Reprinted from
Price Flexibility and Employment, by Oscar Lange.
Cowles Commission Monograph No. 8.
The Principia Press, Bloomington, Indiana, 1944.

APPENDIX

THE STABILITY OF ECONOMIC EQUILIBRIUM*

1. *The Hicksian Conditions*

THE THEORY of stability of economic equilibrium is based on the assumption that an excess demand for a good causes a rise in its price, while an excess supply causes a fall in price. The equilibrium is thus said to be stable when, in the neighborhood of the equilibrium position, a price above the equilibrium price causes excess supply and a price below the equilibrium price causes excess demand. This condition was first stated by Walras. Walras, however, formulated it in a way which limits its applicability to partial-equilibrium analysis. Within the framework of general-equilibrium theory the stability conditions must take into account the repercussions of the change in price of a good upon the prices of other goods as well as the dependence of excess demand (or excess supply) of a good on the prices of the other goods in the system. This has been done by Professor Hicks.[1]

According to Professor Hicks, the economic system is in stable equilibrium if a rise of the price of any good above the equilibrium price causes an excess supply of and a fall of the price below the equilibrium price causes an excess demand for that good, *when the prices of all other goods in the system are so adjusted as to maintain equilibrium in all other markets.* Otherwise the system is either in unstable or in neutral equilibrium. The former is the case when a rise of the price above the equilibrium price produces excess demand and a fall of the price produces excess supply; the latter is the case when no excess demand or excess supply is produced. In both cases adjustment of all other prices maintaining equilibrium in the other markets is presupposed. This formulation of the theory of stability of equilibrium leads to a series of conditions which are best formulated mathematically.

Let there be $n+1$ goods in the economy and let one of them, say the $(n+1)$th, serve as money and *numéraire*. Denote by $p_r (r=1, 2, \cdots, n)$ the price of the rth good; $p_{n+1}=1$ by definition. Write further $D_r(p_1, p_2, \cdots, p_n)$ for the demand function and $S_r(p_1, p_2, \cdots, p_n)$ for the supply function of the rth good. We have then n independent excess-demand functions X_r defined by

$$(1.1)\quad X_r(p_1, p_2, \cdots, p_n) \equiv D_r(p_1, p_2, \cdots, p_n) - S_r(p_1, p_2, \cdots, p_n) \qquad (r = 1, 2, \cdots, n).$$

The system is in equilibrium when $X_r=0$ $(r=1, 2, \cdots, n)$. The equilibrium is stable when, at the equilibrium point,

* Also issued as Cowles Commission Papers, New Series, No. 8.

[1] See *Value and Capital*, pp. 66 ff. and pp. 315–316.

$$
\begin{aligned}
&(1.21) \qquad \frac{dX_r}{dp_r} < 0, \\
(1.2) \qquad & \qquad\qquad\qquad\qquad\qquad\qquad (r \text{ and } s = 1, 2, \cdots, n). \\
&(1.22) \qquad \frac{dX_s}{dp_r} = 0 \quad (s \neq r),
\end{aligned}
$$

The inequality indicates negative excess demand (i.e., excess supply) when the price rises above equilibrium, and positive excess demand when the price falls below equilibrium. The equations indicate that the prices in the other markets are adjusted to maintain equilibrium in these markets.[2]

Let us write

$$(1.3) \qquad a_{sr} \equiv \frac{\partial X_s}{\partial p_r} \qquad (r \text{ and } s = 1, 2, \cdots, n).$$

We have

$$(1.4) \qquad \frac{dX_s}{dp_r} = a_{s1}\frac{dp_1}{dp_r} + a_{s2}\frac{dp_2}{dp_r} + \cdots + a_{sr} + \cdots + a_{sn}\frac{dp_n}{dp_r} \qquad (s = 1, 2, \cdots, n).$$

Exchanging places between dX_s/dp_r and a_{sr}, and taking into account (1.2), (i.e. 1.22) we obtain the equations

$$
\begin{aligned}
-a_{1r} &= a_{11}\frac{dp_1}{dp_r} + a_{12}\frac{dp_2}{dp_r} + \cdots + \quad 0 \quad + \cdots + a_{1n}\frac{dp_n}{dp_r}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(1.5) \qquad -a_{rr} &= a_{r1}\frac{dp_1}{dp_r} + a_{r2}\frac{dp_2}{dp_r} + \cdots - \frac{dX_r}{dp_r} + \cdots + a_{rn}\frac{dp_n}{dp_r}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-a_{nr} &= a_{n1}\frac{dp_1}{dp_r} + a_{n2}\frac{dp_2}{dp_r} + \cdots + \quad 0 \quad + \cdots + a_{nn}\frac{dp_n}{dp_r}.
\end{aligned}
$$

[2] The stability conditions can also be expressed in terms of excess demand for money. The aggregate value of the excess demand in all n markets is the excess supply of money, i.e.,

$$-X_{n+1} \equiv \sum_{s=1}^{n} p_s X_s.$$

All other markets remaining in equilibrium, we have $X_s = 0$ for $s \neq r$ and

$$-\frac{dX_{n+1}}{dp_r} = X_r + p_r\frac{dX_r}{dp_r}.$$

At the equilibrium point $X_r = 0$ and, by virtue of (1.2),

$$(1.3) \qquad \frac{dX_{n+1}}{dp_r} > 0 \qquad (r = 1, 2, \cdots, n).$$

Thus the excess demand for money must become positive when the price of a good other than money rises above equilibrium, and become negative when the price falls below equilibrium.

Write

$$(1.6) \qquad J \equiv \begin{vmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{vmatrix}$$

and denote by J_{rs} the cofactor of a_{rs}. Solving the equations (1.5) we get

$$(1.7) \qquad \frac{dX_r}{dp_r} = \frac{J}{J_{rr}} < 0 \qquad (r = 1, 2, \cdots, n).$$

This is negative because of (1.2).

Amplifying and modifying Professor Hicks's terminology, we introduce the concept of *partial stability* of different order and rank. The system is said to be partially stable of *order m* ($m \leqq n$) if (1.2) is satisfied when only m other prices are adjusted and *the remaining prices are kept constant.*[3] By a procedure analogous to that leading to (1.7) we obtain as a condition of partial stability of order m

$$(1.8) \qquad \left(\frac{dX_r}{dp_r}\right)_{n-m} = \frac{J_{nn,\cdots,n-m}}{J_{nn,\cdots,n-m,rr}} < 0 \quad (r = 1, 2, \cdots, m),$$

where the numerator and the denominator are cofactors of J of order m and $m-1$ respectively. The subscript on the left-hand side indicates which prices are kept constant (namely, $m+1$, $m+2, \cdots, n$). The concept of partial stability is always relative to the prices which are kept constant. The system may be partially stable of order m if certain $n-m$ prices are held constant but may fail to be so if $n-m$ other prices are kept constant. When the system is partially stable of order n (n being the number of goods, exclusive of money) we say that it is *totally* stable. The condition (1.8) then turns into (1.7).

The system is said to be stable of *rank m* (and *unstable or neutral of rank* $n-m$) if it is partially stable of order m but not of any higher order. The rank of the stability of the system is thus the highest order of partial stability it possesses. A totally stable system has stability of rank n.

Partial stability of order m is said to be *perfect* when the system shows partial stability of *all* lower orders with respect to *any* prices being held constant. Otherwise the partial stability is said to be imperfect. This definition of perfect partial stability applies also to partial stability of order n, i.e., to total stability. In virtue of (1.8) the condition for perfect stability of order m can be written

[3] In this case (1.2) holds for r and $s = 1, 2, \cdots, m$.

$$(1.9)\quad a_{11} < 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \cdots, \quad \text{sign} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} = \text{sign}\,(-1)^m,$$

the numeration of the goods being, of course, arbitrary. These are the Hicksian conditions for perfect stability.[4]

2. Dynamic Stability Conditions

The reader will have noticed that in the mathematical formulation of the theory of stability of economic equilibrium the basic assumption of that theory, namely that excess demand for a good makes its price rise and excess supply makes it fall, does not appear explicitly. This assumption, however, is tacitly implied in the choice of the condition that excess demand should occur when the price is below equilibrium and excess supply should occur when it is above equilibrium. In order to clarify all the implications of stability analysis the basic assumption mentioned must be explicitly introduced into the mathematical formulation of the theory of stability equilibrium. When this is done, stability analysis becomes part of a dynamic theory, as was shown recently by Professor Samuelson.[5] The traditional method of treating the stability of economic equilibrium, as applied by Walras, Marshall, and Hicks, is but an implicit (and therefore imperfect) form of dynamic analysis.

The basic assumption of stability analysis, i.e., that excess demand causes the price to rise and excess supply causes it to fall, can be formulated as follows:

$$(2.1)\qquad \text{sign}\,\frac{dp_r}{dt} = \text{sign}\,X_r \qquad (r = 1, 2, \cdots, n),$$

where dp_r/dt is the rate of change of price over time. Let

$$(2.2)\qquad \frac{dp_r}{dt} = F_r(X_r) \qquad (r = 1, 2, \cdots, n)$$

be a set of functions which satisfy the relations (2.1). Then by (2.1) we have

$$(2.3)\qquad F_r(0) = 0 \qquad (r = 1, 2, \cdots, n),$$

as the equilibrium conditions of the system.

In (2.2) we have a normal system of n differential equations which has the

[4] Professor Hicks limits the concept of perfect stability to total stability. The conditions for perfect stability given by him are thus only for the case $m = n$.

[5] "The Stability of Equilibrium: Comparative Statics and Dynamics," *Econometrica*, Vol. 9, April, 1941, pp. 97–120.

solutions $p_r(t)$ $(r=1, 2, \cdots, n)$.[6] The functions $p_r(t)$ are the adjustment paths of the prices and the equilibrium is stable when these paths lead back to the equilibrium prices, unstable when they lead away from them, and neutral when neither is the case.[7] Expressing all prices in terms of deviations from the equilibrium prices, i.e., putting the latter equal zero, we thus have stable equilibrium when

$$\lim_{t=\infty} p_r(t) = 0 \qquad (r = 1, 2, \cdots, n). \tag{2.4}$$

In order to solve the equations we expand, on the right-hand side of (2.2), F_r and X_r by Maclaurin's theorem and retain only the linear part of the expansion. Expanding F_r, we have

$$\frac{dp_r}{dt} = F'(0)X_r \qquad (r = 1, 2, \cdots, n),$$

and then, expanding X_r, we obtain

$$\frac{dp_r}{dt} = F_r' \sum_{s=1}^{n} a_{rs}p_s \qquad (r = 1, 2, \cdots, n), \tag{2.5}$$

where p_s is expressed as a deviation from the equilibrium price $p_s{}^0=0$. $F_r' \equiv F_r'(0) = \text{const.}$ and $a_{rs}{}^0 = a_{rs}(p_1{}^0, p_2{}^0, \cdots, p_n{}^0) = \text{const.}$ We have now a system of linear equations with constant coefficients.

It will be noticed that in view of (2.1)

$$F_r'(0) > 0 \qquad (r = 1, 2, \cdots, n). \tag{2.6}$$

Thus when the functions on the right-hand side of (2.2) are taken as linear in X_r the basic assumption of stability analysis implies necessarily that *the speed of increase of price is the greater the greater the excess demand.* $F_r'(0)$ may serve as a measure of the flexibility of the price p_r. In general it will be said that the price is flexible when $F_r'(0)>0$, inflexible, or rigid, when $F_r'(0)=0$, and negatively flexible when $F_r'(0)<0$. The last two cases are excluded by (2.6).

The solution of the linear system (2.5) is given by the set of functions

$$p_r(t) = \sum_{s=1}^{k} q_{rs}(t)e^{\lambda_s t} \qquad (r = 1, 2, \cdots, n), \tag{2.7}$$

[6] It is assumed that the existence conditions are satisfied. This is always the case when the functions F_r and X_r $(r=1, 2, \cdots, n)$ and their first derivatives are continuous.

[7] These definitions are broader than those on the first page of this Appendix and include the latter as a special case.

where the λ_s $(s=1, 2, \cdots, k)$ are the k $(k \leqq n)$ distinct roots of the characteristic equation[8]

$$f(\lambda) \equiv \begin{vmatrix} F_1'a_{11}^0 - \lambda & F_1'a_{12}^0 & \cdots & F_1'a_{1n}^0 \\ F_2'a_{21}^0 & F_2'a_{22}^0 - \lambda & \cdots & F_2'a_{2n}^0 \\ \cdot & \cdot & \cdot & \cdot \\ F_n'a_{n1}^0 & F_n'a_{n2}^0 & \cdots & F_n'a_{nn}^0 - \lambda \end{vmatrix} = 0, \tag{2.8}$$

and the $q_{rs}(t)$ are polynomials in t of degree one less than the multiplicity of the root λ_s.[9] Of the coefficients of the polynomials n are arbitrary and determined by the initial conditions (i.e., by the initial disturbance of equilibrium), the remaining coefficients are found from a system of homogeneous linear equations with matrix of coefficients as given in (2.8).

Let the roots be complex and write

$$\lambda_s = R(\lambda_s) + I(\lambda_s) \qquad (s = 1, 2, \cdots, k), \tag{2.9}$$

where the two terms on the right-hand side indicate the real and the imaginary part respectively. This includes real roots as a special case in which $I(\lambda_s) = 0$. Writing $I(\lambda_s) = \beta i$, we have

$$e^{\lambda_s t} = e^{R(\lambda_s)t}(\cos \beta t + i \sin \beta t). \tag{2.10}$$

The equilibrium is thus stable, i.e. (2.4) is satisfied, when

$$R(\lambda_s) < 0 \qquad \text{for } s = 1, 2, \cdots, k. \tag{2.11}$$

This is the stability condition which in the dynamic theory replaces the static condition (1.7). If some $R(\lambda_s) > 0$ we get $\lim_{t=\infty} p_r(t) = \pm \infty$ $(r=1, 2, \cdots, n)$, and the equilibrium is unstable. If some $R(\lambda_s) = 0$ and no $R(\lambda_s) > 0$ the equilibrium is neutral.

As in the static theory, we introduce the concepts of *partial stability* of a given order and of rank of stability of the system. The dynamic system is partially stable of *order m* if it is stable when only m prices are allowed to adjust themselves and the other $n-m$ prices are kept constant. This implies that

$$F_r' \equiv 0 \qquad \text{for } r = m + 1, \cdots, n \tag{2.12}$$

and

$$p_s \equiv p_s^0 = 0 \qquad \text{for } s = m + 1, \cdots, n. \tag{2.13}$$

[8] Professor Samuelson (*op. cit.*, pp. 109–110) leaves out the factors F_r' in the characteristic determinant. This can be done only when $F_1' = F_2' = \cdots = F_n'$. His results thus hold only for the special case where the flexibility of all prices in the system is the same.

[9] Thus when λ_s is a simple root the corresponding polynomials $q_{rs}(t)$ $(r=1, 2, \cdots, n)$ reduce to constants.

The system of equations (2.5) turns into

$$\frac{dp_r}{dt} = F_r' \sum_{s=1}^{m} a_{rs} p_s \qquad (r = 1, 2, \cdots, m) \tag{2.14}$$

and the solutions become

$$p_r(t) = \sum_{s=1}^{k} q_{rs}(t) e^{\lambda_s t} \quad (r = 1, 2, \cdots, m; k \leqq m). \tag{2.15}$$

The condition for partial stability of order m is given, as before, by (2.11) except that the λ_s are roots of a characteristic equation of order m. The characteristic determinant of this equation is a principal minor of order m of the characteristic determinant in (2.8).

When the dynamic system is partially stable of order n we say that it is *totally* stable. The highest order of partial stability of the system is called the *rank of the stability* of the system.

When the characteristic determinant is symmetric all roots are real.[10] In order that they be all negative it is necessary and sufficient[11] that the Hicksian conditions (1.9) be satisfied. Dynamic partial stability of order m thus requires and implies *perfect* Hicksian stability of the same order. This is clear: symmetry of the characteristic determinant of order m implies (and requires) symmetry of all its principal minors.

3. *Implications of the Validity of the Hicksian Conditions*

The Hicksian conditions for perfect stability are equivalent to the dynamic stability conditions when the characteristic determinant of order m is symmetric. Let us examine the economic meaning of such symmetry. We have from (2.2)

$$F_r' = \frac{d}{dX_r}\left(\frac{dp_r}{dt}\right) \qquad (r = 1, 2, \cdots, m). \tag{3.1}$$

Taking into account (1.3) we obtain

$$F_r' a_{rs} = \frac{\partial}{\partial p_s}\left(\frac{dp_r}{dt}\right) \qquad (r \text{ and } s = 1, 2, \cdots, m). \tag{3.2}$$

[10] We assume that the $F_r' a_{rs}$ are all real, and apply the well-known theorem about the characteristic (or secular) equation proved in the theory of determinants. Cf., for instance, G. Kowalewski, *Einführung in die Determinantentheorie* (Berlin and Leipzig, 1925), pp. 114 ff.; H. W. Turnbull and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices* (London and Glasgow), p. 101. A very simple proof is given by F. R. Moulton, *Differential Equations* (New York, 1930), pp. 298–299.

[11] This is the fundamental theorem about definite Hermitian forms. Cf. Kowalewski, *op. cit.*, p. 199.

The symmetry $F_r' a_{rs} = F_s' a_{sr}$ thus implies

$$\frac{\partial}{\partial p_s}\left(\frac{dp_r}{dt}\right) = \frac{\partial}{\partial p_r}\left(\frac{dp_s}{dt}\right) \qquad (r \text{ and } s = 1, 2, \cdots, m), \tag{3.3}$$

i.e., *the marginal effect of a change in the price p_s upon the speed of adjustment of the price p_r equals the marginal effect of a change in the price p_r upon the speed of adjustment of the price p_s.*[12]

The symmetry of the marginal effect of a change in one price upon the speed of adjustment of another price can be clarified further by a mathematical consideration. The symmetry conditions (3.3) are the sufficient conditions for the integrability of the total differential equation

$$\sum_{r=1}^{m} \frac{dp_r}{dt} dp_r = 0. \tag{3.4}$$

When conditions (3.3) hold, there exists a function (or rather a class of functions)[13]

$$P[p_1(t), p_2(t), \cdots, p_m(t)] \tag{3.5}$$

[12] It has been held by some economists that, in order that static equilibrium and stability analysis be applicable, the speed of adjustment must be the same in each market. This view was expressed by S. Kohn ("On the Problems of the Modern Theory of Price and Value," *Economista*, 1925, in Polish); by P. N. Rosenstein-Rodan ("Das Zeitmoment in der Mathematischen Theorie des wirtschaftlichen Gleichgewichtes," *Zeitschrift für Nationalökonomie*, Vol. 1, 1930, pp. 129–142, and "The Role of Time in Economic Theory," *Economica*, N.S., Vol. 1, February, 1934, pp. 90–91); and by Simon Kuznets ("Equilibrium Economics and Business-Cycle Theory," *Quarterly Journal of Economics*, Vol. 44, February, 1930, p. 404). As shown above, this is wrong. The condition of applicability of static analysis is not equality of the speed of price adjustment in each market, but the symmetry of the cross effects of a change in one price upon the speed of adjustment of the other, as indicated in (3.3). This symmetry is similar to the Hotelling conditions in the pure theory of demand or supply without budget limitations ("Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions," *Journal of Political Economy*, Vol. 40, October, 1932, pp. 591 and 594). These conditions are

$$\frac{\partial D_r}{\partial p_s} = \frac{\partial D_s}{\partial p_r}$$

and $$\frac{\partial S_r}{\partial p_s} = \frac{\partial S_s}{\partial p_r} \qquad (r \text{ and } s = 1, 2, \cdots, n).$$

If these conditions are satisfied we have, on account of (1.1) and (1.3), $a_{rs} = a_{sr}$ (r and $s = 1, 2, \cdots, n$). If $F_r' = F_s'$ (r and $s = 1, 2, \cdots, n$), this implies the fulfillment of the condition (3.3). Thus, when the flexibility of all prices is the same, the condition of applicability of static-equilibrium and stability analysis is identical with the Hotelling conditions for demand and supply functions.

[13] If P is a solution of the equation then any function $\phi(P)$ such that $\phi'(P) \neq 0$ is also a solution.

such that

$$\frac{dp_r}{dt} = \frac{\partial P}{\partial p_r(t)} \qquad (r = 1, 2, \cdots, m), \tag{3.6}$$

i.e., such that the speeds of adjustments are its partial derivatives. The equation (3.4) can be interpreted as the maximum condition of this function (or class of functions[14]). The adjustment paths $p_r(t)$ $(r=1, 2, \cdots, m)$ are then co-ordinated into a consistent system maximizing this function. The function P may, therefore, be called the *adjustment potential,* and a dynamic system for which an adjustment potential exists will be called an *integrated system; m* will be called the *order of integration* of the system. From (3.3) we see that when the system is integrated of order m it is also integrated in all lower orders. The Hicksian conditions provide the sufficient[15] conditions of (partial) stability (of order m; $m \leqq n$) for integrated (of order m) dynamic systems.

The economic meaning of an integrated system can be illustrated as follows. Suppose that the m adjustment paths $p_r(t)$ $(r=1, 2, \cdots, m)$ are determined by a planning authority that wants to maximize at each moment the total welfare of the community. The adjustment paths must then satisfy the maximum conditions of a function like (3.5). As atomistic competition automatically produces (though not without important qualifications) maximum total welfare within a static system, similarly a dynamic system *may,* under appropriate circumstances, imply the maximization of a potential function which serves as an indicator of total welfare.

4. Homogeneous Systems

Consider a system consisting of $n+1$ goods and suppose that the $(n+1)$th good functions as money. Let the excess-demand functions of m goods other than money $(m<n)$ be homogeneous of zero degree in the prices of these goods,[16] and let the excess-demand functions of the remaining $n-m$ goods

[14] The second-order maximum conditions are given by the Hicksian inequalities (1.9). In order to satisfy these, the functions $\phi(P)$ must be restricted to cases where $\phi'(P)>0$.

[15] The conditions of integrability of (3.4) are that $\frac{1}{2}(m-1)(m-2)$ equations of the form

$$\frac{dp_r}{dt}\left[\frac{\partial}{\partial p_t}\left(\frac{\partial p_s}{\partial t}\right) - \frac{\partial}{\partial p_s}\left(\frac{\partial p_t}{\partial t}\right)\right] + \frac{dp_s}{dt}\left[\frac{\partial}{\partial p_r}\left(\frac{\partial p_t}{\partial t}\right) - \frac{\partial}{\partial p_t}\left(\frac{\partial p_r}{\partial t}\right)\right] + \frac{dp_t}{dt}\left[\frac{\partial}{\partial p_s}\left(\frac{\partial p_r}{dt}\right) - \frac{\partial}{\partial p_r}\left(\frac{\partial p_s}{\partial t}\right)\right] = 0$$

are satisfied; for this, (3.3) is sufficient but not necessary.

[16] A function $f(x_1, x_2, \cdots, x_m; x_{m+1}, \cdots, x_n)$ is said to be homogeneous of the kth degree *in the variables* $x_1, x_2, \cdots, x_m$ if, for every k, $f(kx_1, kx_2, \cdots, kx_m; x_{m+1}, \cdots, x_n) = kf(x_1, x_2, \cdots, x_m; x_{m+1}, \cdots, x_n)$.

other than money be homogeneous of first degree in the same prices. We shall prove that such a system has the following properties:

(1) The excess-demand function for money is homogeneous of first degree in the same m prices.

(2) The system is neutral of rank not less than one and the rank of stability of the system does not exceed $n-1$;

(3) The equilibrium value of one of the m prices in which the excess-demand functions are homogeneous of zero degree is arbitrary and the equilibrium values of the other $m-1$ of these prices are proportional to the arbitrary equilibrium price.

In order to fix ideas assume that the excess-demand functions $X_1, X_2 \cdots, X_m$ are homogeneous of zero degree in the prices $p_1, p_2, \cdots, p_m$ and that the excess-demand functions $X_{m+1}, X_{m+2}, \cdots, X_n$ are homogeneous of first degree in the same variables. We observe that the relation

$$\sum_{r=1}^{m} p_r X_r + \sum_{r=m+1}^{n} p_r X_r + X_{n+1} \equiv 0 \tag{4.1}$$

holds between the $n+1$ excess-demand functions. This relation is an identity in the p's and may be called *Walras' law*.[17] If the prices $p_1, p_2, \cdots, p_m$ are multiplied by an arbitrary number k and the prices $p_{m+1}, \cdots, p_n$, are kept constant, each of the expressions under the summation sign in (4.1) is increased k-fold, for in the first expression the p's are increased k-fold and the X's are unchanged, while in the second expression the p's are unchanged and the X's are increased k-fold. It follows from the identity that X_{n+1} is also increased k-fold. This proves the first property of our system.

Applying Euler's theorem, we have

$$\sum_{s=1}^{m} a_{rs} p_s = 0 \qquad \text{for } r = 1, 2, \cdots, m \tag{4.2}$$

and

$$\sum_{s=1}^{m} a_{rs} p_s = X_r \quad \text{for } r = m+1, m+2, \cdots, n, \tag{4.3}$$

where a_{rs} is defined as in (1.3). Putting the equilibrium prices p_r^0 ($r=1, 2, \cdots, n$) into (4.2) and (4.3) and remembering that $X_r(p_1^0, p_2^0, \cdots, p_n^0) = 0$ ($r=1, 2, \cdots, n$), we obtain

[17] For a special case (the foreign-exchange markets) this relation was known already to Cournot (cf. *Researches into the Mathematical Principles of the Theory of Wealth*, trans. by T. Bacon; New York: Macmillan Co., 1927, pp. 33–34). Walras, however, was the first to give it a general mathematical formulation and to recognize its importance for the theory of prices. See his *Eléments d'économie politique pure* (édition définitive; Paris and Lausanne, 1926), pp. 120–121.

$$\sum_{s=1}^{m} a_{rs}{}^0 p_s{}^0 = 0 \qquad (r = 1, 2, \cdots, n), \tag{4.4}$$

where $a_{rs}{}^0 = a_{rs}(p_1{}^0, p_2{}^0, \cdots, p_n{}^0)$.

Consider now the determinant

$$J^0 = \begin{vmatrix} a_{11}{}^0 & a_{12}{}^0 \cdots a_{1n}{}^0 \\ a_{21}{}^0 & a_{22}{}^0 \cdots a_{2n}{}^0 \\ \cdots & \cdots \\ a_{n1}{}^0 & a_{n2}{}^0 \cdots a_{nn}{}^0 \end{vmatrix}. \tag{4.5}$$

Multiply the first column by $p_1{}^0$, add the second column multiplied by $p_2{}^0$, etc., finally add the mth column multiplied by $p_m{}^0$. The result is the determinant

$$\begin{vmatrix} \sum_{s=1}^{m} a_{rs}{}^0 p_s{}^0 & a_{12}{}^0 \cdots a_{1n}{}^0 \\ \sum_{s=1}^{m} a_{rs}{}^0 p_s{}^0 & a_{22}{}^0 \cdots a_{2n}{}^0 \\ \cdots & \cdots \\ \sum_{s=1}^{m} a_{rs}{}^0 p_s{}^0 & a_{n2}{}^0 \cdots a_{nn}{}^0 \end{vmatrix} = p_1{}^0 J^0. \tag{4.6}$$

On account of (4.4) this determinant vanishes and so does J^0, because the origin of the price co-ordinates can always be chosen so that $p_1{}^0 \neq 0$. Thus J^0 is at most of rank $n-1$. The same procedure cannot be repeated with all of the first minors of J^0 and it is impossible to show that they must all vanish. They may vanish, of course, but need not do so. All that can be asserted is, therefore, that the rank of J^0 cannot exceed $n-1$.

The determinant

$$D^0 = \begin{vmatrix} F_1'a_{11}{}^0 & F_1'a_{12}{}^0 \cdots F_1'a_{1n}{}^0 \\ F_2'a_{21}{}^0 & F_2'a_{22}{}^0 \cdots F_2'a_{2n}{}^0 \\ \cdots & \cdots \\ F_n'a_{n1}{}^0 & F_n'a_{n2}{}^0 \cdots F_n'a_{nn}{}^0 \end{vmatrix} = F_1'F_2' \cdots F_n'J^0, \tag{4.7}$$

where $F_r' = F_r'(0) > 0$ $(r = 1, 2, \cdots, n)$ by virtue of (2.6), is at most of the same rank as J^0, i.e. $n-1$.

The characteristic equation (2.8) can be written in the polynomial form

$$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + \cdots + (-1)^n S_n = 0, \tag{4.8}$$

where S_r $(r = 1, 2, \cdots, n)$ is the sum of all principal minors of order r in

D^0. D^0 being of rank not higher than $n-1$, at least the last term of the polynomial vanishes and we have

$$\lambda[\lambda^{n-1} - S_1\lambda^{n-2} + S_2\lambda^{n-3} + \cdots + (-1)^{n-1}S_{n-1}] = 0. \tag{4.9}$$

The characteristic equation thus has at least one root $\lambda = 0$ and the system is, therefore, neutral at least of rank one. Since at least one of the roots equals zero, at most $n-1$ roots can have negative real parts, i.e., the order of stability of the system cannot be higher than $n-1$. This proves the second property of our system.

The equilibrium equations are

$$X_r(p_1, p_2, \cdots, p_n) = 0 \qquad (r = 1, 2, \cdots, n). \tag{4.10}$$

In view of the fact that $X_1, X_2, \cdots, X_m$ are homogeneous of zero degree and $X_{m+1}, X_{m+2}, \cdots, X_n$ are homogeneous of the first degree in the variables $p_1, p_2, \cdots, p_m$, the equations can be written in the form

$$\begin{aligned} &\Phi_r\left(1, \frac{p_2}{p_1}, \cdots, \frac{p_m}{p_1}; p_{m+1}, \cdots, p_n\right) = 0 \\ &\qquad\qquad\qquad \text{for } r = 1, 2, \cdots, m, \\ &p_1\Phi_r\left(1, \frac{p_2}{p_1}, \cdots, \frac{p_m}{p_1}; p_{m+1}, \cdots, p_n\right) = 0 \\ &\qquad\qquad\qquad \text{for } r = m+1, m+2, \cdots, n. \end{aligned} \tag{4.11}$$

We see immediately that if the set of prices $p_1^0, p_2^0, \cdots, p_m^0, p_{m+1}^0, \cdots, p_n^0$ is a solution of (4.11),[18] the set of prices $kp_1^0, kp_2^0, \cdots, kp_m^0, p_{m+1}^0, \cdots, p_n^0$, where k is an arbitrary number, is also a solution. This proves the third property of our system.

A practical application of the system under discussion is found by interpreting the goods $1, 2, \cdots, m$ as commodities and stocks and the goods $m+1, m+2, \cdots, n$ as fixed-income-bearing securities. Our system then describes the case where the excess-demand functions of commodities and stocks are homogeneous of zero degree in the prices of commodities and stocks, interest rates (or the prices of fixed-income-bearing securities) being constant. Under these circumstances the demand and supply functions, and, consequently, also the excess-demand functions, of fixed-income-bearing securities are homogeneous of first degree in commodity prices, because if all commodity and stock prices increase k-fold the real earning power of the securities mentioned decreases in inverse proportion and it takes k times as many securities to represent the same real earning power as before.[19] The properties of such a system have been discovered by Lord Keynes in his doc-

[18] The existence of a solution of the equilibrium equations is assumed.

[19] Cf. pp. 16 above.

trine of the effect of changes in money wages upon employment and upon product prices.[20] Lord Keynes's theory presupposes a system in which interest rates are kept constant and in which the demand and supply functions of all commodities are homogeneous of zero degree in money wage rates and commodity prices. Professor Hicks has developed further this doctrine in application to general-equilibrium theory under conditions where all price expectations are of unit elasticity.[21] A mathematical proof of Professor Hicks's conclusions was given by Dr. Mosak.[22] Dr. Mosak uses the Hicksian stability conditions in his proof. His proof is, therefore, restricted to systems in which these conditions are valid. The results established in this section contain those of Keynes, Hicks, and Mosak as special cases.

5. The Law of Composition of Goods

The rank of stability of economic equilibrium indicates the maximum number of flexible prices compatible with the stability of the system. To secure stability, the remaining prices must be rigid. Any argument, however, which attaches importance to the number of goods or prices presupposes the existence of a way of classifying goods and determining their number which is not purely arbitrary. From experience we know that there is no unique way of classifying goods. A commodity can be split up into several subcommodities; for instance, wheat into wheat of different grades. On the other hand, several commodities can be combined into one composite commodity. The classification of goods occurring in practical economic life is to a certain degree conventional. In economic science, however, the classification of goods cannot be made on a purely arbitrary basis, because the laws of economics would then be dependent on the particular classification adopted. This would restrict the significance of the propositions of economics to a degree that would make them practically valueless. Each proposition might be changed into its opposite by a mere reclassification of goods. We adopt, therefore, the following *Principle of Invariance:*

The criterion of classification of goods must be such that reclassification of any group of goods in the economic system leaves invariant (1) all propositions of economic theory which relate to the subsystem consisting of the remaining goods, and (2) the formal mathematical structure of the propositions relating to the goods which are reclassified.

In equilibrium and stability theory the criterion required is obtained by

[20] *The General Theory of Employment, Interest and Money* (New York: Harcourt. Brace, 1936), pp. 257–271.

[21] *Op. cit.*, pp. 254–255. It seems, however, that he was not aware of the fact that his analysis and conclusions presuppose a neutral monetary system. Cf. footnote 10 on p. 24 above.

[22] Jacob Mosak, *General-Equilibrium Theory in International Trade*, Cowles Commission Monograph No. 7 (Bloomington, Indiana: Principia Press, 1944), pp. 162–164.

means of the following consideration: Take a system consisting of $n+1$ goods (including money). Let q ($q<n$) goods be such that their prices vary *always* in the same proportion. Combine these goods into one composite good and define the price of the composite good as a linear combination of the prices of the q goods. Without loss of generality, we can assume that these are the goods $1, 2, \cdots, q$, and the composite good may be represented by the symbol $(1q)$. We have then

$$p_r(t) \equiv b_r p_q(t) \qquad (r = 1, 2, \cdots, q-1). \tag{5.1}$$

where $b_r = \text{const.} > 0$ ($r=1, 2, \cdots, q-1$). Denoting the price of the composite good by $p_{(1q)}$, we shall write

$$p_{(1q)}(t) \equiv \sum_{r=1}^{q} w_r p_r(t) \qquad (w_r = \text{const.} > 0). \tag{5.2}$$

Combining (5.1) and (5.2) we find

$$p_r(t) \equiv c_r p_{(1q)}(t) \qquad (r = 1, 2, \cdots, n), \tag{5.3}$$

where

$$c_r = \frac{b_r}{\sum_{s=1}^{q} w_s b_s} > 0 \qquad (b_q = 1). \tag{5.4}$$

The excess demand $X_{(1q)}$ for the composite good $(1q)$ will be defined by the relation

$$p_{(1q)} X_{(1q)} \equiv \sum_{r=1}^{q} p_r X_r. \tag{5.5}$$

Together with (5.2), this leads to the relations

$$X_r = w_r X_{(1q)} \qquad (r = 1, 2, \cdots, q). \tag{5.6}$$

Taking into account (5.3), we write this in the form

$$\begin{aligned} X_r(p_1, p_2, \cdots, p_q; p_{q+1}, \cdots, p_n) &\equiv X_r[c_1 p_{(1q)}, c_2 p_{(1q)}, \cdots, c_q p_{(1q)}; p_{q+1}, \cdots, p_n] \\ &\equiv w_r X_{(1q)}[p_{(1q)}, p_{q+1}, \cdots, p_n]. \end{aligned}$$

Following our previous notation, let us write

$$a_{(1q)s} = \frac{\partial X_{(1q)}}{\partial p_s} \qquad [s = (1q), q+1, q+2, \cdots, n], \tag{5.7}$$

and we obtain the relations

$$(5.8)\qquad \sum_{s=1}^{q} a_{rs}c_s = w_r a_{(1q)(1q)} \qquad (r = 1, 2, \cdots, q).$$

$$a_{rs} = w_r a_{(1q)s} \qquad \text{for } s = q+1, q+2, \cdots, n.$$

Consider the system of differential equations

$$(5.9)\qquad \frac{dp_r}{dt} = F_r' \sum_{s=1}^{n} a_{rs}{}^0 p_s \qquad (r = 1, 2, \cdots, n),$$

i.e., the system (2.5) discussed above. Because of (5.3) and (5.8) this system can be written in the following form:

$$(5.10)\qquad \begin{aligned} \frac{dp_{(1q)}}{dt} &= \frac{F_r' w_r}{c_r}\left[a_{(1q)(1q)}{}^0 p_{(1q)} + \sum_{s=q+1}^{n} a_{(1q)s}{}^0 p_s\right] && \text{for } r = 1, 2, \cdots, q, \\ \frac{dp_r}{dt} &= F_r' \sum_{s=1}^{n} a_{rs}{}^0 p_s && \text{for } r = q+1, q+2, \cdots, n. \end{aligned}$$

Since the system (5.10) is equivalent to the system (5.9) the prices of the goods $q+1$, $q+2, \cdots, n$ are not affected by the combination of the goods $1, 2, \cdots, q$ into a composite good. We see from (5.10) that the differential equations for $s=q+1$, $q+2, \cdots, n$ are not affected either. The prices $p_1, p_2, \cdots, p_q$ are transformed into $p_{(1q)}$ through multiplication by a constant. By writing

$$(5.11)\qquad F_{(1q)}' = \frac{F_r' w_r}{c_r} \qquad (r = 1, 2, \cdots, q),$$

the system (5.10) can be written in the reduced form

$$(5.12)\qquad \frac{dp_r}{dt} = F_r' \sum_{s} a_{rs}{}^0 p_s \qquad [r \text{ and } s = (1q), q+1, q+2, \cdots, n].$$

Comparing this reduced system with the original system (5.9) we find that the first q differential equations in (5.9) are reduced to one equation which retains the mathematical structure of the original equations (i.e., is a linear equation with constant coefficients). We see also that the composite good behaves exactly as if it were a single good and that the composition does not affect the other goods in any way.

The passage from the system (5.9) to the system (5.10) or (5.12) is equivalent to subjecting the system (5.9) to the algebraic transformations

$$(5.13)\qquad \begin{aligned} \frac{dp_r}{dt} &\equiv \frac{c_r}{w_r}\frac{dp_{(1q)}}{dt} && \text{for } r = 1, 2, \cdots, q, \\ \frac{dp_r}{dt} &\equiv \frac{dp_r}{dt} && \text{for } r = q+1, q+2, \cdots, n, \end{aligned}$$

and

$$(5.14) \qquad \begin{aligned} p_s(t) &\equiv c_s p_{(1q)}(t) \qquad \text{for } s = 1, 2, \cdots, q, \\ p_s(t) &\equiv p_s(t) \qquad \text{for } s = q+1, q+2, \cdots, n. \end{aligned}$$

These transformations are nonsingular and can be inverted. In economic terms the inverse transformations mean the splitting-up of the composite good $(1q)$ into q separate goods. The inverse transformations change neither the prices of the goods $q+1$, $q+2$, $\cdots$, n nor the corresponding differential equations. The prices of the separated goods $1, 2, \cdots, q$ are obtained by multiplying the price of the composite good by a constant and the corresponding differential equations retain the mathematical structure of the original equation.

Thus the transformations (5.13) and (5.14) as well as their inverses satisfy our *Principle of Invariance*. This consideration leads us to the following criterion of classification of goods:

Any goods the prices of which always vary in the same proportion can be combined into one composite good; and, conversely, any good can be split up into an arbitrary number of separate goods with prices varying always in the same proportion.

We shall call it the *law of composition of goods*. By application of this law the number of goods in the theoretical system can be reduced to a certain minimum. This minimum is attained when no two goods in the system are such that their prices vary always in the same proportion. In this case the theoretical system will be said to be *canonical*. In a canonical system the number of goods is uniquely determined. In a noncanonical system the number of goods is arbitrary and need not even be finite. For any good can be split up into several goods with prices always varying proportionally. By successive application of transformations of this kind the number of goods can be increased indefinitely.

Constant prices are a special case of prices which always vary in the same proportion, namely in the same proportion as the price of money, which equals unity by definition. Thus all goods with rigid prices can be combined with money into one composite good. In a canonical system the introduction of rigid prices is synonymous with a reduction of the number of goods. This suggests an interpretation of the rank of stability of economic equilibrium. Stability of rank $n-q$ of a system containing $n+1$ goods (including money) means that q prices must be kept rigid in order to secure stability. This means that the corresponding canonical systems cannot contain more than $n-q+1$ goods and still be stable. The instability is due to there being q goods too many. In order to secure stability q goods must be combined with money into one composite good. Thus stability short of total stability can be interpreted as indicating an excessive number of goods in the canonical system.

6. Imperfect Competition

With some reinterpretation of the economic meaning of symbols, our analysis can be extended to systems containing forms of imperfect competition where sellers or buyers are confronted with determinate and differentiable demand or supply functions. These forms are monopoly and monopsony, monopolistic and monopsonistic competition.[23] This presupposes that each seller deals with atomistic buyers and each buyer deals with atomistic sellers. Each nonatomistic seller or buyer must be regarded as dealing in a separate good. Equilibrium obtains in the system when all prices are such that every seller and every buyer maximizes his profit or utility. If perfectly competitive markets are present, excess demand must vanish in them.

That atomistic buyers and sellers maximize their profit or utility is implied in the construction of their demand and supply functions. The demand and supply functions of the atomistic buyers and sellers being given, the profit or utility U_r, which the nonatomistic seller or buyer of the good r maximizes, can be considered as a function of the prices, i.e., $U_r \equiv U_r(p_1, p_2, \cdots, p_n)$. Of these prices the nonatomistic seller or buyer controls only p_r, and, under the forms of imperfect competition under consideration, he does not take into account a possible influence of a change in p_r upon other prices. We define now for each nonatomistic seller and buyer a function $X_r(p_1, p_2, \cdots, p_n)$, such that

(6.1) $X_r \equiv \dfrac{\partial U_r}{\partial p_r}$ (r running through any values of the sequence $1, 2, \cdots, n$).

We shall call it the *marginal-gain function.*

$X_r = 0$ when the nonatomistic seller or buyer of the good r maximizes his profit or utility. The second-order maximum condition requires that $X_r \gtrless 0$ according as his price is less or greater than the price which maximizes his profit or utility. Thus when $X_r > 0$ the nonatomistic seller or buyer raises his price. He lowers his price when $X_r < 0$. The functions X_r thus conform to the equations (2.1) and, consequently, the differential equations (2.2) and (2.5).[24] In these equations the functions X_r can, therefore, be interpreted as excess-demand functions when the market for the good r is subject to perfect competition, and as marginal-gain functions when competition is imperfect. In this way our analysis can be extended to systems which contain imperfections of competition of the type mentioned. The conclusions of Sections 1–3 and 5 hold fully for such systems.

[23] Oligopoly and oligopsony based on group behavior are excluded because the demand or supply functions, though determinate, are not differentiable at the point of the conventionally established price.

[24] They also satisfy the inequalities (1.2) which are Professor Hicks's conditions for "imperfect" stability.

The properties of homogeneous systems established in Section 4 hold in systems which contain imperfect competition in any of the goods $1, 2, \cdots, m$ (i.e., commodities and stocks), provided the nonatomistic buyers and sellers are firms.

Suppose that the assumptions of Section 4 are satisfied in the atomistic markets. Since in nonatomistic markets excess demand is always zero, irrespective of whether these markets are in equilibrium or not, the corresponding terms in identity (4.1) vanish. This identity is thus restricted to terms relating to atomistic markets and the first property of homogeneous systems follows immediately.

Suppose further that in each atomistic market the demand function confronting the monopolist or the supply function confronting the monopsonist is homogeneous of zero degree in the prices $p_1, p_2, \cdots, p_m$. Denote the demand function or supply function confronting the nonatomistic seller or buyer of the rth good by $D_r(p_1, p_2, \cdots, p_n)$ or $S_r(p_1, p_2, \cdots, p_n)$, respectively. The firm's profit can be expressed in the form

$$U_r(p_1, p_2, \cdots, p_n) \equiv p_r D_r + \sum_{s \neq r} p_s q_s \tag{6.2a}$$

or

$$U_r(p_1, p_2, \cdots, p_n) \equiv -\, p_r S_r + \sum_{s \neq r} p_s q_s, \tag{6.2b}$$

according as the firm sells or buys the rth good in a nonatomistic market. The q_s are quantities of goods sold or bought in atomistic markets and can be any of the goods $1, 2, \cdots, m$. The q_s which stand for goods bought are negative. Given all prices except p_r, the quantities q_s are chosen so as to maximize the firm's profit. These quantities are thus determined by the set of equations

$$\frac{\partial U_r}{\partial q_s} = \frac{\partial D_r}{\partial q_s}\left(p_r + D_r \frac{\partial p_r}{\partial D_r}\right) + p_s = 0 \tag{6.3a}$$

or $\qquad (s \neq r).$

$$\frac{\partial U_r}{\partial q_s} = -\frac{\partial S_r}{\partial q_s}\left(p_r + S_r \frac{\partial p_r}{\partial S_r}\right) + p_s = 0 \tag{6.3b}$$

In these equations $\partial D_r/\partial q_s$ or $\partial S_r/\partial q_s$ is derived from the firm's transformation function and is the marginal rate of transformation of the sth into the rth good, or vice versa. $\partial p_r/\partial D_r$ or $\partial p_r/\partial S_r$ is the reciprocal of the partial derivative of the demand function or supply function, respectively.

Since D_r or S_r is homogeneous of zero degree in $p_1, p_2, \cdots, p_m$, $\partial p_r/\partial D_r$ or $\partial p_r/\partial S_r$ is homogeneous of first degree in the same variables ($\partial D_r/\partial p_r$ or $\partial S_r/\partial p_r$ is homogeneous of degree -1). The prices p_r and p_s being among the variables $p_1, p_2, \cdots, p_m$, the equations in (6.3) are invariant under a proportional change of these variables. Consequently, the quantities q_s, which are the solutions of these equations, are not affected by a proportional change in the prices $p_1, p_2, \cdots, p_m$. It follows that the expression (6.2) is homogeneous of first degree in $p_1, p_2, \cdots, p_m$, because the q_s as well as D_r or S_r remain constant when p_r and the p_s all change in the same proportion. The marginal-gain function $X_r \equiv \partial U_r/\partial p_r$ is, therefore, homogeneous of zero degree in $p_1, p_2, \cdots, p_m$. The second and third property of homogeneous system follow from the results of Section 4 by mere reinterpretation of symbols.