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Some Models of a Sales Organization*

by

C. B. McGuire

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1. Introduction

The models presented in this paper grew out of an attempt to apply the theory of teams of Marschak and Radner [1], [2], [3], [4] to the organization of the sales force in a typical wholesale bakery. As organizations go, the one we have chosen to analyze is an extremely simple one, even when -- as is quite obviously not the case in what follows -- it is viewed in its full complexity. At the risk of achieving results of quite limited general interest, a "simple," easily quantified subject was deliberately chosen as the best place to begin to apply a theory that pretends to prescribe optimum decisions of a team in a precise way.

We shall not here go beyond a mere discussion of some mathematical models which deal only with the day-to-day problem a bakery sales force faces in attempting to "properly" supply its regular customers with a single product. Problems of advertising, of price policy, of product design, and of obtaining new customers are ignored. For the present discussion all the reader needs to know is that the sales force consists of truck-driver salesmen who daily visit each of their given customers (i.e. grocery stores) leaving, on consignment, an amount of bread to be decided by the salesman. At the end of the day the salesman returns to the plant and submits an order for the next day. For our purposes here the "organization" is characterized by the way in which these orders are jointly formed.

A suggestion of Radner's, [3], [4] that certain organization problems can be formulated in linear programming terms, has been followed assiduously. The present examples, insofar as they are successful, hint that the applicability of this approach may be quite wide.

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* Many of the points discussed in this paper first arose in my discussion with Jacob Marschak and Martin Beckmann. I am heavily indebted and grateful

2. The Team Problem of Marschak and Radner

2.1. A General Formulation

In the Marschak-Radner formulation of the team problem, decision (or "action") functions $\alpha_i(x)$, $(i=1, \ldots, N)$, are to be determined which tell each of N team members how to act when the state of the world is x. If each member is perfectly informed about the value of x, then the payoff to the team as a whole for given x and given functions α_i is a real number $u[\alpha_1(x), \ldots, \alpha_N(x); x]$ minus whatever costs are incurred in making the value of x known to each of the members. If observation and communication are costly it will generally not be best to fully inform each member. In order to specify systems less costly in this respect let the function $\eta_i(x)$, $(i=1,\ldots, N)$, denote the information about x that is made available to the $i^{\underline{th}}$ team member. Thus if x is an M-tuple $(x_1, \ldots, x_M)$, one particularly simple example of an information function might be $\eta_i(x_1, \ldots, x_M) = x_i$. More generally η_i is some function from the set X of states of the world to the set of all subsets of X. With each of these <u>information structures</u> $\eta = (\eta_1, \ldots, \eta_N)$ is associated a cost $k(\eta)$. With a given structure then and a known probability distribution over X of states of the world the N-tuple of decision functions $\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_N)$ is to be found which maximizes the expected payoff to the team *

(1) $$E \quad u[\alpha_1(\eta_1(x)), \ldots, \alpha_N(\eta_N(x)); x] \quad - k(\eta) .$$

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* Notice that $k(\eta)$, the cost of the information structure, is here taken to be a function of η independent of the state of the world. This need not be so. Whenever costs of communication depend on the intensity of use of certain communication links -- as in the case of long-distance telephoning -- k will depend also on x and will occur under the expectation sign in (1).

Once a procedure is found for determining, for any given structure η, the corresponding optimum set of decision functions $\hat{\alpha}$, then the goodness of different structures can be compared.

2.2 A More Specific Formulation

In this section a more specific, but still quite general formulation of the team problem is proposed. Let the state of the world be represented by a (2N+1)-tuple, $x = (z_1, \ldots, z_N; y_1, \ldots, y_N; w)$. The y_i are demand parameters in each of several markets. No particular real interpretation of the z_i will be adhered to; we suppose only that they are statistically related to the respective y_i and yet are very much easier to observe. That different states of the world can affect matters of production is recognized by distinguishing such states by the single variable w. (Through most of this paper w is fixed and known.)

Each of N salesmen will have an information function η_i $(i = 1, \ldots, N)$ which will be a function only of the $z_1, \ldots, z_N$, and the $y_i, \ldots, y_N$. The latter are supposed to be prohibitively expensive to observe. In this limited context a completely coordinated organization would imply an information structure η in which $\eta_i(z_1, \ldots, z_N; y_1, \ldots, y_N) = (z_1, \ldots, z_N)$, or more briefly $\eta_i(x) = z$, for each i; that is, where every salesman knows all that can "reasonably" be known about the demand parameters in all markets. The most obvious example of η in an uncoordinated system is $\eta_i(x) = z_i$ for every i. Since our investigation of the set of information structures will not in any case be exhaustive, we shall be concerned with picking out these "obvious" structures for analysis.

The action variables $a_i = \alpha_i[\eta_i(x)]$ will be interpreted as supply actions or "orders" by the salesmen: in response to the information $\eta_i(x)$, Salesman i supplies a_i units of product to his market.

The payoff to the team will be taken to be the expected profit of the firm: expected revenue from sales minus expected cost of production minus cost of information structure. More specifically the payoff is

$$(2) \quad ER(a_1, \ldots, a_N; y_1, \ldots, y_N) - EC(a_1, .., a_N; w) - k(\eta)$$

where R is a revenue function and C a production cost function. It is assumed that the z_i influence profit only through their rôle as arguments in the action functions. The y_i do not influence production costs, and w does not influence demand.

For a team problem to be something more than a collection of N one-person problems there must be a source of what Marschak has called "interaction" among the actions of the N members: the effect of one man's decision of the payoff must not be quite independent of the actions of all of the other men (in continuous terminology, $\partial^2\mu|\partial a_i \partial a_j$ is not always zero for all i and j). In the present instance interaction among decisions comes only from the production cost relationship. It will be supposed that production costs depend only on the sum of the individual orders; revenue from one market will be assumed to depend only on the state of the world in that market (y_i in market i) and on the supply decision in that market. The payoff function (2) can therefore be further specified:

$$(3) \quad \sum_{i=1}^{N} ER_i(a_i, y_i) - EC(\sum_{i=1}^{N} a_i, w) - k(\eta)$$

where R_i is the revenue function in market i.

Payoff function (3) represents the general situation to be investigated in this paper: one plant supplying N markets. Three analogous problems suggest themselves. We shall do no more than mention them.

The first is the "opposite" of (3): N plants supplying one market, with payoff

$$(4) \qquad ER\left(\sum_{i=1}^{N} a_i,\ y\right) - \sum_{i=1}^{N} EC_i(a_i,\ w_i) - k(\eta)$$

where the state-of-the-world variables are appropriately and analogously redefined. One would expect a strong similarity between the results that analysis of (3) yields and results from (4).

The second is N plants supplying N markets with no plant impinging on another's territory:

$$(5) \qquad \sum_{i=1}^{N} ER_i(a_i,\ y_i) - \sum_{i=1}^{N} EC_i(a_i,\ w_i) - k(\eta).$$

The team aspect has disappeared, leaving N one-person problems.

The third analog is N plants jointly supplying N markets:

$$(6) \qquad ER\left(\sum_{i=1}^{N} a_i,\ y\right) - EC\left(\sum_{i=1}^{N} a_i,\ w\right) - k(\eta)$$

In this very simple form, where the team is indifferent as to which plant supplies a unit of product or in which market a unit is sold, it might appear that again the problem degenerates into a one-person situation. But the very multiplicity of solutions that makes the problem look easy causes trouble in the absence of coordination:* no amount

* The term "coordination" is used here in the sense of a once-and-for-all coordination, as opposed to the coordination supplied day-to-day by a team member (absent in our model) whose actions vary as the states of the world changes. The once-and-for-all type is exemplified by the (legally enforced!) convention among drivers to use their respective right-hand sides of a road. The example is from Marschak, who has emphasized the relation between the need for coordination and the uniqueness of $\hat{\alpha}$.

of information about the external state of the world is sufficient for one team member to act in a way that complements the action of his fellows. If the optimum N-tuple of action functions were unique, coordination would contribute nothing and a one-person problem would result.

3. Model I

One of the simplest interesting models with payoff function of the form (3) is the following one.

Let the random variable y_i be the price in market i , and suppose this price is unaffected by the supply action a_i of salesman i . The total revenue curve in market i will then be some straight line through the origin, the slope of which is, in general, not precisely known at the time the supply decision is made. The random variable z_i is observed by salesman i on the preceding day and is used to predict y_i . If we like, we may regard z_i as the price prevailing in the earlier period and suppose that the price series is autocorrelated. The joint distribution $p(z_1, \ldots, z_N; y_1, \ldots, y_N)$ is known. We will suppose further that the z_i and the y_i vary discretely.

Total cost of production is, in accord with (3), a function of the sum of the supply actions

$$(7) \quad c_a \sum_i a_i + (c_b - c_a) \text{ Max } [\sum_i a_i - w, o]$$

with $w > o$, $c_b > c_a > o$. The parameters c_a, c_b and w are assumed to be fixed and known to all participants, although in a more comprehensive study of this organization it would be necessary to include these three cost parameters in the state-of-the-world vector where they would be observed and communicated at some cost.

3.1. Centralization

If the information structure is such that every member has all obtainable information about demand in all market then, for every i, $\eta_i(x) = z$. Given this structure, the problem is to find a set of action functions $\alpha_i(z)$, i=1, .., N, which maximizes expected profit. In order to write payoff as a linear function of the $\alpha_i(z)$, let us define a non-negative function $b(z)$ with

$$(8) \quad b(z) \geq \sum_i \alpha_i(z) - w .$$

If $b(z)$ is chosen to be as small as possible for any set of $\alpha_i(z)$, then payoff can be written

$$(9) \quad \sum_z \sum_y \quad \Sigma(y_i - c_a)\alpha_i(z) - (c_b - c_a)b(z) \quad p(z,y)$$

where the summation signs on the left stand for $\sum_{z_1=o}^{K} \dots \sum_{z_N=o}^{K}$ and $\sum_{y_1=o}^{K} \dots \sum_{y_N=o}^{K}$ respectively, K being some high integer beyond the ranges of variation of the z_i and the y_i.

We must now select $\alpha_i(z)$ for each i and z and $b(z)$ for each z so as to maximize (9), subject only to constraint (8) on $b(z)$ and non-negativity of $b(z)$ and $\alpha_i(z)$. In case the range of y_i extends above c_b it will be necessary to put an upper bound on the $\alpha_i(z)$, say $\bar{a}$, otherwise the maximum of (9) will be unbounded. Since both the maximand and the constraints are linear in the $b(z)$ and the $\alpha_i(z)$, the necessary and sufficient conditions for the optimum values of these variables can be found by differentiating the Lagrangean expression. Let $\lambda(z) \geq 0$ be the multiplier associated with constraint (8). Then the conditions are

$$(10) \qquad \frac{\partial\mu}{\partial\alpha_i(z)} = \sum_y (y_i - c_a)p(z,y) - \lambda(z) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} 0 \qquad \text{if} \begin{cases} 0 = \alpha_i(z) \\ 0 < \alpha_i(z) < \bar{a} \\ \alpha_i(z) = \bar{a} \end{cases}$$

and

$$(11) \qquad \frac{\partial\mu}{\partial b(z)} = -(c_b - c_a) \sum_y p(z,y) + \lambda(z) \begin{Bmatrix} \leq \\ = \end{Bmatrix} 0 \quad \text{if} \quad b(z) \begin{Bmatrix} = \\ > \end{Bmatrix} 0$$

with $\lambda(z)$ vanishing if strict inequality holds in (8).

Constructing a solution will be easier if (10) and (11) are rewritten in more familiar terms:

$$(12) \qquad E(y_i|z) \quad \begin{matrix} \leq \\ = \\ \geq \end{matrix} \quad c_a + \frac{\lambda(z)}{p(z)} \qquad \text{if} \quad \begin{matrix} 0 = \alpha_i(z) \\ 0 < \alpha_i(z) < \bar{a} \\ \alpha_i(z) = \bar{a} \end{matrix}$$

and

$$(13) \qquad \lambda(z) \begin{Bmatrix} \leq \\ = \end{Bmatrix} (c_b - c_a)p(z) \quad \text{if} \quad b(z) \begin{Bmatrix} = \\ > \end{Bmatrix} 0$$

where $p(z) \equiv \sum_y p(z,y)$.

Let us fix our attention on a particular value of z. If the conditional price expectations in all markets are below c_a, the lowest extreme of marginal production cost, then intuition and (12) both assert that $\alpha_i(z) = 0$ for all i.

If the highest conditional price expectation exceeds c_a, then, by (12), either $\alpha_i(z) = \bar{a}$ in this high market or $\lambda(z)$ is positive. But by assumption, $\bar{a}$ is a high number so, by (8), $b(z)$ must be positive and hence, by (13), $\lambda(z)/p(z) = c_b - c_a$. In either case a price expection exceeding c_a implies a positive value of $\lambda(z)$. This in turn implies equality in constraint (8), which means that $\sum_i \alpha_i(z) \geq w$.

If the highest expected price falls <u>between</u> c_a and c_b then (12) and $\sum_i \alpha_i(z) \geq w$ can be simultaneously satisfied only by $\alpha_i(z) = w$ in the high market and zero in other markets.

If one or more of the price expectations exceeds c_b then $\alpha_i(z) = \bar{a}$ in all such markets and $\alpha_i(z) = 0$ in the others since the left side of (12) can never exceed c_b.

This elliptical* argument has been presented only in an effort to motivate the following solution, the validity of which is easily checked against (8), (12), and (13):

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* The distinction between < and $\leq$ was ignored for one thing.

Let k denote the maximizer of $\max_i E(y_i|z)$ for a given z .

Then

(14) if $E(y_k|z) \leq c_a$, set $\alpha_1(z) = \ldots = \alpha_N(z) = b(z) = \lambda(z) = 0$;

if $c_a < E(y_k|z) \leq c_b$, set $\alpha_k(z) = w$

$$\alpha_i(z) = 0 \quad \text{for } i \neq k$$

$$b(z) = 0$$

$$\lambda(z) = (c_b - c_a)p(z)$$

if $c_b < E(y_i|z)$ and $E(y_j|z) \leq c_b$ set

$$\alpha_i(z) = \bar{a}$$

$$\alpha_j(z) = 0$$

$$b(z) = \Sigma\, \alpha(z) - w$$

$$\lambda(z) = (c_b - c_a)p(z) \ .$$

The solution is not (quite) unique, the choices between $<$ and $\leq$ having been made arbitrarily in (14). For those values of z however where the delicate issue of $<$ versus $\leq$ in applying (14) never arises the values for the $\alpha_i(z)$ (but not necessarily $\lambda(z)$) are unique. In the case $N=2$ a graphical representation of (14) indicates the areas of non-uniqueness. In Figure 1 the optimum values of $\alpha_i(z)$ are shown over various regions of the space of conditional price expectations. Along the borders common to two regions (or the points common to three or four regions) all interpolated values are optimal. As long therefore as z is in the interior of a region, the intelligent salesman will know how to act

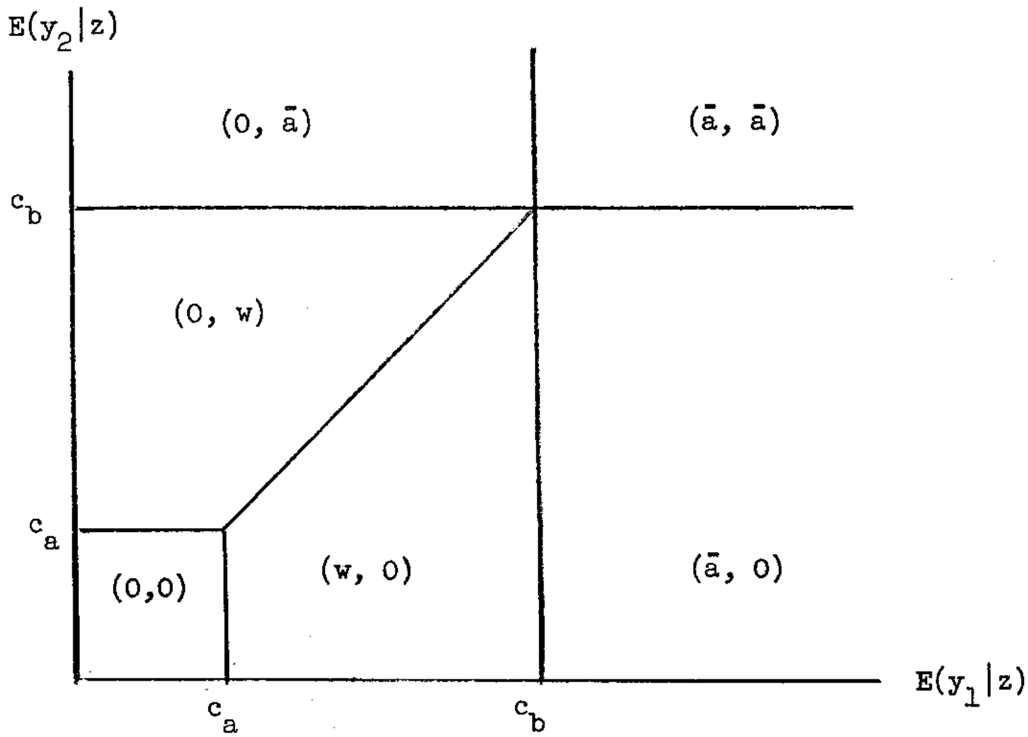

Figure 1. Optimum Supply Actions $(\alpha_1(z), \alpha_2(z))$

optimally, but if z occurs on a border he will not. Complete information about the external state of the world in these cases does not lead to automatic coordination.

3.2 Decentralization

Here we suppose the information structure η to be such that $\eta_i(x)=z_i$ for every i ; each salesman knows his own market predictor, but not those in other markets. The action functions to be determined are now $\alpha_i(z_i)$ instead of $\alpha_i(z)=\alpha_i(z_1, \ldots, z_N)$ as in the complete information case of the preceding section.

The constraint on $b(z)$ is now

$$(15) \qquad b(z) \geq \sum_i \alpha_i(z_i) - w$$

and the function to be maximized (which of course represents the payoff if $b(z)$ is properly chosen) is

$$(16) \quad \sum_{zy}\sum \left\{ \sum_i (y_i - c_a)\alpha_i(z_i) - (c_b - c_a) b(z) \right\} p(z,y).$$

Proceeding just as before we find conditions on the $\alpha_i(z_i)$ and $b(z)$:

$$(17) \quad \frac{\partial u}{\partial \alpha_i(z_i)} = \sum_{z_1} \cdots \sum_{z_{i-1}} \sum_{z_{i+1}} \cdots \sum_{z_N} \left[\sum_y (y_i - c_a) p(z,y) - \lambda(z) \right] \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} 0 \text{ if } \begin{cases} 0 = \alpha_i(z_i) \\ 0 < \alpha_i(z_i) < \bar{a} \\ \alpha_i(z_i) = \bar{a} \end{cases}$$

and for $\frac{\partial u}{\partial b(z)}$ condition (11) again. And as before, when (17) is translated into terms of conditional expectation we have

$$(18) \quad E(y_i | z_i) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} c_a + \sum_{z_1} \cdots \sum_{z_{i-1}} \sum_{z_{i+1}} \cdots \sum_{z_N} \frac{\lambda(z)}{p(z_i)} \quad \text{if} \begin{cases} 0 = \alpha_i(z_i) \\ 0 < \alpha_i(z_i) < \bar{a} \\ \alpha_i(z_i) = \bar{a} \end{cases},$$

where $p(z_i)$ stands for $\sum_{z_1} \cdots \sum_{z_{i-1}} \sum_{z_{i+1}} \cdots \sum_{z_N} p(z)$.

For convenience, we repeat

$$(13) \quad \lambda(z) \begin{Bmatrix} \leq \\ = \end{Bmatrix} (c_b - c_a) p(z) \qquad \text{if } b(z) \begin{Bmatrix} = \\ > \end{Bmatrix} 0 .$$

In attempting to construct a solution, we start by concentrating on a given i and given value of z_i . We assume that a solution exists in which the $\lambda(z)$ take only the values zero or $(c_b - c_a)p(z)$ (sometimes of course this assumption is violated, as will become clear in a moment -- in any case no harm is done by investigating the construction it leads to).

If $E(y_i|z_i) < c_a$ then $\alpha_i(z_i)$ must be zero from (18). If $E(y_i|z_i) \geq c_a$ then the sum on the right of (18) must be evaluated. From the above assumption about the possible values of the $\lambda(z)$ we know that when z is such that $\Sigma\alpha$ exceeds w, $(c_b-c_a)p(z)$ will enter the sum for $\lambda(z)$, and when z is such that Σa falls short of w, no contribution results. The right-hand sum in (18) therefore can be written $(c_b-c_a)p'$, where p' is the conditional probability given z_i that $\Sigma\alpha > w$. We have therefore that

$$(19) \qquad \frac{E(y_i|z_i)-c_a}{c_b-c_a} \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} p' \qquad \text{if} \quad \begin{cases} 0 = \alpha_i(z_i) \\ 0 < \alpha_i(z_i) < \bar{a} \\ \quad \alpha_i(z_i) = \bar{a} \end{cases}$$

with the qualification that the $\lambda(z)$ must take their extreme values only.

A graphical interpretation of the uses to which (19) can be put will be given for the very special case where $N=2$ and when $E(y_i|z_i) = z_i$. Suppose one suspects that both action functions $\alpha_i(z_i)$ are non-decreasing in z_i. If this were so then $z_2' > z_2$ and $\alpha_1(z_1) + \alpha_2(z_2) > w$ would imply $\alpha_1(z_1) + \alpha_2(z_2') > w$ and similarly for $z_1' > z_1$. All the positive $\lambda(z)$ would therefore occur in the Northeast part of the z space of Figure 2. Condition (19) then suggests the definition of a function $g_1(z_1)$ by*

$$(20) \qquad \sum_{z_2=g_1(z_1)}^{K} p(z_1,z_2) \equiv \frac{z_1 - c_a}{c_b - c_a} \qquad\qquad (c_a \leq z_1 < c_b)$$

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* In the present discussion we can ignore the discreteness of the z_i.

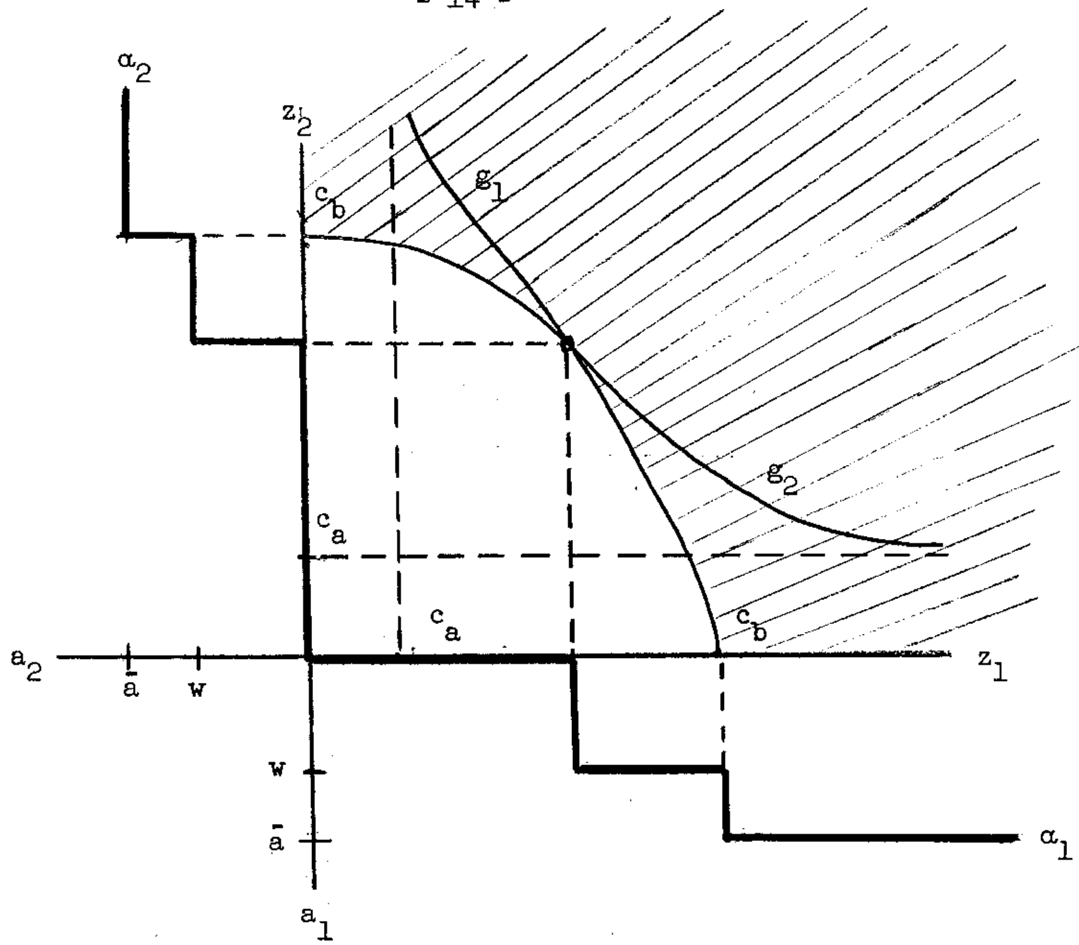

Figure 2

and a function $g_2(z_2)$ in the same fashion. When $z_1 = c_a$, g_1 must be at the "top" of the conditional z_2 distribution, and when $z_1 = c_b$, g_1 must be at the "bottom". Figure 2 shows a pair of very well behaved g functions -- well behaved in the sense that they intersect only once and are monotonic. Functions of this simple kind will result if z_1 and z_2 are independent, or if they are inversely correlated and $p(z)$ is not too misshapen.

Now set $\lambda(z) = (c_b - c_a)p(z)$ when z is in the shaded area of Figure 2 and $\lambda(z) = 0$ otherwise. Next define three-step action functions as in Figure 2 with ordinates 0, w, and $\bar{a}$, with the first steps occurring at the coordinates of g_1-g_2 intersection, and the second steps at c_b. That the three functions $\lambda(z)$, $\alpha_1(z_1)$, and $\alpha_2(z_2)$ so defined are optimum is easily established by checking against (18) and (13). Also, it should be noted, the solution is unique.

When z_1 and z_2 are positively correlated the procedure just described will quite often not work, because the g curves intersect in more than one place and/or lack the monotonic properties on which the earlier construction rested. Figure 3 portrays one such case. It will be recalled however that the definition of the g curves was prompted

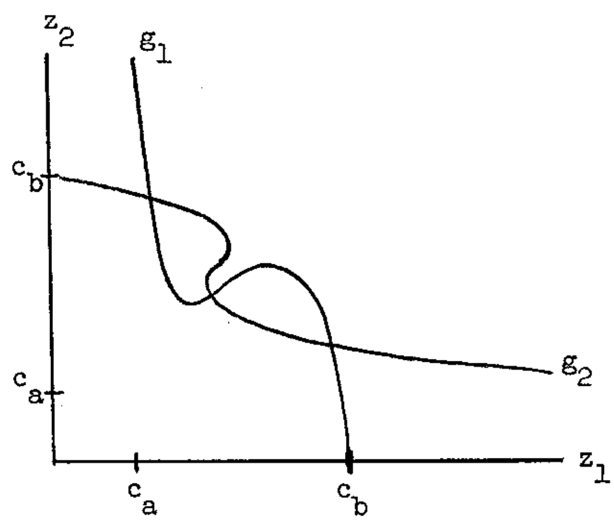

Figure 3

by a hunch that the action functions were non-decreasing. But when z_1 and z_2 are positively correlated, we would expect rather that <u>one</u> of the action functions will be non-decreasing and the other non-increasing. By "reversing" one of the g functions (i.e.,

$\sum_{z_2=0}^{g_1(z_1)} p(z_1,z_2) \doteq \frac{z_1-c_a}{c_b-c_a}$) and making the obvious changes in the construction (i.e., the positive $\lambda(z)$ are put in the Northwest or Southeast corner of the z space) similarly simple step-function solutions for these positively correlated distributions can sometimes be found.

Still another class of "easy" solutions occurs with positively correlated discrete distributions which are nearly symmetric about the line $z_1 = z_2$. In these cases the action functions are constant in the interval $[c_a, c_b]$ and zero and $\bar{a}$ respectively below and above this interval. The middle levels a_i need only be chosen so that $a_1 + a_2 = w$; here of course the need, discussed earlier, for coordination in selecting complementary action functions is again acute.

The geometric methods suggested above for constructing solutions to the decentralized problem are obviously very limited: the changes required to replace z_i with the original $E(y_i | z_i)$ are minor, but the extension to $N > 2$ raises quite new questions. Moreover, it will have been observed that maximizing (16) subject to (15), with the restriction that the variables be non-negative, is a standard problem of linear programming; the armory of established techniques for these problems can be put to use. The point is however that the number of variables (and constraints) in our linear programming formulation is so extremely high as to make ordinary computational techniques quite impracticable for many interesting problems. The formulation is uneconomic in the sense that it makes no use of regularity properties the $p(z,y)$ distribution might have (e.g., a single mode); but rather is ready to deal with bizarre checkerboard distributions which never occur. One hopes that geometric arguments like those presented will provide a time-saving guide for machine computation, both by suggesting computational methods geared to the peculiarities of the problems, and by classifying problems.

3.3. A Mixed Case - Partial Spread of Information

Let $N=2$, $\eta_1(x)=z_1$, and $\eta_2(x)=(z_1,z_2)$. The maximand becomes

$$(21)\quad \sum_z \sum_y \left\{(y_1-c_a)\alpha_1(z_1)+(y_2-c_a)\alpha_2(z_1,z_2)-(c_b-c_a)b(z)\right\} \; p(z,y)$$

and the constraint on $b(z)$

$$(22)\qquad b(z) \geq \alpha_1(z_1) + \alpha_2(z_1,z_2) - w \, .$$

The optimum condition on $\alpha_1(z_1)$ is

$$(23)\quad \sum_{z_2} \left\{\sum_y (y_1-c_a)p(z,y) - \lambda(z)\right\} \qquad = 0 \text{ , etc.}$$

$$\text{or} \quad E(y_1|z_1)-c_a - \sum_{z_2} \frac{\lambda(z_1, z_2)}{p(z_1)} \qquad = 0 \text{ , etc.}$$

And for $\alpha_1(z_1,z_2)$

$$(24)\quad \sum_y (y_2-c_a)p(z,y) - \lambda(z) \qquad = 0, \text{ etc.}$$

$$\text{or} \quad E(y_2|z_1,z_2)-c_a - \frac{\lambda(z_1,z_2)}{p(z_1,z_2)} = 0, \text{ etc.}$$

If in the middle range of z we can manage to keep $\Sigma\alpha=w$ then $\lambda(z)$ can take any non-negative value not exceeding $(c_b-c_a)p(z)$, from (13). In particular it can take the value it has when equality holds in (24). Substituting this value in (23) we get

$$(24)\quad E(y_1|z_1)-c_a - \sum_{z_2} \left\{ E(y_2|z_1,z_2)-c_a \right\} \frac{p(z_1,z_2)}{p(z_1)}$$

or

$$E(y_1|z_1)-E(y_2|z_1) = 0 \text{ , etc.}$$

Thus when z_1 is in the interval $[c_a,c_b]$, Man 1 supplies w whenever his conditional expectation of y_1-y_2 exceeds zero. When z_2 is in the $[c_a,c_b]$ interval Man 2 supplies w when Man 1 is not acting and nothing when he is. The total then is always w in the middle z range, so the substitution for $\lambda(z)$ in (24) was permissible. As before both men order zero and $\bar{a}$ respectively below and above the $[c_a,c_b]$ interval.

An interesting question suggested by this result is whether the function $\alpha_1(z_1)$ just derived differs from the $\alpha_1(z_1)$ of the centralized case of the preceding section. In other words, does Man 1 act differently when Man 2 has more information? I cannot prove the two functions are the same, but neither can I prove that they are different. In the event that they are the same, the formidable computation problem of the last section is immensely simplified.

4. Model II

In the market facing a bread salesman price is constant* and the quantity of fresh bread demanded at this price is one of the random variables characterizing the state of the world x . Normalizing our money measure, we can let price in this model be unity. The demand in

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* The wholesale prices of baked goods do change of course, but only as a result of relatively infrequent and ponderous decisions at the highest levels of management. To ignore these price changes is not to deny their importance as an element of a theory of organization. We simply wish to limit ourselves here to a smaller more manageable problem - one which almost surely can be safely "factored out" of the grander price-decision problem.

Market i will be written y_i and whatever advance information the salesman can learn about demand z_i. Ignoring the production cost parameters, we have then $x = (z,y)=(z_1, \ldots, z_N; y_1, \ldots, y_N)$. Just as in Model I, the salesman's <u>action</u> $a_i = \alpha_i(\eta_i(z,y))$ is the amount of fresh bread he supplies his market. Lacking precise knowledge of the demand y_i that will materialize, he will sometimes supply too much and sometimes too little.

Throughout this section we shall assume that bread delivered fresh to the market in the morning remains saleable as "fresh" bread only to the end of the day -- i.e., its "shelf-life" is one day.* Any excess

* "Shelf-life" is of course a policy decision, not a technological parameter. Like price, we suppose it to be given.

e_i of the amount a_i supplied by the salesman over the amount y_i demanded by consumers must be picked up by the salesman at the beginning of the next day and returned to the plant where it is sold in an (N+1)th market at a price $1-r$, with $1-c_a < r < 1$. Demand for "stale" bread in this special market will be supposed infinitely elastic.

If Salesman i "undersupplies" his market the reaction of disappointed consumers will be supposed to bring about a diminution of the firm's future profits the present discounted value of which is directly proportional to the deficiency d_i of supply. Let such loss per unit of deficiency be the same in all markets and denote it q.

To simplify the analysis, both r and q will be regarded as given; in fact, they vary* from time to time and from place to place,

* Variation in r would arise if the assumption of elasticity in the demand for stale bread were dropped. Failure to meet the demand of a particularly good customer, or of an ordinary customer on an occasion particularly important to him, could give rise to changes in q .

but probably less than the y_i .

In these terms total revenue in Market i is

(25) $$a_i - re_i - qd_i$$

$$\text{where } e_i = \max (a_i - y_i, 0)$$

$$d_i = \max (y_i - a_i, 0) .$$

In Figure 4 total revenue curves are shown for two different values of y_i .

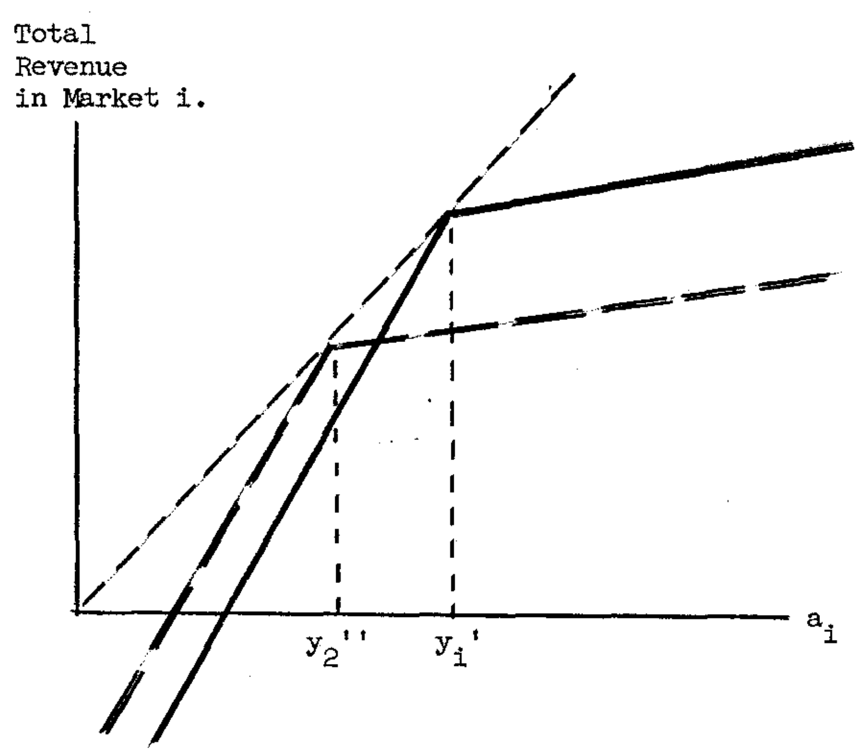

Figure 4. Total Revenue in Market i as Function of Amount Supplied

From a formal standpoint the present problem now appears more closely analogous to the Model I problem than one might have suspected at first. Once again the problem is to determine how each salesman should behave in the face of a shifting total revenue curve. To be sure, the curves are somewhat more complicated in the present case than in Model I, but the analysis based on them can proceed in the same fashion. For purposes of the analysis, the interpretation given these revenue curves is unimportant: whether we say that when the amount supplied exceeds y_i marginal revenue is lower because the stale returns are higher or because the "price" must be lower to clear the market makes no difference. The only new feature of general interest introduced by Model II is the dependence of marginal revenue on the amount supplied to the market; in Model I marginal revenue was unaffected by the supply decision.

4.1. Centralization

Let the information structure be specified by $\eta_i(x) = z$. The problem then is to select non-negative $\alpha_i(z)$, $d_i(z,y_i)$, $e_i(z,y_i)$ for each i , and $b(z)$ so as to maximize

$$(26) \quad \sum_z \sum_y \left\{ \sum_i [(1-c_a)\alpha_i(z) - re_i(z,y_i) - qd_i(z,y_i)] - (c_b - c_a)b(z) \right\} p(z,y)$$

subject to the constraints

$$(27) \quad d_i(z,y_i) \geq y_i - \alpha_i(z) \quad , \quad (i=1, \ldots, N)$$

$$(28) \quad e_i(z,y_i) \geq \alpha_i(z) - y_i \quad , \quad (i=1, \ldots, N)$$

$$(29) \quad b(z) \geq \sum_i \alpha_i(z) - w \quad .$$

Let $\mu_i(z,y_i)$, $\gamma_i(z,y_i)$ and $\lambda(z)$ be the Lagrange multipliers respectively associated with (27), (28), and (29). The partial derivatives of the Lagrangean maximand with respect to $\alpha_i(z)$, $d_i(z,y_i)$, $e_i(z,y_i)$, and $b(z)$ are respectively

$$(30) \quad (1-c_a)p(z) - \lambda(z) + \sum_{y_i} [\mu_i(z,y_i) - \gamma_i(z,y_i)] = 0 \text{ , etc.}$$

$$(31) \quad \mu_i(z,y_i) - qp(z,y_i) = 0 \text{ , etc.}$$

$$(32) \quad \gamma_i(z,y_i) - rp(z,y_i) = 0 \text{ , etc.}$$

$$(33) \quad \lambda(z) - (c_b - c_a)p(z) = 0, \text{ etc.}$$

Let us first look at (27) and (28) in relation to (31) and (32) for given i. If $y_i > \alpha_i(z)$ then $d_i(z,y_i) > 0$, so equality must hold in (31) and strict inequality in (28), in turn implying $\gamma_i(z,y_i) = 0$. If $y_i < \alpha_i(z)$ the same argument yields equality in (32) and $\mu_i(z,y_i) = 0$. If $y_i = \alpha_i(z)$ then <u>both</u> μ_i and γ_i can be positive.

With these results the third term in (30) can be written

$$(34) \quad \left\{ q - (r+q) S_i[\alpha_i(z)] \right\} p(z)$$

where $S_i[\alpha_i(z)]$, which we shall call the "(conditional) probability of no sell-out," is defined by

$$(35) \quad \text{Prob}[y_i < \alpha_i(z) | z] \leq S_i[\alpha_i(z)] \leq \text{Prob}[y_i \leq \alpha_i(z) | z] \text{ .}$$

We can now rewrite (30) as

$$(36) \quad \frac{1+q - c_a - \frac{\lambda(z)}{p(z)}}{r+q} \left\{ \begin{matrix} \leq \\ = \end{matrix} \right\} S_i[\alpha_i(z)] \text{ if } \alpha_i(z) \left\{ \begin{matrix} = \\ > \end{matrix} \right\} 0 \text{ .}$$

With knowledge of his own demand distribution a salesman can easily choose his $\alpha_i(z)$ so as to put S_i at any prescribed level between zero and one. Suppose the $\alpha_i(z)$ are tentatively set at values which make

$$(37) \qquad S_i[\alpha_i(z)] = \frac{1+q-c_a}{r+q} \quad .$$

For all z such that $\Sigma\, \alpha_i(z) < w$, these tentative $\alpha_i(z)$ are optimal, because in these cases $\lambda(z) = 0$ and (36) is the same as (37).

Now, for the remaining z , set the $\alpha_i(z)$ tentatively at values which make

$$(38) \qquad S_i[\alpha_i(z)] = \frac{1+q-c_b}{r+q} \quad .$$

These values will be correct for all z for which $\Sigma\, \alpha_i(z) > w$, since now (36) is the same as (38).

For the still remaining z , the $\alpha_i(z)$ must be selected so as to make $\Sigma\alpha = w$ and $S_i = S_j$, at a level intermediate between (37) and (38).

The solution is easy to see in a graph for the case N=2. To simplify the picture let $z_1 = E(y_i|z)$. The solid lines divide the z space into the three regions just described. In a practical case of centralization one might wish to compute a whole family of contours of the function $\lambda(z)$; the dotted line in Figure 5 is an intermediate member of this family. Notice that all any individual salesman need know in order to act optimally is the value of $\lambda(z)$; the particular z that gave rise to this $\lambda(z)$ is of no importance. This suggests that from the computational

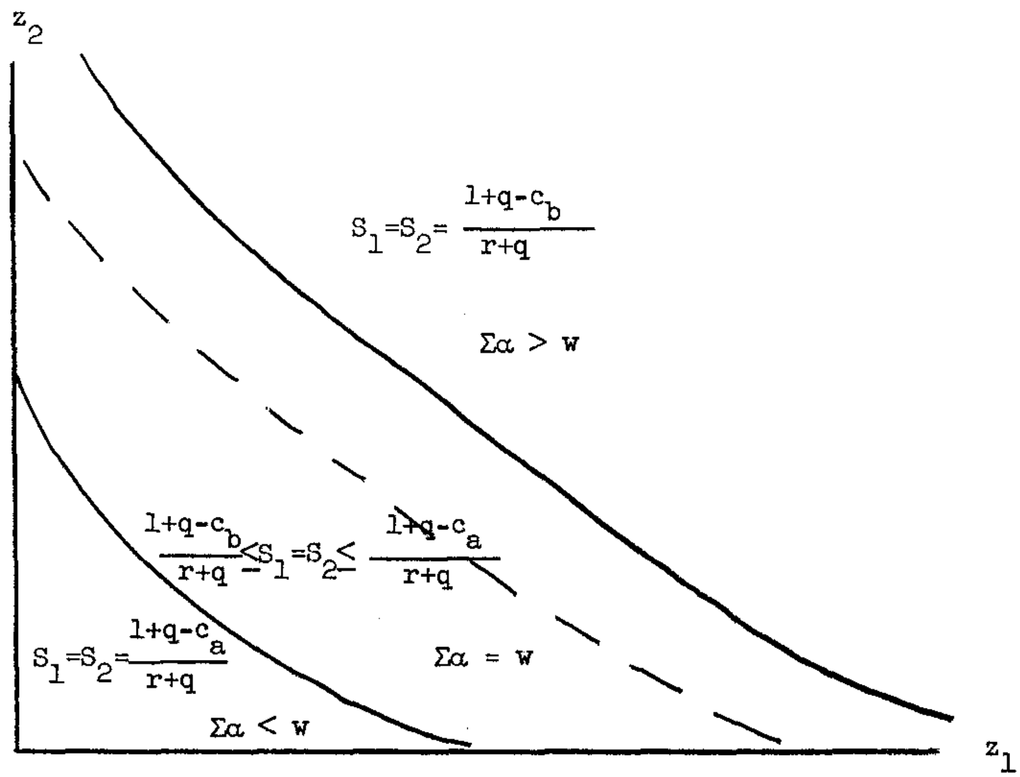

Figure 5

point of view the optimal set of actions given a z, might best be found, not by tabulating $\lambda(z)$ and informing all members of z, but rather by playing a "market game" in which individuals repeatedly submit tentative orders on the basis of their own z_i and a tentative value of λ. A custodian (of overtime labor) then revises λ in the light of $\Sigma\alpha$, the procedure continuing until a "centralized" solution satisfying (36) is found.

4.2. Decentralization

The information structure is now $\eta_i(x) = z_i$. The condition analogous to (36), namely,

$$(39) \qquad \frac{1+q-c_a - \dfrac{\lambda(z_i)}{p(z_i)}}{r+q} \left\{ \begin{matrix} \leq \\ = \end{matrix} \right\} S_i[\alpha_i(z_i)] \quad \text{if} \quad \alpha_i(z_i) \left\{ \begin{matrix} = \\ > \end{matrix} \right\} 0$$

follows by exactly the same argument. In (39) the abbreviated notation for sums, used up to now only for probabilities, is applied to the $\lambda(z)$:

$$\lambda(z_i) \equiv \sum_{z_1} \cdots \sum_{z_{i-1}} \sum_{z_i} \cdots \sum_{z_N} \lambda(z) .$$

In looking for a method of computation we can, as in the decentralization case of Model I, fall back on the standard techniques available for linear programming problems. But just as in Model I, this alternative cannot be regarded as much more than a last resort -- the number of variables is so high. Again we wish to make use of whatever special characteristics the problem may have that were not incorporated in the mathematical formulation.

Results -- or rather weak computational hints -- analogous to those obtained for Model I are slightly harder to derive and, once derived, are probably of less direct usefulness. Since they are suggestive, however, they follow.

Letting p and p' denote the conditional probabilities that $\Sigma\alpha < w$ and $> w$, respectively, for a given z_i, the left side of (39) can be written*

$$(40) \qquad p\left(\frac{1+q-c_a}{r+q}\right) + p'\left(\frac{1+q-c_b}{r+q}\right).$$

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* Recall (35), which allows us to ignore occasions where $\lambda(z)$ has an "intermediate" value.

An optimal action function is therefore seen to be one which results from setting $S_i[\alpha_i(z_i)]$ equal to a linear interpolation between the extreme values of (39), the weights being the conditional probabilities p and p'.

Designate the two hypothetical action functions that would be optimal when $w=0$ and $w=\infty$ by α_i^0 and α_i^∞, respectively (if $w=0$ then c_b always prevails; if $w=\infty$, c_a always prevails). For any given z_i, the value $\alpha_i(z_i)$ of the function we seek is a non-linear interpolation, via $S_i[\alpha_i(z_i)]$, between $\alpha_i^0(z_i)$ and $\alpha_i^\infty(z_i)$.

Figure 6, a four quadrant diagram analogous to Figure 2, shows the functions referred to for the case $N=2$, $z_i=E(y_i|z_i)$.

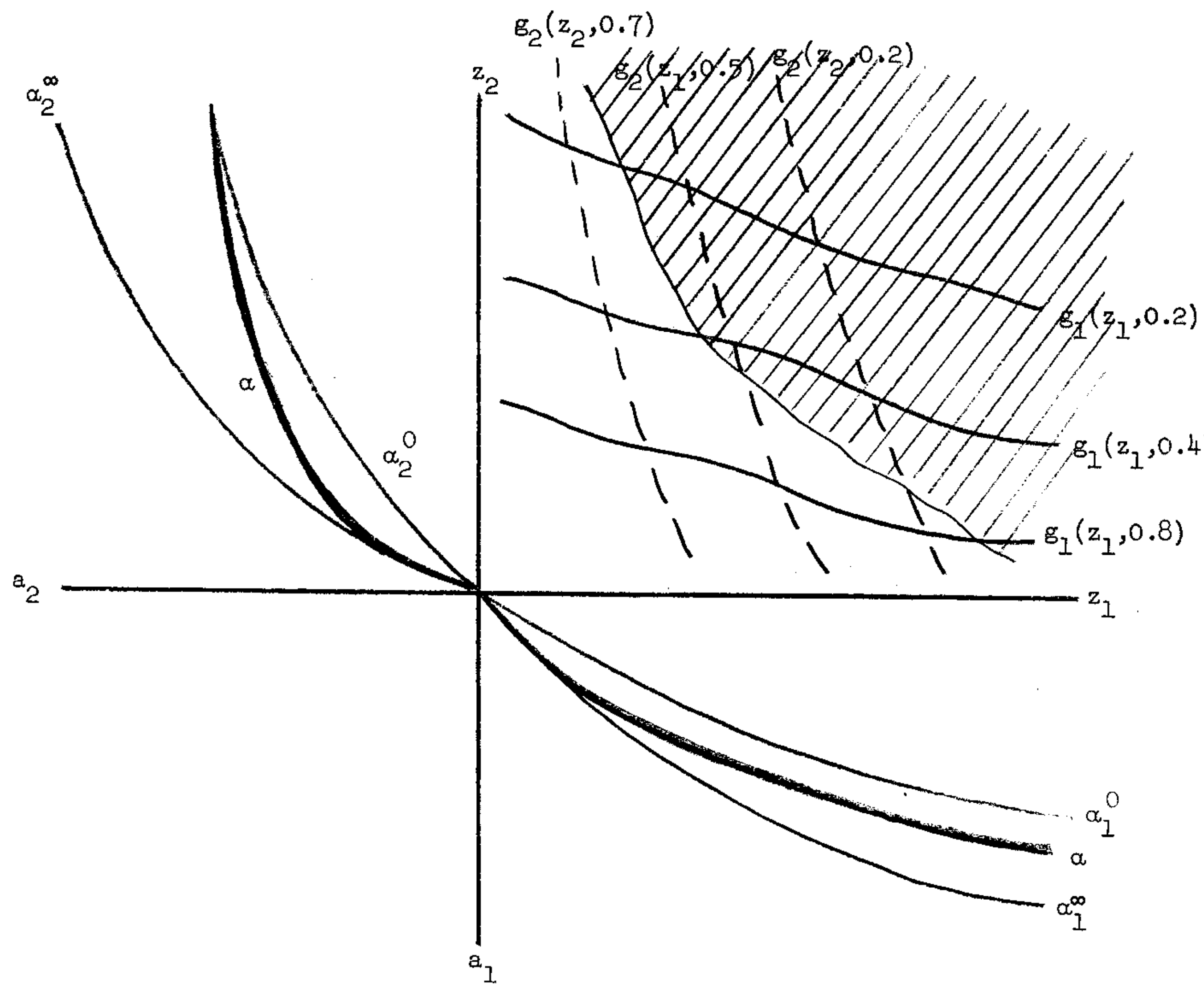

Figure 6

If z_1 and z_2 are independent or negatively correlated, then both α_1 and α_2 are probably non-decreasing, so the positive $\lambda(z)$ will occur in the Northeast region of the z space. Define a <u>family</u> of g_1 functions as follows:

$$\sum_{z_2=g_1(z_1,t_1)}^{\infty} p(z_1,z_2) \equiv t_1 \qquad\qquad 0 \leq t_1 \leq 1$$

and a similar family of g_2 functions. Just as in Model I, much will depend on how well behaved these g functions are. If they are monotonic as in Figure 6 , and as would be expected with a "reasonable" probability distribution with no positive correlation, then the following procedure will yield a solution without too much difficulty.

Pick a z . Observe the t_1 and t_2 associated with the g curves through z . Select tentative $\alpha_i(z_i)$ so as to make

$$S_i[\alpha_i(z_i)] = \frac{1+q-c_a-t_i(c_b-c_a)}{r+q}$$

. If $\Sigma\alpha > w$, mark z with a "+".

If $\Sigma\alpha < w$, mark z with a "0". Next, if z is "+" then so is any $z' \geq z$ (in the vector sense); if z is "0" then so is $z' \leq z$. When either plus or zero has been associated with all z (except those dividing the two regions, which we will ignore), we set $\lambda(z)=(c_b-c_a)p(z)$ if z is "+", and $\lambda(z)=0$ if z is "0". Set the final values of $\alpha_i(z_i)$ by the values of t_i along the border of the "+" region (shaded in Figure 5).

That the solution so constructed satisfies (39) is easily verified.

Notice that by no means all the z need be investigated, and hence not all the g curves need be constructed. It is here that the saving of effort over a straightforward linear programming calculation is most evident. The qualifications mentioned earlier about extensions to $N > 2$ still hold, of course. Furthermore, the difficulties of adapting the procedure (e.g. by making one of the g functions cumulate probability from the bottom of the z_i distributions instead of the top) to positive correlation seems more difficult than in Model I.

5. Model III

In most wholesale bakeries the shelf-life of the leading product is set at two days rather than -- as in Model II -- one day. In this section we attempt to investigate the consequences of changing Model II in this one single respect.

No longer is one day's profit independent of another day's profit; the supply actions taken on Tuesday affect not only Tuesday's profit but also Wednesday's since some of the bread supplied fresh Tuesday morning may not be taken by consumers on Tuesday. This carryover, not yet "stale", affects the probability of sell-out on Wednesday, hence the optimum amount to be supplied fresh on that day, hence the carry-over into Thursday and the profit on that day, etc.

In order that the profit function be bounded, we must introduce either a discount factor or a beginning and end to the period considered. Since Sundays* are the salesman's day off, they provide a natural break

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* And perhaps Wednesdays also in plants on a five-day week.

in the interrelatedness of profits on successive days, so the second alternative will be chosen here.* Wherever necessary, variables will

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* If -- as is the case for some products -- shelf-life were three or four days, a discount factor would be unavoidable.

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be labeled with superscript t to denote day of the week ($t=1, \ldots, T$).

We shall speak of three classes of bread on any one day: fresh, day-old, and stale, corresponding respectively to ages less than one day, between one and two, and greater than two. The consumer will be supposed indifferent or unable to distinguish between fresh and day-old. We assume however that a "FIFO" rule of consumption prevails: that the consumer always takes the oldest loaf of bread from the store shelf.*

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* The alternatives are "LIFO" or random selection. The first could just as well have been used here, but it leads to problems of non-concavity in the problem (not treated in this paper) of optimum allocation of a given total order among stores. This was pointed out to me by Robert Summers. Random selection is probably realistic, interesting, and complicated.

The definition of "excess" and "deficiency" must be extended to day-old bread. In accordance with the FIFO assumption, demand will be first applied to the stock of day-old to determine $\bar{e}_i^t$ and $\bar{d}_i^t$, the excess and deficiency of day-old, and $\bar{d}_i^t$ will be the "demand" that is applied to the stock of fresh bread to determine e_i^t and d_i^t, the excess and deficiency of fresh bread.

The state-of-the-world vector now becomes a TN-tuple of the z_i^t and the y_i^t. As the week of length T days progresses a salesman learns more and more about the true x: at the beginning of Day t we suppose he has observed $z_i^1, \ldots, z_i^t$ and $e_i^1, \ldots, e_i^{t-2}$ in his own market and, in the centralized case, the same variables in the other markets. If we follow the formulation of Model II the e_i^t are pseudo action functions, optimum values of which must be determined in the profit maximizing problem. Yet here we find they also play the rôle of information variables given by x and information structure η, supposedly fixed. This conflict is resolved by supposing that the salesman on Day t learns yesterday's demand y_i^{t-1} instead of the carryover e_i^{t-1} from yesterday. We then make an assumption that y_i^{t+1} is statistically unrelated to all variables but z_i^{t+1}, thus ensuring that the extra information that y_i^{t-1} represents over e_i^{t-1} is useless to the salesman. Thus we can still regard the e_i^t as pseudo action functions without violating the assumption of fixed η.

A convenient formalism is to regard Salesman i on Day t as a team member distinct from Salesman i on Day (t+1). For each "member" then there is an information function η_i^t.

Proceeding just as in Model II we have the constraints

(40) $$b^t \geq \sum_i \alpha_i^t$$

(41) $$\bar{d}_i^t \geq y_i^t - e_i^{t-1}$$

(42) $$\bar{e}_i^t \geq e_i^{t-1} - y_i^t$$

(43) $$d_i^t \geq \bar{d}_i^t - \alpha_i^t$$

(44) $$e_i^t \geq \alpha_i^t - \bar{d}_i^t$$

where the information vectors upon which the variables depend have been omitted for brevity. (For clarity we emphasize that given i and given t each of the constraints above is still a vector constraint, with a component of the vector for each value assumed by the information vector.)

The maximand of the problem, again leaving out the information variables is

$$(45)\qquad \sum_x \left\{ \sum_t \sum_i [(1-c_a)\alpha_i^{\,t} - r\bar{e}_i^{\,t} - qd_i^{\,t}] - (c_b - c_a)\sum_t b^t - r\sum_i e_i^{\,T} \right\} p(x)$$

Just as in Model II, (45) *represents gross profit only if in the solution* the pseudo action variables take on values consistent with the interpretations of them that determine their rôle in (45). Thus in (45) $\bar{e}_i^{\,t}$ represents the excess of day-old on Day t. If in the solution *non*-zero $\bar{e}_i^{\,t}$ were found for which strict inequality held in (42), then (45) would be incorrect; extra day-old bread would have been generated mathematically.

In Model II this trouble never arose: $d_i^{\,t}$ and $e_i^{\,t}$ always correctly represented deficiencies and excesses. That the trouble is present here is easily seen. Set $\bar{d}_i^{\,t} = \alpha_i^{\,t} = \infty$ and $e_i^{\,t} = d_i^{\,t} = \bar{e}_i^{\,t} = 0$ for all i, t, and x. Constraints (40)-(44) are satisfied and (45) is unbounded -- but of course none of the pseudo action variables play correctly the rôles assigned them in (45).

Throughout this paper we have been maximizing piecewise linear concave functions of unconstrained (except for positivity) variables by first finding equivalent linear functions of linearly constrained variables. The trouble in the present case seems to be that the true gross profit function, in terms of unconstrained action variables, is not concave. If so, this is somewhat of a surprise. It need not mean however that hope of an "easy" approach to a solution must be abandoned. Modified Model II techniques should first be tried. One can, for instance, blithely go ahead with the argument as before, meanwhile forcing the pseudo action variables when positive to satisfy equality in the constraints. A reasonable looking solution results but its relation to the optimum is not clear.

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