector p , $\text{Prob}(Z = 1) = f(p)$. Now

$$\text{Prob}(Z = 1) = \text{Prob}(Z = 1 | p \in T) \mathcal{P}(p \in T) + \text{Prob}(Z = 1 | p \notin T) \mathcal{P}(p \notin T) .$$

$$\text{Hence } \text{Prob}(Z = 1) \geq \frac{\Theta}{4} + 0$$

$$\text{and } \text{Prob}(Z = 1) \leq \Theta + 1/2(1 - \Theta) = 1/2 + \frac{\Theta}{2} .$$

Thus Z is a binominal chance variable with mean $\mu = \text{Prob}(Z = 1)$

where

$$(10.3) \quad \frac{\Theta}{4} \leq \mu \leq 1/2 + \frac{\Theta}{2}$$

Let Z_1, \dots, Z_m be independent observations on Z , each based on a different trifle (a,b,c). Let $\bar{Z} = \sum_{k=1}^m Z_k/m$. Our first test is based on the following considerations.

1) Let $0 \leq \theta' \leq 1$. If $\theta \geq \theta'$ then $\mu \geq \theta'/4$. Let V' be a binomial chance variable with $\text{Prob}(V' = 1) = \theta'/4$. Then $\text{Prob}(Z_k = 0) \leq \text{Prob}(V' = 0)$. Let $t' \leq \theta'/4$. Then $\text{Prob}(\bar{Z} \leq t') \leq \text{Prob}(\bar{V}' \leq t')$ where \bar{V}' is the mean m values of V' . If a confidence level α' is specified one can choose m large enough so that $\text{Prob}(\bar{V}' \leq t') \leq \alpha'$. Hence:

$$(10.4') \quad \text{if } \theta \geq \theta', \text{ Prob}(\bar{Z} \leq t') \leq \alpha'.$$

As remarked in our coin example above, θ' cannot be chosen before the statistic t' has been obtained from observations: for $\theta' \geq 4t'$.

2) Similarly, consider θ^0 , with $0 \leq \theta^0 < 1/2$. If $\theta \leq \theta^0$ then $\mu \leq 1/2 + \frac{\theta^0}{2}$. Let V^0 be a binominal random variable with $\text{Prob}(V^0 = 1) = 1/2 + \frac{\theta^0}{2}$. Then $\text{Prob}(\bar{Z}_k = 1) \leq \text{Prob}(V^0 = 1)$. Let $t^0 \geq 1/2 + \frac{\theta^0}{2}$. Then $\text{Prob}(\bar{Z} \geq t^0) \leq \text{Prob}(\bar{V}^0 \geq t^0)$ where \bar{V}^0 is the mean of m values of V^0 . Again with a specified confidence level α^0 we can choose large enough m to make $\text{Prob}(\bar{V}^0 \geq t^0) \leq \alpha^0$.

Hence

$$(10.4^0) \quad \text{if } \theta \leq \theta^0, \text{ Prob}(\bar{Z} \geq t^0) \leq \alpha^0.$$

Summarizing: If $\bar{Z} \geq t^0$ we assert $\theta > \theta^0$; if $\bar{Z} \leq t'$ we assert $\theta < \theta'$; if $t' < \bar{Z} < t^0$ we assert nothing. The consequences of this are shown in the following table of probabilities of occurrence:

Assertion	True state:	
	$\theta \leq \theta^0$	$\theta \geq \theta'$
$\theta > \theta^0$	α^0	---
$\theta < \theta'$	---	α'
Nothing	---	---

where the dashes indicate probabilities that we have not estimated. This procedure protects us against extreme errors of classification but leaves open the possibility of coming to no decision.

If m is large enough so that the normal approximation to the binomial may be used, then the constants involved in the test are determined by the following equations:

$$(10.5) \quad F(s^0) = 1 - \alpha^0, \quad F(s') = 1 - \alpha',$$

where F is the normal distribution function with zero-mean and unit-variance, and

$$(10.6') \quad s' = (\Theta' - 4t') \cdot \sqrt{m} / 2\sqrt{\Theta'(1 - \Theta'/4)}$$

$$(10.6^0) \quad s^0 = (2t^0 - 1 - \Theta^0) \cdot \sqrt{m} / \sqrt{(1 + \Theta^0)(1 - \Theta^0)}.$$

The experimenter may specify the parameters Θ^0 , Θ' , t^0 , t' either before or after the experiment. Let us first discuss the case where the choice is made after he has the data. Suppose \bar{Z} turns out to be quite high. He is then seeking to make an assertion $\Theta > \Theta^0$. He can do this with confidence level $1 - \alpha^0$ where α^0 is determined from (10.6⁰), provided that $t^0 \leq \bar{Z}$ and $\Theta^0 \leq 2t^0 - 1$. He would like Θ^0 to be as high as possible and at the same time $1 - \alpha^0$ to be as high as possible. As he takes Θ^0 closer to $2t^0 - 1$, s^0 tends to zero and α^0 tends to $1/2$, which is not a convincing confidence level. Thus to keep the confidence level $1 - \alpha^0$ high he must take Θ^0 much less than $2t^0 - 1$; on the other hand the smaller he takes Θ^0 the weaker the impact of his assertion $\Theta > \Theta^0$. Thus it is clear that he should choose t^0 as large as possible, namely \bar{Z} . Now if the experimenter

wants to work with a fixed confidence level $1 - \alpha^0$ he should adjust θ^0 low enough so that he achieves, by (10.6⁰) his desired confidence level. If, on the other hand, there is a fixed "tolerance level" θ^0 which the experimenter regards as the significant indicator of inconsistency, then he would use this fixed θ^0 and again with $t^0 = \bar{Z}$ find his confidence level by (10.6⁰). More generally, he can strike what balance he wishes, pushing θ^0 up while driving $1 - \alpha^0$ down, keeping of course $t^0 = \bar{Z}$. If, instead, \bar{Z} is quite low then he wishes to assert $\theta < \theta'$. Here he would like θ' as small as possible and $1 - \alpha'$ as large as possible. Considerations similar to those above, applied now to (10.6') show that he should choose $t' = \bar{Z}$ and similarly make an arrangement between θ' and $1 - \alpha'$.

It may be however, that the experimenter is doing many experiments, and would like to have a fixed procedure, regardless of the data. He can specify t^0 and θ^0 in advance, subject to $\theta^0 \leq 2t^0 - 1$ and knows that his confidence level is given by (10.6⁰); (the larger $2t^0 - 1 - \theta^0$, the larger the confidence level, but the larger he takes t^0 the more likely he will have nothing to assert, while the smaller he takes θ^0 the weaker are the assertions he does make; this balance he settles in advance). Similarly he selects θ' , t' in advance. Now the data from many experiments can be handled by a fixed procedure; if $\bar{Z} \geq t^0$ or $\bar{Z} \leq t'$ an assertion is made, while if $t' < \bar{Z} < t^0$ no assertion. This procedure will be more convenient, but clearly it involves a loss of one or more of the following: confidence level, strength of assertion, likelihood of making an assertion, as compared with the method of selecting the parameters after the experiment. However if the experimenter is testing a large number of individuals in order to select a subset for further testing, he might find it better, as far as the data-processing is concerned to fix the parameters in advance.

Our second method is to find a confidence interval for θ . Let $0 < \delta < 1$ and $0 < \alpha < 1$ be given. Then it follows that for m sufficiently large $P(-\delta < \bar{Z} - \mu < \delta) > 1 - \alpha$; i.e.

$P(\bar{Z} - \mu < \delta \text{ and } \bar{Z} - \mu > -\delta) > 1 - \alpha$, or $P(\bar{Z} < \delta + \mu \text{ and } \bar{Z} > \mu - \delta) > 1 - \alpha$; from (10.3) it follows that $P(\bar{Z} < \delta + 1/2 + \frac{\theta}{2} \text{ and } \bar{Z} > \frac{\theta}{4} - \delta) > 1 - \alpha$. Thus $P(2\bar{Z} - 2\delta - 1 < \theta < 4\bar{Z} + 4\delta) > 1 - \alpha$.

While it is true that the length of the confidence interval $(2\bar{Z} - 2\delta - 1, 4\bar{Z} + 4\delta)$ is $2\bar{Z} + 6\delta + 1 > 1$, it is not centered at $1/2$ so that if \bar{Z} is near zero or unity the effective length of the interval may be quite small; e.g. if $\bar{Z} = 0$ the conclusion is $-2\delta - 1 < \theta < 4\delta$ which has an effective length of only 4δ ; while if $\bar{Z} = 1$ the conclusion is $1 - 2\delta < \theta < 4 + 4\delta$ which has an effective length of 2δ . Since for a prescribed significance level $1 - \alpha$ we do not know δ explicitly we can use the estimate $\delta = k(\alpha)\sigma = k(\alpha)\sqrt{\frac{pq}{m}} \leq \frac{k(\alpha)}{2\sqrt{m}}$. Thus $P(2\bar{Z} - \frac{k(\alpha)}{\sqrt{m}} - 1 < \theta < 4\bar{Z} + \frac{2k(\alpha)}{\sqrt{m}}) > 1 - \alpha$.

If the normal approximation for the binomial is used $k(\alpha)$ is determined from the equation

$$(10.7) \quad F(k(\alpha)) = 1 - \alpha/2,$$

Where F is the normal distribution with zero mean and unit-variance.

The confidence interval may also be used for making decisions, e.g., let $0 < \theta^* \leq \theta^{**} < 1$. If the confidence interval is contained in the interval $(0, \theta^*)$ assert that $\theta \leq \theta^*$; if the confidence interval is contained in the interval $(\theta^{**}, 1)$ assert that $\theta \geq \theta^{**}$; otherwise assert nothing. In other words if $\bar{Z} < \frac{\theta^*}{4} - \frac{k(\alpha)}{2\sqrt{m}}$ assert that $\theta \leq \theta^*$; if $\bar{Z} > 1/2 \left(\theta^{**} + 1 + \frac{k(\alpha)}{\sqrt{m}} \right)$ assert

that $\theta \geq \theta^{**}$ otherwise assert nothing. We then get the following table of probabilities*

* Naturally, much stronger tests are obtained if some a priori restrictions are admitted. Thus, in the Davidson-Marschak paper [1] two alternative hypotheses were formulated to test a stochastic constraint, viz., the transitivity conditions (t) and (t_s): 1) the probability-triples (ac, bc, ca) are distributed uniformly over $[0,1]^3$; 2) they are distributed uniformly over the region defined by condition (t) or (t_s), respectively. Mathematical simplicity is about the only serious claim in favor of these restrictions unless one invokes Laplace's "principle of ignorance."

	True State		
	$0 \leq \theta < \theta^*$	$\theta^* \leq \theta \leq \theta^{**}$	$\theta^{**} < \theta \leq 1$
Assertion: $\theta \leq \theta^*$	---	< α	< α
$\theta \geq \theta^{**}$	< α	< α	---
Nothing	---	---	---

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