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Linear Programming and Sequential Decision Models*

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Summary

This paper is designed to show how a typical sequential probabilistic model may be formulated in linear programming terms. In contrast with Dantzig [4] and Radner [10], the time horizon here is an infinite one. For another very closely related study, the reader is referred to a paper by R. Howard [7].

The essential idea underlying this linear programming formulation is that the "state" variable <u>i</u> and the "decision" variable <u>j</u> are introduced as subscripts to the unknowns x_{ij}. These unknowns x_{ij} represent the <u>joint</u> probabilities with which the state variable takes on the value of <u>i</u> and the decision variable the value of <u>j</u>. Although the particular application described is a rather specialized one, there seem to be quite a number of other cases where the technique should be an efficient alternative to the functional equation approach of Bellman [1].

Formulation of the problem

This is a single-item inventory control problem in which the initial stock on hand at the beginning of each "month" is, in Bellman's terminology, the "state variable." [2, p. 81] The size of initial inventory will be indicated by the subscript <u>i</u>. The quantity produced within the month is

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* The author is indebted for suggestions made by Martin Beckmann, Gerard Debreu, and Jacob Marschak. Their comments have made it possible to improve considerably upon an earlier draft dated August 13, 1958.

the "decision variable," and the amount produced will be indicated by the subscript j. Our problem is to obtain an optimal sequential decision rule - that is, to specify a value of j for each value taken on by i.

The sum of initial inventory plus the quantity produced will be known as the "available stock," and its size will be denoted by k.

The quantity demanded during the month is a serially independent random variable, n. The symbol p_n represents the probability with which n units will be demanded.

The size of month-end terminal inventories will be indicated by t. If backlogs of demand are to be ruled out, $t = \max(0, k-n)$.

It will be assumed that the objective is one of minimizing average monthly costs over an indefinitely long time horizon. These costs are the sum of the expected value of three components: (1) those costs related to the initial inventory levels i, (2) those related to the production quantities j, and (3) those related to the shortage levels (n-k). Symbolically, total costs are expressed as follows:

$$\mathcal{E}C_1(i) + \mathcal{E}C_2(j) + \mathcal{E}C_3(n-k) \qquad (1)$$

No convexity restrictions of any sort are imposed upon any of the three functions $C_1(i)$, $C_2(j)$, and $C_3(n-k)$.* Convexity is, in effect, brought

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* It is a serious limitation of the Holt-Modigliani-Simon production smoothing model that all cost functions must be of a quadratic nature. [6] No such assumption is required in the case discussed here.

about by supposing that mixed strategies are available - i.e., that whenever the initial inventory is at level i, it is possible to assign a positive probability to the choice of two or more of the alternative actions j.*

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* At a later point, it will be shown that even though probability mixtures are permissible, there will always be an optimal solution consisting solely of "pure" strategies.

Some fairly light restrictions are imposed upon the quantities i, j, k, n, and t. First, they must be integers. Second, there must exist positive integers K and N such that:

$$0 \leq i \leq K$$

$$0 \leq i+j = k \leq K$$

$$0 \leq t \leq K$$

$$0 \leq n \leq N$$

The linear programming problem described below will involve $K+1$ equations. In order for the simplex computations to be carried out with present-day electronic machine programs, it would be necessary to redefine units so that the integer K does not exceed something of the order of 200.

Some definitions

DF: y_i = probability that the initial monthly stock equals i. ($\sum_i y_i = 1$.)

DF: y'_t = probability that the terminal monthly stock equals t. ($\sum_t y'_t = 1$.)

Statistical equilibrium requires:

$$y_i = y'_t \qquad (i = t) \qquad (2)$$

DF: x_{ij} = joint probability with which the initial stock equals i and the production quantity equals j.

$$\therefore \sum_j x_{ij} = y_i \qquad (i = 0, 1, \ldots, K) \tag{3}$$

and $$\sum_{i,j} x_{ij} = 1 \tag{4}$$

DF: z_k = probability that the available stock equals k

$$\therefore z_k = \sum_{\substack{i,j:\\ i+j=k}} x_{ij} \qquad (k = 0, 1, \ldots, K) \tag{5}$$

DF: p_n = probability that n units are demanded within the month.

N.B. The probabilities p_n are independent of any choices made by the decision-maker. The probabilities x_{ij}, y_i, y'_t, and z_k, however are directly under his control.

Relationships between the individual probabilities

Since the random variable n is independent of the available stock k, and since $t = \max(0, k-n)$:

$$\left.\begin{aligned} y'_0 &= \sum_{\substack{k,n:\\ k-n \leq 0}} p_n z_k \\ y'_t &= \sum_{\substack{k,n:\\ k-n=t}} p_n z_k \qquad (t = 1, 2, \ldots, K) \end{aligned}\right\} \tag{6}$$

By (5):

$$\left.\begin{aligned} y'_0 &= \sum_{\substack{i,j,n:\\ i+j-n \leq 0}} p_n x_{ij} \\ y'_t &= \sum_{\substack{i,j,n:\\ i+j-n=t}} p_n x_{ij} \qquad (t = 1, 2, \ldots, K) \end{aligned}\right\} \tag{7}$$

By (2) and (3), we finally arrive at the interdependence relationships between the individual unknowns x_{ij}:

$$\sum_j x_{0j} = \sum_{\substack{i,j,n: \\ i+j-n \leq 0}} p_n x_{ij} \tag{8.0}$$

$$\sum_j x_{1j} = \sum_{\substack{i,j,n: \\ i+j-n=1}} p_n x_{ij} \tag{8.1}$$

$$\vdots$$

$$\sum_j x_{Kj} = \sum_{\substack{i,j,n: \\ i+j-n=K}} p_n x_{ij} \tag{8.K}$$

Equations (8.0) - (8.K) may each be interpreted as a requirement of statistical equilibrium. The left-hand side measures the probability with which the initial monthly inventory level will be i, and the right-hand side the probability with which the terminal level will equal t. Statistical equilibrium implies that if i=t , these two probabilities must coincide.

The unknowns in the linear programming model are the joint probabilities x_{ij}. The constraints consist of the usual non-negativity conditions, together with equations (4) and (8.1) - (8.K). Equation (8.0) is redundant, and need not be included explicitly within the constraint set.

Expected costs

The cost coefficient associated with each of the x_{ij} will be known as c_{ij} . The total cost expression to be minimized in the simplex computation is as follows:

$$\sum_{i,j} c_{ij} x_{ij} \tag{9}$$

How do we assign values to the coefficients c_{ij} so as to be consistent with the minimand given previously by expression (1)? Note that:

$$\mathcal{E}C_1(i) = \sum_i y_i C_1(i) = \sum_{i,j} x_{ij} C_1(i)$$

$$\mathcal{E}C_2(j) = \sum_{i,j} x_{ij} C_2(j)$$

$$\mathcal{E}C_3(n-k) = \sum_{i,j} x_{ij} \sum_n p_n C_3(n-i-j)$$

The cost coefficient c_{ij} associated with the unknown x_{ij} is therefore constructed as follows:

$$c_{ij} = C_1(i) + C_2(j) + \sum_n p_n C_3(n-i-j) \qquad (10)$$

<u>A numerical example</u>

In order to construct a numerical example, it is necessary to assign values to the demand probability distribution, to the three cost functions, and to the upper limit placed upon inventory accumulation. For illustrative purposes, we will work with the following:

$p_0 = 2/3$ $\qquad C_1(i) = i$ $\qquad K = 3$

$p_1 = 0$ $\qquad C_2(j) = 2j$

$p_2 = 1/3$ $\qquad C_3(n-i-j) = \max\,[0,\ 6(n-i-j)]$

In addition, it will be assumed that the production capacity is at most one unit per month (i.e., j = either 0 or 1). Note that the mean demand level amounts to only 2/3 of this capacity limit. There is, however, a 1/3 probability that demand will actually amount to twice the production limit.

Table 1 contains a calculation of the cost coefficients for this problem, and Table 2 indicates the constraint matrix in detached coefficients form. In transcribing equations (8.1)-(8.3) into this matrix, the right-hand side shown earlier in the text has been subtracted from the left-hand side. Equation (8.1), for example, has been transformed as follows:

$$\sum_{j} x_{1j} - \sum_{\substack{i,j,n: \\ i+j-n=1}} p_n x_{ij} = 0$$

Also shown in Table 2 is the optimal linear programming solution to the problem. According to this solution, the initial inventory will be at a zero level during 1/3 of the months, at a unit level 2/9 of the time, and at a level of two during the remaining 4/9.* Whenever the initial inventory has dropped to a level of either zero or unity, one unit of production is ordered. At higher initial levels, no production takes place at all. Note that no mixed strategies are indicated by the solution - despite the fact that this option was built into the model.

* The average monthly cost associated with this solution equals (1/3) (4) + (2/9) (3) + (4/9) (2) = 27/9. It is of some interest to compare this cost level with that of the do-nothing basic feasible solution - one in which the unknown x_{00} equals unity, all other unknowns are set at zero, and the resulting monthly costs amount to 4.

Further comments:

(1) There are two possible lines along which to sketch out a proof that it will always be optimal to adopt pure strategies. One would be to

Table 1. Calculation of the cost coefficients c_{ij}

Identification subscripts (i, j)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)
Inventory costs $= C_1(i) = i$	0	0	1	1	2	2	3
Production costs $= C_2(j) = 2j$	0	2	0	2	0	2	0
Shortage costs $= \sum_n p_n C_3(n-i-j) = \sum_n p_n \max[0, 6(n-i-j)]$	4	2	2	0	0	0	0
Total cost coefficient $= c_{ij}$	4	4	3	3	2	4	3

Table 2. Detached coefficients matrix

Identification subscripts (i,j)	(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	Constant terms
Equation (4)	1	1	1	1	1	1	1	= 1
Equation (8.1)	0	-2/3	1/3	1	0	-1/3	-1/3	= 0
Equation (8.2)	0	0	0	-2/3	1/3	1	0	= 0
Equation (8.3)	0	0	0	0	0	-2/3	1/3	= 0
Optimal activity levels, x_{ij}	-	1/3	-	2/9	4/9	-	0*	

* Activity in basis at a zero level.

follow the reasoning by which Dvoretzky, Kiefer, and Wolfowitz dismiss mixed strategies in a problem of this sort - one in which the demand probabilities p_n are known in advance to the decision-maker, and do not have to be estimated by him. [5, p. 191 n.] This is the same as saying that in a two-person game in which you have "found out" your opponent's strategy, it will never hurt to restrict your own choice of strategies to pure ones.*

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* I am indebted to J. Marschak for having pointed out the applicability of this line of reasoning to the problem at hand.

An alternative way to demonstrate this proposition would be to construct a proof along the same lines as Samuelson's "substitution" theorem concerning input-output models. [11] There, a mixture of activities for producing a given product i is permissible, but only one activity for each product will ever be needed in order to attain economic efficiency. Here, the corresponding statement is that no more than one activity associated with the i th initial inventory level will ever appear in a basis.

It looks as though the absence of mixed strategies could greatly simplify the problems of numerical analysis associated with a model of this sort. Substituting equation (8.0) in place of (4), one observes that each basis will be of the form (I-A), where A is a non-negative square matrix with column sums not exceeding unity. To obtain $(I-A)^{-1}$, all the usual theorems concerning Frobenius-Leontief matrices may be invoked. [12]

(2) The choice of an upper limit, K, upon inventory accumulation is admittedly an arbitrary one. If, after finding an optimal solution for a given value of K, and observing that $x_{K-j,j} = 0$ for all j, it is entirely

possible that a further increase in the value of K will lower the minimand still further. It is a simple matter to construct pathological cost functions that will yield this result. It seems doubtful, however, that this will constitute a serious obstacle in analyzing empirical problems.

(3) It is not altogether legitimate to have brushed aside the question of initial conditions for our Markov process. If the optimal basis in the linear programming solution is a "decomposable" one [12, pp. 33-35], the initial conditions will clearly govern the ultimate statistical equilibrium. The most direct way to circumvent this paradox would be to assume that the initial conditions lie within the control of the decision-maker - at least to the extent that he may choose them so as to start off within any one of the subsystems into which the larger system splits up.

(4) It is possible to attach a useful economic interpretation to the implicit prices (dual variables) associated with the linear programming solution. They represent the amount by which total costs would be altered if the initial inventory were at the i th level rather than at zero.*

* For the numerical solution shown in Table 2, the implicit prices associated with equations (8.1)-(8.3) are, respectively -5/3, -8/3, and -4/3. These values serve to measure the comparative advantage of having an initial inventory level i of 1, 2, or 3 units.

Collectively, they are equivalent to the solution of Bellman's functional equation for the inventory problem. [2, pp. 159-164] This being so, it should be a comparatively simple matter to use them in order to link together a non-stationary finite-horizon model with a stationary one having an infinite horizon.

Generalizations

Among the applications that suggest themselves, the following stochastic models would seem to be of the most interest:

(1) Changes in the rate of production. A number of studies have been concerned with systems in which the costs depend not only upon the rate of production (as in the example above), but also upon the rate of change of that level. (E.g., [6].) This kind of problem could be attacked through the same methods outlined here by defining the "state variable" i as a *pair* of numbers: one representing the initial inventory level and the other the rate of production during the immediately preceding period. With this one change in interpretation, things would proceed in essentially the same way that has been suggested here. The only serious difficulty might arise from the computational costs involved in an increase in the number of equations within the linear programming model. Instead of just one equation for each of the (K+1) levels of inventory, there would now be r equations - one for each of the r discrete rates of production that were considered. Altogether, the programming matrix would contain $r \cdot (K+1)$ rows.

(2) Seasonal storage of inventories. Several recent papers have been focussed upon the problem of optimization under conditions of seasonally fluctuating demands (e.g., the demand for heating oil [3]) or of supplies (e.g., the supply of water for hydroelectric installations [9]). In order for a linear programming model to reflect such seasonal fluctuations in the probability distribution of demands or of supplies, the state variable i would again have to represent a pair of numbers - the first indicating the season of the year and the second the inventory level at the beginning

of the particular season. The conditions of statistical equilibrium would then imply equality between probabilities for the terminal inventories of one season and the initial inventories of the one following. With $\underline{s}$ seasons and (K+1) inventory levels in each, a total of $s(K+1)$ equations would be involved. Even with time subdivided into 12 individual months and with 10 levels of inventory considered during each month, the computational requirements would still remain modest - a 120-equation system.

(3) Multi-location inventory problems. In the event that inventories are scattered among several geographical locations, it may no longer be appropriate to describe the system in terms of a single state variable - the aggregate quantity held in stock. Instead, a separate quantity must be specified for each location.* If, then, there are stocks held at ℓ different locations, the state variable $\underline{i}$ will have to be regarded as an ℓ-tuplet of numbers. With (K+1) alternative inventory levels at each individual location, the linear programming model would contain no less than $(K+1)^{\ell}$ distinct equations. As far as any realistic problems are concerned, it must be conceded that this number of equations could become hopelessly large. Even with just four locations and five inventory levels at each, the system would contain 625 equations!

(4) Delivery lags. Each of the cases described thus far has been based upon the assumption that delivery lags are short - that any production ordered at the beginning of a period will be available to satisfy

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* Essentially the same problem arises if, instead of one commodity in several locations, we are concerned with planning for several different commodities at a single location.

whatever demand takes place within the period. With long delivery lags, these models hardly seem to be appropriate.

A number of authors [1, 8] have shown, however, that there is a simple way to analyze a problem in which there are long but fixed delivery lags - that is, no randomness in the time required for delivery. (In addition to non-random delivery lags, these authors also assume that a shortage in supply is reflected in a temporary backlog rather than in a permanent loss of demand. This formulation guarantees that all currently outstanding orders will have been received prior to the arrival of any order placed currently.)

With these assumptions, the appropriate state variable required in order to describe the system is no longer the actual inventory on hand, but rather the sum of that inventory plus all outstanding orders. To adapt this suggestion to the linear programming models discussed here, all that needs to be done is to reinterpret the state variable $\underline{i}$ as "stock on hand plus orders outstanding." The probability p_n would be regarded as the probability that $\underline{n}$ units were demanded during whatever time interval is required for the delivery of an order. Note that this interpretation is equally well adapted to the case in which time is regarded as a discrete or as a continuous parameter.

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