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Assignment Problems and the Location of Economic Activities**

by

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"...one of the last set of Tuesdays, you know". Alice was puzzled. "In our country", she remarked, "there's only one day at a time", The Red Queen said, "That's a poor thin way of doing things. Now here, we mostly have days and nights two or three at a time, and sometimes in the winter we take as many as five nights together -- for warmth, you know."

Lewis Carroll,
Through the Looking Glass.

SUMMARY

Two problems in the allocation of indivisible resources are discussed, which can be interpreted as problems of assigning plants to locations. The first problem, in which cost of transportation between plants is ignored, is found to be a linear programming problem, with which is associated a system of rents that sustains an optimal assignment. The recognition of cost of inter-plant transportation in the second problem introduces complications which call for more laborious and largely unexplored computations and which also appear to defeat the price system as a means of sustaining an optimal assignment.

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Assignment Problems and the Location of Economic Activities

1. The allocation of indivisible resources.

There are several important areas of economic analysis in which progress depends on the development of methods for solving or analyzing problems in the efficient allocation of indivisible resources. In the first place, there are "direct" practical decision problems of this type, such as the combining of suitable numbers of machine tools of various kinds within a plant, or the choice of number and sites of dams in river valley development. Furthermore, indivisibilities in the more highly specialized human or material factors of production are at the root of the phenomenon of increasing returns to the scale of production, whether it arises within the plant or firm, or in relation to a cluster of firms through so-called "external economies". Thus, strong "indirect" interest in a better understanding of the effects of indivisibilities derives from the fact that if in an industry increasing returns to scale are still present at a scale of production comparable to total demand in the relevant market, this precludes perfect competition and thus presumably reduces the effectiveness of the price system in efficiently allocating resources. Finally, and more specifically again, the theory of location of economic activities has no chance of explaining such interesting realities as large and small cities without recognizing indivisibilities in the processes of production, and of mere existence.

In the light of the practical and theoretical importance of indivisibilities, it may seem surprising that we possess so little in the way of successful formal analysis of production problems involving indivisible resources. However, the mathematical difficulties that arise in attempts to construct a general theory of allocation of indivisible resources have so far seemed quite formidable. Perhaps the best chance of progress lies in isolating for detailed study a few rather limited but well defined problems, proceeding gradually from crude simplicity to more realistic complexity. The present paper is offered with such a motivation.

2. The linear assignment problem.

A relatively simple problem in the allocation of indivisible resources is that of matching two sets of an equal number n of objects, by making up pairs of objects consisting of one object from each set. Objects belonging to the same set are similar in kind but not identical. For each of the n^2 possible pairs a score or value is given. The problem is to find a matching (or assignment to each other) of objects for which the sum of the scores of pairs matched is as high as possible.

There are a variety of practical decision problems of which this is an idealization. The problem of assignment of persons to jobs or job categories on the basis of performance scores in psychological tests was discussed originally by Thorndike [1950] in these terms. Another rather elementary interpretation, brought to our attention by I. N. Herstein, is that of assigning cabinets to desks in a room in such a way that the sum of the walking distances between matched desks and cabinets is minimized.

Because of an underlying interest in location theory, we shall here discuss the problem in terms of assigning industrial plants to locations. Each plant, still on the drawing board, is supposed to have a given expected profit in each location, different locations having different suitabilities for the production processes to be carried out. Transportation costs of primary inputs or final outputs to or from the location in question may also enter into the profitability comparisons. However, for the moment we rule out any consideration of transportation of intermediate commodities between plants to be assigned, or any other circumstances that could make the profitability of any plant at any location depend on the manner in which the remaining plants and locations are matched. The problem then is to find an assignment that makes the sum of the profits obtainable from all plant-location combinations selected as large as possible. Needless to say, the problem is fully defined by its mathematical formulation, given below, independently of the locational interpretation that interests us in particular.

The profitabilities of the n^2 pairs can be set out in the form of a square matrix, the typical element a_{ki} representing the profit expected from the operation of plant k in location i. A possible <u>profitability matrix</u> of the order $n = 4$, that is, referring to 4 plants and locations, would be

$$
\text{(2.1)} \qquad
\begin{array}{c|cccc|}
 & \multicolumn{4}{c}{\underline{\text{locations}}} \\
\underline{\text{Plants}} & 1 & 2 & 3 & 4 \\
1 & 25 & \boxed{20} & 5 & 19 \\
2 & 18 & 3 & \boxed{0} & 12 \\
3 & \boxed{22} & 4 & 2 & 12 \\
4 & 16 & 7 & -2 & \boxed{10}
\end{array}
= \begin{bmatrix} a_{ki} \end{bmatrix} = A
$$

The framed entries represent a maximal assignment, with a total profit of 52 units, as can readily be checked by the reader. Note that the most profitable pair, plant 1 in location 1, does not occur in a maximal assignment. Prescribing it would diminish the maximum profitability attainable by suitable assignment of the other three plants to 46 units.

The unknown assignment with which our problem is concerned can itself be represented by a so-called <u>permutation matrix</u>. This is a matrix $P = [p_{ki}]$ of which each row and each column contains a single element 1, while all other elements are 0. The particular permutation matrix that represents the solution indicated in (2.1) to that problem is

$$
(2.2)\qquad \begin{array}{c|cccc|}
 & \multicolumn{4}{c}{\underline{\text{locations}}} \\
 & 1 & 2 & 3 & 4 \\
\underline{\text{plants}} & & & & \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
3 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1
\end{array} = \left[\hat{p}_{ki}\right] = \hat{P} .
$$

Each row shows the location assigned to the corresponding plant by the place of the unit element. The profit from that assignment can be written as a double sum

$$
(2.3)\qquad \pi = \sum_{k,i=1}^{4} a_{ki}\,\hat{p}_{ki} = 22 + 20 + 0 + 10 = 52,
$$

in which the positions of the unit elements in $\hat{P}$ indicate which elements of A are included in the addition. We note in passing that the permutation obtained from $\hat{P}$ by interchange of the locations of plants 2 and 4 also presents a solution to the problem. We also note that the double sum formula $\sum_{ki} a_{ki}\, p_{ki}$ can of course serve to indicate the total profit from the selection of any other permutation P. (From here on we shall use the terms "permutation" and "permutation matrix" as interchangeable.)

The mathematical formulation of our problem then is to find a permutation matrix $\hat{P} = [\hat{p}_{ki}]$ of the same order n as the given matrix A, such that (omitting summation limits $k, i = 1, \ldots, n$)

$$
(2.4)\qquad \sum_{k,i} a_{ki}\, p_{ki} \leqq \sum_{k,i} a_{ki}\, \hat{p}_{ki}
$$

for all permutation matrices $P = [p_{ki}]$ of that order. The problem is, of course, symmetric as between plants and locations: these two words are interchangeable in all interpretations to be given.

It has been observed by von Neumann [1953] that, if we modify the matrix A of given profitabilities by adding any (positive or negative) constant to any row (or column), both members of (2.4) are increased by that same constant, because exactly one element out of that row (or column) is selected by any permutation for incorporation in the summation. Hence any solution P of the original problem remains a solution of the modified problem. This expresses the obvious fact that, if the conditions of the problem do not permit us to withhold any of the plants from assignment, only the differences in the profitability of each plant between locations matter, not the absolute amounts of profit. For some purposes, it will be convenient to use this freedom to modify the original problem so as to make the profits from all combinations positive.

$$(2.5) \qquad a_{ki} \qquad 0, \quad \text{for} \quad k, i = 1,\ldots,n.$$

Our problem can be looked upon as a pure problem in indivisible resources. We have ruled out any possibility of subsequent choice of continuous variables such as output levels or factor combinations. Hence the only choice studied is to select one out of the $n!$ permutations. This being a finite (though possibly large) number, the brute force method of listing all permutations, evaluating the maximand for each and selecting a permutation with highest value of the maximand is "in principle"available. This consideration led a mathematician approached by Thorndike [1950] to declare the problem trivial. However, there remains considerable mathematical challenge in the problem of finding short cuts that will extend the range of values of the "problem size" n for which we can hope to solve actual problems by computation on available computing equipment from limited budgets. There is also a challenge to the economist in the question whether a price system is possible which will sustain an optimal assignment if locational decisions are made independently by n entrepreneurs, in response to prices, and on the basis of their own knowledge of profitabilities of given production processes in alternative locations.

In both of these aspects, our problem has been decisely helped forward by von Neumann [1953], who observed that a mathematical theorem due to Birkhoff [1946] is the clue to an important simplification of the linear assignment problem. von Neumann used this clue to construct a zero-sum two-person game which is equivalent to the linear assignment problem, and which we shall briefly describe in section 5 below. However, we shall in the next section study more closely an equivalent linear programming problem derived from the same clue.

3. An equivalent linear programming problem.

This problem is obtained by blandly ignoring the indivisibilities of plants, and admitting the assignment of <u>fractional plants</u> to locations in our model, even though this is supposed to be meaningless from a realistic point of view. The profit obtained from a fraction x_{ki} of plant k at location i is "postulated" to be $a_{ki}\, x_{ki}$, that is, that same fraction x_{ki} of what the profit from the entire plant k would be at location i. The earlier assumption that there is no interaction between the profitabilities of plants at different locations is now extended to an "assumption" of no interaction between fractions of plants whether at the same or at different locations. Accordingly, the maximand in this fictitious problem is again of the form

$$\sum_{k,i} a_{ki}\, x_{ki} \,. \tag{3.1}$$

However, the unknowns x_{ki} are no longer restricted to the values 0 or 1. They are subject only to the "milder" restrictions[1/]

1/ One of the restrictions (3.2.1), (3.2.2) is redundant and can be derived from the others by simple additions and subtractions based on the identity

$$\sum_k \sum_i x_{ki} = \sum_i \sum_k x_{ki} \,.$$

$$
(3.2)\begin{cases}
(3.2.1) & \sum_i x_{ki} = 1, & k = 1,\dots, n; \\
(3.2.2) & \sum_k x_{ki} = 1, & i = 1,\dots, n; \\
(3.2.3) & x_{ki} \geqq 0 & k, i = 1,\dots, n.
\end{cases}
$$

The first of these expresses that precisely one plant of each kind is to be assigned. The second expresses that precisely one location of each kind is available, and that a location is fully taken up when the sum of the fractions of all plants assigned to it equals one. The last restriction precludes the assignment of negative amounts of plant.

Consider the set R of points in the Euclidean space with n^2 coordinates x_{ki}, $k,i = 1,\dots,n$, that satisfies the restrictions (3.2). It is the geometrical image of the set of all possible assignments, in integral units or by fractions, of n plants to n locations. It is formed by the intersection of 2n-1 hyperplanes (of n^2-1 dimensions each) with n^2 halfspaces. Since the resulting set is bounded (each x_{ki} is necessarily wedged between 0 and 1 as attainable lower and upper bounds), the set R is a convex polyhedron. Birkhoff's theorem identifies all its vertices. It establishes that these are precisely the n! permutation matrices[2/] $[p_{ki}]$. Thus for n = 2, the set R is a line segment with endpoints in (Small print)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ For n = 3, it is a four-dimensional polyhedron with vertices in the six points

2/ For a proof see Appendix A. For mathematical discussion and references relating to the linear assignment problem, see Motzkin [195..].

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},$$

respectively.

It is intuitively obvious, and not hard to prove[3/], that a linear function defined on a convex polyhedron will reach its maximum in a vertex. If this maximum is not reached in any other vertex, then it is not reached in any other point of the polyhedron either. If it is reached in more than one vertex, then it is also reached in all points of a "face" of the polyhedron, that is, a polyhedron contained in the boundary[3a/] of the original polyhedron, and having as its vertices all those vertices of the original polyhedron where the function reaches its maximum. The two cases are illustrated in Figure 1.

[end of small print]

These simple facts of geometry clarify the relations between the linear assignment problem, and the linear programming problem (3.1), (3.2) that we have substituted for it. Any solution of the former is necessarily a solution of the latter. If the former problem has only one solution (one optimal assignment), then this is also the unique solution of the latter. If the former has more than one solution, then the latter has as its solution all points of a polyhedron which is spanned by the solutions of the former as vertices. Thus, by permitting fractional assignments, we have not lost from the set of solutions any of the integral,(i.e., non-fractional) assignments that solve our problem.

3/ See, for instance, Koopmans [1951], p. 88, footnote 17.

3a/ Provided the given linear function is not a constant.

Even so, the reader will insist on knowing what is the advantage of the enlargement of our problem obtained by admitting fractional assignments. Of the two important advantages, the first was pointed out by von Neumann. A great simplification in the computation of solutions is obtained, which brings a large class of problems of respectable magnitude within the power of available computation equipment.

The straightforward, brute force method requires the evaluation of the function in (2.4) in each of $n!$ permutations. For $n = 10$, the number $n!$ is about 3.6 million. On the other hand, the linear programming problem (3.1), (3.2) contains n^2 unknowns (100 if $n = 10$) subject to $2n - 1$ restraints (19 if $n = 10$). Moreover, as was observed by Votaw and Orden [1952], this linear programming problem is an especially simple representative of the category of transportation problems [Koopmans and Reiter, 1951], itself a category with special features that permit solution by a straightforward algorithm [Dantzig, 1951].

The transportation problem is the problem of shipping a homogeneous commodity, available in given amounts e_k at geographical points E_k, $k=1,\dots,m$, respectively, to meet stated requirements f_i at points F_i, $i=1,\dots,n$. If $d_{ki} > 0$ represents the cost of transporting one unit of the commodity from E_k to F_i, and if transportation cost on each route is proportional to amount shipped, then the transportation problem can be written as the minimizing of

$$(3.3) \qquad \sum_{k=1}^{m} \sum_{i=1}^{n} d_{ki} \, x_{ki}, \quad \text{where} \quad d_{ki} > 0,$$

subject to the restrictions

$$(3.4) \qquad \begin{cases} \sum_{i=1}^{n} x_{ki} = e_k, & \text{where } e_k \geqq 0, \; k = 1, \dots m, \\ \sum_{k=1}^{n} x_{ki} = f_i, & \text{where } f_i \geqq 0, \; i = 1, \dots, n, \\ x_{ki} \geqq 0, & k, i = 1, \dots, n. \end{cases}$$

These restrictions can be satisfied only if $\sum_{k=1}^{m} e_k = \sum_{i=1}^{n} f_i$.

The special case of the linear (fractional) assignment problem is now obtained by making the number of sources equal to the number of receiving points $(m = n)$, the amounts associated with all sources and receiving points equal[4] $(e_k = f_i = 1; \; k,i = 1,\ldots,n)$, and setting[5] $a_{ki} = -d_{ki}$.

The second advantage that can be derived from the consideration of fictitious fractional assignments is the subject of the next section.

4. A price system associated with a solution of the linear assignment problem.

It is well known that with each solution of a linear programming problem (and more generally with each efficient point in a linear activity analysis problem[6]) one can associate a price system for the commodities involved, which has the property of preserving the optimal (or efficient) point under decentralized profit-maximizing decision-making. The underlying mathematical fact can be put in the form of the Minkowski-Farkas lemma for linear inequalities [see Gale, Kuhn and Tucker, 1951], which in turn can be derived from a separation theorem

4/ In the language of this interpretation, Birkhoff's theorem becomes very plausible. It says that, if the availabilities at all sources and the requirements of all receiving points are equal, one can always find a cheapest routing that utilizes precisely one source for each receiving point.

5/ If we also wish to satisfy (2.5), we must set $a_{ki} = A - d_{ki}$, where A is a sufficiently large number.

6/ See Koopmans [1951].

for convex sets [see, for instance, von Neumann and Morgenstern, 1947, sec. 16.3]. In the case of the transportation model (and hence also of the linear assignment problem), a simple constructive proof of the properties of such a price system can also be given [Koopmans and Reiter, 1951]. Here we shall merely state the proposition in question as applied to the linear assignment problem, and derive it from the Minkowski-Farkas lemma in Appendix B.

Let an optimal assignment be given, and let plants and locations be renumbered in such a way that in that optimal assignment each plant is matched with the location bearing the same number. Then, so says our theorem, there exists a system of rentals q_k, $k = 1,\ldots,n$, on all plants and r_i, $i = 1,\ldots,n$, on all locations such that

$$(4.1)\quad \begin{cases} (4.1.1) \quad a_{kk} = q_k + r_k, & k = 1,\ldots,n, \\ (4.1.2) \quad a_{ki} \leq q_k + r_i, & k, i = 1\ldots,n. \end{cases}$$

Conversely, if such a system of rentals exists, then the matching of plants to locations bearing the same number is an optimal assignment.

(The latter, converse, statement does not depend on the Minkowski-Farkas lemma. It follows directly from the fact that, for any permutation matrix p_{ki}, if (4.1) holds.

$$\sum_{k,i} a_{ki}\, p_{ki} \leq \sum_{k,i} (q_k + r_i)\, p_{ki} =$$

$$= \sum_k q_k \left(\sum_i p_{ki}\right) + \sum_i r_i \left(\sum_k p_{ki}\right) = \sum_k q_k + \sum_i r_i = \sum_k (q_k + r_k) = \sum_k a_{kk},$$

which is the value of the maximand for the "k-to-k" assignment assumed optimal).

The first condition (4.1.1) states that the profit from each plant-location pair in an optimal assignment can be split into two parts, one a rental imputed to the plant, the other a rental imputed to the location.

The second condition (4.1.2) then states that the rental imputed to a plant is the highest amount that could be "earned" by this plant in any location, if its share is computed by subtracting from the profit it can make in that location the rental imputed to that location. Symmetrically, the rental imputed to a location is the highest income that could be secured for this location by attracting any plant to it and subtracting the rental imputed to that plant from the profit so obtained. Assuming rents to be known and regarded as given, the effect of rents with the properties stated would be that no plant owner or landlord would be better off with any location or tenant plant other than that with which he is provided under the optimal assignment in question. In this sense, that assignment may be said to be sustained by a market mechanism operating through profit-maximizing response to a system of rentals.

The price conditions (4.1) that permit this interpretation as an optimum-sustaining market mechanism are stated in complete symmetry as between plants and locations. However, response to rents on only one side of the market is sufficient for the price system to operate. Suppose for instance that plants are not available for rent, but are owned by entrepreneurs who decide on their location. Then, only location rents need to be quoted. Conditions equivalent to (4.1) in terms of location rents alone are easily found to be

$$(4.2) \qquad a_{ki} - a_{kk} \leq r_i - r_k, \qquad k, i = 1, \ldots, n.$$

These are derived from (4.1) by elimination of the q_k. Conversely, if location rents r_k are found that satisfy (4.2), then these same rents supplemented by plant rents defined through $q_k = a_{kk} - r_{kk}$ will satisfy (4.1). It follows that profit-maximizing plant owners responding to location rents r_k satisfying (4.2) will have no reason to depart from the "k-to-k" assignment, and that this is an optimal assignment. No plant rents need enter into their comparisons.

An important characteristic of the market mechanism just described is the way in which its information requirements are distributed. Each plant owner needs to know only the rents on locations, and the profitabilities of his own plant in each location. The latter information (his own row in the matri $[a_{ki}]$) is likely to be more accessible to him than to anyone else. The location rents may thus be looked upon as a condensation of the information present in the entire matrix of profitabilities a_{ki} to that smaller number of data that is sufficient, in combination with his own row of the matrix, to enable each plant owner to hold his place in an optimal assignment. We meet here with the informational decentralization made possible by a price system, which has been stressed by many economists as one of the main merits of resources allocation through competitive markets.[7/]

It should be added that we have here established only that a competitive market can sustain an optimal assignment once it and an associated system of rents have been established, and that such a market cannot sustain any non-optimal assignment. Whether a competitive market can find an optimal assignment through a process of alternating adjustments in prices and in choices of locations cannot be answered without specify-dynamic characteristics of the market processes in question. Since this subject is outside the scope of the present article, we shall merely remark here that there is a possible parallel between the iterative computation methods for the transportation problem, referred to above, and the market adjustment processes mentioned here.

Owning to special features of the present problem, the imputation of rents to plants and locations under an optimal assignment is by no means unique, even if there is only one optimal assignment. In the first place any constant (positive or negative) amount Δ may be taken out of all plant rents and added to all location rents.

7/ See, for instance, F. A. Hayek [1945].

$$(4.3) \qquad q_k^* = q_k - \lambda \qquad r_i^* = r_i + \lambda, \quad k,i = 1,\ldots\ldots,n,$$

and the new rentals will satisfy the conditions (4.1) if the old ones did. Moreover, within limits imposed by the consideration that no non-optimal pair should be allowed to become profitable after payment of rents, similar transfers can often be made between the two rentals within each individual pair in an optimal assignment.

$$(4.4) \quad q_k^* = q_k - \lambda_k, \quad r_k^* = r_k + \lambda_k, \quad a_{ki} \leqq q_k - \lambda_k + r_i + \lambda_i, \quad k,i = 1,\ldots\ldots,n,$$

without violating the conditions. Thus, allocative considerations alone do not fully determine the price system in the present case. Such ranges of indeterminacy can be expected in problems where indivisible resources have only a finite number of alternative uses. In the present model, the indeterminacies are increased by the fact that the number of plants and locations is evenly matched, so that no competition from unused resources brings one or more prices down to zero. At the same time, the indeterminacies noted leave the present model adaptable for embedding in various more general models that recognize alternative uses for plants in a given location, or alternative methods for manufacturing the plants themselves from more basic scarce resources, etc.

By slightly sharper reasoning, it can be shown that if all profitabilities a_{ki} are positive, then a system of non-negative rents q_k, r_i can always be found that meets the price conditions (4.1). To obtain this conclusion, one permits the withholding of fractions of plants and of locations from assignment. The restrictions (3.2) then take the more inclusive form

$$(4.5) \begin{cases} (4.5.1) & \sum_i x_{ki} \leqq 1, \quad k = 1,\ldots,n, \\ (4.5.2) & \sum_k x_{ki} \leqq 1, \quad i = 1,\ldots,n, \\ (4.5.3) & x_{ki} \geqq 0, \quad k,i = 1,\ldots,n. \end{cases}$$

Now since every plant in every location brings some profit, an optimal assignment under these restrictions will in fact completely assign all plants and locations. Hence the enlarging of the restraint set (4.5) does not add any new solutions. However, the Minkowski-Farkas lemma applied to the new restrictions (4.5) implies that a system of non-negative rents satisfying the price conditions (4.1) can be associated with an optimal assignment.

Before concluding the discussion of optimizing price systems, we should point out that, where no market mechanism exists or can be created, a price system as described can still be an important aid in computation of an optimal assignment. It further reduces the number of unknowns from n^2 quantities x_{ki} to only 2n rents q_k, r_i. These rents can be looked upon as numbers which, by subtraction from the corresponding rows and column, respectively, of the matrix $[a_{ki}]$ of profitabilities, produce an equivalent assignment problem, characterized by a matrix

$$(4.6) \qquad a^*_{ki} = a_{ki} - q_k - r_i,$$

of which the solution can be read off immediately by selection of a maximal element from each row and column. To illustrate this idea, prices associated with the solution of the problem (2.1) there indicated are

$$(4.7) \qquad [q_k] = [10 \quad 3 \quad 6 \quad 1], \qquad [r_i] = [16 \quad 10 \quad -3 \quad 9],$$

and the matrix of the equivalent problem is (with a solution marked)

$$(4.8) \qquad [a^*_{ki}] = \begin{bmatrix} -1 & \boxed{0} & -2 & 0 \\ -1 & -10 & \boxed{0} & 0 \\ \boxed{0} & -12 & -1 & -3 \\ -1 & -4 & 0 & \boxed{0} \end{bmatrix}$$

Algorithms based on this principle have been discussed by Tornquist [1953] for both the linear and the quadratic assignment problem.

Finally, we register a few straightforward yet interesting implications of the inequalities (4.2) for the connections between location rent differences and the profitability differences between locations encountered by the plants. In the first place, we have

$$(4.9) \qquad a_{ki} > a_{kk} \text{ implies } r_i > r_k ,$$

that is, if location i is more profitable for plant k than the location k to which it is optimally assigned, then the rent of location i exceeds that of location k. This obvious statement is as yet of little help unless the optimal assignment is known. However, it implies a weaker statement that does not depend on what is the optimal assignment:

$$(4.10) \qquad a_{ji} > a_{ki} \text{ for all } j \text{ implies } r_i > r_k .$$

In words, if location i is more profitable than location k for every plant, the rent of location i must exceed that of location k.

Secondly, again from (4.2),

$$(4.11) \qquad r_i < r_k \text{ implies } a_{ki} < a_{kk} ,$$

that is, if location k rents higher than location i, the plant optimally assigned to location k is more profitable there than at location i. In particular, the plant assigned to the location that rents highest of all (if there is a single such location) is more profitable there than at any other location. A similar statement holds for the plant assigned to the next highest renting location in comparison with all locations renting still lower. It follows that, if plants are ordered by descending rents of their optimal locations, the optimal assignment can be reconstructed by first locating the first plant so as to maximize its profitability, next locating the second plant so as to maximize its profitability among the remaining locations, etc. One can therefore also look upon the ordering of plants by descending rent of their optimal locations as the unknown of the assignment problem.

Finally, (4.2) permits us to place lower and upper bounds on the range of location rents without requiring knowledge of the optimal assignment. In the first place, from (4.2)

$$\min_{j} \max_{i,k} (a_{ji} - a_{jk}) \leqq \max_{i,k} (a_{ki} - a_{kk}) \leqq \max_{i,k} (r_i - r_k)$$

Secondly, again from (4.2) after interchanging i and k and changing signs,

$$\max_{i,k} (r_i - r_k) \leqq \max_{i,k} (a_{ii} - a_{ik}) \leqq \max_{j} \max_{i,k} (a_{ji} - a_{jk}) .$$

Taking these relations together,

$$(4.12) \quad \min_{j} \max_{i,k} (a_{ji} - a_{jk}) \leqq \max_{i,k} (r_i - r_k) \leqq \max_{j} \max_{i,k} (a_{ji} - a_{jk}) .$$

we see that the range of location rents is comprised between the smallest and the largest of the profitability ranges encountered by individual plants as between different locations.

Of course, all of the relations (4.9) - (4.12) can be translated into corresponding relations between plant rent differences and profitability differences as between alternative plants, encountered by the locations.

5. Von Neumann's equivalent game.

The decisive step in the reduction of the linear assignment problem to manageable proportions for moderate values of n is the application of Birkhoff's theorem. Beyond that several roads are open, of which we have followed one leading to an equivalent linear programming problem. Von Neumann [1953] has chosen another road leading to an ingenious zero-sum two-person game which is likewise equivalent to the assignment problem. We shall briefly describe this game (and its solution) because there is a connection between it and the price system discussed above. This section can be passed over by readers unfamiliar with the theory of games.

The rules of the game are as follows: Player I selects a field (k,i) in a checkerboard of n rows and n columns and communicates his selection to a referee. Player II guesses either the row k or the column i in which this field is found, indicating also whether he is guessing by row or by column. If the guess is correct, player II receives a payoff

$$\frac{1}{\bar{a}_{ki}} \tag{5.1}$$

from player I (it is presupposed that [2.5] has been satisfied). If the guess is wrong, there is no payment.

The solution of the game is as follows: Player I chooses a strategy of selecting only fields corresponding to an optimal assignment in the linear assignment problem defined by the matrix a_{ki}, and selecting any given field occurring in that assignment with a probability proportional to $\frac{1}{\bar{a}_{ki}}$. If more than one optimal assignment exists, he may choose any probability mixture of the strategies of this type associated with optimal assignments. The value of the game to player II equals the reciprocal

$$(5.2) \qquad \frac{1}{\sum_{k,i} a_{ki} \hat{p}_{ki}} ,$$

of the maximal value of the sum of scores attained in an optimal assignment $\hat{p}_{ki}$. Player II chooses a strategy of selecting row k and column i with probabilities proportional to any set of non-negative prices q_k , r_i associated, by (4.1), with any one or more optimal assignments.8/

6. The quadratic assignment problem.

For an understanding of the complexities of locational decisions, both in reality and from an optimizing point of view, the assumption that the benefit from an economic activity at some location does not depend on the uses of other locations is quite inadequate. The literature contains many references to "direct" interactions, such as the benefits of improvements extending to adjacent lots, or the detrimental effects of noise, vibration, and air or water pollution stemming from surrounding activities9/. Of these direct interactions nothing more will be said here except the obvious remark that they often tend to favor the conglomeration of similar activities in the same neighborhood. They have drawn attention in the literature mainly as examples of discrepancies between social cost (or benefit) and private cost which cause failure of the price system as a mediator of efficient allocative decisions.

8/ This statement, not contained in Von Neumann's article, was communicated to one of the authors by Von Neumann. Its verification is left to the reader.

9/ See, for instance, A. Pigou [1920] Chapter IX of Part II (4th ed.).

Our main point in this discussion is that one does not even need to look for these phenomena of "direct" physical interaction between production and/or consumption processes to find reasons for such failure of the price system. The mere fact that scarce resources need to be utilized for the transportation of intermediate commodities between plants appears to be sufficient to deprive the price system of its ability to induce or preserve efficient decentralized allocative decisions. In order to press this point, we shall now introduce the quadratic assignment problem.

We again consider n plants and n locations, and a matrix a_{ki} of which the element a_{ki} now represents a "semi-net" revenue from the operation of plant k at location i, that is, gross revenue less cost of primary inputs, but before subtracting cost of transportation of intermediate products between plants. Having adopted this definition, we maintain the assumption that "semi-net" revenue a_{ki} is independent of the assignment of other plants to other locations.

In order to express interplant transportation cost, let a set of non-negative numbers $b_{k\ell}$, $k \neq \ell$, $k,\ell = 1,\ldots,n$, represent required commodity flows (in weight units) from plant k to plant ℓ, and a set of positive numbers c_{ij}, $i \neq j$, $i,j = 1,\ldots,n$, represent the cost of transportation for the unit flow from location i to location j. The flow coefficients $b_{k\ell}$ are assumed independent of the locations assigned, and the transportation cost coefficients c_{ij} are assumed independent of the plant assignments and applicable to all amounts and compositions of flows.

If each plant were assigned to the location bearing the same number, total interplant transportation cost would be given by

$$(6.1) \qquad \sum_{k,\ell} b_{k\ell} \, c_{k\ell} \, ,$$

provided we set

$$(6.2) \qquad b_{kk} = 0, \quad c_{kk} = 0, \quad k = 1,\ldots,n.$$

For any other assignment, each $b_{k\ell}$ must instead be multiplied, before the summation, with that c_{ij} connecting the locations to which the plants k and ℓ are assigned, respectively. If the assignment in question is defined by the permutation matrix $[p_{ki}]$, it is readily seen that this leads to the expression

$$(6.3) \qquad \sum_{k,\ell} \sum_{i,j} b_{k\ell} \; p_{ki} \; c_{ij} \; p_{\ell j} \, .$$

It follows that total net revenue for the agglomeration of plants is represented by

$$(6.4) \qquad \sum_{k,i} a_{ki} \; p_{ki} \; - \sum_{k,\ell} \sum_{i,j} b_{k\ell} \; p_{ki} \; c_{ij} \; p_{\ell j} \, .$$

The quadratic assignment problem is the problem of maximizing this expression by suitable choice of a permutation matrix $[p_{ki}]$. It is called quadratic[10/] because the maximand contains a term of second degree in the unknown permutation.

The computational difficulties of finding a solution of this problem for moderate values of n (say $n = 10$) have so far been unsurmountable.

10/ There is some arbitrariness in this designation. A given discrete problem can often be converted to different mathematical forms depending on the choice of the space in which the given discrete alternatives are embedded in order to make methods depending on continuity applicable. Thus, in Section 8, we shall encounter an equivalent formulation of the "quadratic" assignment problem which is "linear" but for one extra non-linear restriction on the values of the unknowns.

To our knowledge, the only sizable example that has been computed through so far relates to a special case of the quadratic assignment problem, known as the traveling salesman problem. This case is obtained by setting $a_{ki} = 0$ and taking

$$(6.5) \qquad b_{k\ell} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & & & . \\ 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & 0 & . & . & 0 \end{bmatrix},$$

which is itself a permutation matrix. The one and only "intermediate commodity" now is a traveling salesman who is required to call once at each location and return to his point of departure. The assignment problem is the problem of so choosing the order of points of call that the total cost of transportation of the salesman for the tour is minimized. Dantzig, Fulkerson and Johnson [1954] computed the minimum cost tour through forty-nine cities in the United States, one in each state and Washington, D. C. They equated cost with road distance, and proved the validity of their solution. The methods used were ad hoc methods that were rewarding in the problem at hand, but do not necessarily carry over to other cases of the quadratic assignment problem.

In view of the computational complexity of the quadratic assignment problem, the question of the possibility of a price system preserving an optimal assignment is very pertinent. The unavailability of a practicable computation method is less of an obstacle to good use of resources if market processes can be relied upon to sustain, and possibly even to lead to, an optimum assignment. Unfortunately, our tentative and heuristic exploration into this problem, to be reported on in the next section, also meets with a negative outcome.

7. Can optimal interdependent locational decisions be sustained by a price system?

By analogy with the functioning of a price system in the linear assignment problem and in the transportation problem, it is of interest to examine whether an optimal assignment in the quadratic assignment problem could be sustained by a system of rents on locations and/or plants, and of prices on each intermediate commodity that depend on the location at which the commodity is quoted.

By a line of reasoning that may also have some interest in itself, one can disprove this possibility for what might appear at first sight to be the most natural way of defining the operation of such a price system. This reasoning again utilizes the device of fictitious fractional plants sharing locations.

We assume that the "semi-net" revenue from a given fractional plant k in a given location i, as well as the input and output flows of intermediate goods to and from that plant, are proportional to the size x_{ki} of the fraction. To simplify matters further, we assume that the commodity (or commodity bundle) which flows from plant k to plant ℓ is specific to this pair of plants. It, or any of its component commodities, cannot be drawn from any type of plant other than (fractions of) plant k, and cannot be used by any type of plant other than ℓ. We shall therefore speak of this commodity bundle as a single commodity. It is fully distinguishable by the combination of indices (k,ℓ), in that order.

Within the stipulation given, we allow and indeed require most economic routing of each distinct commodity (k,ℓ) between fractional plants. In particular, if in keeping with (6.2) we regard transportation between fractional plants in the same location as costless, this will favor the assignment to the same location of combinations of fractional plants that supply each other with intermediate commodities.

Let us denote by $x_{k\ell,ij}$ the flow from location i to location j of the commodity which is supplied by plant k to plant ℓ. To avoid irrelevant in determinacy, we shall say that $x_{k\ell,ii} = 0$. Then our problem has become that of maximizing

$$(7.1) \qquad \sum_{k,i} a_{ki}\ x_{ki} - \sum_{k,\ell} \sum_{i,j} c_{ij}\ x_{k\ell,ij}$$

by suitable choice of the x_{ki} and $x_{k\ell,ij}$ subject to the restrictions

$$(7.2) \begin{cases} (7.2.1) & x_{ki}\, b_{k\ell} + \sum_j x_{k\ell,ji} = x_{\ell i}\, b_{k\ell} + \sum_j x_{k\ell,ij}, \quad k,\ell,i = 1,\ldots,n, \\ (7.2.2) & \sum_i x_{ki} = 1, \qquad k = 1,\ldots,n, \\ (7.2.3) & \sum_k x_{ki} = 1, \qquad i = 1,\ldots,n. \\ (7.2.4) & x_{ki} \geqq 0, \quad x_{k\ell,ij} \geqq 0, \quad x_{k\ell,ii} = 0, \quad k,\ell,i,j = 1,\ldots,n. \end{cases}$$

The restrictions (7.2.1) specify that the total inflow of the intermediate commodity (k,ℓ) to the location i added to its production at that location equals the total outflow from plus its consumption at that same location. The other restrictions are the same as before, except that (7.2.4) also specifies the non-negative character of intermediate commodity flows.

It is perhaps worth pointing out that the linear problem defined by (7.1) and (7.2) becomes equivalent to the quadratic assignment problem by adding the simple non-linear restriction

$$(7.3) \qquad x_{ki} = 0 \text{ or } 1, \qquad k,i = 1,..,n.$$

However, if we do not add such a restriction, then the problem is truly linear, and has all the general properties of linear programming problems. Above we have used the property that with a solution of a linear programming problem is associated a system of prices which meets certain conditions that permit the solution to be sustained by decentralized profit-maximization. However, we have also noted a converse property: if with a point (a set of numbers x_{ki}, $x_{k\ell,ij}$) satisfying the restrictions (7.2) one can associate a price system that meets the conditions (still to be enumerated), then this point is a solution to the problem[11/]. In particular, we shall make use of this property in its equivalent negative form: if a point satisfying the restrictions is not a solution to the problem, then there exists no set of prices that meets the conditions in question in association with this point.

Now we can readily specify one particular case in which the nature of the solution(s) of the linear problem (7.1), (7.2) we have substituted for the quadratic assignment problem can be seen directly. This case is obtained by specifying that the "semi-net" revenue from a given plant is independent of its location,

$$(7.4) \qquad a_{ki} = a_k , \qquad i = 1,\dots,n.$$

Under this assumption, the first term in the total profit (7.1) becomes, by (7.2.2) ,

$$(7.5) \qquad \sum_{k,i} a_k \; x_{ki} = \sum_{k} \left(a_k \sum_{i} x_{ki}\right) = \sum_{k} a_k ,$$

a constant independent of the assignment selected. The maximization of total profit thus becomes equivalent to the minimization of total transportation cost,

$$(7.6) \qquad \sum_{k,\ell} \; \sum_{i,j} c_{ij} \; x_{k\ell,ij} ,$$

under the restrictions (7.2) on the assignment of fractional plants.

An obvious solution to this problem is to distribute each plant in equal fractions over all locations,

11/ The proof of this property in the present case follows the same line of reasoning as that given in the paragraph after (4.1).

$$(7.7) \qquad x_{ki} = \frac{1}{n}, \quad x_{k\ell,ij} = 0, \quad i \neq j, \quad k,\ell,i,j = 1,\dots,n.$$

In this case there is no need for transportation, hence no cost thereof.

If every flow coefficient $b_{k\ell}$ with $k \neq \ell$ is positive, this is the only solution. If some $b_{k\ell}$ are zero, additional solutions arise, but as long as at least one of the $b_{k\ell}$ is positive, no integral assignment of plants to locations can be a solution, because the positive coefficient $b_{k\ell}$ would be multiplied by one of the transportation cost coefficients c_{ij}, $i \neq j$, all of which are positive. In particular, any solution of the quadratic assignment problems defined by (6.4) and (7.4), being by definition an integral assignment, is not a solution of the linear problem (7.6), (7.2), hence does not have associated with it a price system meeting the conditions which we shall now write down.

The conditions in question follow from the form of the mimimand and of the restrictions (7.2). With each of the restrictions (7.2.1-3) we associate a price for that commodity of which the conservation is expressed by this restriction. We shall use $u_{k\ell,i}$ for the price of the commodity (k,ℓ) at location i, q_k for the rental of the unit of plant k, and r_i for the rental of the unit of location i. We shall again number plants so that assigning each plant as a whole to the location bearing the same number is an optimal integral assignment. The price conditions which as we have shown can <u>not</u> be satisfied then are

$$(7.8.1) \qquad c_{k\ell} = u_{k\ell,\ell} - u_{k\ell,k}, \qquad k \neq \ell,$$

$$(7.8.2) \qquad c_{ij} \geqq u_{k\ell,j} - u_{k\ell,i}, \qquad i \neq j,$$

$$(7.8.3) \qquad (a_{kk} =)\ a_k = q_k + r_k + \sum_{\ell} b_{k\ell}\, u_{k\ell,k} - \sum_{\ell} b_{\ell k}\, u_{\ell k,k},$$

$$(7.8.4) \qquad (a_{ki} =)\ a_k \leqq q_k + r_i + \sum_{\ell} b_{k\ell}\, u_{k\ell,i} - \sum_{\ell} b_{\ell k}\, u_{\ell k,i},$$

$$k,\ell,i,j = 1,\dots,n.$$

Condition (7.8.1) says that, on the one route ($i=k$, $j=\ell$) on which transportation of the commodity (k,ℓ) is called for in the optimal (integral) assignment in question, the price difference between the point ℓ of requirement and the point k of availability equals the transportation cost. Similarly, (7.8.3) says that for each plant k, the semi-net revenue it can make anywhere just suffices to pay rents on that plant and on the location to which it is optimally assigned, after allowing for the proceeds of the sale of its intermediate outputs and the cost of purchase of its intermediate inputs, evaluated again at the prices quoted for the location to which the plant in question is (optimally) assigned. Condition (7.8.2) says that price quotations on each commodity in all locations are subject to the restrictions that no price difference between two locations shall exceed the transportation cost. Finally, condition (7.8.4) says that no plant can make a positive profit in any location.

Let us recall again that we have shown that these conditions can not be satisfied in an optimal integral assignment, or for that matter in any integral assignment. The consequences of this negative conclusion seem to us to be far-reaching. It means that no price system on plants, on locations and on commodities in all locations, that is regarded as given by plant owners, say, will sustain any assignment. There will always be an incentive for someone to seek a location other than the one he holds. In the case of plants on the drawing board, compatible choices cannot be induced or sustained by such a price system. In the case of actual establishments already located, the cost of moving is the only element of stability. Without this brake on movement, there would be a continual game of musical chairs. Whatever the assignment, prices of intermediate commodities and rents on locations cannot be so proportioned as to give no plant an incentive to seek a location other than the one it holds.

It might be objected that we have ourselves created this difficulty by asking for a price system that discourages not only what is inefficient, but also what is anyhow impossible. We have asked for a price system that would discourage plant owner k from desiring to change to a location now held by plant owner ℓ, without inquiring whether the latter has a similar inducement to make this possible by another move of his own.

Before looking further into this objection, it should be recalled that, in the linear assignment problem, the price system does not invite incompatible choices. In case only one optimal assignment exists, prices can be found such that alternative choices are inferior for all plant owners concerned. If more than one optimal assignment exists, there is always a set of compatible choices such that each plant owner's choice is not inferior, for him, to any alternative. This expresses the fact that efficiency prices in linear activity analysis in general, and in linear programming in particular, not only discourage inefficient use of resources, but also remove the incentive for anyone to claim more of a scarce resource than is available, given the amounts already allocated to others. Hence, if we should have made our problem more difficult by requiring prices to administer scarcities in addition to discouraging inefficiencies, this must be due to the non-linear character of the present problem.

That this is at least part of our difficulty, becomes clear by looking at the very simplest problem of interplant transportation cost minimization for just $n = 2$ plants and locations. If the initial assignment is "k-to-k" and if positive shipments $b_{k\ell}$, $k \neq \ell$, go in both directions between the two plants, (7.8.1) becomes

$$(7.9) \qquad c_{12} = u_{12,2} - u_{12,1}\,, \qquad c_{21} = u_{21,1} - u_{21,2}\,.$$

The relevant terms in the profit of plant 1 are, at location 1,

$$(7.10) \qquad u_{12,1} - u_{21,1} - r_1$$

and, at location 2,

$$(7.11) \qquad u_{12,2} - u_{21,2} - r_2 .$$

The profit at location 1 exceeds that at 2 by

$$(7.12) \qquad u_{12,1} - u_{12,2} - u_{21,1} + u_{21,2} - r_1 + r_2 = c_{12} - c_{21} - r_1 + r_2$$

This expresses the simple fact that, if a move of plant 1 to location 2 is evaluated at prices corresponding to the initial "k-to-k" assignment taken as given, its effect on the profit of plant 1 is the same as if plant 1 were "doubling up" with plant 2 in location 2. The entire interplant transportation cost is then saved. It is, of course, possible to offset this for plant 1 by a large rent differential $- r_1 + r_2$. But this only strengthens the profit incentive

$$(7.13) \qquad -b_{21} - b_{12} - r_2 + r_1$$

for plant 2 to move to location 1. Since the sum of the two incentives (7.12) and (7.13) is $-2\ (b_{21} + b_{12})$, it is impossible, if transportation costs b_{kl} are positive, to find rents r_i that prevent both plant owners from desiring to move. This is so irrespective of whether the initial assignment is optimal or not.

It might be thought, in the light of this example, that all one should require of a price system is that, for any non-optimal permutation out of an optimal assignment, it would make enough plant owners refuse the proposed move to cause the permutation to be blocked.[12/] However, it is hard to see how such a requirement could lead to a determination of rents at all. Unwillingness of a plant owner to participate in an otherwise attractive permutation could be tested by bidding a higher rent

12/ This would require at least one refusal in each cycle of moves into which the permutation can be decomposed.

to the owner of his location. If the idea that rents should reflect the value of each location to its occupant cannot be incorporated, then the search for a price system seems to lose much of its interest.

If this heuristic reasoning is accepted, there remains one other possibility to be explored. One can lay the failure of a price system in the 2-plant case, and perhaps also in the n-plant case considered before, to the fact that a move of one plant is evaluated without considering the effect on everybody's transportation costs of the subsequent moves of other plants that are needed to make room for the first move. This neglect is inherent in the attempt to work with one given system of commodity prices in the various locations. To avoid it, one may give up the search for commodity prices altogether, and look instead for principles that bring changes in total inter-plant transportation cost to bear on individual plant budgets in such a way that, in response to a rent system to be determined in conjunction with these principles, a non-optimal permutation out of an optimal assignment will be rejected by <u>every</u> participant in the permutation. We have made some explorations along these lines, in which we have received substantial help from Professor Theodore Motzkin. While a more detailed report of these attempts would exceed the limits of this article, it may be mentioned here that again no rent system was found that even halfway meets the requirement of informational decentralization.

We are frankly perplexed by the difficulty of establishing meaningful relations between location rents and transportation costs in a problem as elementary, in comparison with the actual complexities of locational markets, as the quadratic locational assignment problem. For this reason, we have delayed publication of our results for several years, in the hope that more conclusive results might be obtained. It now seems better to present such largely negative results as we have obtained concerning the possibilities of pricing in the quadratic assignment problem, because this problem seems to be close to the core of location theory, because of the importance of location theory in itself, and in the hope that an examination of this example of apparent failure of the price system may ultimately lead to better insight in the possibilities and limitations of price systems as means of decentralizing the allocation of indivisible resources.

APPENDIX A

Proof and Application of a Theorem due to Birkhoff

[small print]

We formulate the theorem in question [Birkhoff, 1946] in the form of a minimum statement needed for the present application, and adapt Von Neumann's proof accordingly:

Any matrix $[x_{ki}]$ of order n satisfying the restrictions

$$\text{(A.1)} \quad \sum_{k=1}^{n} x_{ki} = 1, \quad \sum_{i=1}^{n} x_{ki} = 1, \quad x_{ki} \geq 0, \quad k,i = 1,\ldots,n,$$

can be written as a weighted average, with non-negative weights,

$$\text{(A.2)} \quad x_{ki} = \sum_{r=1}^{n!} w_r \, p_{ki}^r , \qquad w_r \geq 0 , \quad \sum_{r=1}^{n!} w_r = 1 ,$$

of the $n!$ permutation matrices p_{ki}^r .

In order to prove this let us call $x_{k_1 i_1}$ an inner element of $[x_{ki}]$ if $0 < x_{k_1 i_1} < 1$, and an outer element if either $x_{k_1 i_1} = 0$ or $x_{k_1 i_1} = 1$. It follows from (A.1) that the number of inner elements in any row or column is either 0 or at least 2. Let ν be the number of inner elements in the given matrix $[x_{ki}]$. If $\nu = 0$ the assertion of the theorem is true. If $\nu > 0$, let $x_{k_1 i_1}$ be an arbitrary inner element, $x_{k_1 i_2}$ another element in row k_1 , $x_{k_2 i_2}$ another inner element in column i_2 , $x_{k_2 i_3}$ another inner element in row k_2 , etc. Since the total number of elements is finite, this construction will at some point cause an element to be repeated. Drop all elements in the sequence preceding the first element to be repeated. The sequence then has one of the two forms

$$\text{(A.3)} \quad x_{k_m i_m}, \; x_{k_m i_{m+1}}, \; x_{k_{m+1} i_{m+1}}, \ldots, \; x_{k_p i_p}, \; p > m \; , \; k_p = k_m, \; i_p = i_m,$$

or

$$x_{k_m i_{m+1}}, x_{k_{m+1} i_{m+1}}, x_{k_{m+1} i_{m+2}}, \ldots\ldots\ldots, x_{k_p i_{p+1}}, p > m,$$

$$k_p = k_m, i_{p+1} = i_{m+1}.$$

The second form can be reduced to the first by dropping the repeated element $x_{k_p i_{p+1}}$ off the end and repeating the preceding element $x_{k_p i_p}$ as $x_{k_m i_m}$, $i_m = i_p$, at the beginning. Taking (A.3) as the notation for the sequence so found, consider the matrix $[x_{ki}(\epsilon)]$ defined by

$$\text{(A.4)} \quad \begin{cases} \left.\begin{aligned} x_{k_q i_q}(\epsilon) &= x_{k_q i_q} + \epsilon, \\ x_{k_q i_{q+1}}(\epsilon) &= x_{k_q i_{q+1}} - \epsilon, \end{aligned}\right\} \quad q = m, m+1, \ldots, p-1, \\ x_{ki}(\epsilon) = x_{ki} \text{ whenever } (k, i) \neq (k_q, i_{q+1}) \text{ for all} \\ \qquad\qquad q = m, \ldots, p-1. \end{cases}$$

We note that $x_{ki}(\epsilon)$ if substituted for x_{ki} in (A.1) satisfies the first two conditions (A.1) for all ϵ, because every addition of ϵ to a row or column sum is offset by a subtraction of ϵ. We note further that the number $\nu(\epsilon)$ of inner elements of $[x_{ki}(\epsilon)]$ is at most that of $[x_{ki}]$ since any outer element $x_{k_0 i_0}$ of $[x_{ki}]$ cannot occur in the sequence (A.3) and hence the corresponding element $x_{k_0 i_0}(\epsilon)$ of $[x_{ki}(\epsilon)]$ is equal to $x_{k_0 i_0}$ and therefore also outer.

The set of values of ϵ for which $[x_{ki}(\epsilon)]$ satisfies also the third condition (A.1) is easily seen to be the interval

(A.5) $\underline{\epsilon} \leq \epsilon \leq \bar{\epsilon}$, where $\underline{\epsilon} = -\min_q x_{k_q i_q}$, $\bar{\epsilon} = \min_q x_{k_q i_{q+1}}$,

the minima being taken over the values $q = m, m + 1, \ldots, p - 1$. Since all $x_{k_q i_q}$ and $x_{k_q i_{q+1}}$ involved are positive,

$$\underline{\epsilon} < 0 < \bar{\epsilon} \ . \tag{A.6}$$

Moreover, since at least one $x_{ki}(\epsilon)$ vanishes at $\epsilon = \underline{\epsilon}$ and at least one at $\epsilon = \bar{\epsilon}$,

$$\nu(\underline{\epsilon}) \ , \ \nu(\bar{\epsilon}) < \nu(0) = \nu \ . \tag{A.7}$$

Finally, because $[x_{ki}(\epsilon)]$ is linear in ϵ ,

$$x_{ki} = x_{ki}(0) = - \frac{\bar{\epsilon}}{\bar{\epsilon}-\underline{\epsilon}} x_{ki}(\bar{\epsilon}) + \frac{\bar{\epsilon}}{\bar{\epsilon}-\underline{\epsilon}} x_{ki}(\underline{\epsilon}) \ . \tag{A.8}$$

In (A.8) the given matrix $[x_{ki}]$ is written as a weighted average, with positive weights, of two matrices, $[x_{ki}(\underline{\epsilon})]$ and $[x_{ki}(\bar{\epsilon})]$, both of which satisfy (A.1) and have a smaller number of inner elements than $[x_{ki}]$ has. As long as that is still a positive number, either of these can in turn be again written as a weighted average of two other matrices satisfying (A.1) and having a still smaller number of inner elements. Since a weighted average of weighted averages is itself a weighted average, any given matrix $[x_{ki}]$ can ultimately be written as a weighted average of matrices satisfying (A.1) with no inner elements, that is, of permutation matrices. The theorem is thereby proved.

In order to apply it to the maximization of the linear function (3.1) let

$$f_r = \sum_{k,i} a_{ki} \, p^r_{ki} \tag{A.9}$$

be the value of the maximand for the assignment represented by the permutation $[p^r_{ki}]$. Then, from (A.2), omitting summation limits, the maximand is

(A.10) $$\sum_{k,i} a_{ki}\, x_{ki} = \sum_{k,i}\left(a_{ki} \sum_{r} w_r\, p^r_{ki}\right) = \sum_{r}\left(w_r \sum_{k,i} a_{ki}\, p^r_{ki}\right) = \sum_{r} w_r\, f_r \; .$$

Without loss of generality, we can renumber the permutations in such a way that

(A.11) $$f_1 = f_2 = \ldots = f_s > f_r \quad \text{for all} \quad r > s, \quad \text{where} \quad s \geqq 1 \; .$$

This means that the first s permutations are optimal, and the remaining ones non-optimal. The maximand (A.10) then becomes

(A.12) $$\sum_{r} w_r\, f_r = \left(\sum_{r=1}^{s} w_r\right) f_1 + \sum_{r=s+1}^{n!} w_r\, f_r \leqq \left(\sum_{r=1}^{n!} w_r\right) f_1 = f_1 \; ,$$

where we have used that $w_r \geqq 0$, and where the equality sign applies only if $w_r = 0$ for all $r > s$. This shows that all weighted averages of optimal permutations, and only such, maximize the given linear function (3.1) subject to the restrictions (A.1).

APPENDIX B

Statement and Application of the Minkowski-Farkas Lemma

[small print]

The lemma referred to in Section 4 is as follows.

In order that

$$\sum_{n=1}^{N} b_n u_n \leqq 0 \qquad \text{(B.1)}$$

for all u_n, $n = 1,\ldots,N$,

satisfying

$$\sum_{n=1}^{N} a_{mn} u_n \leqq 0 , \qquad m = 1,\ldots M' . \qquad \text{(B.2)}$$

it is necessary and sufficient that

$$b_n = \sum_{m=1}^{M'} v_m a_{mn} , \qquad n = 1,\ldots,N , \qquad \text{(B.3)}$$

for some $v = \left[v_m \right]$ such that $v_m \geqq 0$, $m = 1,\ldots M'$

For proofs and further discussion, see Farkas [1901], and references in Gale, Kuhn, and Tucker [1951].

The Price Theorem of Linear Programming [Koopmans, 1951, p. 82, Theorem 5.4.1] says the following:

Necessary and sufficient that

$$\sum_n b_n x_n \leqq \sum_n b_n \bar{x}_n \qquad \text{(B.4)}$$

for all $x = \left[x_n \right]$ that satisfy

$$\sum_n a_{mn} x_n \leqq c_n , \quad m = 1,\ldots M; \quad x_n \geqq 0 , \quad n = 1,\ldots,N; \qquad \text{(B.5)}$$

(B.5) with $m = \bar{M} + 1, \ldots M$ and with $n = \bar{N} + 1, \ldots, N$ are met. However, since all left-hand numbers in (B.9) and (B.10) are linear and homogeneous in the u_n, the requirement $u_n < \delta$ can be omitted from this statement. We can therefore use the Minkowski-Farkas Lemma (necessity clause), with the matrix $[a_{mn}]$ in (B.2) and (B.3) replaced by

$$\begin{bmatrix} a_{11} & \ldots & a_{1\bar{N}} & a_{1,\bar{N}+1} & \ldots & a_{1N} \\ - & - & - & - & - & - \\ a_{\bar{M}1} & \ldots & a_{\bar{M}\bar{N}} & a_{\bar{M},\bar{N}+1} & \ldots & a_{\bar{M}N} \\ -1 & \ldots & 0 & 0 & \ldots & 0 \\ - & - & - & - & - & - \\ 0 & \ldots & -1 & 0 & \ldots & 0' \end{bmatrix}$$

to show the existence of non-negative numbers v_m, $m = 1, \ldots, \bar{M}$ and w_n, $n = 1 \ldots, \bar{N}$, such that

$$\text{(B.11)} \qquad b_n = \sum_{m=1}^{\bar{M}} v_m a_{mn} - w_n \quad \text{for} \quad n = 1, \ldots, \bar{N},$$

$$b_n = \sum_{m=1}^{\bar{M}} v_m a_{mn} \quad \text{for} \quad n = \bar{N}+1, \ldots, N .$$

The necessity clause of the price theorem of linear programming now follows by specifying $\lambda_m = v_m$ for $m = 1, \ldots, \bar{M}$ and $\lambda_m = 0$ for $m = \bar{M} + 1, \ldots M$.

In application to the linear assignment problem, the elements of the score matrix $[a_{ki}]$ in (2.1) make up the vector $[b_n]$ of (B.4) whereas the elements of the matrix $[a_{mn}]$ in (B.5) are ones and zeros so selected that (B.5) corresponds to the restrictions (4.4). For the reasons indicated in connection with (4.4), only the first of the two cases distinguished in (B.7) can now occur, if all scores a_{ki} are positive. We therefore can read the price conditions for the optimality of an assignment from (B.6) alone, in which we equivalently reverse the order of the case distinctions is the existence of numbers λ_m , $m = 1, \ldots, M$, such that

(B.6) $$\sum_m \lambda_m a_{mn} \left\{\begin{matrix} = \\ \geqq \end{matrix}\right\} b_n \quad \text{according as} \quad \bar{x}_n \left\{\begin{matrix} > \\ = \end{matrix}\right\} 0,$$

(B.7) $$\lambda_m \left\{\begin{matrix} \geqq \\ = \end{matrix}\right\} 0 \quad \text{according as} \quad \sum_n a_{mn} \bar{x}_n \left\{\begin{matrix} = \\ < \end{matrix}\right\} c_n .$$

Proof: Sufficiency. Suppose there exists a set of λ_m satisfying (B.5) and (B.6) for some $\bar{x}$. Then, by a sequence of steps derived from the relations indicated at each step ,

$$\sum_n b_n \bar{x}_n \overset{(B.6)}{=} \sum_{m,n} \lambda_m a_{mn} \bar{x}_n \overset{(B.7)}{=} \sum_m \lambda_m c_m \overset{(B.5,\ B.7)}{\geqq} \sum_{m,n} \lambda_m a_{mn} x_n \overset{(B.5,\ B.6)}{\geqq} \sum_n b_n x_n$$

for all x that satisfy (B.5).

Necessity: Let $\bar{x} = [\bar{x}_n]$ satisfy (B.4) for all $x = [x_n]$ that satisfy (B.5). Without loss of generality we can specify that

(B.8) $$\begin{aligned} &\bar{x}_n = 0 \ \text{ for } \ n = 1, \ldots, \bar{N}, \qquad \bar{x}_n > 0 \ \text{ for } \ n = \bar{N} + 1, \ldots, N, \\ &\sum_n a_{mn} \bar{x}_n = c_m \ \text{ for } \ m = 1, \ldots, \bar{M}, \quad \sum_n a_{mn} \bar{x}_n < c_n \ \text{ for } \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m = \bar{M} + 1, \ldots M . \end{aligned}$$

By inserting $x = \bar{x} + u$ in (B.4) and (B.5) our premise is seen to imply that

(B.9) $$\sum_n b_n u_n \leqq 0$$

for all $u_n < \delta$, $n = 1, \ldots, N$, such that

(B.10) $$\sum_n a_{mn} u_n \leqq 0 \ \text{ for } \ m = 1, \ldots, \bar{M} , \qquad u_n \geqq 0 \ \text{ for } \ n = 1, \ldots, \bar{N},$$

provided, δ is a positive number small enough to ensure that the conditions for a more natural interpretation. If $a_{ki} > 0$ for all k,i, an assignment $\bar{x}_{ki}$ is optimal if and only if there exists a set of non-negative rents q_k , r, such that

$$\bar{x}_{ki} \left\{ \begin{matrix} \geqq \\ = \end{matrix} \right\} 0 \quad \text{according as} \quad q_k + r_i \left\{ \begin{matrix} = \\ > \end{matrix} \right\} 0 .$$

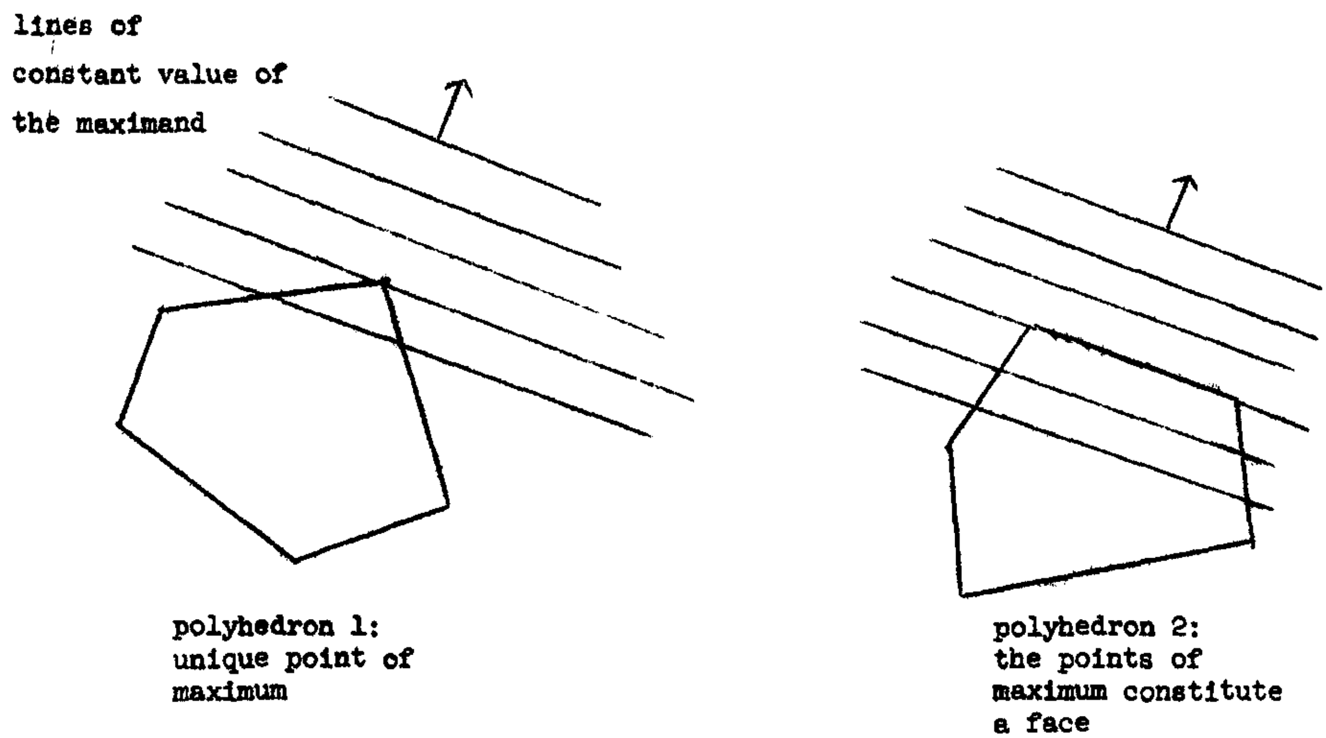

Figure 1

Maximum of a Linear Function on a Polyhedron.

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