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       A Theory of Life Insurance Company Portfolio Selection*

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Financial Institutions have recently enjoyed a higher rate of growth
than has the economic theorizing concerning their investment behavior. The
theory of individual portfolio choice, even if it is a complete theory, can-
not be applied directly to financial institutions; although these institutions
may attempt to invest in accordance with the preferences of their depositors,
the nature of their liability contracts exerts an independent influence on
their investment behavior. Thus, even though a savings bank depositor may
desire to maximize his interest return, the bank will not accommodate his
desire so long as it supplies the joint product, demand liquidity and interest
return. Whether it is more fruitful to derive a theory of institutional invest-
ment from the directors' statements of investment objectives or from the nature
of the institution's liability contract, is at present an unresolved question.
This paper, however, utilizes the latter approach, and for a particular finan-
cial intermediary: life insurance companies.

The paper is composed of three parts: 1) A discussion of market possi-
bilities; the risk and liquidity attributes of market securities. 2) A formu-
lation of the needs of life companies in relation to the nature of their contract
liabilities. 3) A derivation of possible life company investment policies by
matching needs and market possibilities in appropriate manners.

A. Market Possibilities, the attributes of securities . . . . .

The traditional theory of portfolio choice describes the individual as
choosing those securities that maximize the value of one variable, expected

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return. The introduction of uncertainty of future events increases the

number of relevant attributes (variables) related to any security; the

individual must choose securities which yield the optimum combination of attributes

(variables). The ambiguity in "demand" thus introduced is analogous to the

conflicting emotions involved in women's purchase of fur coats -- store labels

inside and fur quality outside. A theory describing price as a function of

the supply and demand for fur quality would seriously mislead. Similarly, if

"widows" purchase long-dated securities in order to gain income security and

corporations purchase short-dated securities in order to gain capital-value

security, relative asset prices cannot be predicted from supply and demand

schedules related to interest yield alone. Rather one must know the supplies

of and demands for the specific attributes. The attributes to be considered

are: (1) return -- interest or dividend, (2) default risk -- the possibility

that interest or principal will become defaulted,[*] (3) marketability or sale-

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[*] Return is defined as interest proceeds adjusted for the expected value
(best guess) of loss through default. Default risk is treated in this manner
because it is of minor interest in this paper.

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ability -- the possibility that a security can be liquidated quickly for small

change of value, (4) capital-value risk -- the probability that the market value

of a security will fluctuate through time, i.e., the degree of price variability

(which is largely proportional to term), (5) income risk -- the fact that

interest income cannot be predicted with certainty beyond the maturity and call

dates of a security. For example, a 3 per cent, forty year bond has a high

degree of income-certainty (little income risk) and a low degree of capital-

certainty (large capital-value risk); conversely a Treasury Bill has the opposite

attributes, high capital-certainty and low income-certainty. It is a conclusion
of the next section that life insurance companies may be viewed as purchasing
a package composed largely of return and income-certainty, with only small
quantities of capital-certainty and marketability.

B.  THE NEEDS OF LIFE COMPANIES, a framework for discussion of life company investment

The Life Insurance Contract. Life companies sell contracts for the future
delivery of specific amounts of dollars at specified dates in the future in
return for agreed inflows of cash in a prescribed manner through time. That the
benefit payments and premium inflows are predictable stems from the single fact
that mortality experience for large groups is predictable. The predictable
nature of these flows makes the following investment discussion possible.

The income or flow position. There are four cash flows of primary
importance; three are inflows and one is an outflow. They are $p_t$ the inflow
of premiums, $r_t$ the inflow of interest income, $A_t$ the inflow from maturing
securities, and $b_t$ benefit payments, the single outflow, which represents all
forms of payment specified in the liabilities. The combination of two of these
flows, benefit payments and premium receipts, will be denoted as $L_t$ ; this
may be viewed, at any point of time, as the independent variable or "given"
to which the life company must adjust its asset portfolio. This net outflow
of cash or "net disbursements curve" is the net liability of the company through
time and is negative if there is a cash inflow, i.e., if premium receipts exceed
benefit payments. The cash inflow is composed of $r_t$ , interest return, and
$A_t$ , maturity flows. Other cash flows, such as costs of operation, are assumed
known with certainty and included within $L_t$ . The maturity of liabilities and
assets is assumed known with certainty.

The balance sheet or stock position. The claims sold against the company are its liabilities; as noted previously, liability designates the net claim on the company. The $\Sigma L_t$ is equal to the net amount of insurance outstanding and the present value of this sum, $\Sigma v^t L_t$ , is denoted in insurance parlance as the reserve or legal reserve. This specifies that the company must have at least this amount of funds today so that, at a given rate of interest, these funds will grow to match $L_t$ over time. The maturity distribution of the assets is denoted by $A_t$ and $\Sigma A_t$ is the current amount of investable funds and is called the Fund. In this simple representation the firm is solvent if the Fund is equal to or greater than the $\Sigma v^t L_t$ at the interest rate guaranteed in the policies.

The determination of the cost of life insurance. When a new group of policies is sold, the premium rates chosen, given correct mortality tables and the existing structure of interest rates, must be such that:  the sum of present value of premium receipts is equal to or greater than the sum of present values of benefit payments over the life of the policy-group.

This can be shown as follows:

Let:  $b_t$ = the time-value distribution of all benefit payments,

$\quad$ $p_t$ = the time distribution of premium receipts; this refers to the number of premiums (pieces of paper), not to the dollar value,

$\quad$ $r$ = the current interest rate, where there is a flat maturity yield structure.

To find:  $\alpha$  where  $\alpha$  equals the value per premium paid in;

$\quad$ $p_t$ equals the unit/time distribution of premiums, whereas  $\alpha p_t$ equals the value-time distribution of  $p_t$ .

In this simplified representation,  $b_t$  is determined by the number of "deaths" (liability maturities) in the  t  considered, while  $p_t$  is related to both the number of "deaths" in the present period (this determines the slope

of $p_t$ at that t) and the number of previous deaths (this determines the height of $p_t$ , the number of people still paying in premiums).

The two distributions are shown in figure 1. The required condition is that $\alpha$ must be chosen so that area [a] (the negative $L_t$) can be invested at r such that it will grow to size [b] over the relevant time period. This means that the excess of premium receipts over benefit payments during $t_o$-$t_p$ (area [a]) must equal the present value of the excess of benefits payments over premiums receipts during $t_p$-$t_z$ (area [b]).
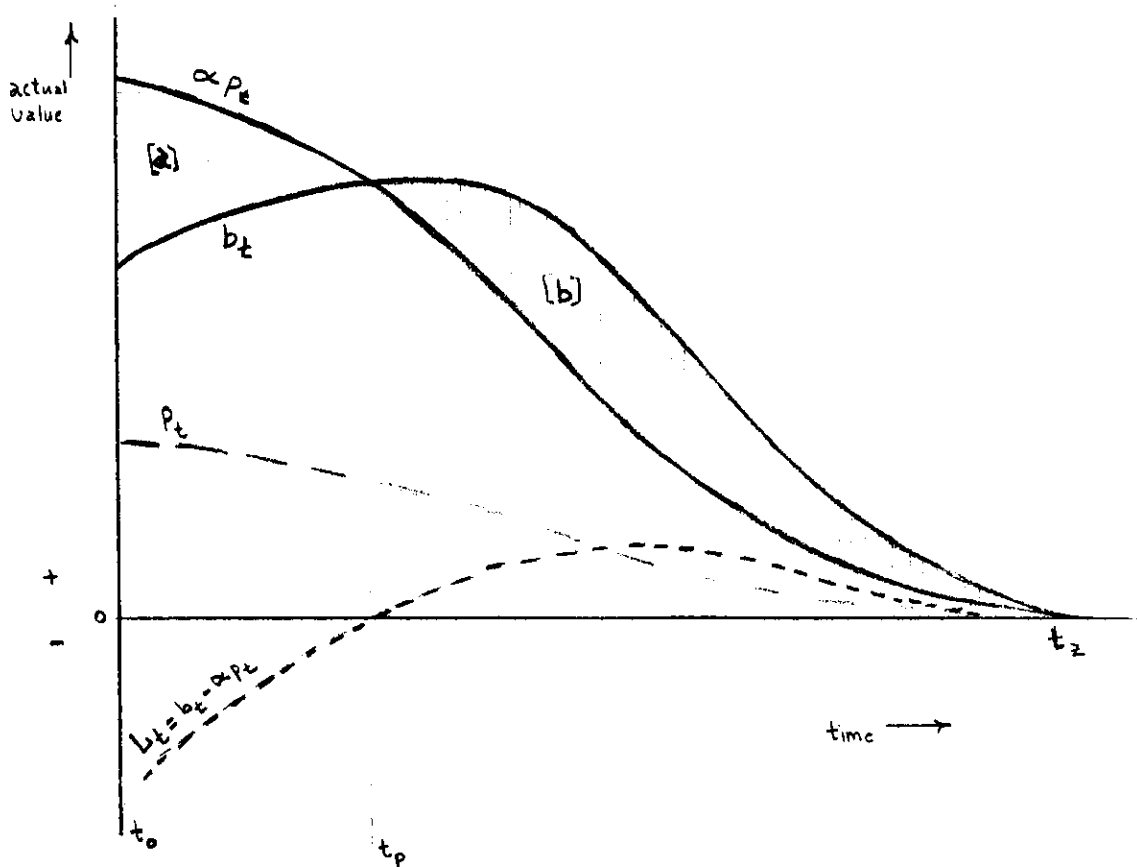


figure 1. Determination of $\alpha p_t$

The larger is $\alpha$ , the larger is area [a] and the smaller is area [b], and conversely. If at $t_o$ the future value of some variable $x_t$ equals

the actual value of another variable $y_t$ , then it is also true that the present value of $x_t$ equals the present value of $y_t$ as long as the same discount rate is used throughout. Hence the above condition is equivalent to the first statement that the sum of present values of premium receipts must equal the sum of present values of benefit payments, $\Sigma v^t \alpha p_t = \Sigma v^t b_t$ .

If the sum of present values of $\alpha p_t$ is greater than the sum of present values of $b_t$ this can be considered the Surplus. If there is an existing stock of assets then the condition is:

$$\sum_{t_o}^{t_z} v^t(A_t + r_t + \alpha p_t) \geqq \sum_{t_o}^{t_z} v^t b_t$$

or, $\Sigma v^t(A_t + r_t) \geqq \Sigma v^t(b_t - \alpha p_t)$

and, by substitution,

$$\Sigma v^t(A_t + r_t) \geqq \Sigma v^t L_t \ .$$

If present $r$ is equal to the coupon on existing $A_t$ , then

$$\Sigma A_t \geqq \Sigma v^t L_t \ ;$$

this is the original solvency condition.

The risk situation of life insurance companies. There are three major forms of risk for a life company: 1) unanticipated variations of mortality experience, 2) unexpected increases of operational expenses, and 3) unexpected changes of interest return resulting in less than anticipated yield on the portfolio. It is assumed throughout that mortality and expense experience occur exactly as anticipated. The yield risk consists of two parts: a) capital-value risk -- the possibility that the liquidity position of the company will

be such that the security must be sold before maturity, entailing capital

loss, and  b) income risk -- the possibility that market interest rates

will change, either during the course of future premium receipts or at

the time securities mature and the funds must be reinvested.  These two

forms of income risk derive from different circumstances.  In the first

case the risk is caused by promising an interest return at  $t_o$  but delaying

the receipt of the premiums, and hence the investment, till some future

date.  This is the unique risk of the life insurance contract.  The second,

or recontract, risk arises because a specific income was desired for a

longer period than the maturity of the asset chosen.  The remainder of the

paper is concerned with how the income risk can be precisely specified and

how the choice of  $A_t$  can reduce its magnitude.

Capital-value risk is herein relegated to a secondary role for the

following reasons.  Capital-certainty demand and marketability are both

closely connected with the liquidity position of a company.  The liquidity

or cash flow position of life companies is accurately predictable for two

reasons.  Since today's/past, the future cash flow is predictable from
premium inflow is largely determined by the sales rate in the

the present rate of sales.  Historically, due to the growth of life insurance

sales, premium inflow has averaged about twice the average benefit outflow.

Secondly, the stability of this ratio is illustrated by the fact that in no

year has there been a net outflow of cash through utilization of the various

liquidity options associated with the policies.  In each year, even in the

depression, premium and interest receipts have exceeded all cash outpayments

of life companies.  If future cash flows are large and predictable there

is little need to schedule maturities in order to provide further liquidity.

Should the sales volume lessen this conclusion might have to be modified.

## C. RISK OBJECTIVES AND INVESTMENT POLICIES . . . . . . .

"Life Insurance provides cover against risk of death or survivance and it provides an investment service involving guarantees of future capital security and of long-term yield" [Kirton and Haynes, 7]. This section is concerned with methods of investment which minimize risk, especially income-risk, associated with the "investment service" provided by life insurance. The objective is not to advocate actual investment policies, but to examine investment possibilities from a very narrow point of view: how could or should life companies invest if minimization of income risk were the sole objective? A majority of the following solutions and examples is taken from the very interesting discussion which has appeared in the actuarial journals of England and Scotland during the last thirty years. The following assumptions will govern the early examples.

1.  All assets have fixed maturities and are non-callable.

2.  All liabilities have predictable maturities; there are no liquidity options such as surrender values, etc.

3.  Mortality and operational expenses occur as anticipated.

4.  There are sufficient numbers of securities having the desired maturity available in the market place.

5.  There is a ready market for all bonds.

6.  Interest compounding is possible; that is, interest income can always be reinvested in the future at the rate it was originally invested (compound interest securities are available).

Concurrent Income and Capital Certainty, (sufficient funds). It is possible to jointly obtain capital and income-certainty when there are sufficient funds at $t_o$ to cover the present value of all existing $L_t$ .

For in this case it is possible to purchase $A_t$ so that $A_t + r_t$ is equal to $L_t$ through time, thus ensuring capital-certainty by matching maturities and ensuring income-certainty by matching inception and maturity dates of assets and liabilities at the interest rate guaranteed in the contracts. No asset need be redeemed before maturity; moreover, the interest guarantee of each policy is immediately hedged by the simultaneous purchase of an asset with identical interest and maturity. This method of marrying assets to liabilities is known as Absolute Matching. Whether sufficient funds are available at $t_0$ to make Absolute Matching possible, depends on size, rates of change, age composition, type and premium inflow of previous policy sales. If the company sells a constant annual amount of policies with identical structure over a long period, then the outstanding value of insurance remains constant. If sales decline, the company "ages" and the amount of insurance outstanding declines.

Consider a constant-sales insurance company in relation to the joint risk objectives of the previous situation. Assume only straight-life policies are written. After $t_0$ no new policies are sold, though the premium inflow from old policies continues as scheduled. After $t_0$ the net insurance liability immediately starts to decrease since it has been the continual sale of new policies in the past (their premium inflow) which has previously prevented this from happening. The $L_t$ is positive and increases and declines in a manner related to age composition and mortality experience associated with the $\Sigma L_t$ at $t_0$. Figure 2b depicts the net insurance liability $L_t$ as a function of time. The solvency condition states that at $t_0$ the Fund $\Sigma A_t$ must at least equal $\Sigma v^t L_t$, the present value of insurance outstanding, $f_0$.
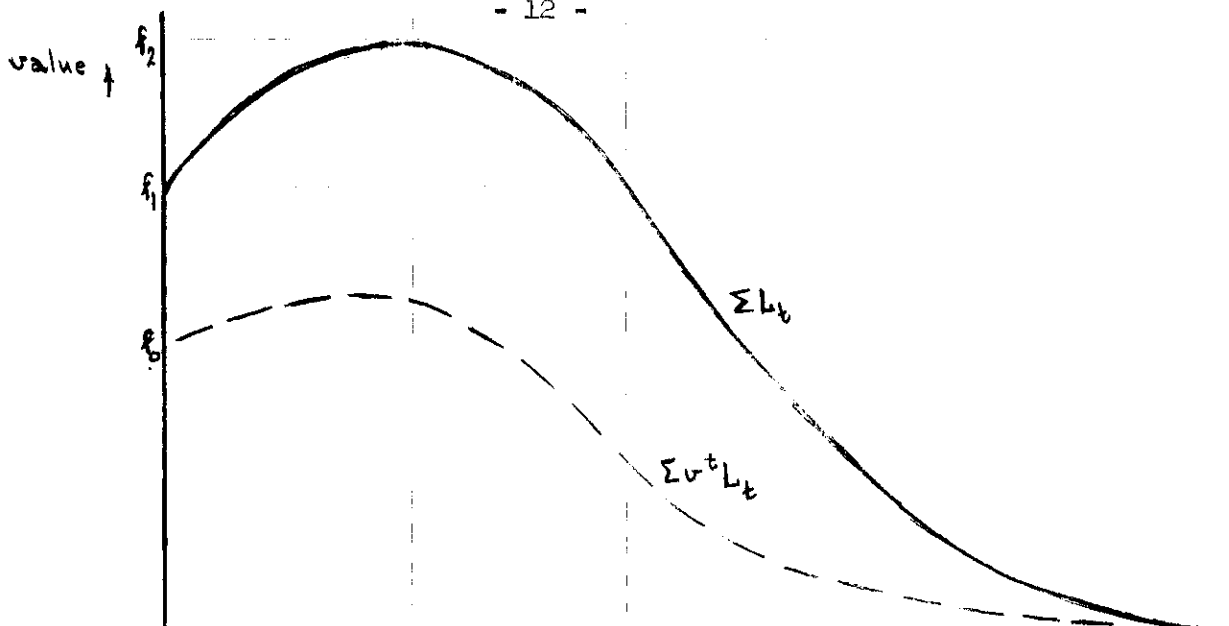
Stock

figure 2a
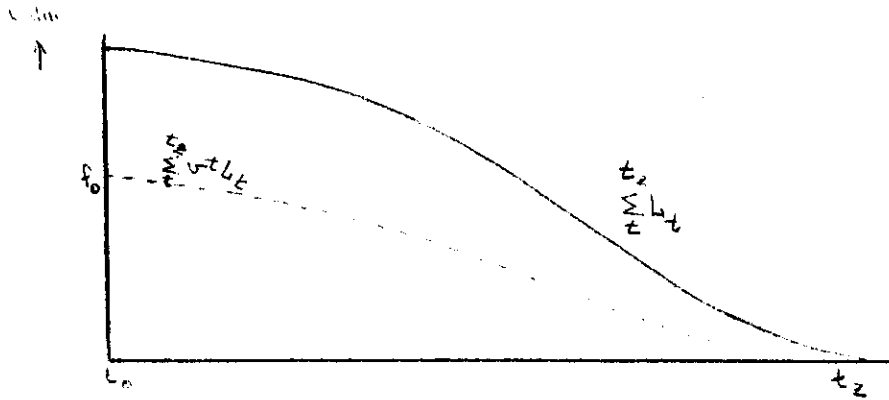
THE DECLINING FUND

Flow

figure 2b
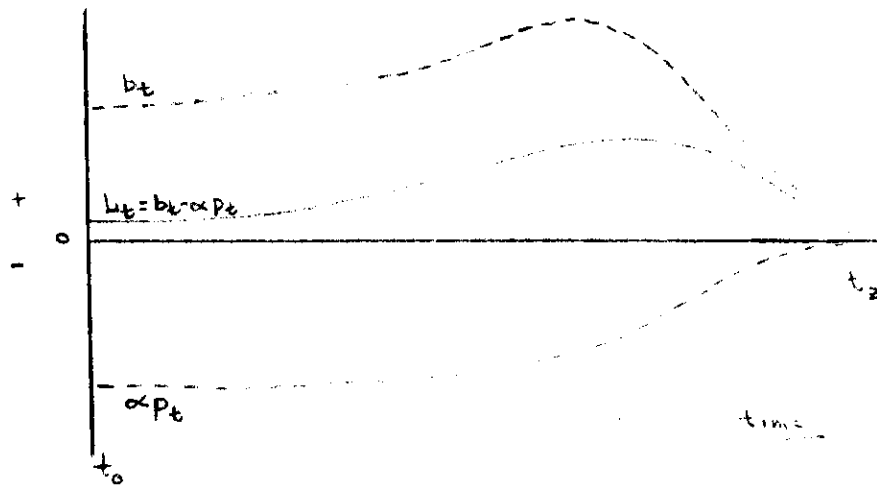


Thus Fund must equal $f_o$ at $t_o$ for the company to be solvent. This is shown in Figure 2a where $f_o$ equals the present value of insurance outstanding. Absolute Matching, which has been discussed by Koopmans [5], Haynes and Kirton [7], and Reddington [8], is applicable here because sufficient funds are available at $t_o$ to equal $\Sigma v^t L_t$ . Since there will be no future net inflow from $L_t$ , premiums will never exceed benefits. The condition for Absolute Matching is to choose $A_t$ so that $A_t + r_t = L_t$ where the r of the Assets equals the r guaranteed in the policy contract. This is equivalent to the previous solvency condition $\Sigma A_t = \Sigma v^t L_t$ since the present value of an asset plus its interest stream is always par value when the coupon rate is the discount rate. Thus the Assets and Liabilities are joined at their inception dates to insure income security and at their maturity dates to insure capital security.

Concurrent Income and Capital Security (insufficient funds). The
Humped Fund, in contrast to the previous constant-sales situation, describes
the more typical life company situation in which the company has been
continually growing in size and consequently (usually) is composed of an
increasing percentage of younger policy holders. These two factors will
cause the premium inflow to exceed benefit outflow for the next five to
ten years in the future even if no new policies are written after $t_o$ .
That $L_t$ will be negative for several years (positive cash inflow),
results from the fact that young people, who do less dying and more paying
of premiums, represent a larger proportion of policy holders than do old
people. Normal experience for American life companies is that premium
receipts exceed benefit payments by a factor of 1.5 or 2. (The only
class of companies in which this is not generally true is the Fraternal
Insurance Societies which have "aged.") Diagramatically this situation
is represented both for the stock of $L_t$ and for the cash flow (rate
of change of $\Sigma L_t$) over time in Figure 3. The term Humped Fund[*] is used

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* The term Humped Fund, used by Haynes and Kirton [7], indicates a
situation in which $\Sigma L_t$ will increase before it declines. The Fund
will also increase if the company remains solvent.

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to indicate that the size of the net insurance liabilities increases
after $t_o$ until the point of time $t_p$ when benefit outflow minus premium
inflow is zero, and subsequently will continually decrease as benefit pay-
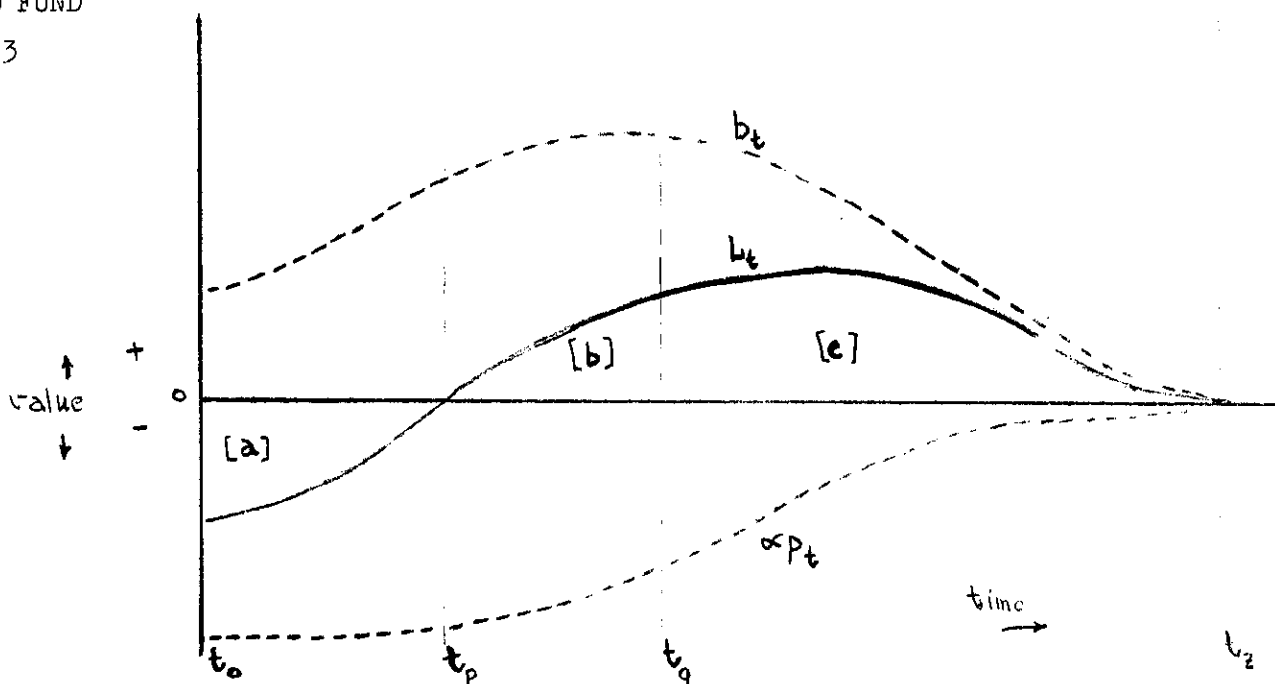ments exceed premium receipts.

Stock

value $f_2$

$f_1$

$f_0$

$\Sigma L_t$

$\Sigma v^t L_t$

THE HUMPED FUND

figure 3

Flow

value

$+$

$0$

$-$

$b_t$

$L_t$

[a]   [b]   [c]

$\propto P_t$

time

$t_o$   $t_p$   $t_q$   $t_z$

The two accompanying diagrams describe the usual life company

situation -- insufficient funds at $t_o$ for Absolute Matching. The

stock diagram charts the time path of $L_t$ and the flow diagram depicts

the rate of change, or derivative, of the stock.

The life history, or Unwind, of such a company in which no policies

are sold after $t_o$ may be divided into periods. During $t_o$-$t_p$ there

is a positive cash inflow as premiums inpayments more than cover benefit

outpayments. The amount of this surplus cash is measured by the area [a]

in the flow diagram and by the distance $f_2-f_1$ in the stock diagram.
After $t_p$ there is continual policy cash outflow. $t_p-t_q$ represents
the period during which area [b] is just equal to area [a] plus the
interest earned on [a]. This is, $\sum\limits_{t_o}^{t_p} (1+i)^t L_t$ equals $\sum\limits_{t_p}^{t_q} L_t$ . Thus if
interest is earned as anticipated in the policy contracts, it will be
true that cash inflow [a] will grow in size to exactly cover the benefit
outflow [b], hence outflow [c] must be covered by assets owned at $t_o$ .
This is the original solvency condition now expressed in terms of the
Humped Fund. Thus if $L_t$ is discounted at the guaranteed rate, $\Sigma A_t$ at
$t_o$ must equal $f_o$ in order for the company to be solvent. [a] will flow
in and be invested and then be paid out to cover [b], after which time the
original assets $f_o$ , now increased in size by addition of $r_t$ , will be
drawn upon and will just cover area [c]. The three time divisions are
arbitrary, but will prove useful. The investment problem may be rephased,
how can the initial funds $f_o$ and the future funds [a] be invested so
that income and capital risk are minimized, under the circumstance that
the interest guarantee on funds [a] is made at $t_o$ and yet some of the
premiums will not be received and invested until the future. How does
one hedge this future interest uncertainty? Two methods of translating
the Humped Liability into a Declining Liability have been suggested, and
in addition a method of minimizing income risk has been given.

Translation, Koopmans [5] has suggested that the Humped Fund can be
translated into a Declining fund by borrowing a quantity of funds at $t_o$
equal to [a]. These can then be invested at the current rates (by definition
the guaranteed rates) so that they will mature over the period [b]. Thus

income and capital risk are completely hedged. As the funds [a] flow in they are used to repay the loan without disturbing the original $A_t$ . The cost of this hedge is the difference between the borrowing and lending rates.

Translation. Haynes and Kirton [7] have noted that the types of policies sold need not be treated as an independent variable, but instead may be chosen to erase the Hump by a reshuffle of cash flows. They suggest, as one possibility, selling sufficient single-payment, immediate annuities (lump sums paid in, the annuities start immediately) to offset the Hump caused by selling too many life contingency and endowment policies. By controlling Sales Policy and cash flow the Humped Fund is translated into a Declining Fund.

The Near Hedge. Koopmans [5] and Haynes and Kirton [7] have suggested the following policy for situations where it is not possible or feasible to borrow large sums or to control sales policy. This policy, in effect, attempts to maintain capital-certainty and then, within this constraint, minimize income risk. The essence of this method is to invest the present fund $f_o$ as "long" as possible in matching $L_t$ , and the future incoming funds as "short" as possible. Specifically, the fund at $t_o(f_o)$ is invested so that $A_t + r_t$ matches $L_t$ from $t_q$ to $t_z$ , area [c]. This gives the maximum income security obtainable with these funds as well as capital security since the securities will be redeemed at maturity. The incoming cash flow [a] is invested starting with maturity $t_q$ and moving toward $t_p$ . By $t_p$ this provides $A_t$ such that area [b] is Absolutely Matched if interest rates did not change during the first period of cash inflow and investment. The difference between actual yield and expected yield is determined by the extent that interest rates do actually change and the duration one receives this shortfall or overage of interest proceeds. This

method minimizes the time interval during which assets will be invested
at future uncertain rates of return under the constraint of maintaining
capital-certainty.

Income Security Only. Because borrowing is distasteful and perhaps
infeasible in large amounts,and because the control of sales policy within
a competitive market would be costly for any one firm, the two translation
techniques are impracticable. In addition, the Near Hedge solution is
imperfect in regard to income risk for two reasons. First, the positive
cash inflow from excess premiums over benefits, area [a], is typically
large and increasing due to increasing sales of deferred group annuities; [*]

* This is only a tentative conclusion, for the United States, Canada and
Great Britain, because there has also been an increasing sale of group
term insurance. See Hood and Main [11] and Clayton and Osborn [ 12 ].

this exposes a growing portion of the portfolio to income risk. Second,
given the factors causing large and predictable internal cash flows
mentioned in part A, there seems little reason to constrain the investment
policy by maintaining the capital-certainty requirement. Thus the under-
lying presumption of this final investment policy is that the "cost" of
giving up capital-certainty is more than compensates by the gain in income-
certainty. No attempt is made in the following section to match maturity
dates, attention is oriented only toward "covering the interest guarantee"
by use of $A_t$ policy.

Immunization. It is significant that this investment policy suggested
by Reddington [8] is called Immunization rather than Matching. Previously,
risk was eliminated by marrying inception and maturity dates of assets and

liabilities through borrowing, sales policy, and covering longest $L_t$ first. Here the objective is different: insure that sufficient income is available to meet the interest guarantee of the contract by choosing an $A_t$ such that no future course of interest rates can jeopardize the interest guarantee of existing policies at $t_o$.

The basic proposition of Immunization is that the risk for a life company is the uncertainty of the interest rate at which future cash premium inflows from current policies can be invested. Because this investment method is designed to offset future interest rate fluctuations, it makes unnecessary the original assumption that future interest proceeds can always be reinvested at current rates.

The aim of this investment policy is to assure that the present value of the asset flow, $A_t + r_t$, remains equal to the present value of the liability flow $L_t$ through time, regardless of the course of future interest rates. This is accomplished by choosing assets with maturities longer than those of the liabilities, so that any future change of interest rates, which will cause either a shortage or overage of interest income, will be exactly offset by an equivalent increase or decrease in the capital value of existing assets.

Reddington's two necessary conditions for an Immunized portfolio are stated below. His derivation, which is the solution of the first three terms of a Taylor Expansion, is summarized in Appendix A.

(1) equality of mean terms: $$\frac{\Sigma[t \cdot v^t \cdot (A_t + r_t)]}{\Sigma[v^t(A_t + r_t)]} = \bar{t}_{A+r} = \bar{t}_L = \frac{\Sigma[t \cdot v^t \cdot L_t]}{\Sigma[v^t \cdot L_t]}$$

(2) equality of variances: $$\frac{\Sigma[(t-\bar{t})^2 \cdot v^t \cdot (A_t + r_t)]}{\Sigma[v^t(A_t + r_t)]} \geqq \frac{\Sigma[(t-\bar{t})^2 \cdot v^t \cdot L_t]}{\Sigma[v^t \cdot L_t]}$$

The two necessary and sufficient conditions for Immunization are that the mean terms of the present values of outflow and inflow should be equal and that the variance of the inflow must be equal to or exceed the variance of the outflow.

Given a change of interest rate, the effect on the present value of a security depends only on its maturity. Thus there is plausibility in Reddington's solution requiring the mean term and the dispersion of the present value of the two cash flows to be equal. Intuitively, this means that the similarity (in terms of present values) of the two time distributions ($L_t$ and $A_t + r_t$) make them equally sensitive to a change of interest rates. If $A_t + r_t$ is "longer" than $L_t$, then the $A_t + r_t$ flow is more sensitive than $L_t$ and a change of interest will cause a "profit."

Since the distribution $r_t$ occurs before the $A_t$ inflow, $A_t$ must be somewhat longer than $L_t$, in order for $\bar{t}_{A+r}$ to $= \bar{t}_L$. To find how much longer $A_t$ must be to counterbalance the shorter $r_t$ for any $L_t$ Reddington provides the proximate rule: if $\bar{t}_L = a_{\overline{n}}$, "the present value of one dollar per annum at compound interest," then "n" equals the correct $\bar{t}_A$. Thus the immunizing $A_t$ is determined by the mean term of the liabilities $\bar{t}_L$, the previous interest coupons on the assets $r_t$, and the current interest rate used in $v^t$.

Before applying these rules to life company investment, the risk situation can be illustrated by an example of an individual investor. Assume a person knows today that he will need exactly $10,000 in ten years to finance his son's college education. He now has $8,000 and the market rate of interest is 2 1/4 per cent. How can he invest this $8,000 today so that he is assured of having $10,000 in ten years though it is

currently impossible to know the interest rates at which forthcoming

interest proceeds can be reinvested. The first condition of the immunized

solution, equality of mean terms, states that by investing now in a 11.46

year maturity he will guarantee having \$10,000 in year ten.[*]

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[*] This is not strictly correct since Immunization through time requires
changing $A_t$ solutions. If the 11.46 year asset is held throughout and
$r_{10} = r_o$ , but in interveening years $r < r_o$ , then the father will have
less than 10,000 in year 10. This will be discussed under transitional
Immunization.

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$$\bar{t}_L = 10.0 = a_n \quad , \ n = 11.46 = \bar{t}_A$$

if the 11.46 is the correct term, then $\bar{t}_{A+r} = \bar{t}_L = 10.0$

$$\bar{t}_{A+r} = \frac{v^1(1)\ (180) + v^2(2)(180) + \ldots + v^{11.46}(11.46)(180/2) + v^{11.46}(11.46)(8,000)}{v^1(180) + v^2(180) + \ldots + v^{11.46}(180) + v^{11.46}(8,000)}$$

Consider the case of greater income uncertainty: The father has \$5,000 now

and knows he will receive \$1,023, \$1,046, and \$1,069, in the three succeeding

years. How should he invest when not only the rate pertaining to reinvested

interest receipts is uncertain, but also the rate on the \$3,138? The mean

term of $L_t$ is in this case greater than 10 because of the positive $L_t$ in

three future years:

$$\bar{t}_L = \frac{v^1(-1,023) + 2v^2(-1,046)\ P\ 3v^3(-1,069) + 10v^{10}(10,000)}{v^1(-1,023) + v^2(-1,046) + v^3(-1,069) + v^{10}(10,000)} = 14.8$$

This means that the $A_t$ necessary to equalize the now longer $\bar{t}_L$ must be

further in the future, namely, 18.13 years. The effect of delaying the

receipt of income further into the future is to increase the amount of

income-risk. This is offset by buying longer-term assets. If the future

sums to be received are even further in the future, say, $\$1,169_{(7)}$ ,

$\$1,195_{(8)}$ , and $\$1,222_{(9)}$ , then $\bar{t}_L$ is decreased to 11.2 years, which

requires an asset maturity of 13.1 years to immunize the $L_t$ . As the

funds are received closer to the expected time of utilization, unanticipated

future interest returns are received for a shorter duration; consequently

there is less income risk and shorter assets are purchased.

Insurance companies are in essentially the same risk situation as

this frenectic father. The investment method suggested for this situation

offsets future interest uncertainty with equivalent capital-value uncertainty.

Thus there is no need to assume that compound-interest securities are available.

Immunization (sufficient funds). Assume a life company has one $L_t$

equal to $\$10_{(10)}$ with r = 2.25 per cent. Since it is the average of both

$A_t$ and $r_t$ which must equal $\bar{t}_L$ (10), the average maturity of assets

must be greater than 10 years by the exact amount to counter-balance the

stream of interest income which occurs before $t_{10}$ and thus cause the mean

of the combined interest and maturity income in present values to equal

10. Using the proximate rule for finding $A_t$ , the immunizing asset term

is $A_{11.48}$. This situation is identical to the first example of the investing

father.

Should interest rates decline in the future, the capital value of the

asset after ten years will be above par by the precise amount to offset

the previous shortfall of interest income on reinvested interest proceeds.

(This is correct if the changing Immunization solution through time, new

$\bar{t}_A$ , is followed.)

Immunization (insufficient funds). The more interesting application of Immunization, and the one for which it was designed, is that of the Humped Fund. This is analogous to the previous example in which the father had future income for investment. The presence of a large future cash receivable greatly increases the income risk, and hence correspondingly increases the mean asset term necessary for immunization. Mathematically, this follows because the negative term is always multiplied by a relatively small t in the numerator, while it is accepted into the denominator without a weight which causes the denominator to decrease much more than the numerator. Furthermore, the positive cash receivable reduces the dispersion of maturity among assets required to match the $L_t$ dispersion. This is because the negative values in the numerator receive a high dispersion weight as they are now more distant from the increased mean maturity. If the dispersion of the asset flow is greater then the dispersion of the liability flow, the company will make a profit if interest rates change in either direction.

The following example, which is representative of British life companies, shows the operation of an immunized investment portfolio. The process whereby "long" assets hedge against the uncertain course of future interest rates is best shown by illustration.

Example of immunization for a representative life company. Three situations are examined: a) effect of interest changes at $t_o$ , b) effect of interest changes at $t_5$ after the new funds have been invested, c) effect

of interest changes at $t_5$ immediately before new funds are invested:

Given: $L_t = -30_{(5)},\ 60_{(15)},\ 50_{(25)},\ 20_{(35)}$

$\phantom{Given:\ }A_t = 25.155_{(40)},\ 25.155_{(65.9)}$ $\Bigg\}$ at $t_o$

$\phantom{Given:\ }r = 2.50$ per cent

Mean terms of the Immunized portfolio:

$$\bar{t}_L = \frac{\Sigma\ tv^t L_t}{\Sigma v^t L_t} = 28.98 \text{ years}$$

if $\bar{t}_L = a_{\overline{n}|} = 28.98$, then $n = 48.96 = $ required $\bar{t}_A$

One possible solution of $A_t$ , giving $\bar{t}_A = 48.96$, is the $A_t$ shown above:

$$\frac{\Sigma(v^t \cdot t \cdot A_t)}{\Sigma(v^t \cdot A_t)} = \frac{(40 \cdot 9.368) + (66 \cdot 4.930)}{14298} = 48.96$$

the dispersion of $L_t = \dfrac{\Sigma[(\bar{t}-t)^2 v^t L_t]}{\Sigma[v^t L_t]} = -127.1$

Because there is a negative $L_t$ at an extreme distance from $\bar{t}_A$ the
dispersion measure is negative. No particular meaning should be attached
to this except that a single $A_t$ suffices to cover this negative dispersion.
The two assets in this portfolio guarantee a "profit" when the interest
rate changes since the second condition of immunization states that if
dispersion of the present values of $A_t + r_t$ is greater than $L_t$ this
causes a "profit" for any interest rate change.

    a) At $t_o$ what is the effect of changes of market interest rate
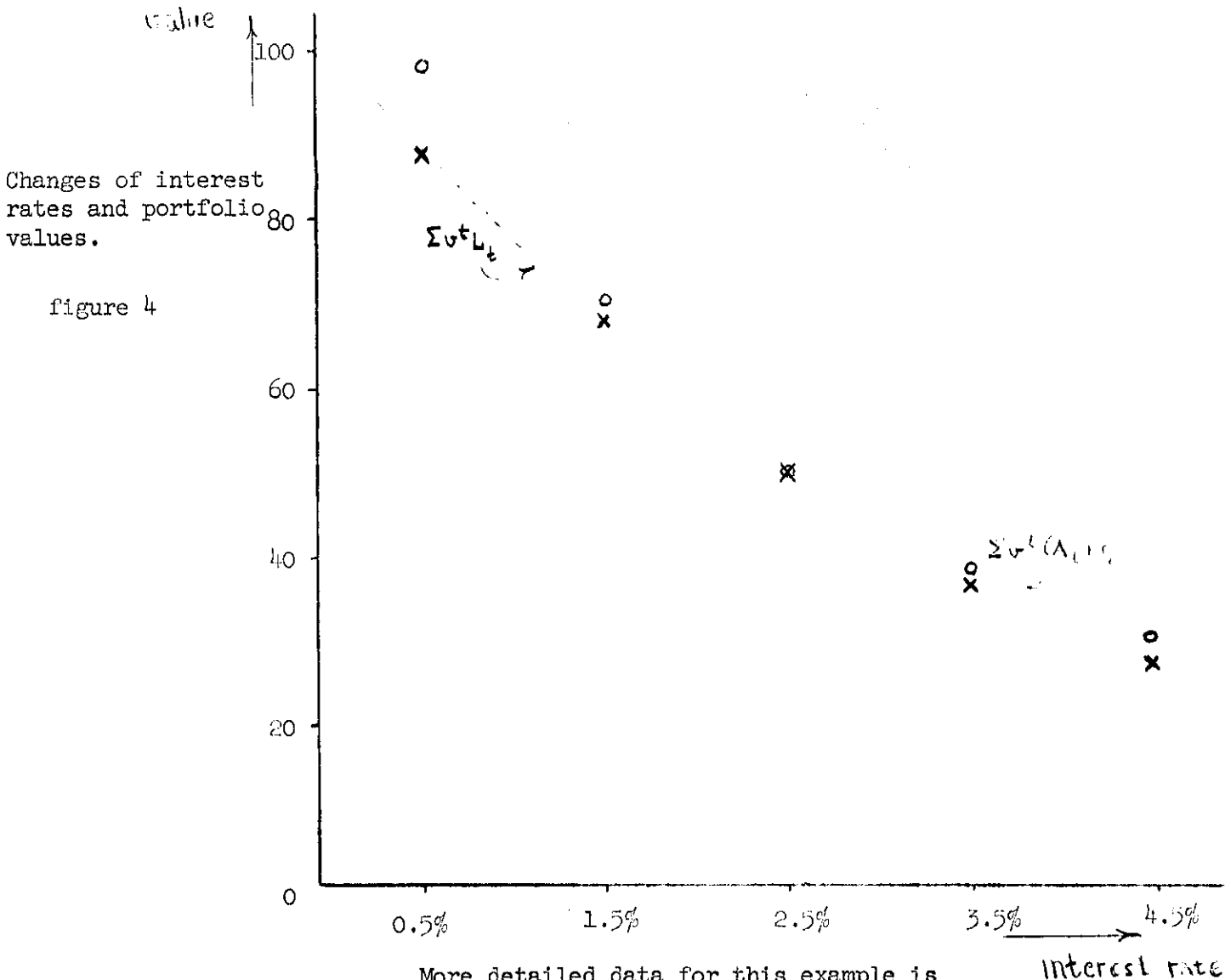       on portfolio values, defined as discounted flows?

value

100

Changes of interest
rates and portfolio
values.

80

figure 4

60

$\Sigma v^t L_t$

40

$\Sigma v^t (A_t + r_t)$

20

0

0.5%    1.5%    2.5%    3.5%    4.5%

Interest rate

More detailed data for this example is
given in Appendix B.  At a point of time
this Immunized Portfolio would be affected
by interest rate changes as shown.  The values
of $\Sigma v^t(A_t + r_t)$ at different interest rates are
shown by the circles, the values of $\Sigma v^t L_t$ are
shown by x's.

b)  Immunization through time requires new $\bar{t}_A$ as $\bar{t}_L$ changes.  In

this case assume no further sales, then $L_t$ simply ages and at $t_5$ is:

$60_{(10)}$, $50_{(20)}$, and $20_{(30)}$.  The negative $L_5$ has now just been received

by the company and must be invested.  In addition assume that at this point

all interest due (compounded) is also received.  The coupon equals

.025 (50.305) = 1.258 per year.  The sum of these annual coupons compounded,

$\sum_{t=1}^{5} (1 + .025)^t 1.258$, equals 6.778.  Thus the funds to be invested equal 36.778.

The new $L_t$ at $t_5 = 15.70$ years $= a_{\overline{n}|}$ ; $n = 20.0 =$ desired $\bar{t}_A$

for the entire portfolio. At what term should the new funds, 36.778,

be invested so that, without shifting the old assets, the $\bar{t}_A$ of the

portfolio will be 20.0 years? It is known that the marginal investment

must be short-term since the new desired $\bar{t}_A$ is only 20. The selection

of the right marginal term is a process of approximation:

if $t = 7$, then $\dfrac{\Sigma[7v^7 \cdot 36.779 + 35.v^{35} \cdot 25.155 + 61v^{61} \cdot 25.155]}{\Sigma[v^7 \; 36.779 + v^{35}25.155 + v^{61} \; 25.155]} = 19.68$

if we try $t = 8$, then $\dfrac{\Sigma[v^t \cdot t \cdot A_t]}{\Sigma[v^t A_t]} = 20.66$ years

Thus $t = 7$ is chosen as the closer approximation, although 7.4 is more

accurate. If 36.78 is invested in a seven year maturity the portfolio

will again be immunized; no future change of interest can cause the company

to default on its $L_t$ .

c) A more crucial question is, what would happen if the interest

rate changed before the 36.78 were invested? Assume the original $A_t$

aged 5 years at $t_5$ , when the interest rate decreases from 2.5 to 1.5

per cent, before the premium sum is received. Figures are also given

for other changes of interest rate. The 36.78 must now be invested at

a lesser interest rate than was anticipated and guaranteed in the policies.[*]

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[*] Since a different maturity solution is required for each interest rate,
a new $\bar{t}_A$ should be found for each interest possibility, e.g. 1.5, 3.5,
etc. This is not done in order to keep the example simple, see Appendix B.

Table Ia

| New interest rates | $L_t$ | | | $\Sigma L_t v^t$ | $\Sigma v^t(A_t+r_t)$ | $35.78_{(7)}$ | $A_t$ | | $\Sigma v^t A_t$ | $\Sigma v^t r_t$ |
| | $60_{(10)}$ | $50_{(20)}$ | $20_{(30)}$ | 130 | | | $25.155_{(35)}$ | $25.155_{(6)}$ | (87.089) | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5% | 57.07 | 45.25 | 17.77 | 119.99 | 129.59 | 35.51 | 21.12 | 18.55 | 75.18 | 54.39 |
| 1.5% | 51.70 | 37.12 | 12.79 | 101.51 | 103.77 | 33.13 | 14.93 | 10.14 | 58.10 | 45.67 |
| 2.5% | 46.86 | 30.51 | 9.53 | 86.90 | 87.07 | 30.94 | 10.60 | 5.57 | 47.11 | 39.96 |
| 3.5% | 42.53 | 25.13 | 7.12 | 74.78 | 75.72 | 28.90 | 7.54 | 3.08 | 39.52 | 36.20 |
| 4.5% | 38.63 | 22.82 | 5.34 | 66.78 | 67.89 | 27.02 | 5.39 | 1.71 | 34.15 | 33.74 |

Table la lists the present value sums for different interest rates. The sums of the new interest flows are shown in Table 1b. The 50.3 originally invested at a coupon of 2.5 earns 1.258/year independent of any interest rate changes at $t_5$ . To 1.258 must be added the new interest proceeds, $r(36.779)$ to find $r_t$ where $t < 7$ . These $r_t$ flows and their present values are shown below.

Table 1b

| Interest rates | $t_1 \longrightarrow t_7$ | | $t_8 \longrightarrow t_{35}$ | | $t_{36} \longrightarrow t_{61}$ | | $\Sigma v^t r_t$ |
| | r per yr. | $\Sigma v^t r_t$ | r per yr. | $\Sigma v^t r_t$ | r per yr. | $\Sigma v^t r_t$ | |
|---|---|---|---|---|---|---|---|
| 0.5% | 1.439 | 9.874 | 1.258 | 31.66 | .629 | 12.85 | 54.39 |
| 1.5% | 1.807 | 11.922 | 1.258 | 25.75 | .629 | 7.99 | 45.67 |
| 2.5% | 2.177 | 13.810 | 1.258 | 20.88 | .629 | 5.29 | 39.97 |
| 3.5% | 2.545 | 15.549 | 1.258 | 17.46 | .629 | 3.18 | 36.20 |
| 4.5% | 2.911 | 17.152 | 1.258 | 14.55 | .629 | 2.04 | 33.75 |

In general it is seen that even though $r_t$ is small for low interest rates, the change of capital values is sufficient to offset the low $r_t$. The two center columns provide a quick comparison of the liability and asset sums. For all interest rates $\Sigma v^t(A_t + r_t)$ exceeds $\Sigma v^t L_t$.

Perhaps the manner in which these flows offset each other can be seen more clearly by examining a specific interest change, from 2.5 to 1.5 per cent.

1) The shortfall in interest proceeds caused by the fall of interest rates, using a discount rate of 1.5 per cent, is:

$$\sum_{t=1}^{7} v^t(.015 - .025)(36.779) = -2.425$$

2) The increase of value of $r_t$, if coupon rate had <u>not</u> changed would have been +.542 for $t_1 \to t_7$ (from 13.882 to 14.364). For the other two periods the values of $r_t$ did increase, as shown, by +7.596. Hence, the change of discount rate alone (no change of coupon) causes a total change of value of $r_t$ equal to +8.138.

3) The increase of value of $A_t$, caused by the decrease of the interest rate, equals + 11.100. Change of $A_t + r_t = 19.238$.

4) The increase of value of $L_t$ equals +14.706. Thus the net excess of increase of $A_t + r_t$ over $L_t$ equals 19.238 - 14.706 = 4.532. This is sufficient to cover the interest proceeds shortfall of -2.425 (caused by the decrease of coupon rate on the 36.779 invested) and still leave a profit of 2.100. This is equivalent to the difference of aggregate values shown on the chart, 103.893 - 101.621. Thus it is essentially the greater maturity values of $A_t$ which causes $\Sigma v^t(A_t + r_t)$ to increase more than $\Sigma v^t L_t$ and thereby cover the shortfall in interest proceeds.

D.  MODIFICATIONS OF THE IMMUNIZED PORTFOLIO . . . . . . . .

The previous discussion has specified the formal conditions of a

"no-income-risk" position.[*]  This is important, for without agreement

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[*]  This terminology is somewhat misleading.  Actually, income-risk is
eliminated only by acquiring sufficient portions of capital-value risk
as a counterpoise.  Thus it is a "no-income-risk" position only if one
is willing to concede that capital-value risk is unimportant to life
companies.

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on this point it is impossible to speak of a portfolio being "too long

or short," containing more or less income risk.  Though it is essential

to define a benchmark for measuring income risk, there must be infinite

disputation over the actual value of the benchmark, and, over the extent

and conditions under which a life company should depart from its safe

position.

Ultimately, the answers to these questions rely on preferences; the

following presents some considerations which impinge on the income-risk

position.  This is accomplished by relaxing some of the assumptions and

by considering the institutional setting of the companies.  The following

are considered:  1) the assumption that life insurance portfolios are

largely non-marketable, 2) equity considerations in relation to dividend

policy of mutual companies, 3) maturity uncertainty, and 4) the necessity

of obtaining a "good" return within the competitive situation.

1.  Non-marketable securities and transitional immunization:  The

first of two separate considerations concerning the applicability of

immunization as the "correct" definition of the "no-income-risk" position

for United States companies concerns the ability of these companies to maintain an immunized position through time. The second consideration relates to the methods of accounting-valuation in use. The conclusion is reached that some degree of immunization is always possible, even though Full Immunization is unobtainable.

Immunization is a "point-of-time" solution. It states that with the correct $A_t$ , and no change of $L_t$ , no future change in the interest rate can affect the ability of the company to meet its liabilities. Through time $L_t$ shortens if no further sales are made; this requires a switching of asset maturities or an investment of new funds in order to decrease $\bar{t}_A$ by more than the decrease in $\bar{t}_L$ . The assets must do more than just age. This was shown in the previous example in which the $36.7 was invested for a short term of seven years in order to sufficiently decrease the mean asset term. The relevant question is whether it is possible to change $\bar{t}_A$ sufficiently if the existing assets of the company are non-marketable. As long as the company is within the negative $L_t$ range, the combination of premium and interest inflow, if invested correctly, contribute toward maintaining the immunized position. Whether these funds are sufficient to maintain the immunized position without reinvesting port- folio assets depends on many factors including the structure and rate of growth of sales and the extent of interest rate changes. When $L_t$ is positive, $A_t$ could be arranged for perfect matching, if the previous necessary shifts have been made in $A_t$ . If continuing sales are made, the $L_t$ may remain constant or even increase, thus entailing all new cash flows be invested long in order to maintain Immunization. Thus, even though a

major portion of the company's assets are non-marketable, it is certain
that so long as insurance sales continue to increase at least an approximation
of the Immunized position can be obtained by correct placement of net premium
inflows.

The accounting-valuation procedures used in the United States - asset
values are stable through amortization; liabilities are valued at the
interest rates guaranteed in the policies - in no way affect the definition
of the "no-income-risk" position for a life company. Immunization may be
used as a measure of and defense against income risk regardless of the
system of accounts used for public display. Two sets of accounts might
be carried if there were valid, but separate, reasons.

Even if Immunization is deemed inappropriate as the measure of the
"no-income-risk" position because the position is unobtainable (perhaps
the necessary maturities are not available), it should still be noted that
the tests which companies should apply to themselves should contain some
measure of income risk. For instance, the legal solvency requirement in
the United States at present is basically: $\Sigma A_t \geq \Sigma v_j^t L_t$ where $j$ denotes
the interest policy guarantees. This is a capital value or "salvage"
solvency. It fails to consider both the time dimension, which is the
essence of the life company situation, and the relationship between coupons
on investments (r's) and interest policy guarantees (j's). The previous
inequality, with the constraint that $\bar{t}_A = \bar{t}_L$ and $\bar{r} = \bar{j}$ , takes income
risk as well as capital-value risk explicitly into account. These conditions
assert that solvency for life companies consists not only of having sufficient
value of assets, but also having a coupon rate of sufficient size and duration
to meet the comparable dimensions of the liabilities. This is not a formula

for investment but rather a yard-stick of "income-risk" which is applicable
along with other necessary tests to the portfolio.[*]

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[*]   The current income tests used by life companies (the comparison of
tabular interest and investment income) is somewhat similar to the
comparison of the  r's  and  j's  above, but it does not compare the
time duration of the guarantees attached to the  r's  with those attached
to the  j's.  To neglect these time dimensions is either to unknowingly
expose oneself to income risk or to knowingly speculate on the future
course of interest rates.

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The conclusion reached is that the unique position of a life company,
in which it guarantees to earn an interest rate on funds which it has not
yet received, and the method whereby this risk can be eliminated are not
affected by the particular accounting-valuation systems used.  Though the
applicability of such an investment policy may be diminished if all assets
are non-marketable, it is still possible to obtain some immunization by
appropriate investment of the cash flows.

2.  <u>Equity considerations</u>.  The policy which a company adopts in regard
to dividend policy will influence the desired  $\bar{t}_A$ .  No immunization policy
is independent of the dividend policy adopted for allocating the current
year's shortfall or overage of investment income among time-policy groups.
The policy may range from giving all the current year's gain or loss of
investment income to the current year's sales group to the opposite extreme
of sharing the loss or gain equally among time-policy groups.  Thus dividend
policy may be designed to be either independent of or dependent on future
interest rate changes; the former requires longer-term securities while the
latter requires shorter-term securities.  Three possibilities will be briefly
considered, with emphasis on conclusions rather than the proofs of these
conclusions.

a.  Stability of dividend (non-responsive).  In the Reddington
solution of Full Immunization the dividend rate is guaranteed at the
rate obtainable under the interest rate prevailing at inception of the
policy.  By investing long, this rate of dividend will be maintained
no matter the future course of interest rates.  In effect, this treats
all policies as non-participating since the dividend will not be affected
by future interest rate changes.  The dividend rate for a single time-
group of policies is completely stable over time while at a point of time
each different time-group of policies may earn a different dividend rate.

b.  Consistency of dividend (semi-responsive).  Bayley and Perks [9]
suggest the equity norm:  "future premiums on existing policies should
secure at least as favorable benefits as new business."  The question then
to be decided is how responsive to current interest rates should the
dividend be?  The solution proposed by Bayley and Perks makes the dividend
depend on,  1) past and current rates with the weights determined by the
ratio of premiums received in current year to total premiums received thus
far, and 2) the difference between earned interest return and guaranteed
interest return.

$$\left[ \frac{\alpha p_{t_*}}{\sum\limits_{t=t_o}^{t_*} [\alpha p_t]} \cdot (r_t - r_g) + 1 \right] \cdot (\text{dividend}_{t_* - 1})$$

As each premium payment is paid in, this is invested to immunize the
"paid-up-life-value" of the policy at that time.  This effectively guarantees
a dividend rate proportional to the current difference between guaranteed

and market rates of interest throughout the future life of that policy. As the policy ages the dividend is more and more determined by "sunk" investment; the additional investment can change the dividend rate only marginally even if interest rates should change dramatically. This method simply invests the current funds at existing rates for required terms and lets dividends emerge as they will, that is, as future interest rates decide. The implicit and important assumption of this method is that dividend loadings are sufficient so that future interest changes influence dividend payments, but are not large enough to imperil the sum insured. This is a necessary condition since this method always exposes the fund to future interest changes on forthcoming premium payments and assumes that the future interest fluctuations can be absorbed by changes in the dividend rate. The contrast between these two methods illustrates very clearly the clash between stability of dividend and complete income-security, and responsiveness of dividend and, by necessity, less income-security.

The implications of this method, Paid-up Immunization, are that the desired $\bar{t}_A$ should be about one half that for Full Immunization. The Fund is invested much shorter so that interest changes are reflected in future dividend payments.

c. Equality of dividend rates (fully responsive). It is easy to see that the desire for dividend responsiveness carried to its logical conclusion requires equality of dividend rates on all time-policy groups. The particular history of interest rates associated with a time-policy group no longer differentiates the dividend of that group from the dividend of other groups.

Anderson and Binns [10] have suggested this equality criterion.  Under the two previous methods, if the interest rate changed the bonus rate earned on new policies would differ from that earned on existing policies. By this criterion the bonus on old policies should change sufficiently to equal that on new policies.  Furthermore, when new coupon rates are earned next year, the dividends of all time-policy groups should remain equal though of different size than in the current year.

Since a high degree of responsiveness to current interest rates is desired, the Fund must be invested short so that the Fund "turns over" frequently to maintain touch with current interest rates.  The cost of this high degree of dividend responsiveness is an extreme degree of income-risk.  Equal Bonus Immunization requires a mean asset term about one-half that for Paid-up Immunization, and about one-fourth that for Full Immunization.

3.  The uncertainty regarding "maturity".  The assumption that Assets and Liabilities have fixed maturities which are predictable with certainty must now be relaxed.  Liabilities are encumbered with numerous liquidity and settlement (terminal) options, while assets have call and prepayment options.  In fact, the prepayment provisions attached to assets during the postwar period, both prescheduled and at borrowers option, have increased so extensively that even the usefulness of the maturity concept has become blurred.  Yet, in spite of this, there are observable regularities in the degree of utilization of options.  One example is the disinclination with which individuals have used their life policies as primary liquidity defense.

From the observed historical regularities one could form an Expected Value for $A_t$ and $L_t$ , taking into account the options existing on outstanding policies and historical experience. Then the portfolio could be immunized on the basis of these expected distributions. Since these options are usually taken when it is disadvantageous to the companies, it is probable that a longer than required portfolio would be desired to offset the risk associated with this likelihood.

4. The competitive situation of life insurance. Defining the "no-income-risk" position of life companies under various objectives of capital-certainty and dividend policy has been the primary objective of this paper. This objective stands,quite apart from any questions relating to whether Immunization should be adopted as an actual investment policy. Necessarily, the life company is simultaneously involved with considerations in many dimensions and, as is so often the case, the cost of obtaining perfection in any one direction is usually prohibitive in terms of alternatives foregone. Thus it may be useful to conclude by noting the obstacles in the way of adopting Immunization as an operating policy. Three relevant issues concern the supply of assets, the yield structure and the objective of maximum yield.

Given the present size and rate of growth of U.S. companies, the existing total supply of long-term debt is insignificant in relation to the demand requirements of Immunization. Even if companies were willing to accept lower return in order to obtain lower risk, there is the very real problem of the magnitude of new flotations and refundings necessary to provide the desired supply of maturities.

Immunization assumes that when policies are written at a given rate, the company is able to purchase assets at that rate for any required term. This implies a flat yield function, or at least a yield function of stable differentials over time. The changing shapes and differentials in the marketplace present a real problem.

Finally, a severe cost of Immunization is that it hedges equally against profits as well as losses. Even though there is an obtainable "no-income-risk" position, it may not be prudent to attain such a position for two reasons. First, profit maximization may be at variance with income risk; the Expected Return from investing in short assets (five to ten years) may be sufficient to compensate the directors for assuming the income risk.*

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* Default risk also has a bearing on the term selected. Consider two trends in the postwar capital markets. On the demand side, the length of maturities "desired" has probably increased due to the large flow of funds into pension trusts, life companies, etc. Yet on the supply side, the average term of new flotations has noticeably shortened, except for public utility and special revenue bonds. While these trends seem at variance with the analysis presented, the explanation serves to emphasize that life companies operate in two dimensions of risk, income and default; a departure from the no-risk position in either case requires a reward of increased expected return. This paper has been confined to the ramifications of the income risk dimension, while the explanation of the opposing maturity trends depends, in part, on both dimensions. The primary reason for the shortening of maturities has probably been the attempt by life companies to broaden their lending markets in order to absorb increased placements without an accompanying rise in default risk. By specifying rigorous amortization schedules, an automatic "testing" of the borrowers is regularly accomplished with small administrative cost. Moreover, it appears that the life companies have required a higher return on amortizable industrial loans to compensate for the increased income risk. It is also probable that some of the explanation for the shorter term of securities lies on the supply side in the composition and longevity of the capital investment. Today a smaller share of investment goes into long-term durables such as railroads than did in the prewar period. These are conjectures; research is necessary to explore these institutional impacts on the capital market.

This is particularly true if the elasticity of expectation for long rates is near zero. Second, if "buffers" such as large Surplus and Dividend Loading exist, there may be no risk of insolvency for wide ranges of future interest rates. For instance, a Mutual Company with high dividend loading in its premiums may be able to meet almost any adverse interest trend by drawing down Surplus and omitting divided payments. In the opposite case of a young, stock company with small surplus there is a strong presumption in favor of Immunization.

In summary, this analysis implies that the low capital-value risk on short maturities is of little utility to life companies while the large income risk on these securities is very undesirable. Hence there is a strong presumption in favor of long-term securities in life insurance investment.

This implies a declining maturity yield function which is opposite to that suggested by current interest theory. It is commonly accepted, as Hicks [16] says, "that most people would prefer to lend short," i.e., most people prefer capital certainty. This implies that higher interest rates must be offered on long-term securities in order to induce people to depart from short, safe securities. This may be a correct interpretation of individuals' preferences; it is clearly an invalid interpretation for certain institutions. For life companies and pension trusts there are strong motivations, deriving from the nature of their liability contracts, in favor of long maturities as the riskless asset. Hence the predictable impact of these institutions acting alone (apart from expectations) is a downward sloping maturity yield function.

## Appendix A

The following is Reddington's [8] derivation of the conditions for an Immunized portfolio.

Let $V_A = \Sigma \, v^t (A_t + r_t)$

$V_L = \Sigma \, v^t L_t$

Given $v = e^{-\delta}$ ; $\delta = \log (1+i) = $ "force of interest"

Assume that these are continuous distributions and that $V_A = V_L$ . Then a change of interest from $\delta$ to $(\delta + \epsilon)$, with a consequent change of $V_A$ and $V_L$ to $V_A'$ and $V_L'$ , is given by Taylor's Theorem:

$$V_A' - V_L' = (V_A - V_L) + \epsilon \, \frac{d(V_A - V_L)}{d\delta} + \frac{\epsilon^2}{2!} \cdot \frac{d^2(V_A - V_L)}{d\delta^2} + \ldots$$

The first term of the expansion vanishes since $V_A = V_L$ . It is clear that if there is to be no profit or loss whatever from the change in the force of interest then all the successive derivatives must vanish. In practice the first derivative is the most important for small changes of the rate of interest and therefore the Fund is defined as immunized if the assets are so invested that $\dfrac{d(V_A - V_L)}{d\delta}$ is zero.

If the second derivative is positive, then, since the coefficient $\dfrac{\epsilon^2}{2!}$ is positive whether $\epsilon$ is positive or negative, any change in the force of interest will result in a profit to the Fund so long as the change is not so large that the higher terms in the expansion begin to take effect. It is desirable, therefore, ... that $\dfrac{d^2(V_A - V_L)}{d\delta^2}$ should be positive.

A satisfactory immunization policy can, therefore, be expressed symbolically in two equations.

(1) $\quad \dfrac{d(V_A - V_L)}{d\delta} = 0$

(2) $\quad \dfrac{d^2(V_A - V_L)}{d\delta^2} > 0$

These can then be expanded into the two equations given in the text.

## Appendix B:  the example of Immunization

a) Table IIa lists the detailed figures from which figure 5 is derived. Table IIb lists the detailed figures for the present values of the interest flow.  These charts refer to part (a) of the example.  There is no change of coupon rates since all assets are assumed invested at the time of the supposed interest changes.

### Table IIa

| Interest rates | $L_t$ $-30_{(5)}$ | $60_{(15)}$ | $50_{(25)}$ | $20_{(35)}$ | $\Sigma v^t L_t$ 100 | $\Sigma v^t(A_t+r_t)$ | $A_t$ $25.2_{(40)}$ | $25.2_{(66)}$ | $\Sigma v^t A_t$ 50.3 |
|---|---|---|---|---|---|---|---|---|---|
| 0.5% | -29.26 | 55.67 | 44.14 | 16.79 | 87.35 | 96.69 | 20.60 | 18.09 | 38.70 |
| 1.5% | -27.85 | 47.99 | 34.46 | 11.88 | 66.48 | 68.34 | 13.87 | 9.42 | 23.28 |
| 2.5% | -26.51 | 41.43 | 26.97 | 8.43 | 50.31 | 50.31 | 9.37 | 4.93 | 14.30 |
| 3.5% | -25.26 | 35.81 | 21.16 | 6.00 | 37.71 | 38.524 | 6.35 | 2.60 | 9.13 |
| 4.5% | -24.08 | 31.00 | 16.66 | 4.29 | 27.88 | 30.49 | 4.32 | 1.38 | 5.70 |

### Table IIb

| Interest rates | $t_1 \longrightarrow t_{40}$ r per yr. | $\Sigma v^t r_t$ | $t_{41} \longrightarrow t_{66}$ r per yr. | $\Sigma v^t r_t$ | $\Sigma v^t r_t$ |
|---|---|---|---|---|---|
| 0.5% | 1.258 | 45.50 | .629 | 12.48 | 57.99 |
| 1.5% | 1.258 | 37.64 | .629 | 7.42 | 45.05 |
| 2.5% | 1.258 | 31.57 | .629 | 4.44 | 36.01 |
| 3.5% | 1.258 | 26.87 | .629 | 2.53 | 29.39 |
| 4.5% | 1.258 | 23.15 | .629 | 1.64 | 24.79 |

b) Immunization requires a new solution of $\bar{t}_A$ for each change of $r$ as well as for each change of $L_t$ . Hence in parts b and c , the four changes of interest rate were presented, the correct method of solution would be:

     1) find new $\bar{t}_L$ for each new $r$ rate,

     2) find new required $\bar{t}_A$ , given $\bar{t}_L$ ,

     3) find maturity term for the marginal investment

          of the 36.78, such that $\bar{t}_A$ of the entire port-

          folio equals the required $\bar{t}_A$ .

Whenever the portfolio is reimmunized, perhaps every three to five years, the above process must be accomplished. The different required $\bar{t}_A$'s of the interest rate are:

if $r$ = .005 , then $\bar{t}_L$ = 16.67 = $a_{\overline{n}}$ ; $n$ = $\bar{t}_A$ = 17.6

    $r$ = .015 , then $\bar{t}_L$ = 16.17 = $a_{\overline{n}}$ ; $n$ = $\bar{t}_A$ = 18.7

    $r$ = .025 , then $\bar{t}_L$ = 15.70 = $a_{\overline{n}}$ ; $n$ = $\bar{t}_A$ = 20.1

    $r$ = .035 , then $\bar{t}_L$ = 15.27 = $a_{\overline{n}}$ ; $n$ = $\bar{t}_A$ = 22.2

    $r$ = .045 , then $\bar{t}_L$ = 15.02 = $a_{\overline{n}}$ ; $n$ = $\bar{t}_A$ = 25.8

Thus it is seen that the "correct" term of investment for the 36.78 would have been less than 7 years for low interest rates and greater than 7 years for high interest rates. In this situation the portfolio remained Immunized even without considering this effect.

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