COWLES FOUNDATION DISCUSSION PAPER #84 (Revised)

Note: Cowles Foundation Discussion Papers are preliminary
materials circulated to stimulate discussion and
critical comment. Requests for single copies of a
Paper will be filled by the Cowles Foundation within
the limits of the supply. References in publications
to Discussion Papers (other than mere acknowledgment
by a writer that he has access to such unpublished
material) should be cleared with the author to pro-
tect the tentative character of these papers.

Capacity Expansion and Probabilistic Growth [*]

Alan S. Manne

February 4, 1960

Capacity Expansion and Probabilistic Growth

Alan S. Manne[*]

1. Introduction

This study stems from an optimizing model originally suggested by Hollis Chenery for predicting investment behavior [6]. Like Chenery's paper, this one is concerned with the interplay between economies of scale and an anticipated persistent growth in demand for capacity. The generalizations discussed here are of two types: (a) the use of probabilities in place of a constant rate of growth in demand; and (b) a study of the economies and the penalties involved in accumulating backlogs of unsatisfied demand. The possibility of accumulating such backlogs raises considerable doubt with respect to Chenery's "excess capacity hypothesis."

Surprisingly enough, generalization (b) leads to greater difficulties in analysis than (a). The use of probabilities to describe the growth process does little - if anything - to complicate matters. A probabilistic version of Chenery's model turns out to be closely related to the classical problem of gambler's ruin, and a powerful tool can be borrowed from that area - the Laplace transform for the duration of the game. Thanks to this transform, the zero-backlog probabilistic model becomes no more difficult to study than the corresponding deterministic one. A direct implication is that a probabilistic growth course makes it necessary to incur higher expected costs, and also makes it desirable to install plant capacity of a

somewhat _larger_ size than would be optimal if demand were growing at a steady rate equal to the expected value of the probabilistic increments. Uncertainty, in this sense, has a stimulating effect upon the magnitude of individual investments.

Going beyond Chenery's model to include the possibility of backlogs, it turns out that there is a curious ambiguity in the effects of an increase in the variance of demand. Once the possibility of backlogs is admitted, an increase in variance can even lead to a decrease in the optimal level of costs.

2. The deterministic model - no backlogs in demand

In order to provide a reference point for discussion of the more difficult cases, Chenery's deterministic model itself will first be reviewed. Following this will come the modifications involving (a) probabilistic growth and (b) the possibility of accumulating backlogs in demand. Chenery's model grew out of his studies of the natural gas transmission industry - a sector characterized by rapid growth and by substantial economies of scale in pipeline construction and operation. Much the same situation seems to prevail in the case of oil pipelines [9], the telephone industry [8], highway construction, electric power generation, petroleum refining, and chemicals processing [7].

Figure 1 charts the course of demand and of capacity over time under the following simplifying assumptions: (1) that demand grows linearly over time; (2) that the equipment has an infinite economic life; and (3) that whenever demand catches up with the existing
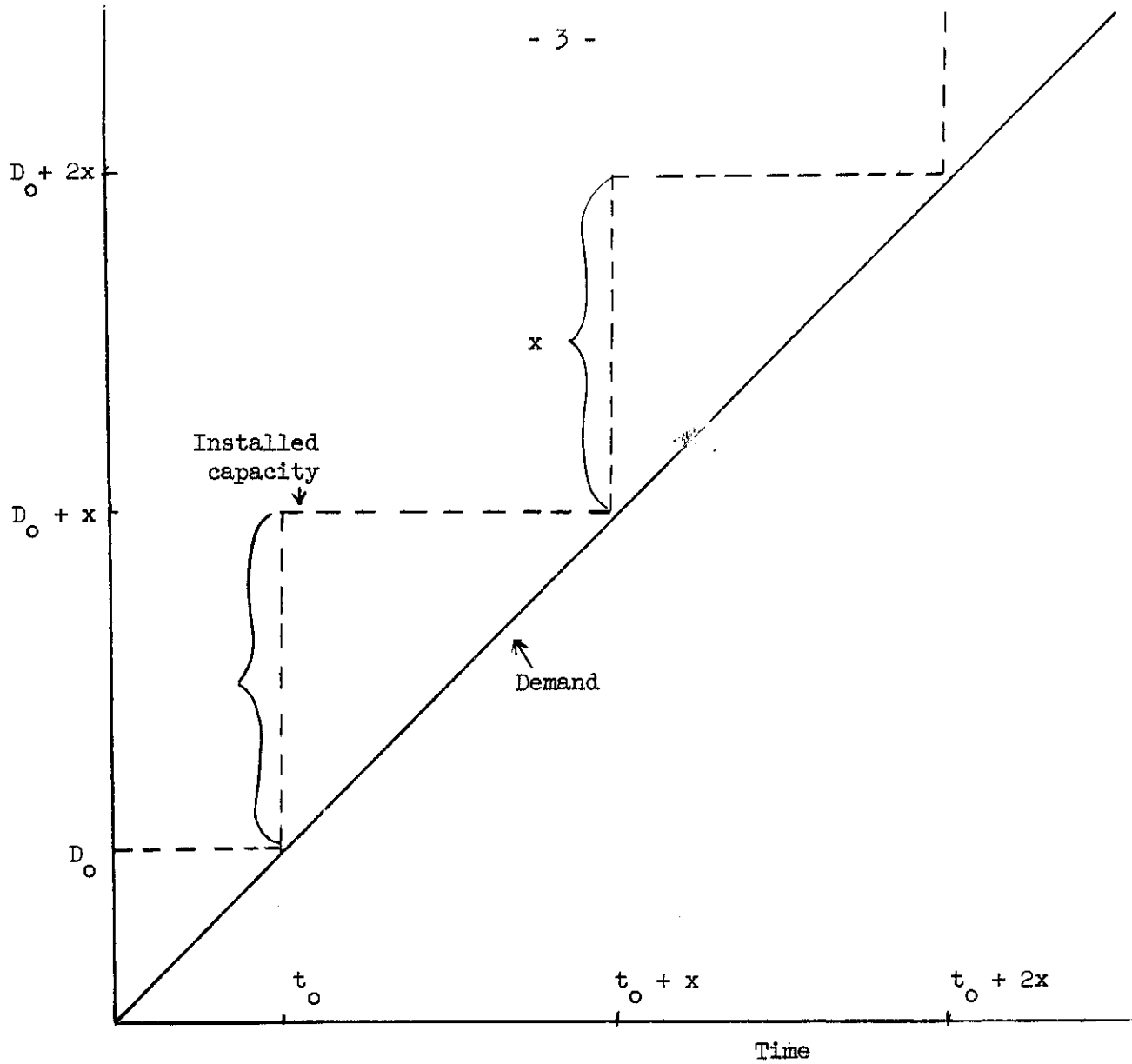
Figure 1. Growth of demand and of capacity over time.

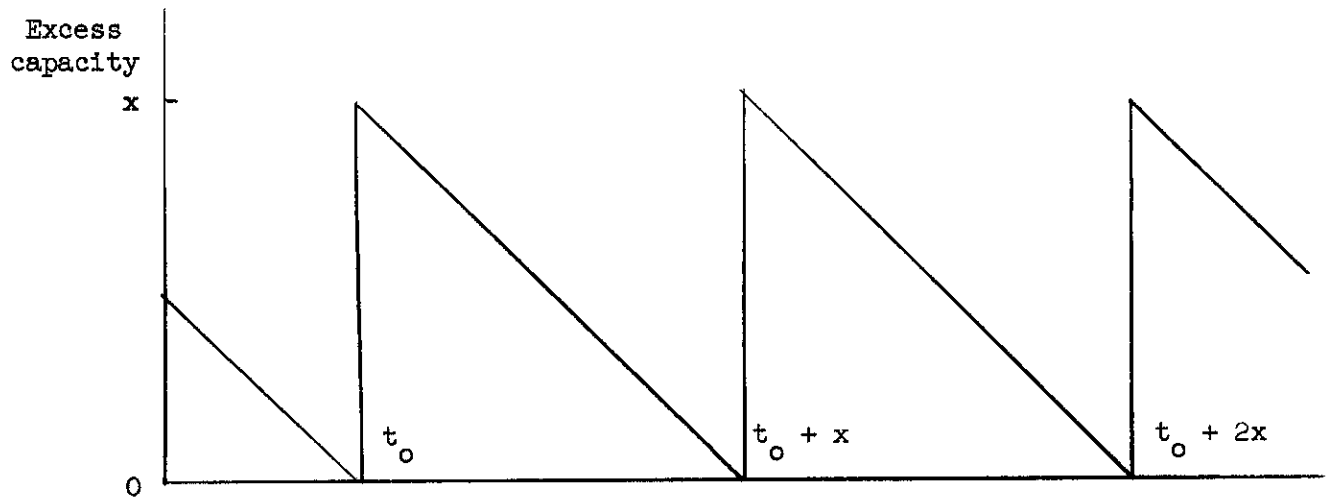(Demand and capacity vs. Time)

- $D_o + 2x$
- x
- Installed capacity
- $D_o + x$
- Demand
- D_o
- t_o, $t_o + x$, $t_o + 2x$

Figure 2. Evolution of excess capacity over time

(Excess capacity x vs. Time; t_o, $t_o + x$, $t_o + 2x$; 0)

capacity, $\underline{x}$ units of new capacity are installed.[*] (The demand at t_o is

------

------

denoted by D_o .) Unlike Chenery, we shall assume that the planning horizon

is infinite, rather than being truncated after an arbitrary finite number

of years. Excess capacity, when plotted on Figure 2, then displays a

sawtooth pattern typical of the closely related Wilson-type inventory model.

[1, pp. 252-255] If, for convenience, the physical unit of capacity and

of demand is set equal to one year's growth in demand, this sawtooth cycle

repeats itself every $\underline{x}$ years.

The installation costs that result from a single capacity increment

of size $\underline{x}$ are assumed to be given by a cost relationship in the form of

a power function:[**]

$$(2.1) \qquad kx^a \qquad (k > 0 \; ; \; 0 < a < 1)$$

If, for example, $a = 1/2$, this cost function says that a pipeline

capable of handling 16 years' worth of growth in demand is only twice as

expensive as one that can accomodate four years' worth. The existence of

such substantial economies of scale implies the desirability of building

new capacity considerably in advance of demand. But how much in advance?

Here the discounting of future costs becomes crucial.

Without discounting, it would be perfectly sensible to spend a dollar

now in order to save a dollar's worth of costs either next year or ten

years from now, or 100 years hence. There is no limit to the size of line

which it pays to build. With discounting, on the other hand, this paradox

can be sidestepped.* Throughout, we shall adopt the expression e^{-rt} as

---

* Gifford Symonds has suggested an additional reason for the discounting
of future costs - the expectation of continuing progress in pipeline
technology. If the general price level remains constant, it is reasonable
to suppose that in, say, 10 years' time the cost of building a line with
a capacity of x units will be significantly cheaper than the cost of
such a line today. The proviso about constancy of the general price level
is important. If one is a believer in the inevitability of creeping in-
flation, the one factor would tend to cancel out the other.

---

the present value of a dollar due t years in the future. $(r > 0.)$ The

quantity r will be referred to as the "discount rate."

As a time origin for subsequent calculations, it will be convenient

to take any such point as t_o or $t_o + x$ or $t_o + 2x$ on Figure 2 - a

time at which the previously existing excess capacity has just been wiped

out. Such a point will be known hereafter as a "point of regeneration."

Note that when we have reached $t_o + x$, the future looks identical with

the way it appeared x units of time previously. Then if we say that

$C(x)$ is a function of $\underline{x}$ that represents the sum of all discounted

future costs looking forward from a point of regeneration, we may write

down the following recursive equation:

$$(2.2) \qquad C(x) = k \cdot x^a + e^{-rx} C(x)$$

The first term on the right-hand side indicates the installation

costs incurred directly at the beginning of the current cycle. (See

equation (2.1).) The second term measures the sum of all installation

costs incurred in subsequent cycles, and discounts these from the next

point of regeneration back to the present one - a difference of $\underline{x}$ years.

From (2.2), it follows directly that:

$$(2.3) \qquad \frac{C(x)}{k} = \frac{x^a}{1-e^{-rx}}$$

Differentiating $\log C(x)$ with respect to $\underline{x}$, and setting the

result equal to zero:

$$\frac{d \log C(x)}{dx} = \frac{a}{x} - \frac{re^{-rx}}{1-e^{-rx}} = 0$$

$$\text{or (2.4)} \qquad a = \frac{r\hat{x}}{e^{r\hat{x}}-1}$$

where $\hat{\underline{x}}$ denotes the optimal size of installation.

The reader can verify for himself that (2.4) is not only a necessary

condition, but also a sufficient one to ensure a unique minimum-cost solu-

tion. With this equation, the optimal capacity increment $\hat{\underline{x}}$ may be

Figure 3.
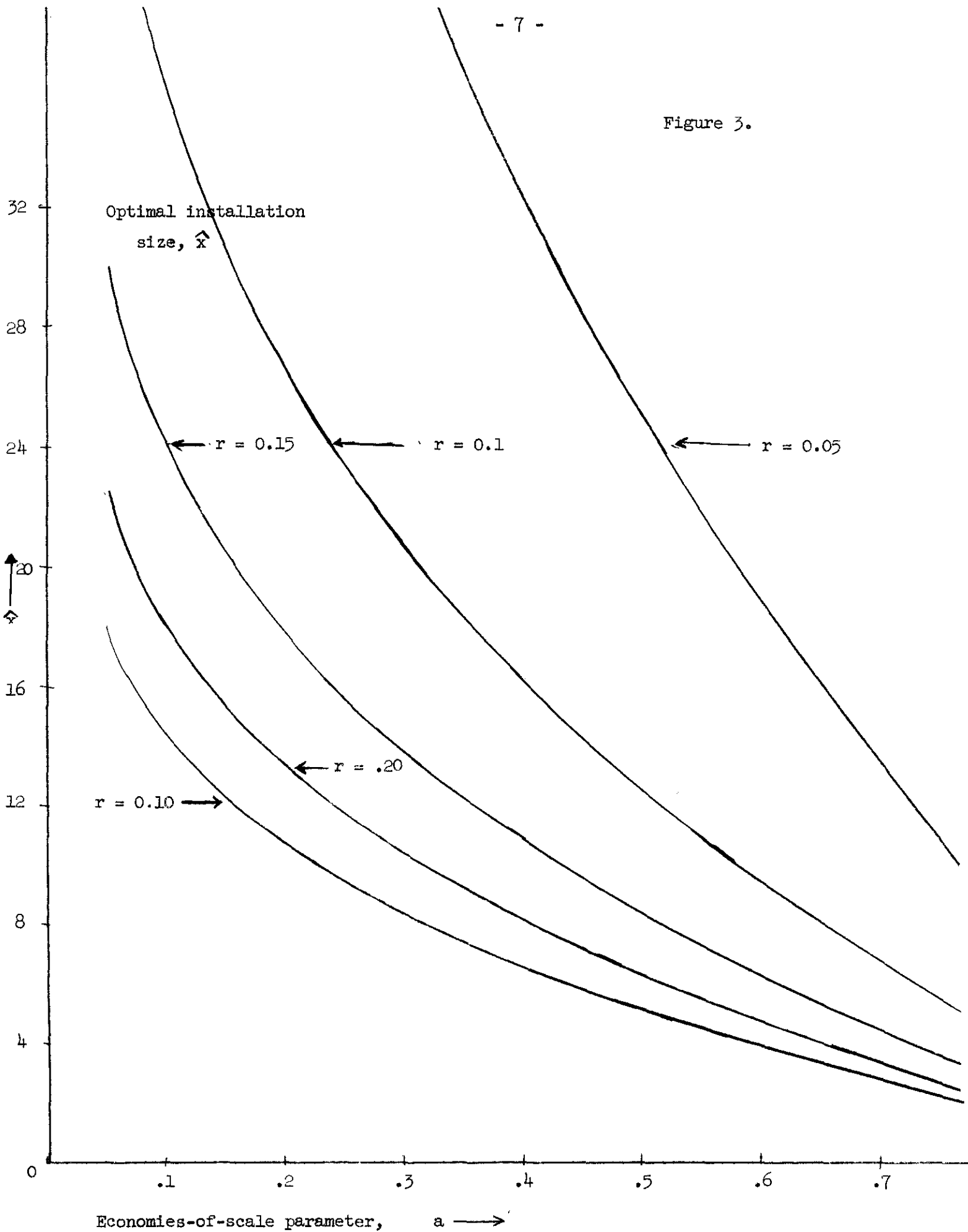
determined for any combination of the two parameters a and r - a cross-plot being provided in Figure 3. From this figure, if one were interested in the economies-of-scale effect, he would observe, say, that when $a = 2/3$ and $r = .15$, the optimal value $\hat{x}$ is approximately 5 years' worth of demand growth. With the discount rate r unchanged, but with the economies-of-scale factor at a level of $1/2$, $\hat{x}$ rises to around 8 years.

This deterministic model lends itself readily to sensitivity-testing. To find out how the optimal level $\hat{x}$ is affected by changes in r , one need only observe that (2.4) is written as a function of the product $r\hat{x}$, and that therefore, for a constant value of a :

$$r d\hat{x} + \hat{x} dr = 0$$

Or (2.5) $\quad \dfrac{d\hat{x}}{dr} = \dfrac{-\hat{x}}{r} < 0$

The derivative $\dfrac{d\hat{x}}{dr}$ is clearly negative for positive values of $\hat{x}$ and of r. The higher the discount rate (i.e., the higher the cost of capital), the smaller will become the optimal size of each installation.

Both to an economist and to an operations researcher, it is likely that the general shape of the cost function $\dfrac{C(x)}{k}$ will be of even greater interest than the optimal value $\hat{x}$ itself. Figure 4 contains a plot for a fairly typical set of parameter values: $a = .50$ and $r = .15$. (Incidentally, these numerical values of a and r will be the ones employed in all subsequent illustrations.) The optimal point indicated by this figure leads to a cost of 4.046 at $x = 8.4$. The figure also gives an indication of how little these costs change within a fairly wide range of values assigned to the decision variable x .

Figure 4.

- 9 -

Installation size, $\underline{x}$
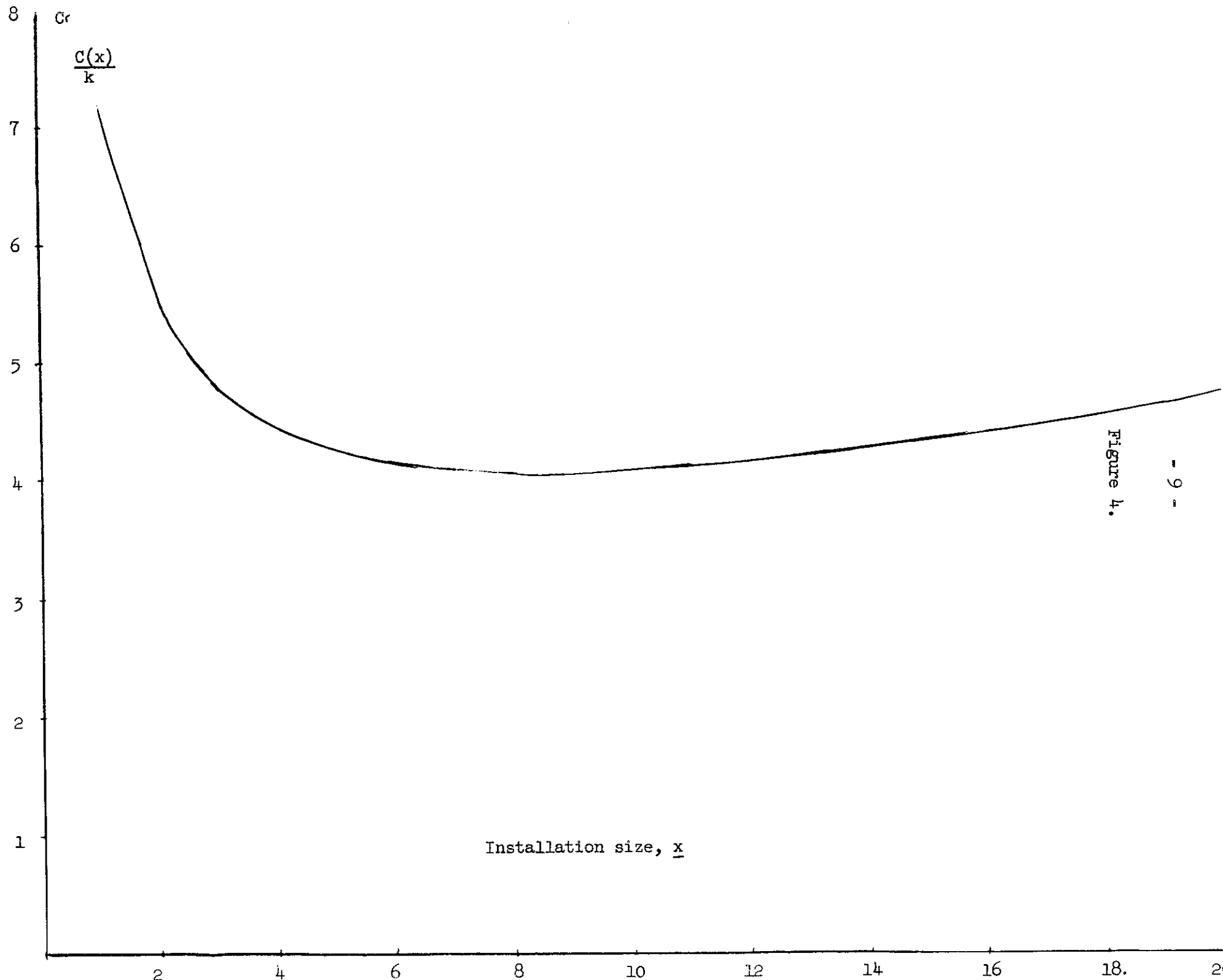
An $\underline{x}$-value as high as 11.0 or as low as 6.0 will increase costs by less than 2%. From the viewpoint of the operations researcher and the business forecaster, this insensitivity is fortunate indeed. Even a substantial error in forecasting will not lead to an egregiously bad choice for the capacity increment.

What is fortunate from the viewpoint of the business executive may be disastrous, however, from the viewpoint of an economist trying to forecast investment choices on the basis of an optimizing model. Even if the economist happens to hit upon the identical values for $\underline{a}$ and $\underline{r}$ that are in the mind of the executive, the latter will suffer no great penalty for deviating from the optimal path predicted by the economist for his behavior.

3. The probabilistic model - no backlogs

With this background, we are in a position to discuss a case of probabilistic growth and minimization of expected costs - still ruling out the possibility of deliberate backlogs in demand. This model represents, of course, just one of many possibilities for describing a growth process in probabilistic terms. The particular structure is one that has been postulated not only for analytical convenience, but also because of its close relationship to the normal distribution.

The model employed here is the Bachelier-Wiener diffusion process in continuous time. Except for notation, our exposition is virtually identical with Feller's. / Feller begins by considering the case in which a dis-
[10,pp.323-27].
continuous random change in demand occurs every Δt units of time, and then examines the limiting form of this process. With probability $\underline{p}$, the discrete change constitutes an increase of ΔD units, and with

probability $q = (1 - p)$, a decrease of ΔD units. In Markov process
terms:

$$(3.1) \qquad D(t) = D(t - \Delta t) + \epsilon(t)$$

where $D(t)$ represents the demand at time $\underline{t}$, and where $\epsilon(t)$ is a
random variable taking on the values of $+\Delta D$ and $-\Delta D$, with respective
probabilities $\underline{p}$ and $\underline{q}$. (It is assumed that each of the $\epsilon(t)$ incre-
ments is distributed identically and independently.) With a change in
demand occurring every Δt units of time, this means that over a fixed
period of, say, $\underline{t}$ years in length, approximately $t/\Delta t$ changes will
have occurred. Quoting Feller directly now:

> Only multiples of ΔD and Δt represent meaningful co-
> ordinates, but in the limit $\Delta D \to 0, \Delta t \to 0$; every displace-
> ment and all times become possible.

> We must not expect sensible results if ΔD and Δt approach
> zero in an arbitrary manner ... Physically speaking, we must keep
> the D- and t- scales in an appropriate ratio or the process
> will degenerate in the limit, the variances tending to zero or in-
> finity. To find the proper ratio note that the total displacement
> during time $\underline{t}$ is the sum of about $t/\Delta t$ mutually independent
> random variables each having the mean $(p - q)\Delta D$ and variance
> $4pq(\Delta D)^2$. The mean and variance of the total displacement in time
> $\underline{t}$ are therefore about $t(p - q)\Delta D/\Delta t$ and $4pqt(\Delta D)^2/\Delta t$, respec-
> tively. To obtain reasonable results we must let ΔD and Δt
> approach zero in such a way that they remain finite for all $\underline{t}$.
> The finiteness of the variance requires that $(\Delta D)^2/\Delta t$ should re-
> main bounded; the finiteness of the mean implies that $(p - q)$
> must be of the order of magnitude of ΔD . This suggests putting

$$(3.2) \qquad \frac{(\Delta D)^2}{\Delta t} = \sigma^2 \,, \quad p = \frac{1}{2} + \frac{\mu\Delta D}{2\sigma^2} \,, \quad q = \frac{1}{2} - \frac{\mu\Delta D}{2\sigma^2}$$

> ... We use the norming (3.2) to pass to the limit $\Delta D \to 0$,
> $\Delta t \to 0$. The total displacement at time $t \approx n\Delta t$ is determined
> by $\underline{n}$ Bernouilli trials, and therefore the limiting form of

$V_{D,n}$ (the probability that after $\underline{n}$ trials, the demand will have grown by a total of exactly $\underline{D}$ units) is given by the normal distribution, $v(D;t)$. For a fixed ΔD the displacement is the sum of finitely many independent variables, and its mean is $t(p - q)\Delta D/\Delta t = \mu t$; its variance $4pqt(\Delta D)^2/\Delta t = \sigma^2 t$. [10, pp. 324-5.]

To sum up: The parameters p , q , ΔD , and Δt enable us to study a discrete stochastic process in which the total growth $\underline{D}$ over a fixed period of $\underline{t}$ years is a random variable $D(t)$. Furthermore, in the limit, for the case of continuous time and a continuous growth path, this process describes the demand increment as a random variable which is normally distributed with mean μt and with a variance of $\sigma^2 t$.

Now in order to make use of this process for the capacity optimization problem, it is going to be necessary to work with a certain probability density function $u(t;x)dt$: the probability with which $\underline{t}$ time units elapse before the point at which demand first exceeds the initial level by $\underline{x}$ units. In other words, $u(t;x)dt$ represents the probability with which $\underline{t}$ units of time elapse between one installation of capacity and the next one. In gambler's ruin terminology, this is the probability with which exactly $t/\Delta t$ "trials" are needed in order for a gambler to go broke - a gambler whose initial capital is $\underline{x}$, and who is playing against an adversary with infinite wealth. At each stage of such a game, the gambler would lose one unit with a probability of $\underline{p}$, and gain one unit with a probability of $q = 1 - p$. [10, pp. 311-21.] The following relationship may therefore be written:

(3.3) $\quad u(t + \Delta t; x) = p\, u(t;\, x - \Delta x) + q\, u(t;\, x + \Delta x) \quad (x > 0;\ 0 \leq t \leq \infty)$

Equation (3.3) says that whatever be the probability of ruin in exactly $(t + \Delta t)/\Delta t$ steps for a gambler with an initial capital of $\underline{x}$, this quantity must equal the weighted sum of the ruin time probabilities in just $t/\Delta t$ steps for a gambler with a capital of $(x - \Delta x)$ and $(x + \Delta x)$ - the respective weights being the transition probabilities $\underline{p}$ and $\underline{q}$.

Expanding according to Taylor's theorem up to terms of second order:

$$(3.4) \quad \Delta t \; \frac{\delta u(t;x)}{\delta t} \; = \; (q-p)\Delta x \; \frac{\delta u(t;x)}{\delta x} \; + \; \frac{(\Delta x)^2}{2} \; \frac{\delta^2 u(t;x)}{\delta x^2}$$

Equating the random variable $D(t)$ to the capacity increment $\underline{x}$, substituting from (3.2), and taking the limit:

$$(3.5) \quad \frac{\delta u(t;x)}{\delta t} \; = \; -\mu \; \frac{\delta u(t;x)}{\delta x} \; + \; \frac{\sigma^2}{2} \; \frac{\delta^2 u(t;x)}{\delta x^2}$$

The Laplace transform of $u(t;x)$ will be indicated by $\bar{u}(r;x)$, and in economic terms is defined as the discounted value of the probabilities $u(t;x)dt$.[*]

$$(3.6) \quad \bar{u}(r;x) \; = \; \int_{t=0}^{\infty} u(t;x)e^{-rt}dt$$

Taking the Laplace transform of each side of (3.5), and recalling the boundary condition that $u(0;x) = 0$, we obtain a second-order linear differential equation with respect to $\underline{x}$:

$$(3.7) \quad r \; \bar{u}(r;x) \; = \; -\mu \; \frac{d\bar{u}(r;x)}{dx} \; + \; \frac{\sigma^2}{2} \; \frac{d^2\bar{u}(r;x)}{dx^2}$$

---

* For a rather different economic application of the Laplace transform, see Blyth [5].

The characteristic equation has two real roots:

$$\lambda_1 = \frac{\mu}{\sigma^2}\left[1 + \sqrt{1 + \frac{2r\sigma^2}{\mu^2}}\right]$$

(3.8)

$$\lambda_2 = \frac{\mu}{\sigma^2}\left[1 - \sqrt{1 + \frac{2r\sigma^2}{\mu^2}}\right]$$

The general solution for the Laplace transform is consequently of the form:

(3.9) $\bar{u}(r;x) = A(r)e^{\lambda_1 x} + B(r)e^{\lambda_2 x}$

where $A(r)$ and $B(r)$ are constants whose values depend upon r and also upon the boundary conditions for $\bar{u}(r;x)$. These boundary conditions are twofold: first, that $\bar{u}(r;x)$ lie between zero and unity, and second, that $\bar{u}(r;0) = 1$. Since $\lambda_1 > 0$, and since $\lambda_2 < 0$, the bounds upon $\bar{u}(r;x)$ can be ensured only by setting the constant $A(r)$ equal to zero. And to have $\bar{u}(r;0) = 1$, the constant $B(r)$ must be unity. With these simplifications, the Laplace transform (3.6) becomes:

(3.10) $\bar{u}(r;x) = e^{\lambda_2 x}$

Note that λ_2 is a function of μ, σ^2, and r alone, and that it is independent of the quantity x.

At last we are ready to employ the Laplace transform $\bar{u}(r;x)$ in the capacity expansion problem. Since $u(t;x)dt$ represents the probability with which exactly t years have elapsed between two successive points of regeneration – points between which the total demand grows by an amount

$\underline{x}$ - the probabilistic analogue of (2.2) may be written:

$$(3.11) \quad C(x) = kx^a + \int_0^\infty u(t;x)e^{-rt} C(x)dt$$

Just as in the earlier deterministic case, the first term on the right-hand side equals the present cost of installing a facility of capacity $\underline{x}$. The second term indicates the probability with which the next point of regeneration will occur in $\underline{t}$ units of time, discounts the corresponding cost back to the present, and integrates over all $\underline{t}$. As in the earlier case, the function $C(x)$ gives the expected present value of all costs incurred over the indefinite future - as measured from a point of regeneration. From (3.11) and from the definition (3.6):

$$\frac{C(x)}{k} = \frac{x^a}{1 - \bar{u}(r;x)}$$

And by (3.10), this becomes:

$$(3.12) \quad \frac{C(x)}{k} = \frac{x^a}{1 - e^{\lambda_2 x}}$$

According to (3.12), the probabilistic model postulated here is no more difficult to analyze than the deterministic one. At no point does it become necessary to make an explicit evaluation of the probability density function $u(t;x)dt$. All that has to be done is to regard the quantity λ_2 as the negative of an adjusted discount rate, and to insert this in place of $\underline{r}$ in equation (2.4) or else in Figure 3.[*] From this

---

[*] When $\sigma^2 = 0$, we have the case of complete certainty - a steady annual increase in demand consisting of μ units. Both the numerator and denominator of the expression for λ_2 vanish when $\sigma^2 = 0$. It is easy, however, to show that as σ^2 approaches zero, the expression for λ_2

approaches the value of $-r/\mu$.

$$\lambda_2 = \mu \left[1 - \sqrt{1 + \frac{2r\sigma^2}{\mu^2}} \right] \div \sigma^2$$

Differentiating the numerator of this expression with respect to σ^2, we obtain $-r/\mu$ when $\sigma^2 = 0$. Differentiating the denominator, the result is $+1$. The ratio of these two derivatives is $-r/\mu$, the limiting value of λ_2 - a result that completely accords with our intuition for the case of zero variance.

---

also, it may be seen that the greater the variance of the growth in demand, the greater will be: (1) the optimal level of expected discounted costs $C(x)$, and (2) the optimal size of the capacity increments $\hat{x}$. A proof may be worked out as follows:

Our object is to make comparisons between cases in which the expected annual change in demand, $\underline{\mu}$, is held constant at unity, but in which the variance of these changes is altered. First we show that $\frac{d\lambda_2}{d\sigma^2} > 0$.* If

---

To prove that $\frac{d\lambda_2}{d\sigma^2} > 0$, observe that when $\mu = 1$:

$$\lambda_2 = \frac{1}{\sigma^2} \left[1 - \sqrt{1 + 2r\sigma^2} \right]$$

$$\text{sgn}\left(\frac{d\lambda_2}{d\sigma^2} \right) = \text{sgn} \left\{ \sigma^2 \frac{d}{d\sigma^2} \left[1 - \sqrt{1 + 2r\sigma^2} \right] - \left[1 - \sqrt{1 + 2r\sigma^2} \right] \right\}$$

$$= \text{sgn} \left\{ 1 + r\sigma^2 - \sqrt{1 + 2r\sigma^2} \right\}$$

Define $f(\sigma^2)$ as follows:

$$f(\sigma^2) = \left\{ 1 + r\sigma^2 - \sqrt{1 + 2r\sigma^2} \right\}$$

When $\sigma^2 = 0$, $f(\sigma^2) = 0$. Furthermore, for $\sigma^2 > 0$,

$$\frac{df}{d\sigma^2} = r \left\{ 1 - \frac{1}{\sqrt{1 + 2r\sigma^2}} \right\} > 0 .$$

Hence, except for the special case of $\sigma^2 = 0$, we have shown that $f(\sigma^2) > 0$, and that $\frac{d\lambda_2}{d\sigma^2} > 0$.

this be granted, then assertion (1) is proved directly - the greater the variance, the greater will be the level of expected discounted costs - regardless of the value of $\underline{x}$. (Note that λ_2 is a negative quantity, and that an increase in variance makes λ_2 less negative.)

In order to prove assertion (2), we return to the sensitivity analysis at the end of the preceding section. According to (2.5), the optimal size of installation increases as the discount rate $\underline{r}$ is lowered. In our probabilistic model, we have already shown that λ_2 may be viewed as nothing but an "adjusted" discount rate. Hence assertion (2): the greater the variance, the lower will be the absolute value of λ_2 , and the higher will be the optimal value $\hat{\underline{x}}$. This completes the proof.

To illustrate these results, Table 1 provides a few calculations for several alternative values of σ^2 . In each of the calculations presented in this table, the expected rate of annual growth in demand is, of course, identical - namely unity. Note that as the variance increases, so does the optimal size of installation, and the minimum value of expected discounted costs.

Other things equal, our model indicates that the riskier the growth in demand, the larger ought to be the amount invested in each installation. To some, this result will seem to fly in the face of common sense. However, to those familiar with models of inventory stockage under conditions of probabilistic demand [e.g., 1, pp. 256-259], this should come as no paradox. In both this capacity model and in many cases of

inventory control, the greater the risk of running out of capacity or out of inventory in a specified period of time, the greater the amount which it pays to invest in order to avert this contingency.

Table 1

Variance of Demand Versus Optimal Capacity Increments

(a = .50, r = .15)

σ^2		0	1	4	16
$\lambda_2 = \dfrac{1}{\sigma^2}\left[1 - \sqrt{1 + 2r\sigma^2}\right]$		-.1500	-.1402	-.1208	-.0880

optimal capacity increment

$= \hat{x}$	8.4	9.0	10.4	14.3

minimum expected discounted costs =

$\dfrac{c(\hat{x})}{k} = \dfrac{\hat{x}^a}{1 - e^{\lambda_2 \hat{x}}}$	4.046	4.185	4.508	5.282

4. The deterministic model - backlogs considered

It is now time to examine the zero backlog assumption in a more critical way, and to explore the implications that result from discarding it. The zero backlog assumption seemed especially appropriate for the industry described by Chenery - natural gas transmission. In the case of this industry, it was reasonable to suppose that since the demand for the de-livered product comes largely from individual homeowners, such individuals - if unable to obtain natural gas fuel at the time their home is initially constructed - would thereafter constitute a rather dubious sales prospect. For an individual homeowner, the initial outlay required for conversion from liquid fuels to natural gas could easily outweigh any benefits that he might conceivably derive from the switch.

This irreversibility phenomenon means that if a natural gas trans-mission line is operating at full capacity and if that capacity is kept unchanged, then demand for the delivered product will also remain constant. Chenery was probably quite right to have assumed that there cannot be negative excess capacity,i.e.,that the gas industry's residential customers cannot be backlogged. Any attempt to do so would only result in a switch in allegiance to an alternative fuel.

Even when customers cannot readily shift over to a competing pro-duct - e.g., the case of telephones, water, and electric power for residential purposes - it may still be sensible for the business enter-prise to plan its investment outlays under the assumption of zero back-logs in demand. Certainly from the public relations standpoint, a utility company would be well advised to keep its capacity ahead of residential

demand - even though its customers cannot easily rig their own tele-
phone lines, dig their own wells, or generate their own power. In
all these cases, the assumption of zero backlogs seems like quite a
reasonable starting point. From this assumption, together with the
economies of scale phenomenon, Chenery derives his "permanent" excess
capacity hypothesis: ". . . excess capacity will occur even with per-
fect forecasting; this may be called 'optimum' overcapacity." [6, p. 2].

Despite the impressive list of sectors just noted, it would be
a mistake to suppose that the assumption of zero backlog possibilities
is a universally valid one. The economist who is accustomed to work
with a downward sloping price-demand curve will certainly find it just
as reasonable to believe that backlogs are admissible, and that they
are accompanied by some kind of penalty cost to the firm. The zero
backlog model then turns out to be a special case - the case in which
backlog costs are infinite. Everything hinges upon the penalty cost
assumption.

To a petroleum transporter, for example, these penalties are far
less than infinite. If he is unable to ship crude or refined products
via a pipeline, there is in almost all cases a transportation alternative
available - tankship, barge, railroad tank-car, or tank truck. The
penalty for failing to have enough pipeline capacity is simply the dif-
ference between the short-run marginal operating cost of the pipeline
and the marginal cost of using the alternative mode of transport. No
irreversibility effects seem significant here. As soon as new pipeline
capacity becomes available, the oil transporter will not hesitate to

switch over from the high-cost mode that is temporarily in use. The change-over costs would be negligible.

This kind of reasoning is surely not confined to petroleum pipelines alone. A steel producer might find that the penalty for being short of capacity in one section of the country would amount to nothing more than an increase in the amount of freight absorption needed to supply the region from a more distant point. Alternatively, the shortage penalty might consist of the profits foregone in being unable to bid on such marginal business as a large construction project or the export market. In none of the examples just cited would it be reasonable to assume that the steel company loses permanent customers. The penalty for being short of capacity is of a temporary nature, and is confined to the period of full-capacity operations.

In graphical terms, the analogy with Figures 1 and 2 is shown on Figures 5 and 6. Just as in the earlier case, we assume that demand grows linearly at the rate of one physical unit per year. Again, $\underline{x}$ units will denote the size of each new installation and the points t_o, $t_o + x$, $t_o + 2x$ . . . still mark the points of regeneration: the points at which excess capacity has just been wiped out. The entire difference between this and the earlier case is that we allow excess capacity to become negative here - in other words, permit backlogs of demand. Once such backlogs become admissible, there is no longer any a priori reason to believe in the necessity of Chenery's excess capacity hypothesis. With sufficiently low penalty costs, it is even conceivable that excess capacity will, on the average, be negative.

Demand and capacity

D_o+2x

D_o+x+y

Installed
capacity

$D_o+ x$

$D_o+ y$

Demand

y

y

D_o

t_o

$t_o+ x$

$t_o+ 2x$
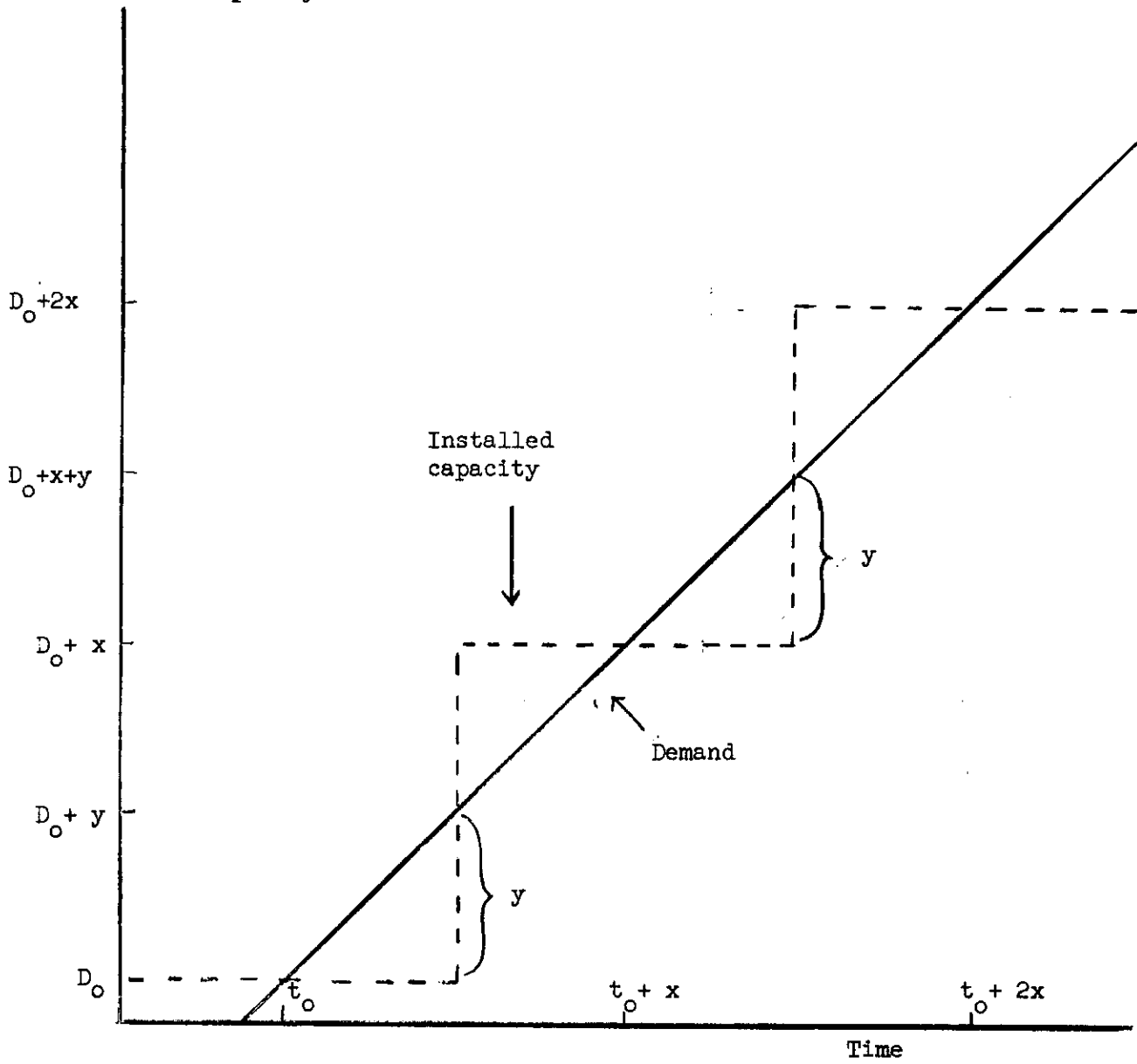
Time

Figure 5. Growth of demand and of capacity over time.

Excess capacity

$x-y$

0

t_o

$t_o+ x$

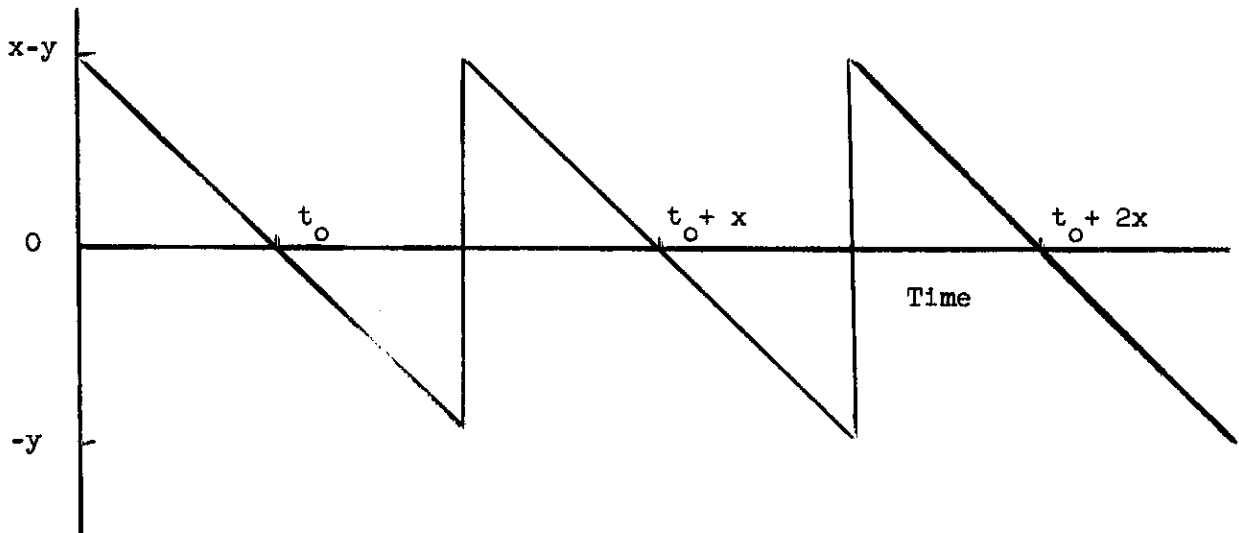$t_o+ 2x$

Time

$-y$

Figure 6. Evolution of excess capacity over time.

Figures 5 and 6 have been drawn on the assumption that whenever the backlog in demand grows to $\underline{y}$ units (that is, whenever excess capacity equals minus $\underline{y}$), a new facility is built - one of size $\underline{x}$. We now have two decision variables: $\underline{x}$, the size of each installation, and $\underline{y}$, the "trigger" level for backlogs in demand.[*] Penalty costs will be assumed strictly proportional to the quantity $\underline{z}$, hereafter employed to denote the size of the backlog.

Looking forward into the future from a point of regeneration, total discounted costs are a function of both $\underline{x}$ and $\underline{y}$. If we denote these discounted costs by $C(x,y)$, the expression that corresponds to (2.2) is as follows:

$$(4.1) \qquad C(x,y) = \int_{z=0}^{y} cze^{-rz}dz + e^{-ry}(kx^a) + e^{-rx} C(x,y)$$

where $\underline{c}$ represents the penalty costs per unit of backlog.

It is easy to see that when demand is growing steadily at the rate of one unit per year, a backlog of size $\underline{z}$ occurs exactly $\underline{z}$ years after a point of regeneration. The first term on the right-hand side of (4.1) therefore measures the discounted sum of all penalty costs incurred during the course of a single construction cycle. The second term measures the installation costs, and discounts them $\underline{y}$ years back to the beginning of the cycle. Finally, the last term indicates the future value

---

* Any reader will note the striking similarity between this and the Ss theory of optimal inventory policy. One important aspect tends to be concealed in the deterministic form of the two models. A replenishment lag is characteristic of the inventory studies, [1,2]. In the interests of simplicity, however, the corresponding feature - a construction lag - is ignored in the present paper.

of all costs incurred in subsequent cycles, and discounts this value back

over a period of $\underline{x}$ years. From (4.1), we readily obtain:

$$C(x,y) = \frac{1}{1-e^{-rx}} \left\{ c \int_{z=0}^{y} ze^{-rz}dz + e^{-ry}(kx^a) \right\}$$

Dividing through by $\underline{k}$ in order to eliminate one parameter, and christening

the ratio c/k with the name $\underline{b}$, we finally have the cost expression to

be minimized:

$$(4.2) \quad \frac{C(x,y)}{k} = \frac{1}{1-e^{-rx}} \left\{ b \int_{z=0}^{y} ze^{-rz}dz + e^{-ry} x^a \right\}$$

$$= \frac{1}{1-e^{-rx}} \left\{ \frac{b}{r^2} [1-e^{-ry}(1+ry)] + e^{-ry} x^a \right\}$$

Expression (4.2) involves three parameters: $\underline{a}$, the economies-of-

scale factor; $\underline{b}$, the penalty factor; and $\underline{r}$, the discount rate.

Minimization of (4.2) with respect to both $\underline{x}$ and $\underline{y}$ could conceivably

have been accomplished by calculus methods as in the earlier one-variable

case, but this approach seemed rather clumsy.[*] Instead, refuge was taken

in numerical methods. An electronic computer[**] evaluated $C(x,y)$ for a

large number of combinations of $\underline{x}$ and $\underline{y}$, and reported the minimum for

each specified set of values of $\underline{a}$, $\underline{b}$, and $\underline{r}$. The results of three

---

[*] If we differentiate (4.2) with respect to $\underline{y}$, we obtain as a necessary
condition:

$$\hat{y} = \frac{r\hat{x}^a}{b}$$

From this it is clear that $\hat{y}$ vanishes only when $\underline{b}$, the unit penalty
cost, becomes infinite.

[**] The machine was the I.B.M. 650 located in the Yale University Computing
Center. E. Uren performed the numerical analysis, with help from M. Davis,
Director of the Center, and also from D. Ciosek.

Table 2

Shortage Penalty Costs, Optimal Backlog

Trigger Levels and Capacity Increments

$(\ a \ = \ .50 \quad \text{and} \quad r = .15 \ )$

shortage penalty costs = b	∞	.50	.10	.05
optimal capacity increment = $\hat{x}$	8	9	12	21
optimal backlog trigger level = $\hat{y}$	0	1	5	14
minimum discounted costs = $\dfrac{C(\hat{x}, \hat{y})}{k}$ (see (4.2).)	4.048	3.791	2.883	2.027

such calculations are shown in Table 2. As in Table 1, the parameters
$\underline{a}$ and $\underline{r}$ were set at .50 and at .15, respectively. One word of caution
about the numerical construction of this table: the decision variables
$\underline{x}$ and $\underline{y}$ were restricted to integer values.

Table 2 and similar calculations strongly suggest one conjecture -
a conjecture which I have not, however, attempted to demonstrate rigorously -
that a decrease in the penalty cost factor $\underline{b}$ can never lead to a decrease
in the optimal levels, $\hat{\underline{x}}$ and $\hat{\underline{y}}$. This conjecture is supported by the
general appearance of the partial derivatives of (4.2) with respect to
$\underline{x}$ and $\underline{y}$.

5. The probabilistic model - backlogs considered

The final stage of this investigation will consist of fitting to-
gether the two kinds of generalizations of Chenery's model: (a) proba-
bilistic growth, and (b) backlogs in demand. Just as in the zero backlog
case, we again assume the operation of a diffusion process such that $D(t)$,
the growth in demand that takes place in $\underline{t}$ years, is a normally distri-
buted random variable - one with a mean of μt and a variance of $\sigma^2 t$.
The particular asymptotic process leading to this result consists of the
cumulation of successive independent changes $\epsilon(t)$. (Refer back to
(3.1).)

As before, we shall let $u(t;x)dt$ represent the probability with
which $\underline{t}$ time units have elapsed at the point when total demand <u>first</u>
exceeds the initial level by $\underline{x}$ units. A similar definition holds for

$u(t;y)dt$. We already know the Laplace transforms for these two probability distributions:

$$(5.1) \qquad \bar{u}(r;x) = \int_{t=0}^{\infty} u(t;x)e^{-rt}dt = e^{\lambda_2 x}$$

$$(5.2) \qquad \bar{u}(r;y) = \int_{t=0}^{\infty} u(t;y)e^{-rt}dt = e^{\lambda_2 y}$$

(Refer back to (3.6), (3.8), and (3.10).)

Now in order to deal with the backlog question, we shall have to introduce one additional piece of notation: $w(z;t,y)dz$. This symbol will denote the probability with which the backlog level equals $\underline{z}$ - given that $\underline{t}$ time units have elapsed since the most recent point of regeneration. Why does the decision variable $\underline{y}$ enter into the definition of this probability? Because the process of building up a backlog will come to an end as soon as demand has increased by a total of y units - that is to say, by an amount large enough to trigger off the construction of a new facility.

In Brownian motion language, $w(z; t,y)dz$ represents the probability with which a particle, starting $\underline{y}$ units above the origin, will at time $\underline{t}$ be $\underline{z}$ units beneath its initial position, without having previously touched the absorbing barrier at the origin. Feller has already provided the generating function for the corresponding probability distribution in the case of discrete time and one-unit movements of the particle. [10, problem 16, p. 336.] The analogous result for the Laplace transform in the continuous

case is as follows:[*]

$$(5.3) \qquad \bar{w}(z;r,y) = \int_{t=0}^{\infty} w(z;t,y)e^{-rt}dt$$

$$= K\,[e^{\lambda_2 z} - e^{\lambda_2 y}\,e^{\lambda_1(z-y)}]$$

---

[*] In order to derive this result, we recall the following definitions:

$v(z;t)dz$ = probability that demand will change by exactly z
units - given that t units of time have elapsed; a normal
density function with mean μt and variance $\sigma^2 t$; (unconstrained
random walk with the particle initially at the origin).

$u(t;y)dt$ = probability that exactly t time units have elapsed
at the time when demand first exceeds its initial level by
y units; (absorbing barrier at the origin, with the particle
initially located y units above the origin).

From these definitions:

$$w(z;t,y)dz = v(z;t)dz - \int_{\tau=0}^{t} u(\tau;y)\,v(z-y;t-\tau)dzd\tau$$

Denote the Laplace transforms of u, v, and w by $\bar{u}$, $\bar{v}$, and $\bar{w}$
respectively. Then:

$$\bar{w}(z;r,y) = \bar{v}(z;r) - \bar{u}(r;y)\,\bar{v}(z-y;r)$$

In order to derive $\bar{v}(z;r)$ and $\bar{v}(z-y;r)$, one follows the same line
of reasoning as in deriving $\bar{u}(r;x)$. One starts with the Fokker-Planck
partial differential equation. (See [10], pp. 325-6.)

$$\frac{\delta v(z;t)}{\delta t} = -\mu\frac{\delta v(z;t)}{\delta z} + \frac{\sigma^2}{2}\frac{\delta^2 v(z;t)}{\delta^2 z}$$

Since the coefficients of this equation are identical with (3.5):

$$\bar{v}(z;r) = K_1(r)e^{\lambda_1 z} + K_2(r)e^{\lambda_2 z}$$

Here we have as our boundary conditions:

$$\int_{-\infty}^{\infty} v(z;t)dz = 1$$

$$\int_{-\infty}^{\infty} \bar{v}(z;r)dz = \int_{t=0}^{\infty}\int_{z=-\infty}^{\infty} v(z;t)e^{-rt}dzdt = \frac{1}{r}$$

In order to satisfy these boundary conditions for all values of $\underline{r}$, and also in order to preserve continuity in the function $\bar{v}(z;r)$:

$$\bar{v}(z;r) = \left\{\begin{array}{c} Ke^{\lambda_2 z} \\[2mm] Ke^{\lambda_1 z} \end{array}\right\} \quad \text{according as} \quad z \left\{\begin{array}{c} \geq 0 \\[2mm] \leq 0 \end{array}\right\}$$

where $K = \dfrac{\lambda_1 \lambda_2}{r(\lambda_2 - \lambda_1)}$

Noting that $z \geq 0$, but that $(z-y) \leq 0$, this completes the derivation of $\bar{v}(z;r)$, $\bar{v}(z-y;r)$, and $\bar{u}(r;y)$, and from these in turn, equations (5.3) and (5.4).

I am much indebted to Gerd Reuter for working out these expressions.

---

where λ_1 and λ_2 are as determined earlier by (3.8) and where the parameter K is given by:

$$(5.4) \qquad K = \frac{\lambda_1 \lambda_2}{r(\lambda_2 - \lambda_1)}$$

The cost equation for the new model may be written down by direct analogy with the deterministic one (4.1):

$$(5.5) \qquad C(x,y) = \int_{z=0}^{y} cz\bar{w}(z;r,y)dz + e^{\lambda_2 y}(kx^a) + e^{\lambda_2 x} C(x,y)$$

Total expected discounted costs, $C(x,y)$, will - as in the preceding cases - be measured from a point of regeneration, a point at which excess capacity equals zero. Now the first term on the right-hand side of (5.5) measures the expected discounted sum of all backlog penalties incurred between this point of regeneration and the point at which the backlog

reaches the critical level, $\underline{y}$. (See Figure 5. Also equation (5.3).)
The penalty cost integration extends over all possible backlog levels
between 0 and $\underline{y}$. Note that it is quite conceivable for the backlog
to become negative at any time after a point of regeneration. Our cost
expression simply says that whenever this happens (that is, whenever
demand drops off enough to create some excess capacity), no additional
outlays are incurred beyond those that were previously committed.

The second term on the right-hand side of (5.5) is the one having
to do with construction costs during the current cycle. These costs are
all incurred at the time of reaching the level $\underline{y}$, and so the appropriate
Laplace transform is (5.2).

Finally, the third term (that measuring the discounted sum of all
costs incurred in subsequent cycles) refers to a cost that is dated as
of the beginning of the following cycle. This cycle will begin whenever
the total demand first increases by $\underline{x}$ units over the current level -
i.e., whenever the $\underline{x}$ units of new capacity are, for the first time,
fully utilized. In the third term therefore the appropriate Laplace
transform is (5.1).

For purposes of numerical analysis, the cost function (5.5) may be
rewritten:

$$(5.6) \quad \frac{C(x,y)}{k} = \frac{1}{1-e^{\lambda_2 x}} \left[bK \left\{ \frac{[1-e^{\lambda_2 y}(1-\lambda_2 y)]}{\lambda_2^2} - \frac{[e^{\lambda_2-\lambda_1}]^y [1-e^{\lambda_1 y}(1-\lambda_1 y)]}{\lambda_1^2} \right\} + e^{\lambda_2 y} x^a \right]$$

Table 3

Variance of Demand Versus Optimal Backlog Trigger Levels
and Capacity Increments

(a = .50, b = .10, and r = .15)

σ^2	0	1	4	16
$\lambda_1 = \dfrac{1}{\sigma^2}\left[1 + \sqrt{2r\sigma^2}\right]$	∞	2.1402	.6208	.2130
$\lambda_2 = \dfrac{1}{\sigma^2}\left[1 - \sqrt{2r\sigma^2}\right]$	-.1500	-.1402	-.1208	-.0880
$K = \dfrac{\lambda_1\lambda_2}{r(\lambda_2 - \lambda_1)}$	-	.8771	.6742	.4152
optimal capacity increment = $\hat{x}$	12	14	16	21
optimal backlog trigger level = $\hat{y}$	5	6	8	11
minimum expected discounted costs = $\dfrac{C(\hat{x}, \hat{y})}{k}$	2.883	2.834	2.831	3.079

(see (5.6).)

where $\underline{b}$ again equals the ratio c/k , as in the deterministic calculations of the preceding section.

The numerical analysis of (5.6) is only slightly more complex than that of (4.2).[*] There are still just two decision variables, $\underline{x}$ and $\underline{y}$, and three economic parameters $\underline{a}$, $\underline{b}$, and $\underline{r}$. The only additional feature is that in the present case we must also take account of the Laplace transform parameters λ_1 , λ_2 , and K . As can be seen from the row headings of Table 3, these last-mentioned parameters all depend directly upon the variance σ^2 . Just as in the zero/backlog case, we now examine the effects of increasing the variance while holding the expected increment in demand constant at unity. Also held constant in Table 3 are the parameters $\underline{a}$, $\underline{b}$, and $\underline{r}$.

Table 3 would be of little interest if it merely confirmed for the backlog case what we already knew about the zero backlog model: that an increase in variance is inevitably accompanied by an increase in $C(\hat{x}, \hat{y})/k$, the minimum level of expected discounted costs.

Instead, it provides an immediate counter-example to this conjecture. Minimum costs keep dropping as σ^2 increases from zero to four times the expected annual increment in demand. Only for the case of $\sigma^2 = 16$ does the level of expected costs increase again.

Lest the reader suppose that it is utterly implausible to find any decreasing relationship between variance and expected costs, it is instructive to consider the following illustrative example based upon discrete rather than continuous quantities: Suppose that every six months there is a discrete change in demand. In the deterministic version of this problem, the increase in demand is exactly 1/2 unit every six months. In

---

* In fact, the same L.B.M. 650 program written to solve (5.6) also handled (4.2).

the stochastic version, the increase is either zero or else one unit with 50-50 probabilities. (The expected annual increase is a total of one unit in either case.)

Now suppose that the capacity increment x and the trigger value y are both set at the arbitrary level of unity. With these values of x and y, a point of zero excess capacity is reached every six months with the probabilistic model and every year with the deterministic one. Because of the resulting annual coincidence in regeneration points, it is sufficient to compare expected discounted costs over a one-year cycle instead of over an infinite horizon.

In order to simplify the cost expressions, let $k = 1$. Let b denote the penalty costs incurred per unit of backlog outstanding at the end of each six-month period. And let ρ denote the present-worth factor for a single six-month period. ($\rho = e^{-r/2}$; $0 < \rho < 1$.) Table 4 then summarizes the expected costs for a one-year cycle.

Note that expected construction costs are always higher in the probabilistic than in the deterministic case - a result that coincides with our previous findings for the zero-backlog situation. For expected penalty costs, however, just the reverse is true. Total expected costs will be higher in the probabilistic case if and only if:

$$\frac{\rho b}{2} + \rho^2(1 + b) < \left(\frac{\rho + \rho^2}{2}\right)\left(1 + b\right)$$

or

$$\rho(1 + b) < 1$$

Table 4. Expected Cost Comparison; One-year Cycle,
Discrete Time Parameter and Discrete Changes in Demand

Version of problem	Change in Demand		Respective Probability of stated changes in demand	Discounted construction costs*		Discounted penalty costs		Total expected discounted costs
	end of first six months	end of second six months		end of first six months	end of second six months	end of first six months	end of second six months	
Deterministic {	0.5	0.5	1.00	0	ρ^2	$\rho(b/2)$	$\rho^2(b)$	} $\frac{\rho b}{2} + \rho^2(1+b)$
Probabilistic {	0.0	0.0	.25	0	0	0	0	} $\left(\frac{\rho+\rho^2}{2}\right)(1+b)$
	1.0	0.0	.25	ρ	0	$\rho(b)$	0	
	0.0	1.0	.25	0	ρ^2	0	$\rho^2(b)$	
	1.0	1.0	.25	ρ	ρ^2	$\rho(b)$	$\rho^2(b)$	

* Since $k = x = 1$, $kx^a = 1$.

6. <u>Summary and generalizations</u>

For the benefit of the reader who has persevered through the sensitivity

analyses that went with each of the four models analyzed here, it seems only

merciful to recapitulate the chief results for the two stochastic cases:

Backlog triggering Assumption	zero	non-zero
Cost equation	(3.12)	(5.6)
$\dfrac{\partial c(\hat{x}, \hat{y})}{\partial \sigma^2}$	> 0	$\begin{matrix} > \\ = \\ < \end{matrix}\ 0$
$\dfrac{\partial \hat{x}}{\partial \sigma^2}$	> 0	?

In each of these cases, the regeneration point technique - when

coupled with the appropriate Laplace transforms - made it a rather simple

matter to relate total costs to the decision variables in a closed analytical

form. There was no need to employ a Bellman-type functional equation [4].

This is not in the least to underrate the importance of the functional

equation approach - only to point out that other tools may also prove use-

ful in sequential dynamic optimization.

Among the problems that seem closely related to the one discussed

here would be models of inventory stockage [1, 2, 11], of equipment replace-

ment [3, 12], and of forestry economics.* Each of these is characterized

---

* The forestry problem was suggested to me in an unpublished paper by
R. F. Keniston of Oregon State College. A similar case was, of course,
investigated a good many years ago by Knut Wicksell [13, pp. 172 - 184].

by a cyclical spurt of build-up activity (either ordering some inventory,
or installing a new piece of equipment, or cutting down and reseeding a
piece of forest land) - a spurt which is followed by a fairly long gest-
ation period (rundown of inventory, accumulation of operating inferiority,
or the growth of new trees) until the beginning of the next cycle. Karlin
has already demonstrated the usefulness of the regeneration point approach
in the case of discrete time inventory problems [11, esp. pp. 280-85].

As intriguing as the research prospects in allied areas might be,
the fact remains that we are far from having exhausted the subject of
capacity expansion in this paper. It is easy to think of certain gen-
eralizations that would be quite straightforward: It would be a fairly
simple matter, for example, to replace the construction cost function (2.1)
with a different one - just as long as the new function also exhibited
economies of scale. (Without economies of scale, the whole rationale
breaks down for the bunching together of investment activity.) Another
fairly simple generalization would consist of replacing the proportional
backlog penalty cost with an arbitrary non-linear function of the backlog
size, z . The Laplace transform (5.3) would still remain valid.

Still another generalization would deal with the case of a construction
time-lag. If one assumes that this lag is fixed rather than random, the
obvious modification is to follow Beckmann and Muth [2, p. 149] and to regard
the state variable as the composite quantity : existing excess capacity
plus construction orders outstanding.

Even if one wished to alter the probability structure underlying the
demand growth pattern, there are certain alterations with which no great
difficulty would be experienced. Instead of representing the time series
of demand by a continuous random-walk process, one might assume that each

change in demand consists of a discrete amount (either +1, 0, or -1 unit) -
and that these changes occur at discrete time intervals. In this case, in-
stead of a linear differential equation such as (3.7) for the Laplace
transform, one would work with a linear difference equation for the generating
function of first-passage times. In other respects, such a discrete version,
though somewhat more awkward, would be quite similar to the continuous one
described here.

Now to enumerate several of the more intractable generalizations:
Suppose that one wished to deal with a discrete case in which, during each
period, demand either increased by 2 units with a probability of p , or
else decreased by 1 unit with a probability of q . Then, no matter what
value is assigned to the "trigger" level of excess demand, there is no
assurance at all that the random-walk process will pass through this point
on each cycle. Unless demand is constrained to move from one point to an
immediately adjacent one, one cannot be sure that the construction process
will be triggered off at exactly the same backlog level during each suc-
cessive cycle. The problem would become the much more difficult one of
calculating the optimal installation size as a function of the backlog
level. In order to attack this class of problems, the functional equation
approach would seem most natural.

One final kind of generalization that is appealing from the viewpoint
of economic realism, but which leads to some analytical difficulties is
the following: Suppose that one wished to replace an arithmetic growth
model with a geometric one. Instead of postulating a random-walk process
for the absolute level of demand, one would be likely to assume such a
process for the logarithm of demand. Even after working out the probability

distributions and Laplace transforms for a model of this sort, there would

still remain the problem of optimization. Neither the optimal backlog

trigger value nor the installation size would be independent of the absolute

level of demand. Again, perhaps the most convenient way to handle this

process would be to resort to the functional equation approach. Here, how-

ever, it would be necessary to relate everything to two state variables -

not only the current level of excess capacity but also to the absolute

level of demand. Using the principle of recursive optimality, it would be

easy enough to calculate numerical results. Analytical results would, I

conjecture, be considerably more difficult to establish than with the

linear probabilistic growth pattern treated here.

References

[1]   Arrow, K. J., T. Harris, and J. Marschak, "Optimal Inventory Policy,"
        Econometrica, July 1951.

[2]   Beckmann, M. J., and R. F. Muth, "An Inventory Policy for a Case of
        Lagged Delivery," _Management Science_, January 1956.

[3]   Bellman, R., "Equipment Replacement Policy," _Journal of the Society
        for Industrial and Applied Mathematics_, September 1955.

[4]   _____, _Dynamic Programming_, Princeton, 1957.

[5]   Blyth, C. A., "The Theory of Capital and Its Time-Measures,"
        Econometrica, October 1956.

[6]   Chenery, H. B., "Overcapacity and the Acceleration Principle,"
        Econometrica, January 1952.

[7]   Chilton, C. H., "'Six-Tenths Factor' Applies to Complete Plant Costs,"
        Chemical Engineering, April 1950, pp. 112-114.

[8]   Clark, P., "The Telephone Industry: A Study in Private Investment,"
        Studies in the Structure of the American Economy, W. W. Leontief, ed.,
        New York, 1953, Ch. 7.

[9]   Cookenboo, L., Jr., "Costs of Operating Crude Oil Pipe Lines," _Rice
        Institute Pamphlet_, April 1954.

[10] Feller, W., _An Introduction to Probability Theory and its Applications_,
        Vol. I, 2nd ed., New York, 1957.

[11] Karlin, S., "Application of Renewal Theory to the Study of Inventory
        Policies," Ch. 15 in _Studies in the Mathematical Theory of In-
        ventory and Production_, by K. J. Arrow, S. Karlin, and H. Scarf,
        Stanford, 1958.

[12] Terborgh, G. W., _Dynamic Equipment Policy_, New York, 1949.

[13] Wicksell, K., _Lectures on Political Economy_, Vol. I, London, 1934.