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An Inventory Model for Repair Parts -- Approximations in the Case of Variable Delivery Time*

Martin J. Beckmann

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The following continuous inventory problem arises in connection with the storage of repair parts. Let demand be for one unit at a time and Poisson, and let delivery time be either fixed or a random variable of known distribution. The size of an order need not be specified until the time of actual production or delivery. The following costs are assumed to be given: A storage cost proportional to quantity of stock and time, a fixed cost of ordering, and a penalty for shortage proportional to the amount of the shortage and its duration. No orders are lost. Alternatively the probability of a loss may be incorporated in the shortage penalty.

The stochastic process which governs the expected value of discounted future costs through time may be described in terms of a pair of differential-difference equations to which an explicit solution can be given, [CFDP 50]. Since the rate of discount is always small, the expected value of future discounted costs is, as a matter of general principle, very nearly equal to the average cost per unit time divided by the discount per unit time. It is sufficient therefore to consider the average cost per unit of time which is given by the following formula:

$$\ell(s,D) = \frac{k + \frac{1}{\lambda} \sum_{i=1}^{D} f(s+i) + \sum_{i=0}^{\infty} f(s-i) \cdot \int_{0}^{\infty} p(i,t)\,[1-Q(t)]dt}{\frac{D}{\lambda} + \int_{0}^{\infty} tq\,dt}$$

where s reordering point

$s + D$ optimal starting stock

k fixed ordering cost

$$f(x) = \begin{cases} hx & h \text{ storage cost} \\ g(x) & g \text{ shortage penalty} \end{cases}$$

λ = average number of demands per unit time

$p(i,t)$ = probability of selling i units during a (random) time interval of length t

$q(t)$ = probability that delivery time is t

$Q(t)$ = probability that delivery time is t or less

Observing that under the assumptions made $p(i,t)$ is the Poisson probability $\frac{(\lambda t)^i}{i!} e^{-\lambda t}$ and substituting for $f(x)$ one obtains after some calculation

$$\ell(s,D) = \frac{\lambda}{D+\lambda\bar{t}} \left[k + \frac{h}{\lambda} sD + \frac{hD(D+1)}{2} + \frac{g\lambda}{2}(\sigma^2 + \bar{t}^2) - gs\bar{t} + (h+g) \sum_{i=0}^{s-1} (s-i)a_i \right]$$

where $\bar{t}$, σ^2 are the mean and variance of the delivery time distribution, respectively, and

$$a_i = \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t} [1-Q(t)]dt \leqq \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t} dt = \frac{1}{\lambda}$$

Now if the delivery time distribution is concentrated at large values of t, then for small i when $p(i,t)$ is concentrated at small values of t one has the approximation

$$a_i \doteq \frac{1}{\lambda}$$

This will not be true any more for $i = s$, but since most of the contribution in $\sum_{i=0}^{s-1} (s-i)\, a_i$ comes from small values of i the percentage error is tolerable in approximating this expression by $\frac{1}{\lambda} \sum_{0}^{s-1} (s-i) = \frac{s(s-1)}{2\lambda} \doteq \frac{s^2}{2\lambda}$.

In order to determine the optimal s we set the first difference of $\ell(s,D)$ with respect to s equal to zero and obtain the exact condition

$$(1) \quad \sum_{0}^{s-1} a_i = \frac{g\bar{t} - \frac{hD}{\lambda}}{g+h} .$$

Using the (rough) approximation $a_i = \frac{1}{\lambda}$ in (1) to obtain

$$s = \frac{\lambda g\bar{t} - hD}{g + h}$$

we see that s is of the order of $\lambda\bar{t}$, the average demand in the least time, the interval between placing order and delivery. A better approximation is obtained by observing that

$$a_i = \int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t}[1-Q(t)]\, dt$$

$$= \int_0^\infty \int_0^t \frac{(\lambda t)^i}{i!} e^{-\lambda t}\, dt \cdot q(t)dt$$

$$= \int_0^\infty F(t)\, q(t)\, dt \quad , \text{say} .$$

Replacing the average of a function by the function at the average we have

$$\doteq F(\bar{t}) = \int_0^{\bar{t}} \frac{(\lambda t)^i}{i!} e^{-\lambda t} dt$$

$$= \frac{1}{\lambda} \left[1 - \sum_{i=0}^{i} \frac{(\lambda\bar{t})^i}{i!} e^{-\lambda\bar{t}}\right]$$

$$= \frac{1 - P(i,\lambda\bar{t})}{\lambda}$$

where $P(i,\lambda\bar{t})$ is the cumulative Poisson distribution with a parameter $\lambda\bar{t}$. Thus

$$(1') \quad \sum_{i=0}^{s-1} [1-P(i,\lambda\bar{t})] \doteq \lambda\bar{t} \cdot \frac{g}{g+h} - \frac{hD}{g+h}$$

The cumulative Poisson distribution has been tabulated. [Molina, E.C. "Poisson's Exponential Binomial Limit," New York, Van Nostrand 1952] If the delivery time distribution is known with sufficient accuracy, better values for s may be obtained, of course, by calculating a_i through numerical integration of $\int_0^\infty \frac{(\lambda t)^i}{i!} e^{-\lambda t} [1-Q(t)]dt$.

For example, when q is the exponential distribution $q(t) = \mu e^{-\mu t}$

$$a_i = \frac{\lambda^i}{(\lambda+\mu)^{i+1}} = \frac{a^i}{\lambda+\mu} \quad \text{say}$$

and $$\sum_0^{s-1} a_i = \frac{1}{\lambda+\mu} \cdot \frac{1-a^s}{1-a} = \frac{1-a^s}{\mu} \doteq \frac{s}{\lambda+\mu} \qquad \text{when } \mu << \lambda .$$

To determine D we set the first difference of $\ell(s,D)$ with respect to D equal to zero.

$$h(s+D)(D+\lambda\bar{t}) - hsD - \frac{h}{2} D(D+1) = k\lambda + (h+g)\lambda \sum_0^{s-1} (s-i)\, a_i + g \frac{\lambda^2}{2} (\sigma^2+\bar{t}^2) - \lambda gs\bar{t}$$

This yields

$$D + \lambda\bar{t} = \sqrt{\frac{2k\lambda}{h} + \frac{g\,\lambda^2\sigma^2}{h} + R}$$

Where $R = \frac{h+g}{h} \left((s-\lambda\bar{t})^2 + 2\lambda \sum_{i=0}^{s-1} (s-i)\, a_i - s^2\right)$

Recalling now that s is somewhat larger than $\lambda\bar{t}$ and that $\sum_{i=0}^{s-1} (s-i)a_i$ somewhat smaller than $\frac{s^2}{2\lambda}$ we see that R is very small.

Hence

$$(2) \qquad D + \lambda\bar{t} \doteq \sqrt{\frac{2k\lambda}{h} + \frac{g\lambda^2\sigma^2}{h}}$$

For an approximate value of $S = D+s$ we obtain therefore

$$S \doteq D + \lambda\bar{t} = \sqrt{\frac{2\lambda k}{h} + \frac{g}{h}\lambda^2\sigma^2}$$

If we now observe that the optimal starting stock S is approximately, and that $D + \lambda\bar{t}$ is exactly, equal to the average size of a shipment (since old stocks will be nearly zero at the time of a delivery on the average), the reason becomes clear why these expressions closely resemble the Wilson lot size formula. In fact, this equation agrees with the Wilson formula, when delivery time is of fixed length, for then $\sigma^2 = 0$.

If delivery times are large and average demand not too small the second term under the square root dominates the first and we have the even simpler approximation

$$(2') \qquad D \doteq \lambda\sigma\sqrt{\frac{g}{h}} - \lambda\bar{t}$$

Substituting in (1) and approximating $g + h$ by g

$$(1'') \qquad \sum_{i=0}^{s-1} a_i = \bar{t} - \sigma\sqrt{\frac{g}{h}}$$

$$(1''') \qquad \sum_{i=0}^{s-1} [1-P(i,\lambda\bar{t})] = \lambda\bar{t} - \lambda\sigma\sqrt{\frac{g}{h}}$$

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