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Economic Theory of Teams*

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Chapter 1

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PART ONE

CHAPTER 1

DECISION UNDER UNCERTAINTY

1. Team. We define an organization as a group of persons whose actions agree with certain rules that further their common interests. We define a team as an organization the members of which have only common interests. We are going to discuss an economic theory of teams.

We just have used terms such as "interests," "rules," "economic" but have not yet defined them. If we first confine our attention to a specially simple case, the one-man team (that is, simply, a single decision-maker) we shall be able not only to present convenient definitions of those terms, but also to introduce some other important aspects of the economic theory of teams. Most of the complications that arise from the presence of several persons in a team will be postponed till Part Two of the book.

2. Economic behavior. It is usual to define economic behavior as one in which best use is made of limited possibilities. In practice, economists confine their study to the use of limited possibilities in producing and consuming so-called commodities, or goods and services, mostly divisible into small physical units. Yet, the best use of limited possibilities is also the subject of disciplines such as military and political science. The adjective "economic" in the title of our book refers to the general meaning; and some of our results should be applicable outside of economics of commodities. It so happens, however, that men's concern with the best use of available possibilities (resources) finds its clearest expression when dealing with divisible commodities.

This divisibility permits the extended use of continuous real variables; and, under certain conditions prevailing in our and certain other cultures, market prices of goods and services can be used to express the complex result of actions as a single continuous quantity such as the monetary profit. Accordingly, our general ideas will often find their simplest illustration in the special domain of commodities and money. The business firm will often furnish the most palpable example of a team concerned with the best use of its possibilities. However, the reader will remain aware that even in a business firm decisions on indivisible things (site of plant, appointment of executives, methods of financing) play a role besides commodities and money.

3. Consistent tastes under certainty. Whether a good use is or is not made of available possibilities would not be ascertainable, and the word "economic" would therefore be meaningless, if the criterion of what the individual considers "good" were not, in some sense, fixed, making his decisions "consistent." Consistency of decisions can be defined operationally (and, moreover, can be made ascertainable from the chooser's actions and not merely from his verbal statements) in the following convenient way:

First, we shall say that a given decision-maker desires the alternative a more than b (and b less than a) - or "prefers a to b" - if he never chooses b when a is available; second, we shall say that he desires a and b equally (or "is indifferent between a and b") if from sets of alternatives containing both a and b he chooses sometimes a , sometimes b ; finally, we say he is consistent if, whenever he desires a not less than b , and b not less than c , he desires a not less than c .

It follows that it is possible to rank in a unique order of desirabilities (also called order of utilities, or preference order), all alternatives that can ever face a given consistent man; with the understanding that equally desirable alternatives have equal rank (a "tie"). We can describe a given consistent man's interests (or tastes) by his particular ordering of alternatives.

It also follows that a consistent man's choices do not depend on "irrelevant alternatives" (Arrow [ ]). If a businessman, told of two available plant sites, one in Alabama, the other near Boston, chooses the former one; and if he is later told that one near Chicago is also available; then, if he is consistent, he will not switch to Boston, although he may switch to Chicago.

4. Description vs. Norm. If taken literally, the above definition of consistency is no doubt too strong to describe real men. The word "never" in the definition of desirability should be replaced by "very seldom" - lest the majority of people be classified either as indifferent between almost all alternatives or as inconsistent. Accordingly, and in agreement with some practices of empirical psychology, a more general theory has been offered. It describes a man's choices in terms of the probability that, when offered a given set of alternatives he will choose a particular one; so that, for example, "indifference between *a* and *b*" is defined as 50-50 chance of choosing *a* out of the pair (*a*, *b*).* We shall not pursue this approach here.

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* Duncan Luce [ ]. Block-Marschak [ ] .

Instead, we shall assume that departures from consistency, as defined, are not serious enough to make it useless as an approximate description of some people and as an attainable norm for all. Following Ramsey [ ] one can regard the theory of consistent choice as a branch of logic, a normative discipline. The theory of consistent choice is perhaps similarly related to the actual choices as ordinary logic - the logic of thought - to the psychology of thought.* But

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* Abelson and Rosenberg [ ] have shown how logical rules can be extended into those of "psycho-logic." See also Bruner [ ].

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just as the logic of thought is a guide, a first approximation to the psychology of thought, so is the logic of choice a guide to the study of actual decision.

5. Description and Norm. This two-fold use of theory, as an approximate description and as a system of prescriptions, or norms, is quite common to all disciplines bearing on practice. Economics deals with descriptive theories as well as with policies for statesmen and businessmen; military science has produced war histories as well as army manuals; medicine describes sick men and has the ideal of a healthy one. Consistency of choices is a possible norm (it is *the* economic norm) of human behavior; but because it also serves as an approximate description of reality, this norm is not completely out of practical reach. As we proceed to develop the consistency concept the reader should keep in mind this double orientation of our book. The team concept itself will be seen to play this double role: to say that a firm is a team of executives is only an approximation to facts, but it is also a useful norm, strived at by an organizer.

6. Actions and outcomes. So far we have considered only one kind of action: the choosing among available alternatives. Here the action and its result coincide: each can be identified with the alternative chosen. In a more general case it is useful to distinguish between an action a and its result (outcome)

$\underline{r} = \rho(\underline{a})$, and to call ρ the outcome function.* Thus $\underline{a}_1$, $\underline{a}_2$ may be two

* As far as possible, we shall use Greek letters for functions and the corresponding Latin letters for the values of the functions.

methods of production and $\underline{r}_1 = \rho(\underline{a}_1)$, $\underline{r}_2 = \rho(\underline{a}_2)$ the respective money profits. There is a preference ordering on sums of money: if r_1 and r_2 were available directly the larger one would be chosen. But it is the actions, not the sums of money, that are available for choice. If the outcome function ρ were changed (e.g., because of a change in the prices of ingredients), the choice may be switched from (say) $\underline{a}_1$ to $\underline{a}_2$; while the preference ordering on the set of results (sums of money) remains the same.

We shall use interchangeably the words decision and action; and the words result and outcome.

7. Environment; uncertainty. The example just given shows that the same action can result in different outcomes, depending on factors not controlled by the decision-makers. These factors can be denoted as a variable $(\underline{x})$, called "nature," "environment," "the external world." We have thus to rewrite the outcome function thus: $\underline{r} = \rho(x, \underline{a})$. In general, the value of $\underline{x}$ is not known to the decision-maker in advance and therefore the result $\underline{r}$ of the action $\underline{a}$ is not known even if the outcome function ρ is known. We say that there is uncertainty about the variable $\underline{x}$; and therefore also about the variable $\underline{r}$, given $\underline{a}$.

For example, suppose that a firm producing for a competitive market is about to set the level of production for the coming month, but there is uncertainty about the price that will prevail during that month. If a denotes the amount produced, $\underline{x}$ the price and $\kappa(a)$ the cost of producing an amount $\underline{a}$, then the resulting profit to the firm is given by the outcome function

$$\rho(x,\ a) = xa - \kappa(a)$$

Here the action variable is the quantity a, and the state of the environment is described by the price x.

What if there is uncertainty about the outcome function itself, as well as about the state of the environment? This difficulty can always be overcome by describing the possible states of the environment in sufficiently greater detail. Referring again to the production example, the firm may not know which of several cost functions will actually apply, e.g., because of the possibility of technological improvements. If we denote the various possible alternative cost functions by κ_1, κ_2, etc., then we can describe the state of the environment by the price p and the number n corresponding to the true cost function. The appropriate outcome function is now

$$\rho(a;\ p,\ n) = pa - \kappa_n(a)$$

the form of the function is once again certain, although the variables p and n describing the state of the environment are not. The environment variable x is now a vector: $x = (p, n)$.

8. Consistent beliefs under uncertainty. Formally, one might maintain the original definition of consistency - even after introducing uncertainty. One might simply say that a consistent man has a preference ordering on his actions, and not inquire for any reasons underlying this ordering just as the economist does not inquire into the reasons for a man's preferring pickles to olives. This approach by Debreu [ ] will not suffice for the purposes of this book.

To study concrete problems of decision under uncertainty it is useful to define, not only consistent interests (tastes), in the sense that, for a given decision maker, certain alternatives (results of actions) are more desirable than others; we have also to define consistent beliefs, in the sense that some states of nature are more probable than others. Thus suppose a farmer knows that a certain crop will fail unless rainfall exceeds x inches per annum; and suppose he decides to plant this crop. It will prove useful to interpret this by saying that he 1) prefers the success of the crop to failure and 2) believes the rainfall more likely to exceed than not to exceed x inches. If the number x is adjusted back and forth till the person shows himself indifferent between the two actions (planting or not planting the crop) we shall say that he assigns equal probabilities to the state "rainfall exceeds x inches" and to its negation. And by making a convention that the probabilities of the event happening and not happening add up to 1, we shall conclude that the man believes the probability of each of the two states to be 1/2. This probability is, then, subjective. Moreover we shall require from a consistent man that this belief, as revealed by his actions, should be independent of the particular pair of outcomes of those actions, provided he prefers one outcome to another (the concept "preferred" being itself defined on the basis of his actions under certainty, as was done in an earlier paragraph). Thus, if he is indifferent between planting and not planting the crop when the profits and losses contingent upon the chosen action are $900 and -$1000 respectively, he should remain indifferent if (because of increased tax on capital gains, for example) the pair of outcomes became "$800 and -$1000," or if pecuniary outcomes were replaced by non-monetary rewards and punishments, honorary degrees and terms of jail.

This operational concept of a consistent belief into equal probabilities of a state and of its negation can be extended to any number of mutually exclusive states, and thus to a definition of consistent belief into any probability $\frac{m}{n}$ (a rational fraction; the extension to irrational numbers is left to mathematicians). Consider, for example, three states: (I) men will land on the moon in the year x or earlier; (II) it will be reached after year $\underline{x}$ but before $\underline{x+y}$; (III) we'll not be there before year $\underline{x+y}$ if at all. Consider now three bets: (a) you win only if I is true; (b) you win only if II is true; (c) you win only if III is true. Suppose that, after adjusting back and forth the numbers x and y we find that you are indifferent between the three bets. If you are consistent you will remain indifferent regardless of the changes in the nature or amount of the promised gain; and we shall say that you assign probability 1/3 to each of the states I, II, and III; and probability 2/3 to each of the states "I or II," "II or III," "I or III."

To justify the application of the term probability to the numbers we have just defined, one has to show that these numbers have all the properties usually assumed in the mathematical theory of probabilities. In particular, one would have to define the concept of $\underline{p}(\underline{x}|\underline{y})$, the "conditional probability of $\underline{x}$ if y is true" (as being revealed by the particular man's actual choices) as distinguished both from the non-conditional probability $\underline{p}(\underline{y})$ of the event $\underline{y}$ and from the (joint) probability $\underline{p}(\underline{x}, \underline{y})$ of the event pair $\underline{x}, \underline{y}$; and show that $p(x, y) = p(y) \cdot p(x/y)$. This does require stating mathematically some rather plausible properties of consistent behavior. We shall not undertake this here and refer to Ramsey [ ], De Finetti [ ] and Savage [ ].

We have not inquired nor do we have to inquire why the subjective probability you, the reader, have assigned to a given event (as revealed by your choices, provided you are consistent) is a certain number and not another one. With a non-repeatable event like the first landing on the moon the inquiry would be a difficult psychological one, digging into the vague background of your experiences with somewhat similar events, your memories and past associations. With repeatable events, however, a useful question naturally arises whether a man would be well advised to behave as if his subjective probabilities of given events were equal to their "objective probabilities" as estimated from the relative frequencies of their occurrence in the past. In a certain sense the answer is "yes." It corresponds to common sense to say, roughly, that if a man with consistent tastes has a belief in the probability of an event (as just defined), that was originally formed without repeatedly observing the occurrence or non-occurrence of the event, then he will reveal, by a revised decision, a revised probability belief after having been able to make such observations. The more numerous the observations, the more drastic may be this revision; the subjective probability of an event will approach more and more the observed relative frequency of its occurrence, and depend less and less on the original "uninformed" belief. We cannot give here a rigorous formulation and proof, which combines Thomas Bayes' [ ] old idea of "*a posteriori* probabilities" with some properties of consistent behavior discussed here.*

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* See Savage [ ]. Economic thought and terminology has been much influenced by F. Knight [ ] who emphasized the prevalence of non-repeatable events in practical life and reserved the word "uncertainty" for the case of non-repeatable events, using the term "risk" for the case of repeatable events. He took a resigned stand as to the possibility of an economic analysis of behavior under "uncertainty" thus defined. For an excellent survey see Arrow [ ].

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9. Prospects. If the environment $\underline{x}$ is uncertain so must be the result $\underline{r} = \rho(\underline{x}, \underline{a})$ of a given action $\underline{a}$. If the outcome function ρ is fixed, and

the probability $\Phi(x)$ of each possible environment $\underline{x}$ is known, then one can compute the probability $\pi_a(r)$ of each of the possible results, $\underline{r}$, of a given action $\underline{a}$. Thus each action $\underline{a}$ is characterized by a probability distribution: a function π_a (defined on the set of all possible results), also called a prospect. To choose between actions - e.g., between buying one or another stock - is tantamount to choosing between prospects.

Let R be the set of all results of actions of a consistent man; that is, R is the set of all alternatives subjected to his preference ordering. A prospect π is a probability distribution on the set R. The set of all prospects will be denoted by Π, and a preference ordering exists on it; this ordering of prospects is generated by the preferences among actions. In a certain sense, the set R of alternatives is included in Π of prospects; for R can be said to consist of all those distributions that assign probability 1 to some particular result.

10. Probability mixture of prospects. We shall expect, in the case of a consistent person, that by forming a probability distribution of prospects (as when stocks are offered in a lottery) no prospect is created that is not already in the ordered set Π. Suppose, for example, that prospect π' consists in participating in a lottery for which there is a chance of .01 of winning ten dollars, and that prospect π'' consists in participating in a lottery for which there is a chance of .10 of winning ten dollars; suppose further that you agree to throw an ordinary die, to participate in lottery π' if the die comes up with a 1 or a 2, and otherwise to participate in lottery π''. An uncertain consequence defined in two stages this way is called a probability mixture of prospects, in this case it is a mixture of the prospects π' and π'' with probabilities 1/3 and 2/3, respectively.

A little arithmetic shows that, for the mixture, the probability of winning ten dollars is $(1/3)(.01) + (2/3)(.1) = .07$. The mixture is therefore equivalent to a lottery for which there is a chance of .07 of winning ten dollars.* Mixtures of prospects will play an important role in the discussion below. A mixture in which prospect π_1 has probability $\underline{p}$ and prospect π_2 the probability $1-\underline{p}$ will be denoted by $\pi_1 \underline{p} \pi_2$. We shall find the following arithmetical properties of prospects useful:

$$1) \quad \pi_1 1 \pi_2 = \pi_1 ;$$

if $\underline{p}$, $\underline{q}$, $\underline{r}$ are probabilities then

$$2) \quad \pi_2 (1-p) \pi_1 = \pi_1 p \pi_2 ;$$

$$3) \quad (\pi_1 p \pi_2)\ r\ (\pi_1 q \pi_2) = \pi_1 [pr + q(1-r)]\ \pi_2 .$$

Properties 1) and 2) are clear; property 3) was illustrated in the numerical example just given.

11. Theory of expected utility. For most people, preferences between sure results (elements of R) are much clearer than those between prospects (elements of $\overline{\Pi}$) and so it is natural to look for methods of relating the latter preferences to the former. Of those methods that are well articulated with probabilistic description of beliefs the one provided by the theory of expected utility has been most fully explored, and is probably the most successful.

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* We have omitted to state explicitly, as a part of the formal definition, that a consistent man should be able to make such arithmetical calculations. We have also omitted, as trivial, the requirement that he does not make arithmetical or logical errors even though TIME (July 7, 1958) reports that "A West Coast pet-food packer sells more three-can packages priced at 29¢ than three individual cans priced at 9¢ apiece."

The theory of expected utility consists in showing that certain plausible postulates about consistent behavior have the following logical consequences: to each result $\underline{r}$ a (real) "utility" number $\nu(r)$ can be assigned, and to each prospect π a (real) number $\Upsilon(\pi)$ can be assigned, such that the numerical order of all the numbers $\nu(r)$ and $\Upsilon(\pi)$ generates the preference ordering of all results and all prospects; and such that $\Upsilon(\pi)$ is the mathematical expectation of the component utility numbers:

$$(1) \quad \Upsilon(\pi) = \sum_r \nu(r)\ \pi\ (r)$$

The real valued function defined on the set R of results is called the utility function. The real number $\Upsilon(\pi)$ is called the expected utility of the prospect π. The theory of expected utility reduces the comparison between prospects to ordinary arithmetic comparison between numbers. In addition the function Υ has the convenient property of linearity, in the following sense. If π is a mixture of prospects π_1 and π_2 with probabilities $\underline{p}$ and $1-\underline{p}$, respectively, then it follows immediately from (1) that the expected utility of π is the average of those of π_1 and π_2, weighted by the probabilities p_1 and p_2:

$$(2) \quad \Upsilon(\pi) = p_1 \Upsilon(\pi_1) + p_2 \Upsilon(\pi_2)$$

This is easily extended to mixtures of three or more prospects. One can also verify that (1) and (2) apply trivially to the special case of a prospect that attaches probability 1 to a single result. It also follows that, for any given preference ordering of prospects any two utility functions can differ only in origin (the zero point) and scale (the unit of measurement) and furthermore any change of origin and scale transforms one utility functinn into another. Thus utility is measurable in precisely the same sense in which temperature (Fahrenheit, centigrade, etc.) is: When two arbitrary benchmark numbers are attached to the temperatures of freezing and of boiling water, thus determining the origin and the scale, the number to be attached to any other temperature becomes unique.

12. Derivation of expected utility theory. We have said that the theory of expected utility requires some plausible postulates on consistent behavior. Some of these we have already used. When choices under certainty were discussed, we postulated the existence of a preference ordering of (sure) outcomes; we now extend this to the whole set of prospects. When discussing actions under uncertainty we introduced further consistency requirements and showed them to imply the existence of subjective probabilities. Since for that purpose only a pair of outcomes (e.g., two unequal gains) needed to be considered, no scaling of utilities was involved at that stage (apart from two arbitrary benchmarks). Such scaling is provided by the expected utility theory. This theory needs two more postulates: (1) the "Continuity" postulate requires that a person who places three prospects π_1, π_2, π_3 in that order of preference be indifferent between π_2 and some unique mixture $\pi_1 p \pi_3$ between π_1 and π_3; (2) the "Substitutability" postulate* requires that a person who is indifferent between

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* Also called - less appropriately, we think - the "independence" postulate. Our set of the postulates comes closest to that of Herstein and Milnor [ ]. Here as elsewhere only an outline of the proof is given. See also Luce and Raiffa [ ] and the literature quoted there.

π_1 and π_2 should, for any stated fraction p, be also indifferent between the mixtures $\pi_1 p \pi_3$ and $\pi_2 p \pi_3$.

A little thought suffices to see that the Continuity postulate excludes (i.e., brands as inconsistent) the "love of danger" exhibited by, say, an adventurous mountain-climber who prefers a 95% chance of survival to both 0% and 100%; or prefers a 90% to 95% survival chance. For suppose a person prefers $\underline{b}$ ("bliss") to $\underline{a}$ ("agony") yet prefers $\underline{bpa}$ to $\underline{b}$ itself, where $0 < \underline{p} < 1$. Then by the continuity postulate he is indifferent between $\underline{b}$ and some mixture of $\underline{bpa}$ and $\underline{a}$; let this mixture be $\underline{(bpa)ra}$, where $\underline{r}$ is some probability. Clearly this mixture is identical with that mixture of $\underline{b}$ and $\underline{a}$, in which $\underline{b}$ has probability $\underline{pr}$. Hence the person is indifferent between $\underline{b(pr)a}$ and $\underline{b}$; that is, $\underline{pr} = 1$, which is only possible if $\underline{p} = 1 = \underline{r}$, contradicting our supposition. One proves in the same way that $\underline{bpa}$ cannot be worse than $\underline{a}$. Thus $\underline{bpa}$ is, preference-wise, between $\underline{b}$ and $\underline{a}$. From this fact, and applying again the Continuity postulate, one shows in a similar way that if $\underline{q}$ is some probability larger than $\underline{p}$ then $\underline{bqa}$ lies preference-wise between $\underline{b}$ and $\underline{bpa}$.

Now let us see whether the following scaling will provide utility numbers satisfying the expected utility theory. First consider all outcomes and prospects that are worse than $\underline{b}$ and better than $\underline{a}$. Assign numbers 0 or 1 to anything that is equally desirable with $\underline{a}$ or $\underline{b}$, respectively; and assign the number $\underline{p}$ to anything that is equally desirable with the mixture $\underline{bpa}$. As stated at the end of the proceeding paragraph, higher numbers will be thus assigned to the more desirable things, thus agreeing with the person's preference ordering. At the same time, this assignment will order all the prospects in the order of their expected utilities, provided the substitutability postulate is valid. For example, let $\underline{r}_1$ and $\underline{r}_2$ be two results. If both are preference-wise between $\underline{b}$ and $\underline{a}$, they will be equally desirable with some two mixtures

bu_1a and bu_2a, respectively, where u_1 and u_2 are two probabilities. Now consider the following prospect: the mixture r_1pr_2. By the substitutability postulates this is equally desirable with the mixture $(bu_1a)p(bu_2a)$ which, in turn, is identical with the mixture of b̲ and a̲ in which b̲ has probability $pu_1 + (1-p)u_2$. But our assignment procedure has associated r_1 with the number u_1; r_2 with the number u_2; and now it turns out that the mixture r_1pr_2 has associated with it the number $pu_1 + (1-p)u_2$, in agreement with equation (1). This number is the expected utility of the prospect r_1pr_2; and u_1 and u_2 are the utilities of the results r_1 and r_2. The proof can be easily extended to prospects that involve more than two results. To take care of a result r_3 that is preferred to b̲, one forms the mixture r_3pa that is equally desirable with b̲, solves the equation $\nu(b) = p\nu(r_3) + (1-p)\nu(a)$, remembering that $\nu(a) = 0$, $\nu(b) = 1$, and obtains $\nu(r_3) = 1/p$; similarly, if r_4 is worse than a̲, one solves $\nu(a) = q\ (b) + (1-q)\nu(r_4)$, $\nu(r_4) = -q/1-q$. It is then easy to show that the complete scale of utilities thus defined will satisfy (1).

13. Generality of expected utility theory. Although the conditions of Continuity and Substitutability place some restriction on an individual's preferences with respect to prospects, they still leave room for a wide variety of preference orderings. Thus both "risk aversion" (leading a person to a diversification of his investments, taking out insurance, etc.) and "risk preference" (leading him to speculation, participating in lotteries, etc.) are possible modes of behavior under the expected utility theory, and can be related to the shape of the function (called utility function of money) assigning

utility numbers to varying levels of monetary wealth.* On the other hand, the

* See Luce-Raiffa [ ] for further references; also Kemeny ... [ ]. Risk aversion and risk preference require the utility function of money to be non-linear. This has nothing to do with the linearity property (2) of expected utilities of prospects.

expected utility theory is incompatible with "love of danger" which, as defined above, was shown to contradict the continuity postulate. It is an open question how far actual behavior is consistent with this normative exclusion.**

** "Love of danger in this sense may very well be present in what are usually considered economic decisions. The danger of loss, including ruin, though probably shunned in the conservative code or cant of business, has quite possibly added to the zest and desirability of many an historically important venture, in the career of the leaders of mercenary armies, in the financing of great geographic discoveries or, closer to our time, in the financing of inventions and theater plays, and in stock and commodity speculation." Marschak, [ ].

In many practical instances, a simple form has been given to the utility function, and the expected utility theory applied. In the business of large companies it suffices for many purposes, to identify utility with money profit (thus neglecting both risk preference and aversion), at least as long as bankruptcy is not considered possible. Using thus the simple assumption that the company is interested in maximizing the mathematical expectation of its profit, or (the sales value being given) in minimizing the mathematical expectation of its cost, fruitful results have been obtained: for example, the recently developed rules for the control of inventories and of production under uncertainty: see, e.g., Arrow, Karlin and Scarf [ ]. For still more special purposes, one has obtained useful implications of minimizing the mathematical expectation of waiting time (as in the analysis of queues, see [ ]), and of some other physical quantities relevant to a particular aspect of business. This is also true of industrial quality control and other applications of statistics. Modern statistics led by A. Wald [ ] has found in the maximization of expected utility (or minimization of expected "loss") a principle for selecting among alternative procedures of getting and processing observational data. This has been most often exemplified by using utilities that decrease as the estimation error increases.

When replacing utility by some handy monetary or physical variable, one should not be too light-hearted. The wise physician will not completely identify his patient's health with his state of nutrition, or identify good nutrition with the intake of numerous calories. While the technocrats' identification of public welfare with the amount of energy leads to absurdities, the "national income" figure yields itself to cautious fruitful use. If - as pointed out by Hitch [ ] - one forgets that the goal of transatlantic convoys of the two world wars was to increase Europe's supplies, and maximizes instead the more easily computed ratio of U-boats sunk to allied ships sunk, one ends up with the ridiculous recommendation that allied destroyers accompany no merchant ships at all. Should then the quantity of goods shipped be used as utility, i.e., its mathematical expectation maximized? Yes, but cautiously; for the overriding goal was global victory not the provision of Europe.

An even more drastically simple approximation to utility is obtained if one classifies all outcomes into "bad" and "good" ones. If, without loss of generality, we put the utility of a bad outcome = 0, and the utility of a good outcome = 1, the expected utility becomes identical with the probability of a good outcome. Thus one maximizes the chance of victory (vs. defeat); of survival (vs. death); of making an error (vs. making no error); of achieving a given aspiration norm, such as a planned output (vs. failure to achieve). Simon [ ] pointed out the frequent occurrence of such two-valued (or sometimes three-valued: good, medium, bad) utility functions in human behavior, and Boulding [ ] suggests that many a firm's day-to-day decisions are guided by the simple norm of preventing the output from exceeding or falling short of certain fixed limits. To the extent to which a two-valued utility function can be used many of the problems of this book are greatly simplified.

Not all authors have found it necessary, or desirable, to accept the expected utility theory (see references in Arrow [ ] and Marschak [ ]). In the case of prospects that are probability distributions of numerical consequences, such as money income, the most common alternative is the suggestion that a prospect be evaluated in terms of the mean and variance of the corresponding probability distribution.* In general such an evaluation will generate

* See Markovitz [ ] for the best developed practical application.

a preference ordering that violates the conditions of the expected utility theory, though there are special cases in which no conflict arises. Other parameters of the probability distribution have also been suggested as being relevant. We shall not follow up the implications of such suggestions here; they are merely mentioned to place the expected utility theory in the proper perspective.

14. Expected payoff of an action. It will be recalled that each outcome $\underline{r}$ is produced by action $\underline{a}$ and environment $\underline{x}$. Thus $\underline{r} = \rho(\underline{x}, \underline{a})$, where ρ is the outcome function. The probability distribution of outcomes - the prospect π - generated by a given action, is determined by the probability distribution Φ of the states of the world. Thus the expected utility of the prospect π is

$$\nu(\pi) = \sum_r \nu(r)\pi(r) = \sum_x \nu[\rho(x, a)]\,\Phi(x) = \Omega(a; \nu, p, \Phi)\text{, say.}$$

The expression on the extreme right brings out that the expected utility depends on the person's action only, given the functions ν, ρ, Φ. Those functions summarize the factors beyond his control: his beliefs in the probabilities Φ, his tastes ν, and his idea of the "physical" relation ρ that states how outcomes are determined by himself and by the environment. His action $\underline{a}$, on the other hand, is under his control. He chose that action that generates a prospect with the greatest expected utility.

Since $r = \rho(x,a)$ and the utility of r is $\nu(r) = u$ (say) we can express utility of outcome directly as a function of $\underline{x}$ and $\underline{a}$:

$$u = \nu(r) = \nu[\rho(x, a)] = \omega(x,a) .$$

ω will be called the payoff function; it is equivalent to the successive application of the outcome function and the utility function. It is thus a combined expression of a person's tastes and of his explanation of the outcome as determined by his action and the environment. We can now rewrite the expected utility of an action in a simpler form:

$$E\omega(x, a) = \sum_x \omega(x, a)\Phi(x) = \Omega(a; \omega, \Phi) .$$

We call this quantity the (expected) payoff of the action a. Again, the symbols ω, Φ summarize the non-controlled conditions.

15. Consistent behavior under uncertainty. To sum up: We shall say that a person is consistent under uncertainty if his behavior agrees with the expected utility theory. That is, for a consistent man, there exists a probability distribution Φ on the set of the states of environment; an outcome function ρ by which he explains the outcome as the joint result of his and nature's action; and a utility function ν on the set of outcomes, with the following property: the action chosen is one that makes the expression $\sum_x \nu[\rho(x, a)] \Phi(x)$ $= \sum_x \omega(x , a)\Phi(x)$ as large as possible.

Although sophisticated in appearance this description of consistent behavior is accepted as obvious common sense in much of practical life. When you bet $2 against $1 on a race it is usual to interpret your action as follows: you assign a probability larger than 2/3 to the event of winning, and the utility of money gains is simply measured by the amount gained. Then the action "betting" has an expected payoff that is larger than $2/3 \cdot 1 + 1/3 \cdot (-2) = 0$; while non-betting has the expected utility 0 . You therefore bet.* Again, one prefers airplanes

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* The approach to probabilities as revealed by actions is quite old; one can find it in Bayes, one of the founders of probability theory [ ]. When the outcomes are not gains of money (or of some other single commodity), or when to measure utility of money gain by its amount contradicts ascertained behavior (e.g., because of "risk aversion" mentioned above) a more refined approach is needed, such as proposed by Ramsey [ ], De Finetti [ ] and Savage [ ], and as used in the preceding pages.

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that can withstand every weather except the most rare kind of storm. Let the possible states of weather be three: $\underline{x}, \underline{y}, \underline{z}$. If it is known that airplane $\underline{a}$ will collapse at $\underline{z}$ only, while airplane $\underline{b}$ is safe only at $\underline{x}$, we choose $\underline{a}$, and here is the interpretation:

$$u = \text{utility of safe trip} = \omega(x, a) = \omega(y, a) = \omega(x, b) ;$$

$$v = \text{utility of accident} = \omega(z, a) = \omega(y, b) = \omega(z, b) .$$

Denoting by $\Phi(x), \Phi(y), \Phi(z)$ the probabilities of the three events we have the expected payoffs:

$$\Omega(a) = u\Phi(x) + u\Phi(y) + v\Phi(z) ;$$

$$\Omega(b) = u\Phi(x) + v\Phi(y) + v\Phi(z) .$$

Hence $\Omega(a) - \Omega(b) = (u-v)\Phi(y)$. Since safe passage is preferred to accident, $u > v$; and hence, $\Phi(y)$ being non-negative, $\Omega(a) \geq \Omega(b)$. If the consistent man always chooses $\underline{a}$ we conclude that he considers intermediate weather $\underline{y}$ not entirely improbable. Though trivial, the example may have helped the reader.