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Returns to Scale in Business Administration*

by

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Enterpreneurial capacity is often regarded as the limiting factor in determining the size of the firm, as it undoubtedly is for the traditional one man enterprise [Kaldor, Knight, and Robinson]. The modern corporation has a device in its hierarchy of executives for overcoming this hurdle. But it is thought that such hierarchies operate under diseconomies to scale. The diseconomies are of two kinds:** delays in decision making

** Some writers believe that a further inevitable diseconomy of size is <u>inefficiency</u> bred by "Say's Law": that a supply (of offices) creates its own demand (of shan work). An amusing version of a dynamic model of this law has gained much popularity under the name of "Parkinson's Law" [<u>The Economist</u>]. In theory, at any rate, there is no basis for the assertion that inefficiency is a necessary attribute of organizational size. For business firms it may be argued on the contrary that the relatively efficient firms will tend to be the large ones. We have therefore disregarded this somewhat elusive type of inefficiency in the analysis that follows.

through bureaucratic "sluggishness"; and increasing cost of administration per worker. Thus it is concluded, that even where increasing returns prevail in production activities, at some level -- possibly beyond that which is required at the present -- the diseconomies of administration will catch up with the economies of production. There exists then an optimal size of the firm at which both effects are optimally balanced.

* This paper owes much to J. Marschak's earlier "Remarks on the Rationale of Leadership" CFDP 6 (October 1955) and to his stimulating comments.

In this paper we shall examine the supposed diseconomies of scale in management by means of a theoretical analysis of the underlying structure of administrative hierarchies.

1. We shall first estimate the total outlay in wage and administrative salaries per worker engaged in production.

We consider a well defined hierarchy of consecutive levels m counted from the top.

Let L_m be the number of employees at the m^{th} level and let

W_m be the wage rate at that level.

<u>Assumption 1.1.</u> $L_m/L_{m-1} \geqq a > 1$.

In a given enterprise the number of subordinates at the next level per executive is never less than a certain constant $\underline{a}$ which is greater than one.

<u>Assumption 1.2.</u> $1 < W_{m-1}/W_m \leqq b$

An executive's salary is greater than that of any immediate subordinate but by not more than a ratio of $b > 1$.

<u>Assumption 1.3.</u> $a > b$

The ratio of salaries from level to level is smaller than the (inverse) ratio of numbers.

Assumption 1.1 is not to be confused with that of a fixed span of control nor assumption 1.2 with that of a fixed ratio salary scale (see section 2 below).

For evidence that $\underline{a}$ tends to be larger than $\underline{b}$ one may compare the numbers and salaries of persons at the highest and lowest levels of a large firm and see that the spread of numbers is larger than that of salaries (see p. 6 below).

We shall now show that the assumptions made above imply that the total administrative cost per worker in a given organization remains below a definite bound no matter how large the organization. In the next section we shall examine the approximation to this bound under rather more special conditions.

Let $$C_m = \sum_{i=1}^{m} L_i W_i$$

be the total wage bill for the upper m levels of the hierarchy. Our assertion is that

$$\frac{C_n}{L_n} \leqq \frac{W_n}{1-\frac{b}{a}}$$

Proof. By (1.1)

$$L_{m-1}/L_m \leq \frac{1}{a} < 1 ,$$

Multiplying by (1.2)

$$L_{m-1}W_{m-1}/L_m W_m \leq \frac{b}{a} ;$$

Therefore

$$\frac{C_n}{L_n} = \frac{1}{L_n} \left(L_n W_n + L_{n-1}W_{n-1} + \ldots + L_1 W_1\right)$$

$$\leq \frac{1}{L_n} \left[L_n W_n + L_n W_n \cdot \frac{b}{a} + L_n W_n \left(\frac{b}{a}\right)^2 + \ldots + L_n W_n \left(\frac{b}{a}\right)^{n-1}\right]$$

$$= \frac{1}{L_n} \cdot L_n W_n \cdot \frac{1-\left(\frac{b}{a}\right)^n}{1-\frac{b}{a}} \leq \frac{W_n}{1-\frac{b}{a}} .$$

It may also be interesting to consider the <u>number</u> of employees per worker. An estimate is obtained by simply setting $W_m = 1$ and $b = 1$ in our previous formula

$$\frac{1}{1 - \frac{1}{a}}$$

To obtain the number of administrators per worker the number one (of the worker himself) must be subtracted

$$\frac{1}{1 - \frac{1}{a}} - 1 = \frac{1}{a-1}$$

In an organization where each administrator has not less than <u>a</u> subordinates at the next level the total number of administrators per worker is therefore not more than

$$\frac{1}{a - 1}$$

no matter how large the organization.

There is some empirical evidence that the administrative cost per worker is independent of the scale of operations in the Electric Power Industry of this country [McNulty].*

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*I am indebted to A. Manne for this reference.

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2. An extreme case is that in which the bounds a and b are taken on at each level, that is: $L_{m+1}/L_m = a$, $W_m/W_{m+1} = b$, and therefore $L_m W_m / L_{m+1} W_{m+1} = b/a$. We shall want to study the approximation of $\frac{C_n}{L_n}$ to the ceiling given under the above conditions. In regard to b , the factor of steepness in the salary scale, there exists some empirical evidence for such a geometric progression [Simon, 1957]. The situation is more complex with respect to <u>a</u> , the so-called span of control. [cf. Simon 1947 , Polanyi] While it is

irrelevant for our purposes that this parameter varies from industry to industry [Koontz and O'Donell], it must be noted that it also varies from level to level in many cases. Thus "Students of Management have found [the span of control] to exist usually at numbers of 4 to 8 subordinates at the upper level of organizations and from 8 to 15 or more at the lower levels [Koontz and O'Donell, p. 88]. However, the recommendation of experts on management is that the span of control should be held between narrow limits and near 4 [Lyndall]. Apart from its simplicity the case of a constant span of control has therefore at least some theoretical interest.

In terms of the functions L_m and W_m this case is characterized by

$$L_m = a^{m-1}$$

$$W_m = W \cdot b^{n-m}$$

where n as before denotes the total number of levels in the organization. Now

$$
\begin{aligned}
C_n &= \sum_{m=1}^{n} L_m W_m = W \sum_{m=1}^{n} a^{m-1}\, b^{n-m} \\
&= \frac{W}{a}\; b^n \sum_{m=1}^{n} \left(\frac{a}{b}\right)^m = \frac{W}{a}\, b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} - 1\right] \\
&= W\, b^{n-1}\, \frac{\left(\frac{a}{b}\right)^n - 1}{\frac{a}{b} - 1} = W \cdot \frac{a^n - b^n}{a - b}
\end{aligned}
$$

Raising the level of the organization from n to $n+1$ increases the labor force by $L_{n+1} - L_n = a^n - a^{n-1} = a^{n-1}(a-1)$ and the wage bill by $C_{n+1} - C_n = \frac{W}{a-b} \; [a^{n+1} - a^n - b^{n+1} + b]$.

The increased cost per worker is therefore

$$\frac{C_{n+1} - C_n}{L_{n+1} - L_n} = \frac{W}{a-b} \cdot \frac{1}{a^{n-1}(a-1)} \cdot [a^n(a-1) - b^n(b-1)]$$

$$= \frac{W}{1-\frac{b}{a}} \cdot [1 - (\tfrac{b}{a})^n \; \frac{b-1}{a-1}]$$

The percentage difference from our previous estimate $\frac{W}{1-\frac{b}{a}}$ is therefore $(\frac{b}{a})^n \; \frac{b-1}{a-1}$. We shall estimate the size of this term presently. That $\underline{b}$ is well below $\underline{a}$ follows from a rough appraisal of our large corporations. A company employing in the neighborhood of 100,000 workers has approximately 6 levels in its hierarchy: foreman, engineer, manager, general manager, vice-president, president. The average "span of control" is then 6.8. But the salary of a president is 100 rather than 100,000 times that of a worker, yielding an average value of $b = 2.5$.

With these values let us now estimate the size of the term by which the incremental cost per worker differs from constancy. We have

$$\frac{(b-1)}{(a-1)} \cdot (\tfrac{b}{a})^n = \frac{1.5}{1.8} \; (\tfrac{2.5}{6.8})^n \simeq 0.258(0.368)^n$$

$$= 1.28\% \quad \text{for} \quad n = 3.$$

For medium and large sized firms administrative cost per worker is thus very close to a constant, if the assumed values of the span of control and the salary progression are at all realistic. For these values the "wage-multiplier" turns out to be

$$\frac{1}{1 - \frac{b}{a}} = 1.74.$$

To summarize: If at the higher levels of administration the span of control and the salary progression are nearly constant and in the neighborhood of the typical values $a = 6$; $b = 2.5$, then the incremental administrative cost per worker due to the addition of one level in the hierarchy is also nearly constant and of the order of the direct wage cost W. In any case, since the span of control will always be greater than the salary ratio, the administrative cost per worker remains below a definite and predictable limit no matter how large the organization.

3. What is the economic phenomenon behind a constant span of control? Let us define a measure of "the amount of decision" required by an amount L of labor in production, as the number of hours that some standard person would need to make the necessary decisions.* It may be objected that the

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* Analysis of the decision process may equip us in the future with a better measure of the "amount of decision making" than recourse to empirical standards of performance. The decision making process may be regarded as a succession of simple steps designed to identify the state of the world and to find the appropriate response. This response may have been determined once and for all and be stored in a memory where the decision maker must locate it. Or, more generally the decision maker is in possession of a program which prescribes the operations that produce the right response.

Assuming that the elementary steps in the decision process are standardized in some fashion -- e.g., as choices among a fixed number of alternatives -- a measure of the amount of decision involved in a given task would then be given by the expected number of steps in the decision process. Of course, the average number of steps required to solve a problem will depend on the efficiency of the program for tasks of this kind. Since in this view decision making is formally equivalent to a coding process, information theory tells us that the decision work can never be less than the "entropy" or "amount of information" of the system, a measure of the improbability of the decision outcomes. But this is a matter for further exploration.

The problem of measuring amounts of decision is further complicated by the fact that decisions are not qualitatively homogeneous but differ by the expected amount of payoff they can contribute to the organization's objective function.

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amount of decision may itself depend on the organizational setup. This we shall circumvent by separating out from decision making proper all coordinating and assignment activities which we shall label "supervision of subordinates."

Assumption 3.1. Each employee requires a constant amount e of supervision by his superior, measured in units of working time.

Assumption 3.2. Each person is utilized at the same level c of standard working time.

A constant span of control means therefore that each administrator is charged with an amount ea of supervision and an amount $c-ea$ of decision making.

The total amount of decision making performed in a n-level organization is therefore

$$D_n = (c-ea) \sum_{i=1}^{n-1} a^{i-1} = (c-ea) \frac{a^{n-1}-1}{a-1}$$

The amount of decisions per worker is then

$$\frac{D_n}{L_n} = \frac{c-ea}{a-1} (1 - a^{-(n-1)}) \approx \frac{c-ea}{a-1}$$

We have thus shown: Under the assumptions 3.1 and 3.2 a constant span of control implies an (almost) constant requirement of decisions per worker. If, in addition, it is assumed that physical output, suitably measured, is proportional to the labor force at the lowest echelon, we have the conclusion that a constant span of control implies a fixed input coefficient of decisions into output.

The utilization of staffs by decision makers would not alter the picture materially. For it is not decision making that is delegated to staffs but the preparation of these decisions through information processing, etc. Only through the line can decision makers be made responsible for their work. But the work of line executives can be expedited by staffs. Our previous considerations apply provided the rate of expedition is the same at all levels.

4. On the basis of the last model let us consider also the matter of delays incurred in decision making. In this model each administrator does an equal amount of decision making. (This if anything is an unfavorable assumption because it tends to overstate the amount of work assigned to the top levels.) We shall be interested in the average delays to decisions. The proportion of decisions that move through n levels to the president's desk is now 1 divided by the size of the administrative staff,

$$\frac{1}{\frac{a^n-1}{a-1}} = \frac{a-1}{a^n-1} \approx \frac{a-1}{a^n}$$

These incur an average delay of at most $n\bar{t}$ moving one way if $\bar{t}$ is the average time for passing on a decision. Allowing for the possibility that a task may arise at the production level, be decided by the president and then handed back down again to the production level, the maximal average delay is $2n\bar{t}$. Now at the second highest level a times this number of decisions are made with a maximal average delay of $2(n-1)\bar{t}$, etc. The average delay of all decisions is thus no more than

$$2\bar{t}\ \frac{\sum_{m=0}^{n-1} m\left(\frac{1}{a}\right)^m}{\sum_{m=0}^{n-1}\left(\frac{1}{a}\right)^m} = 2\bar{t}\left(\frac{1}{a-1} - \frac{n}{a(a^n-1)}\right) < \frac{2\bar{t}}{a-1}$$

Not only is the average delay limited independent of the size of the organization, but for medium and large sized firms it is very nearly constant, the term $\frac{n}{a(a^n-1)}$ being less than 1 percent for $a=6$ and $n \geq 2$. The bound $\frac{2\bar{t}}{a-1}$ can be shown to apply to any organization for which the number of subordinates per person is never less than a constant $\underline{a}$.

If there are no absolute limitations and no substantial diseconomies to the scale of enterpreneurial activity, as we are forced to conclude, then this eliminates one of the basic props to the theory of an optimal size of the firm. The alternative must be given serious consideration, namely that firms are equally efficient over a whole range of sizes so that in any industry considerable variations in size may exist at any time. The next question is obviously, whether there are at least stable distributions of size understandable as the outcome of some selective mechanism operating on size. But this is another subject.

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