Supplemental Material for
IDENTIFICATION-ROBUST SUBVECTOR INFERENCE

By

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Contents

10 Outline

References to sections with section numbers less than 10 refer to sections of the main paper "Identification-Robust Subvector Inference." Similarly, all theorems and lemmas with section numbers less than 10 refer to results in the main paper.

Section 11 generalizes the asymptotic results in Section 8 for the moment condition model from i.i.d. observations to strictly stationary strong mixing time series observations.

Sections 12-14 of this Supplemental Material (SM) provide high-level sufficient conditions for the parts of Assumptions B, C, and OE (stated in Section 5) that concern (i) the estimator set, (ii) the first-step AR CS, and (iii) the data-dependent second-step significance level, respectively, in the moment condition model. Sections 15-17 provide high-level sufficient conditions for the parts of Assumptions B, C, and OE that concern the second-step C(α)-AR, C(α)-LM, and C(α)-QLR1 tests, respectively, in the moment condition model. Section 18 amalgamates the conditions in Sections 12-17 for the two-step AR/AR, AR/LM, and AR/QLR1 subvector tests. Sections 19 and 20 prove Theorems 8.1 and 8.2 by using primitive conditions to verify the sufficient conditions in Section 18.

Section 21 proves the times series results in Theorem 11.1.

Section 22 provides some additional simulation results to those presented in the main paper. For illustrative purposes, Section 23 defines C(α) versions of the CQLR tests in Andrews and Guggenberger (2015) and I. Andrews and Mikusheva (2016), but does not verify the high-level conditions in Section 5 for these tests.

In this SM, a null sequence S is defined as in (5.2), i.e., $S := \{(\theta_{*n}, F_n) : (\theta_{1*n}, F_n) \in \mathcal{F}_{SV}$, $\theta_{2*n} = \theta_{20}, n \geq 1\}$, except in some sections where S is defined with a specific parameter space, such as $\mathcal{F}_{AR/AR}$, in place of the generic parameter space $\mathcal{F}_{SV}$. For an assumption that is stated for a sequence S, we say that it holds for a subsequence S_m if the subsequence version of the assumption holds.

Throughout the SM, we use the following notational convention when considering tests of $H_0 : \theta_2 = \theta_{20}$. For any function $A(\theta)$ of $\theta = (\theta_1', \theta_2')'$, we define

$$A(\theta_1) := A(\theta_1, \theta_{20}) \text{ and } A := A(\theta_{1*n}, \theta_{20}), \tag{10.1}$$

where θ_{20} is the null value of θ_2 and θ_{1*n} is the true value of θ_1.

We let $B(\theta_1, \varepsilon)$ denote a closed ball in R^{p_1} centered at θ_1 with radius $\varepsilon > 0$.

We let τ_{jsn} for $s = 1, ..., p_j$ denote the singular values of $\Omega_n^{-1/2} G_{jn}$ written in nonincreasing order, for $j = 1, 2$, where $G_{jn} \in R^{k \times p_j}$ and $\Omega_n \in R^{k \times k}$ are (nonrandom) population matrices that correspond to the sample Jacobian (wrt θ_j) $\widehat{G}_{jn} \in R^{k \times p_j}$ and sample variance matrix $\widehat{\Omega}_n$, respectively. Let $G_n := [G_{1n} : G_{2n}] \in R^{k \times p}$. Because they arise frequently below, for notational simplicity, we let

$$\begin{aligned} \tau_{jn} &:= \tau_{jp_jn} = \text{smallest singular value of } \Omega_n^{-1/2} G_{jn} \text{ for } j = 1, 2 \text{ and} \\ \tau_n &:= \text{smallest singular value of } \Omega_n^{-1/2} G_n. \end{aligned} \tag{10.2}$$

The quantity τ_{jn} is a measure of the (local) strength of identification of θ_j at $\theta = (\theta'_{1*n}, \theta'_{20})'$ and τ_n is a measure of the (local) strength of identification of θ at $\theta = (\theta'_{1*n}, \theta'_{20})'$.

Similarly, we let τ^{Φ}_{jsn} for $s = 1, ..., p_j$ denote the singular values of $\Omega_n^{-1/2} G_{jn} \Phi_{jn}$ written in nonincreasing order, for $j = 1, 2$, where $\Phi_{jn} \in R^{p_j \times p_j}$ is the (nonrandom) population matrix that corresponds to $\widehat{\Phi}_{jn} \in R^{p_j \times p_j}$ defined in (7.5). For notational simplicity, we let

$$\begin{aligned} \tau^{\Phi}_{jn} &:= \tau^{\Phi}_{jp_jn} = \text{smallest singular value of } \Omega_n^{-1/2} G_{jn} \Phi_{jn} \text{ for } j = 1, 2 \text{ and} \\ \tau^{\Phi}_n &:= \text{smallest singular value of } \Omega_n^{-1/2} G_n Diag\{\Phi_1, \Phi_2\}. \end{aligned} \tag{10.3}$$

The quantity τ^{Φ}_{jn} is another (equivalent) measure of the (local) strength of identification of θ_j at $\theta = (\theta'_{1*n}, \theta'_{20})'$.

In Sections 12 and 13 in this SM, the results are designed to hold not just for the moment condition model, but also for minimum distance models and moment condition models where the moments may depend on $n^{1/2}$-consistent and asymptotically normal preliminary estimators. But, the definitions of $G_{jn}(\theta)$ for $j = 1, 2$ and $\Omega_n(\theta)$ differ across these models. In consequence, for generality, $G_{jn}(\theta)$ and $\Omega_n(\theta)$ are defined in these sections by the conditions they must satisfy in the various results given, rather than by explicit expressions. For sample moments $\widehat{g}_n(\theta)$ (without any preliminary estimators), this leads to $G_{jn}(\theta) = E_{F_n} \widehat{G}_{jn}(\theta)$ and $\Omega_n(\theta) = Var_{F_n}(n^{1/2} \widehat{g}_n(\theta))$. The latter definitions are employed in Section 15 and the sections that follow it, which consider only the moment condition model.

Many results in this SM are stated to hold for both a sequence S and a subsequence S_m. For brevity, we only prove these results for a sequence S. For a subsequence S_m, the proofs only require the minor notational adjustment of changing n to m_n.

11 Time Series Observations

In this section, we generalize the results of Theorems 8.1 and 8.2 from i.i.d. observations to strictly stationary strong mixing observations. In the time series case, F denotes the distribution of the stationary infinite sequence $\{W_i : i = ..., 0, 1, ...\}$. Asymptotics under drifting sequences of true distributions $\{F_n : n \geq 1\}$ are used to establish the correct asymptotic size of the two-step AR/AR, AR/LM, and AR/QLR1 tests. Under such sequences, the observations form a triangular array of row-wise strictly stationary observations. In the time series case, we define $\Omega_F(\theta)$ differently from its definitions in (8.1) for the i.i.d. case:

$$\Omega_F(\theta) := \sum_{m=-\infty}^{\infty} \left(E_F g_i(\theta) g_{i-m}'(\theta) - E_F g_i(\theta) E_F g_i(\theta)' \right). \tag{11.1}$$

Note that $\Omega_F(\theta) = \lim Var_F(n^{-1/2} \sum_{i=1}^n g_i(\theta))$. We let $\{\alpha_F(m) : m \geq 1\}$ denote the strong mixing numbers of the observations under the distribution F.

The time series analogue $\mathcal{F}_{TS,AR/AR}$ of the space of distributions $\mathcal{F}_{AR/AR}$, defined in (8.8), is

$$\begin{aligned} \mathcal{F}_{TS,AR/AR} := \{&(\theta_1, F) : E_F g_i(\theta_1) = 0^k,\ \theta_1 \in \Theta_{1*},\ \{W_i : i = ..., 0, 1, ...\} \text{ are stationary} \\ &\text{and strong mixing under } F \text{ with } \alpha_F(m) \leq Cm^{-d} \text{ for some } d > (2+\gamma)/\gamma, \\ &E_F ||g_i(\theta_1)||^{2+\gamma} \leq M,\ E_F ||vec(G_{1i}(\theta_1))||^{2+\gamma} \leq M,\ E_F \xi_{1i}^{2+\gamma} \leq M, \\ &\lambda_{\min}(\Omega_F(\theta_1)) \geq \delta,\ Var_F(||G_{1si}(\theta_1)||) \geq \delta\ \forall s = 1, ..., p_1\} \end{aligned} \tag{11.2}$$

for some $\gamma, \delta > 0$ and $M < \infty$, where $\Omega_F(\theta)$ is defined in (11.1). For the two-step AR/LM and AR/QLR1 tests with time series observations, we use the parameter space $\mathcal{F}_{TS,AR/LM,QLR1}$, which is defined as in (8.10), but with $\mathcal{F}_{TS,AR/AR}$ in place of $\mathcal{F}_{AR/AR}$.

For CS's, we use the parameter spaces $\mathcal{F}_{\Theta,TS,AR/AR}$ and $\mathcal{F}_{\Theta,TS,AR/LM,QLR1}$, which are defined as in (8.9) and (8.11), respectively, but with $\mathcal{F}_{TS,AR/AR}(\theta_2)$ in place of $\mathcal{F}_{AR/AR}(\theta_2)$, where $\mathcal{F}_{TS,AR/AR}(\theta_{20})$ denotes $\mathcal{F}_{TS,AR/AR}$ with its dependence on θ_{20} made explicit.

Next, we define the second-step C(α)-AR, C(α)-LM, and C(α)-QLR1 tests in the time series context. To do so, we let

$$\begin{aligned} V_F(\theta) &:= \lim Var_F \left(n^{-1/2} \sum_{i=1}^n \begin{pmatrix} g_i(\theta) \\ vec(G_i(\theta)) \end{pmatrix} \right) \\ &= \sum_{m=-\infty}^{\infty} E_F \begin{pmatrix} g_i(\theta) - E_F g_i(\theta) \\ vec(G_i(\theta) - E_F G_i(\theta)) \end{pmatrix} \begin{pmatrix} g_{i-m}(\theta) - E_F g_{i-m}(\theta) \\ vec(G_{i-m}(\theta) - E_F G_{i-m}(\theta)) \end{pmatrix}'. \end{aligned} \tag{11.3}$$

The second equality holds for all $(\theta, F) \in \mathcal{F}_{\Theta,TS,AR/AR}$.

The test statistics depend on an estimator $\widehat{V}_n(\theta)$ of $V_F(\theta)$. This estimator (usually) is a heteroskedasticity and autocorrelation consistent (HAC) variance estimator based on the observations

$\{f_i(\theta) - \widehat{f}_n(\theta) : i \leq n\}$, where $f_i(\theta) := (g_i(\theta)', vec(G_i(\theta))')'$ and $\widehat{f}_n(\theta) := (\widehat{g}_n(\theta)', vec(\widehat{G}_n(\theta))')'$. There are a number of HAC estimators available in the literature, e.g., see Newey and West (1987) and Andrews (1991b). The asymptotic properties of the tests are the same for any consistent HAC estimator. Hence, for generality, we do not specify a particular estimator $\widehat{V}_n(\theta)$. Rather, we state results that hold for any estimator $\widehat{V}_n(\theta)$ that satisfies the following consistency conditions. The Assumptions V and V-CS that follow are applied with two-step tests and CS's, respectively.

Assumption V. $\forall K < \infty$, $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{V}_n(\theta_1) - V_{F_n}(\theta_1)|| \to_p 0$ under any null sequence $\{(\theta_{1*n}, F_n) \in \mathcal{F}_{TS,AR/AR} : n \geq 1\}$ for which $V_{F_n}(\theta_{1*n}) \to V_\infty$ for some pd matrix V_∞.

Assumption V-CS. $\forall K < \infty$, $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{V}_n(\theta_1, \theta_{*2n}) - V_{F_n}(\theta_1, \theta_{*2n})|| \to_p 0$ under any sequence $\{(\theta_{*n}, F_n) \in \mathcal{F}_{\Theta,TS,AR/AR} : n \geq 1\}$ for which $V_{F_n}(\theta_{*n}) \to V_\infty$ for some pd matrix V_∞.

We write the $(p+1)k \times (p+1)k$ matrix $\widehat{V}_n(\theta)$ in terms of its $k \times k$ submatrices:

$$\widehat{V}_n(\theta) = \begin{bmatrix} \widehat{\Omega}_n(\theta) & \widehat{\Gamma}_{11n}(\theta)' & \cdots & \widehat{\Gamma}_{2p_2n}(\theta)' \\ \widehat{\Gamma}_{11n}(\theta) & \widehat{V}_{G_{11}n}(\theta) & \cdots & \widehat{V}'_{G_{p1}n}(\theta) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\Gamma}_{2p_2n}(\theta) & \widehat{V}_{G_{p1}n}(\theta) & \cdots & \widehat{V}_{G_{pp}n}(\theta) \end{bmatrix}, \tag{11.4}$$

where the subscripts on $\widehat{\Gamma}_{jsn}(\theta)$ run from $(j,s) = (1,1), ..., (1,p_1), (2,1), ..., (2,p_2)$.

In the time series case, for the two-step C(α)-AR, C(α)-LM, and C(α)-QLR1 tests, we use the same definitions as in Section 3 for the moment condition model and Section 7, but with $\widehat{\Omega}_n(\theta)$ and $\widehat{\Gamma}_{jsn}(\theta)$ for $j = 1, ..., p_j$, $j = 1, 2$ defined as in Assumption V and (11.4), rather than as in (3.6) and (7.9). The two-step C(α)-AR, C(α)-LM, and C(α)-QLR1 CS's in the time series case are defined using (4.4), the definitions just given for the corresponding tests, and Assumption V-CS in place of Assumption V.

In the time series case, we employ the following assumption in addition to Assumption SI.

Assumption SI-TS. (i) For the null sequence S, the strong mixing numbers satisfy $\sum_{m=1}^{\infty} \alpha_{F_n}^{1/q-1/r}(m) < \infty$ for some $q = \max\{p_1 + \delta_1, 2\}$ and $r = q + \delta_1$ for some $\delta_1 > 0$, where r is as in Assumption SI, and

(ii) $\sup_{\theta_1 \in \Theta_1} ||\widehat{\Omega}_n(\theta_1) - \Omega_{F_n}(\theta_1)|| = o_p(1)$ for $\widehat{\Omega}_n(\theta)$ and $\Omega_F(\theta)$ defined in (11.4) and (11.1), respectively.

For the time series case, the asymptotic results are as follows.

Theorem 11.1 *Suppose the two-step* AR/AR, AR/LM, *and* AR/QLR1 *tests and* CS's *are defined as in this section and Assumption* V *or* V-CS *holds. Then, the results of Theorems* 8.1 *and* 8.2 *hold* *with the parameter spaces* $\mathcal{F}_{TS,AR/AR}$, $\mathcal{F}_{TS,AR/LM,QLR1}$, $\mathcal{F}_{\Theta,TS,AR/AR}$, *and* $\mathcal{F}_{\Theta,TS,AR/LM,QLR1}$ *in place of* $\mathcal{F}_{AR/AR}$, $\mathcal{F}_{AR/LM,QLR1}$, $\mathcal{F}_{\Theta,AR/AR}$, *and* $\mathcal{F}_{\Theta,AR/LM,QLR1}$, *respectively, and with Assumption* SI *augmented by Assumption* SI-TS (*everywhere Assumption* SI *appears in Theorems* 8.1 *and* 8.2).

Comment: Theorem 11.1 shows that the results of Theorems 8.1 and 8.2 for i.i.d. observations generalize to strictly stationary strong mixing observations, provided the spaces of distributions are adjusted suitably and the variance estimator $\widehat{V}_n(\theta)$ of $V_F(\theta)$ is defined appropriately.

12 Verification of Assumptions on the Estimator Set $\widehat{\Theta}_{1n}$

12.1 Estimator Set Results

The estimator set (ES), $\widehat{\Theta}_{1n}$, is defined in (7.3). The following lemma verifies Assumption C(i) under high-level assumptions on $\widehat{\Theta}_{1n}$ including the assumption that there exist $n^{1/2}$-consistent solutions $\{\overline{\theta}_{1n} : n \geq 1\}$ to the FOC's given in (7.3) (which implies that θ_1 is locally strongly identified given θ_{20}). Lemma 12.2 below provides sufficient conditions for the existence of such solutions $\{\overline{\theta}_{1n}\}$.

Assumption ES1. For the null sequence S, there exist solutions $\{\overline{\theta}_{1n} \in \Theta_1 : n \geq 1\}$ to the FOC's given in (7.3) that satisfy $n^{1/2}(\overline{\theta}_{1n} - \theta_{1*n}) = O_p(1)$.

Assumption ES2. For the null sequence S, (i) $\widehat{g}_n(\theta_1)$ is differentiable on $B(\theta_{1*n}, \varepsilon)$ for some $\varepsilon > 0$ (for all sample realizations) $\forall n \geq 1$, (ii) $\widehat{g}_n = O_p(n^{-1/2})$, (iii) $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon)} ||\widehat{G}_{1n}(\theta_1)|| = O_p(1)$ for some $\varepsilon > 0$, (iv) $nc_n \to \infty$ for $\{c_n : n \geq 1\}$ in (7.3), and (v) $\widehat{W}_{1n}$ is a symmetric psd $k \times k$ matrix that satisfies $\widehat{W}_{1n} = O_p(1)$.

Lemma 12.1 *Suppose* $\widehat{\Theta}_{1n}$ *is defined in* (7.3) *and* $\widehat{Q}_n(\theta)$ *is the criterion function defined in* (7.2). *Let* S *be a null sequence* (*or* S_m *a null subsequence*) *that satisfies Assumptions* ES1 *and* ES2. *Then,* $\widehat{\Theta}_{1n}$ *is non-empty wp→1 and Assumption* C(i) *holds for the sequence* S (*or subsequence* S_m).

The following lemma provides sufficient conditions for Assumption ES1 for sequences S that satisfy $\lim \tau_{1n} > 0$ (i.e., for θ_1-locally-strongly-identified sequences). Let $\theta_1 = (\theta_{11}, ..., \theta_{1p_1})'$.

Assumption FOC. For the null sequence S and some $\varepsilon > 0$, (i) $\liminf_{n\to\infty} \tau_{1n} > 0$, (ii) $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon)} ||\widehat{g}_n(\theta_1) - g_n(\theta_1)|| = o_p(1)$ for some nonrandom R^k-valued functions $\{g_n(\cdot) : n \geq 1\}$, (iii) $g_n = 0^k$ $\forall n \geq 1$, (iv) $\theta_{1*n} \to \theta_{1*\infty}$ for some $\theta_{1*\infty} \in \Theta_1$, (v) $\widehat{g}_n(\theta_1)$ is twice continuously differentiable on $B(\theta_{1*n}, \varepsilon)$ (for all sample realizations) $\forall n \geq 1$, (vi) $\widehat{g}_n = O_p(n^{-1/2})$, (vii) $g_n(\theta_1)$ is twice continuously differentiable on $B(\theta_{1*n}, \varepsilon)$ $\forall n \geq 1$, (viii) $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon)} ||\widehat{G}_{1n}(\theta_1) - G_{1n}(\theta_1)|| = o_p(1)$ for some nonrandom $R^{k\times p_1}$-valued functions $\{G_{1n}(\cdot) : n \geq 1\}$, (ix) $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon)} ||G_{1n}(\theta_1)|| =$

$O(1)$, (x) $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon_n)} ||G_{1n}(\theta_1) - G_{1n}|| = o(1)$ for all sequences of positive constants $\varepsilon_n \to 0$, (xi) $G_{1n} \to G_{1\infty}$ for some matrix $G_{1\infty} \in R^{k \times p_1}$, (xii) $G_{1n}(\theta_1) = (\partial/\partial\theta_1')g_n(\theta_1)$ $\forall \theta_1 \in B(\theta_{1*n}, \varepsilon), \forall n \geq 1$, (xiii) $\liminf_{n\to\infty} \lambda_{\min}(\Omega_n) > 0$, (xiv) $\widehat{\xi}_{1n} = O_p(1)$, where $\widehat{\xi}_{1n} := \max_{s,u \leq p_1} \sup_{\theta_1 \in B(\theta_{1*\infty},\varepsilon)} ||(\partial^2/\partial\theta_{1s}\partial\theta_{1u})\widehat{g}_n(\theta_1)||$, (xv) $\xi_{1n} = O(1)$, where $\xi_{1n} := \max_{s,u \leq p_1} \sup_{\theta_1 \in B(\theta_{1*\infty},\varepsilon)} ||(\partial^2/\partial\theta_{1s}\partial\theta_{1u})g_n(\theta_1)||$, and (xvi) $\widehat{W}_{1n}$ is symmetric and psd and $\widehat{W}_{1n} \to_p W_{1\infty}$ for some nonrandom nonsingular matrix $W_{1\infty} \in R^{k \times k}$.

In the moment condition model (where $\widehat{g}_n(\theta_1) = n^{-1}\sum_{i=1}^n g(W_i, \theta_1)$), we have $g_n(\theta_1) = E_{F_n}\widehat{g}_n(\theta_1)$ in Assumption FOC(ii), $G_{1n}(\theta_1) = E_{F_n}\widehat{G}_{1n}(\theta_1)$ in Assumption FOC(viii), and Assumptions FOC(vii), (xii), and (xv) (with ε replaced by $\varepsilon/2$ in Assumptions FOC(xii) and (xv), which does not matter in Lemma 12.2 below) are implied by Assumptions FOC(v) and (xiv) and $E_{F_n}\widehat{\xi}_{1n} = O(1)$ (for $\widehat{\xi}_{1n}$ defined in Assumption FOC(xiv)).[13] In this case, by Assumption FOC(xii), $G_{1n}(\theta_1) = E_{F_n}\widehat{G}_{1n}(\theta_1) = (\partial/\partial\theta_1')E_{F_n}\widehat{g}_n(\theta_1)$.

Lemma 12.2 *Let S be a null sequence (or S_m a null subsequence) that satisfies Assumption* FOC. *Then, Assumption* ES1 *holds for the sequence S (or subsequence S_m).*

Comments: (i). The result of Lemma 12.2 is similar to classical results in the statistical literature, see Cramér (1946), Aitchison and Silvey (1959), and Crowder (1976). The proof of Lemma 12.2 follows that of van der Vaart (1998, Thm. 5.42, p. 69).

(ii) The estimator $\widehat{\Omega}_n$ plays no role in Lemma 12.2. Nevertheless, Assumption FOC(xiii) involves Ω_n because Assumptions FOC(i) and (xiii) imply that the limit of the smallest singular value of G_{1n} is positive (since τ_{1n} is the smallest singular value of $\Omega_n^{-1/2}G_{1n}$).

The next lemma provides sufficient conditions for Assumption OE(i).

Lemma 12.3 *Let S be a null sequence (or S_m a null subsequence) for which* (i) $d_H(\theta_{1*n}, \widehat{\Theta}_{1n}) = O_p(n^{-1/2})$ *and* (ii) $d_H(\theta_{1*n}, CS_{1n} \cup \{\theta_{1*n}\}) = O_p(n^{-1/2})$. *Then, $d_H(\theta_{1*n}, CS_{1n}^+) = O_p(n^{-1/2})$ holds for the sequence S (or subsequence S_m).*

Comments: 1. Lemmas 12.4 and 13.2 below provide primitive sufficient conditions for conditions (i) and (ii), respectively, of Lemma 12.3.

2. Condition (i) of Lemma 12.3 requires that $\widehat{\Theta}_{1n} \neq \varnothing$ wp→1 by the definition of d_H.

[13] To see this, suppose Assumptions FOC(v) and (xiv) hold and $E_{F_n}\widehat{\xi}_{1n} = O(1)$. Taking expections under F_n in the first line of (12.6) below gives Assumption FOC(vii). By a similar expansion to that in (12.6), but about a point in $B(\theta_{1*n}, \varepsilon/2)$, rather than θ_{1*n}, gives $(\partial/\partial\theta_1')g_n(\theta_1) = E_{F_n}\widehat{G}_{1n}(\theta_1) = G_{1n}(\theta_1)$ $\forall \theta_1 \in B(\theta_{1*n}, \varepsilon/2)$, which implies Assumption FOC(xii), and $(\partial^2/\partial\theta_{1s}\partial\theta_{1\ell})g_n(\theta_1) = E_{F_n}(\partial^2/\partial\theta_{1s}\partial\theta_{1\ell})\widehat{g}_n(\theta_1)$ $\forall \theta_1 \in B(\theta_{1*n}, \varepsilon/2)$, which implies that $\xi_{1n} \leq E_{F_n}\widehat{\xi}_{1n} = O(1)$ and Assumption FOC(xv) holds.

Now we verify $d_H(\theta_{1*n}, \widehat{\Theta}_{1n}) = O_p(n^{-1/2})$ for $\widehat{\Theta}_{1n}$ defined in (7.3) for sequences S with lim $\inf_{n\to\infty} \tau_{1n} > 0$, i.e., for θ_1-locally strongly-identified sequences.

Assumption ES3. For the null sequence S, (i) $\liminf_{n\to\infty} \tau_{1n} > 0$, (ii) $\widehat{\Theta}_{1n}$ is non-empty wp→1, (iii) $\widehat{g}_n(\theta_1)$ is differentiable on $B(\theta_{1*n}, \varepsilon)$ for some $\varepsilon > 0$ (for all sample realizations) $\forall n \geq 1$, (iv) $\widehat{g}_n = O_p(n^{-1/2})$, (v) $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon)} ||\widehat{G}_{1n}(\theta_1) - G_{1n}(\theta_1)|| = o_p(1)$ for some $\varepsilon > 0$ for some nonrandom $R^{k\times p_1}$-valued functions $\{G_{1n}(\cdot) : n \geq 1\}$, (vi) $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon_n)} ||G_{1n}(\theta_1) - G_{1n}|| = o(1)$ for all sequences of positive constants $\varepsilon_n \to 0$, (vii) $G_{1n} \to G_{1\infty}$ for some matrix $G_{1\infty} \in R^{k\times p_1}$, (viii) $\widehat{\Omega}_n - \Omega_n \to_p 0$ for some nonrandom matrices $\{\Omega_n \in R^{k\times k} : n \geq 1\}$, (ix) $\liminf_{n\to\infty} \lambda_{\min}(\Omega_n) > 0$, (x) $c_n \to 0$ for $\{c_n : n \geq 1\}$ in (7.3), and (xi) $\widehat{W}_{1n}$ is symmetric and psd and $\widehat{W}_{1n} \to_p W_{1\infty}$ for some nonrandom nonsingular matrix $W_{1\infty} \in R^{k\times k}$.

Note that Assumption ES3(ii) is implied by Assumptions ES1 and ES2 by Lemma 12.1 and, hence, by Assumptions FOC and ES2 by Lemmas 12.1 and 12.2.

Assumption ES4. For the null sequence S, (i) $\sup_{\theta_1\in\Theta_1} n^{1/2}||\widehat{g}_n(\theta_1) - g_n(\theta_1)|| = O_p(1)$ for some nonrandom R^k-valued functions $\{g_n(\cdot) : n \geq 1\}$ and (ii) $\liminf_{n\to\infty} \inf_{\theta_1 \notin B(\theta_{1*n},\varepsilon)} ||g_n(\theta_1)|| > 0$ $\forall\varepsilon > 0$.

Lemma 12.4 *Suppose $\widehat{\Theta}_{1n}$ is of the form in* (7.3) *with $\widehat{Q}_n(\theta)$ as in* (7.2). *Let S be a null sequence* (*or S_m a null subsequence*) *that satisfies Assumptions* ES3 *and* ES4. *Then, $d_H(\theta_{1*n}, \widehat{\Theta}_{1n}) = O_p(n^{-1/2})$ for the sequence S* (*or subsequence S_m*).

12.2 Proofs of Lemmas 12.1-12.4

Proof of Lemma 12.1. Let $\overline{\theta}_{1n}$ be as in Assumption ES1. If $\overline{\theta}_{1n} \in \widehat{\Theta}_{1n}$ wp→1, then

$$d(\theta_{1*n}, CS^+_{1n}) \leq d(\theta_{1*n}, \widehat{\Theta}_{1n}) \leq d(\theta_{1*n}, \{\overline{\theta}_{1n}\}) = ||\theta_{1*n} - \overline{\theta}_{1n}|| = O_p(n^{-1/2}), \qquad (12.1)$$

where the first two inequalities hold because $\{\overline{\theta}_{1n}\} \subset \widehat{\Theta}_{1n} \subset CS^+_{1n}$ wp→1 using the definition of CS^+_{1n} in (4.1) and the last inequality holds by Assumption ES1. Hence, Assumption C(i) holds and $\widehat{\Theta}_{1n}$ is non-empty wp→1.

It remains to show $\overline{\theta}_{1n} \in \widehat{\Theta}_{1n}$ wp→1. By Assumption ES1, $\overline{\theta}_{1n}$ satisfies the first condition in the definition of $\widehat{\Theta}_{1n}$ in (7.3). Hence, it remains to show that the second condition in the definition of $\widehat{\Theta}_{1n}$ in (7.3) holds for $\overline{\theta}_{1n}$. That is, we need to show

$$\widehat{Q}_n(\overline{\theta}_{1n}) \leq \inf_{\theta_1\in\Theta_1} \widehat{Q}_n(\theta_1) + c_n \text{ wp} \to 1. \qquad (12.2)$$

Element-by-element mean-value expansions of $\widehat{g}_n(\overline{\theta}_{1n})$ about θ_{1*n} give

$$\widehat{g}_n(\overline{\theta}_{1n}) = \widehat{g}_n + \widehat{G}_{1n}(\widetilde{\theta}_{1n})(\overline{\theta}_{1n} - \theta_{1*n}) = O_p(n^{-1/2}) + O_p(1)O_p(n^{-1/2}) = O_p(n^{-1/2}), \quad (12.3)$$

where, as defined above, $\widehat{g}_n := \widehat{g}_n(\theta_{1*n})$, $\widetilde{\theta}_{1n}$ lies between $\overline{\theta}_{1n}$ and θ_{1*n} and may differ across the rows of $\widehat{G}_{1n}(\widetilde{\theta}_{1n})$, the first equality uses Assumption ES2(i), and the second equality holds by Assumptions ES1, ES2(ii), and ES2(iii). Equations (7.2) and (12.3) and Assumption ES2(v) yield $\widehat{Q}_n(\overline{\theta}_{1n}) = O_p(n^{-1})$. Hence, we have

$$\widehat{Q}_n(\overline{\theta}_{1n}) - c_n = O_p(n^{-1}) - c_n < 0 \leq \inf_{\theta_1 \in \Theta_1} \widehat{Q}_n(\theta_1), \quad (12.4)$$

where the strict inequality holds wp→1 because $nc_n \to \infty$ by Assumption ES2(iv). Hence, (12.2) holds. □

Proof of Lemma 12.2. First, we establish the existence of consistent (as opposed to $n^{1/2}$-consistent) solutions to the FOC's. Let

$$\widehat{\Psi}_n(\theta_1) := \widehat{G}_{1n}(\theta_1)'\widehat{W}_{1n}\widehat{g}_n(\theta_1) \text{ and } \Psi_n(\theta_1) := G_{1n}(\theta_1)'W_{1\infty}g_n(\theta_1). \quad (12.5)$$

We use essentially the same argument as in van der Vaart (1998, Thm. 5.42, p. 69), but with $\widehat{\Psi}_n(\theta_1)$ and $\Psi_n(\theta_1)$ in place of van der Vaart's $\Psi_n(\theta)$ and $\Psi(\theta)$, respectively. The main differences are that the population quantity $\Psi_n(\theta_1)$ here depends on n, whereas van der Vaart's population quantity $\Psi(\theta)$ does not; $\widehat{\Psi}_n(\theta_1)$ is a product of three random matrices none of which needs to be a sample average, whereas van der Vaart's $\Psi_n(\theta)$ is a sample average; and $\Psi_n(\theta_1)$ here is the product of three population matrices, whereas van der Vaart's $\Psi(\theta)$ is a single population matrix.

For $\theta_1 \in B(\theta_{1*n}, \varepsilon)$ (with $\varepsilon > 0$ as in Assumption FOC), element-by-element two-term Taylor expansions of $\widehat{g}_n(\theta_1)$ about θ_{1*n} give

$$\begin{aligned} \widehat{g}_n(\theta_1) &= \widehat{g}_n + \widehat{G}_{1n} \times (\theta_1 - \theta_{1*n}) + \frac{1}{2}\sum_{s=1}^{p_1}(\theta_{1s} - \theta_{1*ns})\frac{\partial}{\partial \theta_{1s}}\widehat{G}_{1n}(\widetilde{\theta}_{1n})(\theta_1 - \theta_{1*n}) \\ &= o_p(1) + G_{1n} \times (\theta_1 - \theta_{1*n}) + O_p(1)||\theta_1 - \theta_{1*n}||^2, \end{aligned} \quad (12.6)$$

where $\widetilde{\theta}_{1n}$ lies between θ_{1n} and θ_{1*n} and may differ across rows of $(\partial/\partial\theta_{1s})\widehat{G}_{1n}(\widetilde{\theta}_{1n})$, $\theta_1 = (\theta_{11}, ..., \theta_{1p_1})'$, the $o_p(1)$ and $O_p(1)$ terms (in (12.6) and below) hold uniformly over $\theta_1 \in B(\theta_{1*n}, \varepsilon)$ as $n \to \infty$, $\theta_{1*n} = (\theta_{1*n1}, ..., \theta_{1*np_1})'$, the first equality uses Assumption FOC(v), and the second equality uses Assumptions FOC(vi), (viii), and (xiv).

Similarly, for $\theta_1 \in B(\theta_{1*n}, \varepsilon)$, element-by-element two-term Taylor expansions of $g_n(\theta_1)$ about

θ_{1*n} give

$$g_n(\theta_1) = \frac{\partial}{\partial \theta_1'} g_n \times (\theta_1 - \theta_{1*n}) + \frac{1}{2} \sum_{s=1}^{p_1} (\theta_{1s} - \theta_{1*ns}) \frac{\partial^2}{\partial \theta_{1s} \partial \theta_1'} g_n(\widetilde{\theta}_{1n})(\theta_1 - \theta_{1*n})$$
$$= G_{1n} \times (\theta_1 - \theta_{1*n}) + O(1)||\theta_1 - \theta_{1*n}||^2, \quad (12.7)$$

where the $O(1)$ term (in (12.7) and below) holds uniformly over $\theta_1 \in B(\theta_{1*n}, \varepsilon)$ as $n \to \infty$, the first equality uses Assumption FOC(vii) and $g_n = 0^k$ (by Assumption FOC(iii)), and the second equality holds by Assumptions FOC(xii) and (xv).

For $\theta_1 \in B(\theta_{1*n}, \varepsilon)$, element-by-element mean-value expansions of $\widehat{G}_{1n}(\theta_1)$ about θ_{1*n} give

$$\widehat{G}_{1n}(\theta_1) = \widehat{G}_{1n} + \sum_{s=1}^{p_1} \frac{\partial}{\partial \theta_{1s}} \widehat{G}_{1n}(\theta_{1n}^\dagger)(\theta_{1s} - \theta_{1*ns}) = G_{1n} + o_p(1) + O_p(1)||\theta_1 - \theta_{1*n}||, \quad (12.8)$$

where $\theta_{1n}^\dagger$ lies between θ_1 and θ_{1*n} and may differ across rows of $(\partial/\partial\theta_{1s})\widehat{G}_{1n}(\theta_{1n}^\dagger)$, the mean-value expansions use Assumption FOC(v), and the second equality uses Assumptions FOC(viii) and (xiv).

For $\theta_1 \in B(\theta_{1*n}, \varepsilon)$, element-by-element mean-value expansions of $G_{1n}(\theta_1)$ about θ_{1*n} give

$$G_{1n}(\theta_1) = G_{1n} + \sum_{s=1}^{p_1} \frac{\partial}{\partial \theta_{1s}} G_{1n}(\theta_{1n}^\triangle)(\theta_{1s} - \theta_{1*ns}) = G_{1n} + O(1)||\theta_1 - \theta_{1*n}||, \quad (12.9)$$

where $\theta_{1n}^\triangle$ lies between θ_{1n} and θ_{1*n} and may differ across rows of $(\partial/\partial\theta_{1s})G_{1n}(\theta_{1n}^\triangle)$, the first equality uses Assumptions FOC(vii) and (xii) and the second equality holds using Assumption FOC(xv) (since $(\partial/\partial\theta_{1s})G_{1n}(\theta_1) = (\partial^2/\partial\theta_{1s}\partial\theta_1')g_n(\theta_1)$ by Assumption FOC(xii)).

Combining (12.5), (12.6), and (12.8) gives: For $\theta_1 \in B(\theta_{1*n}, \varepsilon)$,

$$\widehat{\Psi}_n(\theta_1) = (G_{1n} + o_p(1) + O_p(1)||\theta_1 - \theta_{1*n}||)'(W_{1\infty} + o_p(1))$$
$$\times (o_p(1) + G_{1n} \times (\theta_1 - \theta_{1*n}) + O_p(1)||\theta_1 - \theta_{1*n}||^2) \quad (12.10)$$
$$= G_{1\infty}' W_{1\infty} G_{1\infty} \times (\theta_1 - \theta_{1*n}) + o_p(1) + o_p(1)||\theta_1 - \theta_{1*n}|| + O_p(1)||\theta_1 - \theta_{1*n}||^2,$$

where the first equality uses Assumption FOC(xvi) and the second equality uses Assumption FOC(xi).

Combining (12.5), (12.7), and (12.9) gives: for $\theta_1 \in B(\theta_{1*n}, \varepsilon)$,

$$\begin{aligned}\Psi_n(\theta_1) &= (G_{1n} + O(1)||\theta_1 - \theta_{1*n}||)' W_{1\infty}(G_{1n} \times (\theta_1 - \theta_{1*n}) + O(1)||\theta_1 - \theta_{1*n}||^2) \\ &= G_{1n}' W_{1\infty} G_{1n} \times (\theta_1 - \theta_{1*n}) + O(1)||\theta_1 - \theta_{1*n}||^2 \\ &= G_{1\infty}' W_{1\infty} G_{1\infty} \times (\theta_1 - \theta_{1*n}) + o(1)||\theta_1 - \theta_{1*n}|| + O(1)||\theta_1 - \theta_{1*n}||^2, \end{aligned} \quad (12.11)$$

where the third equality uses Assumption FOC(xi).

Differentiability of $g_n(\theta_1)$ and $G_n(\theta_1)$ on $B(\theta_{1*n}, \varepsilon)$ holds by Assumptions FOC(vii) and (xii). This implies differentiability of $\Psi_n(\theta_1)$ on $B(\theta_{1*n}, \varepsilon)$. The derivative matrix of $\Psi_n(\theta_1)$ is

$$\begin{aligned}\frac{\partial}{\partial \theta_1'} \Psi_n(\theta_1) &= G_{1n}(\theta_1)' W_{1\infty} G_{1n}(\theta_1) \\ &\quad + \left[\left(\frac{\partial}{\partial \theta_{11}} G_{1n}(\theta_1) \right)' W_{1\infty} g_{1n}(\theta_1), ..., \left(\frac{\partial}{\partial \theta_{1p_1}} G_{1n}(\theta_1) \right)' W_{1\infty} g_{1n}(\theta_1) \right]. \end{aligned} \quad (12.12)$$

This matrix is uniformly continuous on $B(\theta_{1*n}, \varepsilon)$ by Assumption FOC(vii).

Now we show that for some $\varepsilon_1 \in (0, \varepsilon]$,

$$\liminf_{n\to\infty} \inf_{\theta_1 \in B(\theta_{1*n}, \varepsilon_1)} \lambda_{\min} \left(\frac{\partial}{\partial \theta_1'} \Psi_n(\theta_1) \right) > 0. \quad (12.13)$$

We have $\liminf_{n\to\infty} \lambda_{\min}(G_{1n}' W_{1\infty} G_{1n}) > 0$ (by Assumptions FOC(i), (xiii), and (xvi) because τ_{1n} is the smallest singular value of $\Omega_n^{-1/2} G_{1n}$). Using this and (12.9), we obtain: for some $\varepsilon_2 \in (0, \varepsilon]$,

$$\liminf_{n\to\infty} \inf_{\theta_1 \in B(\theta_{1*n}, \varepsilon_2)} \lambda_{\min}(G_{1n}(\theta_1)' W_{1\infty} G_{1n}(\theta_1)) > 0. \quad (12.14)$$

Next, by (12.7), $g_n(\theta_1) = O(1)||\theta_1 - \theta_{1*n}|| \ \forall \theta_1 \in B(\theta_{1*n}, \varepsilon)$. Hence,

$$\max_{s \le p_1} \left\| \left(\frac{\partial}{\partial \theta_{1s}} G_{1n}(\theta_1) \right)' W_{1\infty} g_{1n}(\theta_1) \right\| = O(1)||\theta_1 - \theta_{1*n}|| \ \forall \theta_1 \in B(\theta_{1*n}, \varepsilon) \quad (12.15)$$

using Assumptions FOC(xii) and (xv). This and (12.14) imply (12.13) for some sufficiently small $\varepsilon_1 > 0$.

Now, by the inverse function theorem applied to $\Psi_n(\theta_1)$, for every sufficiently small $\delta > 0$, there exists an open neighborhood $M_{n\delta}$ of θ_{1*n} such that $M_{n\delta} \subset B(\theta_{1*n}, \varepsilon_1)$ and the map $\Psi_n : cl(M_{n\delta}) \to B(0^{p_1}, \delta)$ is a homeomorphism, where $cl(M_{n\delta})$ denotes the closure of $M_{n\delta}$ and $B(0^{p_1}, \delta)$ is the closed ball at 0^{p_1} with radius δ. The diameter of $cl(M_{n\delta})$ is bounded by a multiple of δ that does not depend on n by the following argument:

$$\sup_{\theta_1 \in M_{n\delta}} ||\theta_1 - \theta_{1*n}|| = \sup_{\eta \in B(0^{p_1},\delta)} ||\Psi_n^{-1}(\eta) - \Psi_n^{-1}(0^{p_1})|| = \sup_{\eta \in B(0^{p_1},\delta)} ||(\partial/\partial \eta')\Psi_n^{-1}(\widetilde{\eta}_n)\eta||$$
$$= \sup_{\eta \in B(0^{p_1},\delta)} ||[(\partial/\partial\theta_1')\Psi_n(\theta_1)|_{\theta_1=\Psi_n^{-1}(\widetilde{\eta}_n)}]^{-1}\eta|| \leq \delta / \inf_{\theta_1 \in B(\theta_{1*n},\varepsilon_1)} \lambda_{\min}((\partial/\partial\theta_1')\Psi_n(\theta_1)) = O(1)\delta \quad (12.16)$$

for some $\widetilde{\eta}_n \in B(0^{p_1}, \delta)$ that may differ across the rows of $(\partial/\partial\eta')\Psi_n^{-1}(\widetilde{\eta}_n)$, where the first equality holds because $\Psi_n : cl(M_{n\delta}) \to B(0^{p_1}, \delta)$ is a homeomorphism and $\Psi_n^{-1}(0^{p_1}) = \theta_{1*n}$ by Assumption FOC(iii), the second equality holds by element-by-element mean-value expansions of $\Psi_n^{-1}(\eta)$ about 0^{p_1}, the third equality holds by the standard formula for the derivative matrix of an inverse function, and the last equality holds by (12.13).

Combining (12.10), (12.11), and (12.16) gives

$$\sup_{\theta_1 \in cl(M_{n\delta})} ||\widehat{\Psi}_n(\theta_1) - \Psi_n(\theta_1)|| = o_p(1) + o_p(1)\delta + O_p(1)\delta^2, \quad (12.17)$$

where the $o_p(1)$ and $O_p(1)$ terms are uniform in δ for δ small. (This equation is analogous to the second displayed equation on p. 69 of van der Vaart (1998). The remainder of the proof of the existence of consistent solutions to the FOC's is the same as in van der Vaart (1998), although for completeness we provide more details here.)

Because $P_{F_n}(o_p(1) + o_p(1)\delta > \delta/2) \to 0 \ \forall \delta > 0$, there exists a sequence $\delta_n \downarrow 0$ such that $P_{F_n}(o_p(1) + o_p(1)\delta_n > \delta_n/2) \to 0$. Let $K_{n\delta} := \{\sup_{\theta_1 \in cl(M_{n\delta})} ||\widehat{\Psi}_n(\theta_1) - \Psi_n(\theta_1)|| < \delta\}$. Then, we have

$$\begin{aligned} P_{F_n}(K_{n\delta_n}) := \ & P_{F_n}(\sup_{\theta_1 \in cl(M_{n\delta_n})} ||\widehat{\Psi}_n(\theta_1) - \Psi_n(\theta_1)|| < \delta_n) \\ = \ & P_{F_n}(o_p(1) + o_p(1)\delta_n + O_p(1)\delta_n^2 < \delta_n) \\ = \ & P_{F_n}(o_p(1) + o_p(1)\delta_n + O_p(1)\delta_n^2 < \delta_n, B_n^c) + o(1) \\ \geq \ & P_{F_n}(\delta_n/2 + O_p(1)\delta_n^2 < \delta_n, B_n^c) + o(1) \\ \to \ & 1, \end{aligned} \quad (12.18)$$

where the second equality uses (12.17), the third equality holds by writing $P_{F_n}(A_n) = P_{F_n}(A_n \cap B_n^c) + P_{F_n}(A_n \cap B_n)$ for $B_n = \{o_p(1) + o_p(1)\delta_n > \delta_n/2\}$ and $P_{F_n}(B_n) \to 0$, the inequality holds using the condition in B_n^c, and the convergence holds because $P_{F_n}(B_n^c) \to 1$ and $P_{F_n}(\delta_n/2 + O_p(1)\delta_n^2 < \delta_n) \to 1$ using $\delta_n \to 0$.

On the event $K_{n\delta}$, the map $\eta \to \eta - \widehat{\Psi}_n(\Psi_n^{-1}(\eta))$ maps $B(0^{p_1}, \delta)$ into itself (because $\forall \eta \in B(0^{p_1}, \delta)$, $||\eta - \widehat{\Psi}_n(\Psi_n^{-1}(\eta))|| \leq ||\eta - \Psi_n(\Psi_n^{-1}(\eta))|| + \sup_{\eta \in B(0^{p_1},\delta)} ||\widehat{\Psi}_n(\Psi_n^{-1}(\eta)) - \Psi_n(\Psi_n^{-1}(\eta))|| \leq \delta$, where the second inequality uses the definition of $K_{n\delta}$ and the fact that $\eta \in B(0^{p_1}, \delta)$ implies that $\Psi_n^{-1}(\eta) \in cl(M_{n\delta})$). This map is continuous. Hence, by Brouwer's fixed point theorem, it possesses a fixed point in $B(0^{p_1}, \delta)$. That is, there exists $\overline{\eta}_n \in B(0^{p_1}, \delta)$ such that $\overline{\eta}_n - \widehat{\Psi}_n(\Psi_n^{-1}(\overline{\eta}_n)) = \overline{\eta}_n$. For $\overline{\theta}_{1n} := \Psi_n^{-1}(\overline{\eta}_n) \in cl(M_{n\delta_n})$, this gives $\widehat{\Psi}_n(\overline{\theta}_{1n}) = 0^{p_1}$. Because the set $M_{n\delta_n}$ contains θ_{1*n}, the diameter of $M_{n\delta_n}$ is bounded by a multiple (that does not depend on n) of δ_n, and $\delta_n \downarrow 0$, we have $\overline{\theta}_{1n} - \theta_{1*n} \to_p 0^{p_1}$. Hence, $\overline{\theta}_{1n}$ is a consistent solution to the FOC's $\widehat{\Psi}_n(\theta_1) = 0^{p_1}$.

Given the consistency of the solutions $\{\overline{\theta}_{1n} : n \geq 1\}$ to the FOC's in (7.3), we now establish the $n^{1/2}$-consistency of $\{\overline{\theta}_{1n} : n \geq 1\}$. The FOC's, mean-value expansions around θ_{1*n}, and $\overline{\theta}_{1n} - \theta_{1*n} \to_p 0$ give

$$\begin{aligned} 0^{p_1} &= \widehat{G}_{1n}(\overline{\theta}_{1n})'\widehat{W}_{1n}\widehat{g}_n(\overline{\theta}_{1n}) = \widehat{G}_{1n}(\overline{\theta}_{1n})'\widehat{W}_{1n}\left(\widehat{g}_n + \widehat{G}_{1n}(\widetilde{\theta}_{1n})(\overline{\theta}_{1n} - \theta_{1*n})\right) \text{ and} \\ \overline{\theta}_{1n} - \theta_{1*n} &= -\left(\widehat{G}_{1n}(\overline{\theta}_{1n})'\widehat{W}_{1n}\widehat{G}_{1n}(\widetilde{\theta}_{1n})\right)^{-1} \widehat{G}_{1n}(\overline{\theta}_{1n})'\widehat{W}_{1n}\widehat{g}_n = O_p(n^{-1/2}), \end{aligned} \tag{12.19}$$

where $\widetilde{\theta}_{1n}$ lies between $\overline{\theta}_{1n}$ and θ_{1*n} and may differ across the rows of $\widehat{G}_{1n}(\widetilde{\theta}_{1n})$ and the last equality holds because $\widehat{G}_{1n}(\overline{\theta}_{1n})'\widehat{W}_{1n}\widehat{G}_{1n}(\widetilde{\theta}_{1n}) \to_p G_{1\infty}'W_{1\infty}G_{1\infty}$ (by Assumptions FOC(viii), (x), (xi), and (xvi) and $\overline{\theta}_{1n} - \theta_{1*n} \to_p 0$), $G_{1\infty}'W_{1\infty}G_{1\infty}$ is nonsingular (since $W_{1\infty}$ is nonsingular by Assumption FOC(xvi) and $G_{1\infty}$ has full column rank p_1 by Assumptions FOC(i), (xi), and (xiii) because τ_{1n} is the smallest singular value of $\Omega_n^{-1/2}G_{1n}$), $\widehat{G}_{1n}(\overline{\theta}_{1n})'\widehat{W}_{1n} = O_p(1)$ (by Assumptions FOC(viii), (x), (xi), and (xvi) and $\overline{\theta}_{1n} - \theta_{1*n} \to_p 0$), and $\widehat{g}_n = O_p(n^{-1/2})$ (by Assumption FOC(vi)). Equation (12.19) completes the proof of the lemma. □

Proof of Lemma 12.3. We have

$$\begin{aligned} d_H(\theta_{1*n}, CS_{1n}^+) &= d_H(\theta_{1*n}, CS_{1n} \cup \widehat{\Theta}_{1n})1(CS_{1n} = \varnothing) + d_H(\theta_{1*n}, CS_{1n} \cup \widehat{\Theta}_{1n})1(CS_{1n} \neq \varnothing) \\ &\leq d_H(\theta_{1*n}, \widehat{\Theta}_{1n})[1(CS_{1n} = \varnothing) + 1(CS_{1n} \neq \varnothing)] + d_H(\theta_{1*n}, CS_{1n})1(CS_{1n} \neq \varnothing) \\ &\leq d_H(\theta_{1*n}, \widehat{\Theta}_{1n}) + d_H(\theta_{1*n}, CS_{1n} \cup \{\theta_{1*n}\}) \\ &= O_p(n^{-1/2}), \end{aligned} \tag{12.20}$$

where the first inequality holds using straightforward manipulations, the second inequality holds because $CS_{1n} \neq \varnothing$ implies $d_H(\theta_{1*n}, CS_{1n}) = d_H(\theta_{1*n}, CS_{1n} \cup \{\theta_{1*n}\})$, and the last equality holds by conditions (i) and (ii) of the lemma. □

Proof of Lemma 12.4. Let $\{\widehat{\theta}_{1n} : n \geq 1\}$ be a sequence in $\widehat{\Theta}_{1n}$ (for all $n \geq 1$ and all sample realizations) for which $||\theta_{1*n} - \widehat{\theta}_{1n}|| = d_H(\theta_{1*n}, \widehat{\Theta}_{1n}) + o_p(n^{-1/2})$. Such a sequence exists wp→1 by Assumption ES3(ii).

Let $q_n := \inf_{\theta_1 \notin B(\theta_{1*n},\varepsilon)} ||g_n(\theta_1)||$. By Assumption ES4(ii), $\liminf_{n\to\infty} q_n > 0$. By the definition of q_n, $\widehat{\theta}_{1n} \notin B(\theta_{1*n}, \varepsilon)$ implies $||g_n(\widehat{\theta}_{1n})|| \geq q_n$ $\forall n \geq 1$. Hence, we have

$$\begin{aligned}
& P_{F_n}(\widehat{\theta}_{1n} \notin B(\theta_{1*n}, \varepsilon)) \\
\leq\ & P_{F_n}(||g_n(\widehat{\theta}_{1n})|| \geq q_n) \\
\leq\ & P_{F_n}(||\widehat{g}_n(\widehat{\theta}_{1n})|| + O_p(n^{-1/2}) \geq q_n) \\
\leq\ & P_{F_n}(\widehat{g}_n(\widehat{\theta}_{1n})'\widehat{W}_{1n}\widehat{g}_n(\widehat{\theta}_{1n})/\lambda_{\min}(\widehat{W}_{1n}) \geq (q_n - O_p(n^{-1/2}))^2) \\
\leq\ & P_{F_n}(\inf_{\theta_1\in\Theta_1} \widehat{g}_n(\theta_1)'\widehat{W}_{1n}\widehat{g}_n(\theta_1) + c_n \geq \xi(q_n - O_p(n^{-1/2}))^2) \\
\leq\ & P_{F_n}(\widehat{g}_n'\widehat{W}_{1n}\widehat{g}_n + c_n \geq \xi(q_n - O_p(n^{-1/2}))^2) \\
=\ & o(1),
\end{aligned} \tag{12.21}$$

where the second inequality holds by Assumption ES4(i) and the triangle inequality, the third inequality uses Assumption ES3(xi), the fourth inequality holds by the definition of $\widehat{\Theta}_{1n}$ in (7.3) and because Assumption ES3(xi) implies that $\lambda_{\min}(\widehat{W}_{1n}) \geq \xi$ wp→1 for some constant $\xi > 0$, the last inequality holds because $\theta_{1*n} \in \Theta_1$, and the equality holds because $\widehat{g}_n'\widehat{W}_{1n}\widehat{g}_n = o_p(1)$ using Assumptions ES3(iv) and (xi), $\xi > 0$, $c_n \to 0$ by Assumption ES3(x), and $\liminf_{n\to\infty} q_n > 0$ by Assumption ES4(ii).

Equation (12.21) implies that $\widehat{\theta}_{1n} - \theta_{1*n} \to_p 0$.

Next, the FOC's in (7.3), mean-value expansions around θ_{1*n}, and $\widehat{\theta}_{1n} - \theta_{1*n} \to_p 0$ give

$$\begin{aligned}
0^{p_1} &= \widehat{G}_{1n}(\widehat{\theta}_{1n})'\widehat{W}_{1n}\widehat{g}_n(\widehat{\theta}_{1n}) = \widehat{G}_{1n}(\widehat{\theta}_{1n})'\widehat{W}_{1n}\left(\widehat{g}_n + \widehat{G}_{1n}(\widetilde{\theta}_{1n})(\widehat{\theta}_{1n} - \theta_{1*n})\right) \text{ and so} \\
\widehat{\theta}_{1n} - \theta_{1*n} &= -\left(\widehat{G}_{1n}(\widehat{\theta}_{1n})'\widehat{W}_{1n}\widehat{G}_{1n}(\widetilde{\theta}_{1n})\right)^{-1}\widehat{G}_{1n}(\widehat{\theta}_{1n})'\widehat{W}_{1n}\widehat{g}_n = O_p(n^{-1/2}),
\end{aligned} \tag{12.22}$$

where $\widetilde{\theta}_{1n}$ lies between $\widehat{\theta}_{1n}$ and θ_{1*n} and may differ across the rows of $\widehat{G}_{1n}(\widetilde{\theta}_{1n})$, the mean-value expansions use Assumption ES3(iii), and the second equality on the second line holds because $\widehat{G}_{1n}(\widehat{\theta}_{1n})'\widehat{W}_{1n}\widehat{G}_{1n}(\widetilde{\theta}_{1n}) \to_p G_{1\infty}'W_{1\infty}G_{1\infty}$, $G_{1\infty}'W_{1\infty}G_{1\infty}$ is nonsingular, and $\widehat{G}_{1n}(\widehat{\theta}_{1n})'\widehat{W}_{1n} = O_p(1)$ by Assumptions ES3(i), (v)–(ix) and (xi), $\widehat{\theta}_{1n} - \theta_{1*n} \to_p 0$ (which implies that there exists a sequence of positive constants $\varepsilon_n \to 0$ for which $P_{F_n}(||\widehat{\theta}_{1n} - \theta_{1*n}|| > \varepsilon_n) \to 0$, so that Assumption ES3(vi) can be applied), and $\widehat{g}_n = O_p(n^{-1/2})$ by Assumption ES3(iv). (Note that $G_{1\infty}$ has full column rank p_1 by Assumptions ES3(i) and (vii)–(ix) because τ_{1n} is the smallest singular value of $\Omega_n^{-1/2}G_{1n}$ and $\liminf_{n\to\infty} \tau_{1n} > 0$.)

Given the definition of $\widehat{\theta}_{1n}$, (12.22) implies that $d(\theta_{1*n}, \widehat{\Theta}_{1n}) = O_p(n^{-1/2})$. $\square$

13 Verification of Assumptions for the First-Step AR CS

13.1 First-Step AR CS Results

First, we provide a result that verifies Assumption B(i) for the first-step (FS) AR CS, defined in (7.1).

Assumption FS1$_{\mathbf{AR}}$. For the sequence S, (i) $\widehat{\Omega}_n - \Omega_n \to_p 0$ for some variance matrices $\{\Omega_n \in R^{k\times k} : n \geq 1\}$, (ii) $\Omega_n \to \Omega_\infty$ for some variance matrix $\Omega_\infty \in R^{k\times k}$, (iii) $n^{1/2}\widehat{g}_n \to_d Z_\infty \sim N(0^k, \Omega_\infty)$, and (iv) $\liminf_{n\to\infty} \lambda_{\min}(\Omega_n) > 0$.

Lemma 13.1 *Let S be a null sequence (or S_m a null subsequence) that satisfies Assumption* FS1$_{AR}$. *Then, under the sequence S (or subsequence S_m), the nominal level $1-\alpha_1$ first-step AR CS, CS_{1n}^{AR}, has asymptotic coverage probability $1-\alpha_1$ and, hence, satisfies Assumption* B(i).

Next, we provide a result, Lemma 13.2, that is useful, in conjunction with Lemmas 12.3 and 12.4, for verifying Assumption OE(i) for the first-step AR CS, which equals

$$CS_{1n} = \{\theta_1 \in \Theta_1 : n\widehat{g}_n(\theta_1)'\widehat{\Omega}_n^{-1}(\theta_1)\widehat{g}_n(\theta_1) \leq \chi_k^2(1-\alpha_1)\}. \tag{13.1}$$

Lemma 13.2 provides conditions under which $d_H(\theta_{1*n}, CS_{1n} \cup \{\theta_{1*n}\}) = O_p(n^{-1/2})$ for a sequence S and the AR CS CS_{1n} in (13.1).

When verifying Assumption OE(i) for a sequence S with CS_{1n} in place of CS_{1n}^{+}, where CS_{1n} is as in (13.1), we use the following global strong-identification condition: For all sequences $\{K_n : n \geq 1\}$ for which $K_n \to \infty$,

$$\lim_{n\to\infty} \inf_{\theta_1 \notin B(\theta_{1*n}, K_n/n^{1/2})} n^{1/2}||g_n(\theta_1)|| = \infty. \tag{13.2}$$

Assumption FS2$_{\mathbf{AR}}$. For the sequence S, (i) $\sup_{\theta_1\in\Theta_1} n^{1/2}||\widehat{g}_n(\theta_1) - g_n(\theta_1)|| = O_p(1)$ for some nonrandom R^k-valued functions $\{g_n(\cdot) : n \geq 1\}$, (ii) (13.2) holds, (iii) $\sup_{\theta_1\in\Theta_1} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| = o_p(1)$ for some nonrandom $R^{k\times k}$-valued functions $\{\Omega_n(\cdot) : n \geq 1\}$, (iv) $\sup_{\theta_1\in\Theta_1} ||\Omega_n(\theta_1)|| = O(1)$, and (v) $\liminf_{n\to\infty} \inf_{\theta_1\in\Theta_1} \lambda_{\min}(\Omega_n(\theta_1)) > 0$.

Lemma 13.2 *Suppose CS_{1n} is of the form in* (13.1). *Let S be a null sequence (or S_m a null subsequence) that satisfies Assumption FS2$_{AR}$. Then, $d_H(\theta_{1*n}, CS_{1n} \cup \{\theta_{1*n}\}) = O_p(n^{-1/2})$ for the sequence S (or subsequence S_m).*

13.2 Proofs of Lemmas 13.1-13.2

Proof of Lemma 13.1. We have

$$n\widehat{g}_n'\widehat{\Omega}_n^{-1}\widehat{g}_n \to_d Z_\infty'\Omega_\infty^{-1}Z_\infty \sim \chi_k^2, \tag{13.3}$$

where the convergence in distribution holds because $n^{1/2}\widehat{g}_n \to_d Z_\infty$ by Assumption FS1$_{AR}$(iii) and $\widehat{\Omega}_n^{-1} \to_p \Omega_\infty^{-1}$ by Assumptions FS1$_{AR}$(i), (ii), and (iv), and the χ_k^2 distribution arises because $Z_\infty \sim N(0^k, \Omega_\infty)$ by Assumption FS1$_{AR}$(iii). Hence, we have

$$P_{F_n}(\theta_{1*n} \in CS_{1n}^{AR}) = P_{F_n}(n\widehat{g}_n'\widehat{\Omega}_n^{-1}\widehat{g}_n \leq \chi_k^2(1-\alpha_1)) \to 1-\alpha_1, \tag{13.4}$$

which establishes the result of the lemma. $\square$

Proof of Lemma 13.2. Let $\{\widehat{\theta}_{1n} : n \geq 1\}$ be a sequence in $CS_{1n} \cup \{\theta_{1*n}\}$ (for all $n \geq 1$ and all sample realizations) for which $||\theta_{1*n} - \widehat{\theta}_{1n}|| = d_H(\theta_{1*n}, CS_{1n} \cup \{\theta_{1*n}\}) + o_p(n^{-1/2})$. Such a sequence always exists because $CS_{1n} \cup \{\theta_{1*n}\}$ is non-empty for all $n \geq 1$.

We establish the result of the lemma by contradiction. Suppose $d_H(\theta_{1*n}, CS_{1n} \cup \{\theta_{1*n}\}) \neq O_p(n^{-1/2})$. Then, by definition of $\widehat{\theta}_{1n}$,

$$||\widehat{\theta}_{1n} - \theta_{1*n}|| \neq O_p(n^{-1/2}). \tag{13.5}$$

If a sequence of random variables $\{\eta_n : n \geq 1\}$ satisfies $\eta_n = O_p(1)$, then $\forall\varepsilon > 0$, $\exists K_\varepsilon < \infty$ such that $\limsup_{n\to\infty} P_{F_n}(|\eta_n| > K_\varepsilon) < \varepsilon$. Hence, (13.5) implies: $\exists\varepsilon > 0$ such that $\forall K < \infty$,

$$\limsup_{n\to\infty} P_{F_n}(||\widehat{\theta}_{1n} - \theta_{1*n}|| > K/n^{1/2}) \geq \varepsilon. \tag{13.6}$$

For $n \geq 1$ and $0 < K < \infty$, define

$$P_n(K) := P_{F_n}(||\widehat{\theta}_{1n} - \theta_{1*n}|| > K/n^{1/2}) \text{ and } L_n(K) := \inf_{\theta_1 \notin B(\theta_{1*n}, K/n^{1/2})} n^{1/2}||g_n(\theta_1)||. \tag{13.7}$$

Let $\{K_n : n \geq 1\}$ be a sequence such that $K_n \to \infty$ as $n \to \infty$, e.g., $K_n = \ln(n)$. Let $m_0 = 0$. For a given positive integer n, let m_n $(< \infty)$ be a positive integer for which $P_{m_n}(K_n) > \varepsilon/2$ and $m_n > m_{n-1}$. Such an m_n always exists because (13.6) can be rewritten as $\limsup_{m\to\infty} P_m(K) \geq \varepsilon$. The subsequence $\{m_n\}$ satisfies

$$P_{m_n}(K_n) > \varepsilon/2 \ \forall n \geq 1. \tag{13.8}$$

By the definition of $L_n(K)$ in (13.7),

$$\theta_1 \notin B(\theta_{1*n}, K/n^{1/2}) \text{ implies } ||n^{1/2} g_n(\theta_1)|| \geq L_n(K). \tag{13.9}$$

Equation (13.9) implies that, for all $n \geq 1$,

$$\begin{aligned}
&\text{(i) } \widehat{\theta}_{1m_n} \notin B(\theta_{1*m_n}, K_n/m_n^{1/2}) \text{ implies } ||m_n^{1/2} g_{m_n}(\widehat{\theta}_{1m_n})|| \geq L_{m_n}(K_n) \text{ and } \widehat{\theta}_{1m_n} \in CS_{1m_n},\\
&\text{(ii) } P_{F_{m_n}}(\widehat{\theta}_{1m_n} \notin B(\theta_{1*m_n}, K_n/m_n^{1/2}))\\
&\quad \leq P_{F_{m_n}}(||m_n^{1/2} g_{m_n}(\widehat{\theta}_{1m_n})|| \geq L_{m_n}(K_n) \ \& \ \widehat{\theta}_{1m_n} \in CS_{1m_n}), \text{ and} \qquad (13.10)\\
&\text{(iii) } \varepsilon/2 < P_{m_n}(K_n) \leq P_{F_{m_n}}(\{||m_n^{1/2} g_{m_n}(\widehat{\theta}_{1m_n})|| \geq L_{m_n}(K_n)\} \ \& \ \widehat{\theta}_{1m_n} \in CS_{1m_n}),
\end{aligned}$$

where we choose to take the subscript on K_n to be n throughout (rather than m_n) because we use (13.8) in the last line, the first line uses $\{\widehat{\theta}_{1m_n} : n \geq 1\}$ is a sequence in $CS_{1m_n} \cup \{\theta_{1*m_n}\}$ by definition and $d_H(\theta_{1*m_n}, \{\widehat{\theta}_{1m_n}\}) = 0$ if $\widehat{\theta}_{1m_n} = \theta_{1*m_n}$ (so $\widehat{\theta}_{1m_n} \notin B(\theta_{1*m_n}, K_n/m_n^{1/2})$ implies $\widehat{\theta}_{1m_n} \in CS_{1m_n}$), the first inequality on the last line holds by (13.8), and the second inequality on the last line holds by the inequality in (ii) and the definition of $P_n(K)$ in (13.7).

Given the definition of the AR statistic, $AR_n(\theta_1)$, in (7.1), we have

$$\begin{aligned}
AR_n^{1/2}(\widehat{\theta}_{1n}) &\geq \inf_{\theta_1 \in \Theta_1} \lambda_{\min}^{1/2}\left(\widehat{\Omega}_n^{-1}(\theta_1)\right) ||n^{1/2} \widehat{g}_n(\widehat{\theta}_{1n})||\\
&\geq \delta ||n^{1/2} g_n(\widehat{\theta}_{1n}) + O_p(1)||\\
&\geq \delta(||n^{1/2} g_n(\widehat{\theta}_{1n})|| - O_p(1)), \qquad (13.11)
\end{aligned}$$

where $\widehat{\Omega}_n(\theta_1)$ is nonsingular $\forall \theta_1 \in \Theta_1$ wp$\to$1 by Assumptions FS2$_{AR}$(iii) and (v), which guarantees that the AR statistic on the left-hand side (lhs) of the first line is well defined wp$\to$1, the second inequality holds for some $\delta > 0$ wp$\to$ 1 by Assumptions FS2$_{AR}$(i), (iii), and (iv) because $\inf_{\theta_1 \in \Theta_1} \lambda_{\min}(\widehat{\Omega}_n^{-1}(\theta_1)) = 1/(\sup_{\theta_1 \in \Theta_1} \lambda_{\max}(\widehat{\Omega}_n(\theta_1)))$ and $\sup_{\theta_1 \in \Theta_1} \lambda_{\max}(\widehat{\Omega}_n(\theta_1)) = \sup_{\theta_1 \in \Theta_1} \lambda_{\max}(\Omega_n(\theta_1)) + o_p(1) = O(1) + o_p(1)$ by Assumptions FS2$_{AR}$(iii) and (iv), and the third inequality holds by the triangle inequality. Hence, for all $n \geq 1$,

$$\begin{aligned}
&P_{F_{m_n}}(AR_{m_n}(\widehat{\theta}_{1m_n}) > \chi_k^2(1-\alpha_1) \ \& \ \widehat{\theta}_{1m_n} \in CS_{1m_n})\\
&\geq P_{F_{m_n}}(\delta(||m_n^{1/2} g_{m_n}(\widehat{\theta}_{1m_n})|| - O_p(1)) > (\chi_k^2(1-\alpha_1))^{1/2}\\
&\quad \& \ ||m_n^{1/2} g_{m_n}(\widehat{\theta}_{1m_n})|| \geq L_{m_n}(K_n) \ \& \ \widehat{\theta}_{1m_n} \in CS_{1m_n})\\
&\geq P_{F_{m_n}}(\delta(L_{m_n}(K_n) - O_p(1)) > (\chi_k^2(1-\alpha_1))^{1/2} \ \& \ ||m_n^{1/2} g_{m_n}(\widehat{\theta}_{1m_n})|| \geq L_{m_n}(K_n)\\
&\quad \& \ \widehat{\theta}_{1m_n} \in CS_{1m_n}). \qquad (13.12)
\end{aligned}$$

We show below that $\lim L_{m_n}(K_n) = \infty$. Note that $\lim L_{m_n}(K_n) = \infty$ implies that $\delta(L_{m_n}(K_n) - O_p(1)) > (\chi^2_k(1-\alpha_1))^{1/2}$ wp→1. This and (13.12) yield

$$
\begin{aligned}
&\liminf_{n\to\infty} P_{F_{m_n}}(AR_{m_n}(\widehat{\theta}_{1m_n}) > \chi^2_k(1-\alpha_1) \;\&\; \widehat{\theta}_{1m_n} \in CS_{1m_n}) \\
&\geq \liminf_{n\to\infty} P_{F_{m_n}}(||m_n^{1/2} g_{m_n}(\widehat{\theta}_{1m_n})|| \geq L_{m_n}(K_n) \;\&\; \widehat{\theta}_{1m_n} \in CS_{1m_n}) \\
&> \varepsilon/2, \qquad (13.13)
\end{aligned}
$$

where the second inequality holds by the last line of (13.10). Equation (13.13) is a contradiction because $AR_{m_n}(\widehat{\theta}_{1m_n}) > \chi^2_k(1-\alpha_1)$ implies that $\widehat{\theta}_{1m_n} \notin CS_{1_{m_n}}$. That is, (13.13) asserts that $0 = \liminf_{n\to\infty} P_{F_{m_n}}(\widehat{\theta}_{1m_n} \notin CS_{1_{m_n}} \;\&\; \widehat{\theta}_{1m_n} \in CS_{1m_n}) > \varepsilon/2 > 0$.

It remains to show that $\lim L_{m_n}(K_n) = \infty$. Given the definition of $L_n(K)$ in (13.7), the condition in (13.2) (i.e., Assumption FS2_{AR}(ii)) states that for all sequences $\{K_n\}$ for which $K_n \to \infty$, $\lim L_n(K_n) = \infty$. Hence, for all sequences $\{K_{m_n}\}$ for which $K_{m_n} \to \infty$, $\lim_{n\to\infty} L_{m_n}(K_{m_n}) = \infty$. Given $\{K_n\}$, we show that there exists a sequence $\{K^*_{m_n}\}$ such that $K^*_{m_n} = K_n$ $\forall n \geq 1$. Because m_n is strictly increasing in n, $n \to m_n$ is a one-to-one map. Let $\phi(m)$ be the corresponding inverse map for $m \in \mathcal{M} := \{m_n : n \geq 1\}$. For any $m \in \mathcal{M}$, define $K^*_m = K_{\phi(m)}$. Then, $L_{m_n}(K_n) = L_{m_n}(K^*_{m_n})$ $\forall n \geq 1$ because $\phi(m_n) = n$. In consequence, $\lim_{n\to\infty} L_{m_n}(K_n) = \lim_{n\to\infty} L_{m_n}(K^*_{m_n}) = \infty$, where the second equality holds by (13.2). □

14 Verification of Assumptions on the Second-Step Data-Dependent Significance Level

14.1 Data-Dependent Significance Level Results

Here we verify Assumptions B(iii) and OE(ii) for the second-step significance level (SL) $\widehat{\alpha}_{2n}(\theta_1)$ defined as in (7.4)–(7.8).

The results in this section and the sections that follow apply only to moment condition models. In consequence, in these sections,

$$
\begin{aligned}
g_n(\theta) &:= E_{F_n}\widehat{g}_n(\theta) \in R^k, \\
G_{jn}(\theta) &:= E_{F_n}\widehat{G}_{jn}(\theta) \in R^{k\times p_j} \text{ for } j = 1, 2, \text{ and} \\
\Omega_n(\theta) &:= Var_{F_n}(n^{1/2}\widehat{g}_n(\theta)) \in R^{k\times k}. \qquad (14.1)
\end{aligned}
$$

We define

$$\widehat{\Gamma}_{jn}(\theta) := n^{-1}\sum_{i=1}^{n} vec(G_{ji}(\theta) - \widehat{G}_{jn}(\theta))g_i(\theta)' \in R^{(p_j k)\times k} \text{ and}$$
$$\Gamma_{jn}(\theta) := n^{-1}\sum_{i=1}^{n} E_{F_n} vec(G_{ji}(\theta) - E_{F_n}G_{ji}(\theta))g_i(\theta)' \in R^{(p_j k)\times k} \text{ for } j = 1, 2. \tag{14.2}$$

Note that $\widehat{\Gamma}_{jn}(\theta) = [\widehat{\Gamma}_{j1n}(\theta)' : \cdots : \widehat{\Gamma}_{jp_jn}(\theta)']'$ for $\widehat{\Gamma}_{jsn}(\theta)$ defined in (7.9) for $s = 1, ..., p_1$ and $j = 1, 2$.

We define

$$G_{jsi}(\theta) := \frac{\partial}{\partial\theta_{js}} g_i(\theta) \in R^k,$$
$$\sigma^2_{jsn}(\theta) := Var_{F_n}(||G_{jsi}(\theta)||) \;\forall s = 1, ..., p_j, \text{ and}$$
$$\Phi_{jn}(\theta) := Diag\{\sigma^{-1}_{j1n}(\theta), ..., \sigma^{-1}_{jp_jn}(\theta)\} \text{ for } j = 1, 2. \tag{14.3}$$

First, we provide a lemma that verifies Assumption B(iii) under high-level conditions. The following assumption is employed when the second-step test is the C(α)-AR test defined in (7.10).

Assumption SL1$_{\mathbf{AR}}$. For the null sequence S, (i) $\lim \tau^{\Phi}_{1n} < K_L$ (where $K_L < \infty$ appears in the definition of $\widehat{\alpha}_{2n}(\theta_1)$ in (7.8)), (ii) $\widehat{G}_{1n} - G_{1n} \rightarrow_p 0$ for $\{G_{1n} := G_{1n}(\theta_{1*n}) : n \geq 1\}$ defined in (14.1), (iii) $\limsup_{n\to\infty} ||G_{1n}|| < \infty$, (iv) $\widehat{\Omega}_n - \Omega_n \rightarrow_p 0$ for $\{\Omega_n := \Omega_n(\theta_{1*n}) : n \geq 1\}$ defined in (14.1), (v) $\liminf_{n\to\infty} \lambda_{\min}(\Omega_n) > 0$, (vi) $\limsup_{n\to\infty} ||\Omega_n|| < \infty$, (vii) $\widehat{\sigma}^2_{1sn} - \sigma^2_{1sn} \rightarrow_p 0$ for $\{\sigma^2_{1sn} := \sigma^2_{1sn}(\theta_{1*n}) : n \geq 1\}$ defined in (14.3) $\forall s = 1, ..., p_1$, and (viii) $\liminf_{n\to\infty} \sigma^2_{1sn} > 0$ $\forall s = 1, ..., p_1$.

The following assumption is employed when the second-step test is the C(α)-LM or C(α)-QLR1 test defined in (7.13) and (7.18), respectively.

Assumption SL1$_{\mathbf{LM,QLR1}}$. For the null sequence S, (i) $\lim \tau^{\Phi}_{n} < K_L$, (ii) Assumptions SL1$_{AR}$(ii)–(viii) hold, (iii) $\widehat{\sigma}^2_{2sn} - \sigma^2_{2sn} \rightarrow_p 0$ for $\{\sigma^2_{2sn} := \sigma^2_{2sn}(\theta_{1*n}) : n \geq 1\}$ defined in (14.3) $\forall s = 1, ..., p_2$, and (iv) $\liminf_{n\to\infty} \sigma^2_{2sn} > 0$ $\forall s = 1, ..., p_2$.

Lemma 14.1 *Suppose $\widehat{\alpha}_{2n}(\theta_1)$ is defined in* (7.4)–(7.8) *for the second-step* C(α)-AR, C(α)-LM, *or* C(α)-QLR1 *test. Let S be a null sequence* (*or S_m a null subsequence*) *that satisfies Assumption* SL1$_{AR}$ *for the second-step* C(α)-AR *test and Assumption* SL1$_{LM,QLR1}$ *for the second-step* C(α)-LM *or* C(α)-QLR1 *test. Then, Assumption* B(iii) *holds for the sequence S* (*or subsequence S_m*).

Next, we provide high-level conditions under which a sequence S satisfies Assumption OE(ii). The following assumption is employed when the second-step test is the C(α)-AR test.

Assumption SL2$_{\mathbf{AR}}$. For the null sequence S, $\forall K < \infty$, (i) $\liminf_{n\to\infty} \tau_{1n}^{\Phi} > K_U$ (where $K_U > 0$ appears in the definition of $\widehat{\alpha}_{2n}(\theta_1)$ in (7.8)), (ii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{G}_{1n}(\theta_1) - G_{1n}(\theta_1)|| \to_p 0$ for $\{G_{1n}(\cdot) : n \geq 1\}$ defined in (14.1), (iii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||G_{1n}(\theta_1) - G_{1n}|| \to 0$, (iv) $G_{1n} = O(1)$, (v) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| \to_p 0$ for $\{\Omega_n(\cdot) : n \geq 1\}$ defined in (14.1), (vi) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Omega_n(\theta_1) - \Omega_n|| \to 0$, (vii) $\liminf_{n\to\infty} \lambda_{\min}(\Omega_n) > 0$, (viii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\widehat{\sigma}_{1sn}^2(\theta_1) - \sigma_{1sn}^2(\theta_1)| \to_p 0$ for $\{\sigma_{1sn}^2(\cdot) : n \geq 1\}$ defined in (14.3) $\forall s = 1, ..., p_1$, (ix) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\sigma_{1sn}^2(\theta_1) - \sigma_{1sn}^2| \to 0$ $\forall s = 1, ..., p_1$, and (x) $\liminf_{n\to\infty} \sigma_{1sn}^2 > 0$ $\forall s = 1, ..., p_1$.

The following assumption is employed with the second-step C(α)-LM and C(α)-QLR1 tests.

Assumption SL2$_{\mathbf{LM,QLR1}}$. For the null sequence S, $\forall K < \infty$, (i) $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U$, (ii) Assumptions SL2$_{AR}$(v)–(x) hold, (iii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{G}_n(\theta_1) - G_n(\theta_1)|| \to_p 0$ for some nonrandom $R^{k\times p}$-valued functions $\{G_n(\cdot) : n \geq 1\}$, (iv) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||G_n(\theta_1) - G_n|| \to 0$, (v) $G_n = O(1)$, (vi) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\widehat{\sigma}_{2sn}^2(\theta_1) - \sigma_{2sn}^2(\theta_1)| \to_p 0$ for some nonrandom real-valued functions $\{\sigma_{2sn}^2(\cdot) : n \geq 1\}$ defined in (14.3) $\forall s = 1, ..., p_2$, (vii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\sigma_{2sn}^2(\theta_1) - \sigma_{2sn}^2| \to 0$ $\forall s = 1, ..., p_2$, and (viii) $\liminf_{n\to\infty} \sigma_{2sn}^2 > 0$ $\forall s = 1, ..., p_2$.

Lemma 14.2 *Suppose $\widehat{\alpha}_{2n}(\theta_1)$ is defined in* (7.4)–(7.8) *for the second-step* C(α)-AR, C(α)-LM, *or* C(α)-QLR1 *test. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumption* SL2$_{AR}$ *for the second-step* C(α)-AR *test and Assumption* SL2$_{LM,QLR1}$ *for the second-step* C(α)-LM *or* C(α)-QLR1 *test. Then, Assumption* OE(ii) *holds for the sequence S (or subsequence S_m).*

14.2 Proofs of Lemmas 14.1 and 14.2

Proof of Lemma 14.1. First, we prove the lemma for the second-step C(α)-AR test under Assumption SL1$_{AR}$. Define $\Phi_{1n} := Diag\{\sigma_{11n}^{-1}, ..., \sigma_{1p_1n}^{-1}\}$. We write a SVD of $\Omega_n^{-1/2} G_{1n} \Phi_{1n}$ as $C_{1n}^{\Phi} \Upsilon_{1n}^{\Phi} B_{1n}^{\Phi\prime}$, where C_{1n}^{Φ} and B_{1n}^{Φ} are $k \times k$ and $p_1 \times p_1$ orthogonal matrices, respectively, and Υ_{1n}^{Φ} is a $k \times p_1$ matrix with the singular values of $\Omega_n^{-1/2} G_{1n} \Phi_{1n}$ on its main diagonal in nonincreasing order and zeros elsewhere. The smallest singular value of $\Omega_n^{-1/2} G_{1n} \Phi_{1n}$ is τ_{1n}^{Φ}, see (10.3), and it appears as the (p_1, p_1) element of Υ_{1n}^{Φ}. Let $\lambda_{1n} \in R^{p_1}$ be such that $||\lambda_{1n}|| = 1$ and $B_{1n}^{\Phi\prime} \lambda_{1n} = e_{p_1} := (0, ..., 0, 1)' \in R^{p_1}$. Then, $\Upsilon_{1n}^{\Phi} B_{1n}^{\Phi\prime} \lambda_{1n} = e_{p_1} \tau_{1n}^{\Phi}$.

We have

$$
\begin{aligned}
& ICS_{1n}^2 \\
&= \lambda_{\min}\left(\widehat{\Phi}_{1n}(\widehat{G}_{1n} - G_{1n} + G_{1n})'\widehat{\Omega}_n^{-1}(\widehat{G}_{1n} - G_{1n} + G_{1n})\widehat{\Phi}_{1n}\right) \\
&= \lambda_{\min}\left((\Phi_{1n} + o_p(1))(\Omega_n^{-1/2}G_{1n} + o_p(1))'\left[\Omega_n^{1/2}\widehat{\Omega}_n^{-1}\Omega_n^{1/2}\right](\Omega_n^{-1/2}G_{1n} + o_p(1))(\Phi_{1n} + o_p(1))\right) \\
&= \inf_{\lambda:||\lambda||=1}\left(\lambda'(\Phi_{1n}G_{1n}'\Omega_n^{-1/2} + o_p(1))\left[I_k + o_p(1)\right](\Omega_n^{-1/2}G_{1n}\Phi_{1n} + o_p(1))\lambda\right) \\
&\leq (\lambda_{1n}'\Phi_{1n}G_{1n}'\Omega_n^{-1/2} + o_p(1))\left[I_k + o_p(1)\right](\Omega_n^{-1/2}G_{1n}\Phi_{1n}\lambda_{1n} + o_p(1)) \\
&= (\tau_{1n}^{\Phi}e_{p_1}'C_{1n}^{\Phi\prime} + o_p(1))'\left[I_k + o_p(1)\right](C_{1n}^{\Phi}e_{p_1}\tau_{1n}^{\Phi} + o_p(1)) \\
&= (\tau_{1n}^{\Phi})^2 + o_p(1), \qquad (14.4)
\end{aligned}
$$

where the second equality uses Assumptions SL1_{AR}(ii), (v), and (vii), the third equality uses Assumptions SL1_{AR}(iii)–(vi) and (viii) (where Assumptions SL1_{AR}(v) and (viii) imply that $\Omega_n^{-1/2} = O(1)$ and $\Phi_{1n} = O(1)$, respectively), the inequality holds with λ_{1n} defined as above, the second last equality holds by the calculations above concerning λ_{1n}, and the last equality holds using Assumption SL1_{AR}(i).

Equation (14.4) implies that $ICS_{1n} \leq \lim \tau_{1n}^{\Phi} + \varepsilon$ wp→1 under the sequence S, $\forall \varepsilon > 0$. Using Assumption SL1_{AR}(i) this implies that $ICS_{1n} \leq K_L$ wp→1 under the sequence S. By the definition of $\widehat{\alpha}_{2n}(\theta_1)$ in (7.7) and (7.8), this implies that $\widehat{\alpha}_{2n} = \alpha_2$ wp→1 under the sequence S. That is, Assumption B(iii) holds for the sequence S under Assumption SL1_{AR}.

Next, we prove the lemma for the second-step C(α)-LM or C(α)-QLR1 test under Assumption $\text{SL1}_{LM,QLR1}$. The proof is the same as that given above for the C(α)-AR test but with all quantities involving $\widehat{G}_n$, G_n, $\widehat{\Phi}_n$, Φ_n, and τ_n^{Φ}, rather than $\widehat{G}_{1n}$, G_{1n}, $\widehat{\Phi}_{1n}$, Φ_{1n}, and τ_{1n}^{Φ}, respectively. These changes require the use of Assumption $\text{SL1}_{LM,QLR1}$(i) (i.e., $\lim \tau_n^{\Phi} < K_L$), rather than Assumption SL1_{AR}(i) (i.e., $\lim \tau_{1n}^{\Phi} < K_L$), and of Assumptions $\text{SL1}_{LM,QLR1}$(iii) and (iv) (to obtain the analogues of the second and third equalities in (14.4) for the C(α)-LM and C(α)-QLR1 test cases). □

Proof of Lemma 14.2. First, we prove the lemma for the second-step C(α)-AR test under Assumption SL2_{AR}. Let $\tau_{1n}^{\Phi}(\theta)$ denote the smallest singular value of $\Omega_n^{-1/2}(\theta)G_{1n}(\theta)\Phi_{1n}(\theta)$, where $\Phi_{1n}(\theta) \in R^{p_1 \times p_1}$ is defined in (14.3). For notational simplicity, let $\widehat{G}_{1n\theta_1}$, $G_{1n\theta_1}$ $\widehat{\Omega}_{n\theta_1}$, $\Omega_{n\theta_1}$, $\widehat{\Phi}_{1n\theta_1}$, $\Phi_{1n\theta_1}$, and $\tau_{1n\theta_1}^{\Phi}$ denote $\widehat{G}_{1n}(\theta_1)$, $G_{1n}(\theta_1)$, $\widehat{\Omega}_n(\theta_1)$, $\Omega_n(\theta_1)$, $\widehat{\Phi}_{1n}(\theta_1)$, $\Phi_{1n}(\theta_1)$, and $\tau_{1n}^{\Phi}(\theta_1)$, respectively. Let $\inf_{\theta_1}$ abbreviate $\inf_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})}$ and likewise with $\sup_{\theta_1}$. Let $o_p(\theta_1, \varepsilon_n)$, $O_p(\theta_1, \varepsilon_n)$, and $o(\theta_1, \varepsilon_n)$ denote $k \times p_1$, $k \times k$, or $p_1 \times p_1$ matrices that depend on θ_1 and are $o_p(\varepsilon_n)$, $O_p(\varepsilon_n)$, and $o(\varepsilon_n)$, respectively, uniformly over $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$ for a sequence of positive constants $\{\varepsilon_n\}$.

First, we show

$$\liminf_{n\to\infty} \inf_{\theta_1} \ \tau^{\Phi}_{1n\theta_1}/\tau^{\Phi}_{1n} = 1, \tag{14.5}$$

Given Assumption SL2$_{AR}$(i), (14.5) holds if

$$\liminf_{n\to\infty} \inf_{\theta_1} \left((\tau^{\Phi}_{1n\theta_1})^2 - (\tau^{\Phi}_{1n})^2\right) = 0 \ \forall K < \infty \tag{14.6}$$

because $\inf_{\theta_1} \left((\tau^{\Phi}_{1n\theta_1})^2 - (\tau^{\Phi}_{1n})^2\right) \le 0 \ \forall n \ge 1$.

Let $\lambda_{1n\theta_1} \in R^{p_1}$ be such that $||\lambda_{1n\theta_1}|| = 1$ and $\lambda_{\min}(\Phi_{1n\theta_1} G'_{1n\theta_1} \Omega^{-1}_{n\theta_1} G_{1n\theta_1} \Phi_{1n\theta_1}) = \lambda'_{1n\theta_1} \Phi_{1n\theta_1} \times G'_{1n\theta_1} \Omega^{-1}_{n\theta_1} G_{1n\theta_1} \Phi_{1n\theta_1} \lambda_{1n\theta_1}$. Let LHS denote the lhs of (14.6). We have

$$\begin{aligned} 0 \ge LHS &= \liminf_{n\to\infty} \inf_{\theta_1} \left(\lambda_{\min}(\Phi_{1n\theta_1} G'_{1n\theta_1} \Omega^{-1}_{n\theta_1} G_{1n\theta_1} \Phi_{1n\theta_1}) - \lambda_{\min}(\Phi_{1n} G'_{1n} \Omega^{-1}_{n} G_{1n} \Phi_{1n})\right) \\ &\ge \liminf_{n\to\infty} \inf_{\theta_1} \left(\lambda'_{1n\theta_1} \Phi_{1n\theta_1} G'_{1n\theta_1} \Omega^{-1}_{n\theta_1} G_{1n\theta_1} \Phi_{1n\theta_1} \lambda_{1n\theta_1} - \lambda'_{1n\theta_1} \Phi_{1n} G'_{1n} \Omega^{-1}_{n} G_{1n} \Phi_{1n} \lambda_{1n\theta_1}\right) \\ &= \liminf_{n\to\infty} \inf_{\theta_1} \lambda'_{1n\theta_1} [\Phi_{1n\theta_1} G'_{1n\theta_1} \Omega^{-1}_{n\theta_1} G_{1n\theta_1} \Phi_{1n\theta_1} - \Phi_{1n} G'_{1n} \Omega^{-1}_{n} G_{1n} \Phi_{1n}] \lambda_{1n\theta_1} \\ &= 0, \end{aligned} \tag{14.7}$$

where the first equality holds because the square of the smallest singular value of an $k \times p_1$ matrix A with $p_1 \le k$ equals the smallest eigenvalue of $A'A$ and the last equality holds by Assumption SL2$_{AR}$(iii), (a) $\sup_{\theta_1} ||\Omega^{-1}_{n\theta_1} - \Omega^{-1}_n|| \to 0 \ \forall K < \infty$, (b) $\sup_{\theta_1} ||\Phi_{1n\theta_1} - \Phi_{1n}|| \to 0 \ \forall K < \infty$, and (c) all of the multiplicands Φ_{1n}, G_{1n}, and Ω^{-1}_n are $O(1)$. Condition (a) holds because

$$\sup_{\theta_1} ||\Omega^{-1}_{n\theta_1} - \Omega^{-1}_n|| = \sup_{\theta_1} || - \Omega^{-1}_{n\theta_1} [\Omega_{n\theta_1} - \Omega_n] \Omega^{-1}_n|| = o(1), \tag{14.8}$$

where the last equality holds by Assumptions SL2$_{AR}$(vi) and (vii) (since Assumptions SL2$_{AR}$(vi) and (vii) imply $\liminf_{n\to\infty} \inf_{\theta_1} \lambda_{\min}(\Omega_n(\theta_1)) > 0$). Condition (b) holds by the same argument as for condition (a) using Assumption SL2$_{AR}$(ix) and (x). This completes the proof of (14.6) and, in turn, (14.5).

Next, we have

$$
\begin{aligned}
&\inf_{\theta_1} ICS^2_{1n}(\theta_1)\\
&= \inf_{\theta_1} \lambda_{\min}\left(\widehat{\Phi}_{1n\theta_1}\left(\widehat{G}_{1n\theta_1} - G_{1n\theta_1} + G_{1n\theta_1}\right)' \widehat{\Omega}^{-1}_{n\theta_1}\left(\widehat{G}_{1n\theta_1} - G_{1n\theta_1} + G_{1n\theta_1}\right)\widehat{\Phi}_{1n\theta_1}\right)\\
&= \inf_{\theta_1} \lambda_{\min}\Big((\Phi_{1n\theta_1} + o_p(\theta_1,1))(\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1} + o_p(\theta_1,1))' \left[\Omega^{-1/2}_{n\theta_1}\widehat{\Omega}_{n\theta_1}\Omega^{-1/2}_{n\theta_1}\right]^{-1}\\
&\quad \times(\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1} + o_p(\theta_1,1))(\Phi_{1n\theta_1} + o_p(\theta_1,1))\Big)\\
&= \inf_{\theta_1} \lambda_{\min}\left((\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1} + o_p(\theta_1,1))'\left[I_k + o_p(\theta_1,1)\right](\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1} + o_p(\theta_1,1))\right)\\
&= \inf_{\theta_1}\inf_{\lambda:||\lambda||=1}(\lambda'\Phi_{1n\theta_1} G'_{1n\theta_1}\Omega^{-1}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1}\lambda + \lambda'\Phi_{1n\theta_1} G'_{1n\theta_1}\Omega^{-1/2}_{n\theta_1} o_p(\theta_1,1)\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1}\lambda\\
&\quad +2\lambda' o_p(\theta_1,1)'\left[I_k + o_p(\theta_1,1)\right]\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1}\lambda + \lambda' o_p(\theta_1,1)'\left[I_k + o_p(\theta_1,1)\right] o_p(\theta_1,1)\lambda)\\
&\geq \inf_{\theta_1}\inf_{\lambda:||\lambda||=1}\lambda'\Phi_{1n\theta_1} G'_{1n\theta_1}\Omega^{-1}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1}\lambda\\
&\quad -\sup_{\theta_1}\sup_{\lambda:||\lambda||=1}|\lambda'\Phi_{1n\theta_1} G'_{1n\theta_1}\Omega^{-1/2}_{n\theta_1} o_p(\theta_1,1)\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1}\lambda)|\\
&\quad -2\sup_{\theta_1}\sup_{\lambda:||\lambda||=1}|\lambda' o_p(\theta_1,1)'\Omega^{-1/2}_{n\theta_1} G_{1n\theta_1}\Phi_{1n\theta_1}\lambda| - \sup_{\theta_1}\sup_{\lambda:||\lambda||=1}|\lambda' o_p(\theta_1,1)\lambda|\\
&= \inf_{\theta_1}\ (\tau^{\Phi}_{1n\theta_1})^2 + o_p(1)\\
&= (\tau^{\Phi}_{1n})^2 + o_p(1),
\end{aligned}
\tag{14.9}
$$

where the second equality holds using Assumptions SL2_{AR}(ii) and (vi)–(x), the third equality holds using Assumptions SL2_{AR}(iii)–(vii), (ix), and (x), the second last equality holds using Assumptions SL2_{AR}(iii), (iv), (vi), (vii), (ix), and (x), the definition of $\tau^{\Phi}_{1n\theta_1}$, and the fact that the square of the smallest singular value of a $k \times p_1$ matrix A with $p_1 \leq k$ equals the smallest eigenvalue of $A'A$, and the last equality holds by Assumptions SL2_{AR}(i), (iv), (vii), and (x) and (14.5) (where Assumptions SL2_{AR}(i), (iv), (vii), and (x) imply that $\{\tau^{\Phi}_{1n} : n \geq 1\}$ is bounded away from 0 and ∞).

Equation (14.9) and Assumption SL2_{AR}(i) imply that $\inf_{\theta_1} ICS_{1n}(\theta_1) \geq K_U$ wp$\to$1. Hence, given the definition of $\widehat{\alpha}_{2n}(\theta_1)$ in (7.7) and (7.8) for the second-step C(α)-AR test, Assumption OE(ii) holds for the sequence S.

Lastly, we prove the lemma for the second-step C(α)-LM and C(α)-QLR1 tests under Assumption $\text{SL2}_{LM,QLR1}$. The proof is the same as that given above but with all quantities involving $\widehat{G}_n(\theta_1)$, $G_n(\theta_1)$, $\widehat{\Phi}_n(\theta_1)$, and $\Phi_n(\theta_1)$, rather than $\widehat{G}_{1n\theta_1}$, $G_{1n\theta_1}$, $\widehat{\Phi}_{1n\theta_1}$, and $\Phi_{1n\theta_1}$, respectively. These changes require the use of Assumption $\text{SL2}_{LM,QLR1}$(i) (i.e., $\liminf_{n\to\infty}\tau^{\Phi}_n > K_U$), rather than Assumption SL2_{AR}(i) (i.e., $\liminf_{n\to\infty}\tau^{\Phi}_{1n} > K_U$) and the use of Assumptions $\text{SL2}_{LM,QLR1}$(iii)–(viii) (to obtain the analogues of the second, third, and last equalities of (14.9) for the C(α)-LM and C(α)-QLR1 test cases). $\square$

15 Verification of Assumptions for the Second-Step C(α)-AR Test

15.1 Second-Step C(α)-AR Test Results

This section verifies Assumptions B(ii) and C(ii)-C(v) for the second-step C(α)-AR test defined in Section 7.4.1.

The following lemma provides conditions under which Assumptions B(ii), C(ii), and C(iii) hold for the second-step AR test for a sequence S (whether $\lim \tau_{1n} > 0$ or $\lim \tau_{1n} = 0$). Assumption C(iv) automatically holds for the second-step AR test provided $p_1 < k$ because its nominal level η critical value is the $1-\eta$ quantile of the $\chi^2_{k-p_1}$ distribution which is nondecreasing in η for $\eta \in (0,1)$ when $p_1 < k$.

For a full column rank matrix $A \in R^{k \times p_1}$, let $M_A = I_k - A(A'A)^{-1}A'$.

We write a singular value decomposition (SVD) of $\Omega_n^{-1/2} G_{1n}$ as

$$\Omega_n^{-1/2} G_{1n} = C_{1n} \Upsilon_{1n} B'_{1n}, \tag{15.1}$$

where $C_{1n} \in R^{k \times k}$ and $B_{1n} \in R^{p_1 \times p_1}$ are orthogonal matrices and $\Upsilon_{1n} \in R^{k \times p_1}$ has the singular values $\tau_{11n}, ..., \tau_{1p_1 n}$ of $\Omega_n^{-1/2} G_{1n}$ in nonincreasing order on its diagonal and zeros elsewhere. We specify the compact SVD of $\Omega_n^{-1/2} G_{1n}$ given in (8.6) with $\theta = (\theta'_{1*n}, \theta'_{20})'$ to be the compact SVD that is obtained from the SVD in (15.1) by deleting the non-essential rows and columns of C_{1n}, Υ_{1n}, and B_{1n}. Suppose $\lim n^{1/2} \tau_{1sn} \in [0, \infty]$ exists for $s = 1, ..., p_1$. Let q_1 ($\in \{0, ..., p_1\}$) be such that

$$\lim n^{1/2} \tau_{1sn} = \infty \text{ for } 1 \leq s \leq q_1 \text{ and } \lim n^{1/2} \tau_{1sn} < \infty \text{ for } q_1 + 1 \leq s \leq p_1. \tag{15.2}$$

Define

$$\begin{aligned} S_{1n} &:= Diag\{(n^{1/2}\tau_{11n})^{-1}, ..., (n^{1/2}\tau_{1q_1 n})^{-1}, 1, ..., 1\} \in R^{p_1 \times p_1} \text{ and} \\ S_{1\infty} &:= Diag\{0, ..., 0, 1, ..., 1\} \in R^{p_1 \times p_1}, \end{aligned} \tag{15.3}$$

where q_1 zeros appear in $S_{1\infty}$. We have $S_{1n} \to S_{1\infty}$. In the case of strong or semi-strong identification of θ_1 given θ_{20}, $q_1 = p_1$ and $S_{1\infty} = 0^{p_1 \times p_1}$. In the case of weak identification of θ_1 given θ_{20}, $S_{1\infty} \neq 0^{p_1 \times p_1}$.

For the second-step (SS) C(α)-AR test, we use the following assumption.

Assumption SS1$_{\mathbf{AR}}$. For the null sequence S, (i) $\lim n^{1/2}\tau_{1sn} \in [0,\infty]$ exists $\forall s \leq p_1$, (ii) $n^{1/2}(\widehat{g}_n', vec(\widehat{G}_{1n} - E_{F_n}\widehat{G}_{1n})')' \rightarrow_d (Z_\infty', Z_{G_1\infty}')' \sim N(0^{(p_1+1)k}, V_{1\infty})$ for some variance matrix $V_{1\infty} \in R^{(p_1+1)k\times(p_1+1)k}$ whose first k rows are denoted by $[\Omega_\infty : \Gamma_{1\infty}']$ for $\Omega_\infty \in R^{k\times k}$ and $\Gamma_{1\infty} \in R^{(p_1k)\times k}$, (iii) Ω_∞ is nonsingular, (iv) $\widehat{\Gamma}_{1n} \rightarrow_p \Gamma_{1\infty}$ for $\Gamma_{1\infty}$ as in condition (ii), (v) $\widehat{\Omega}_n - \Omega_n \rightarrow_p 0^{k\times k}$ for $\{\Omega_n := \Omega_n(\theta_{1*n}, \theta_{20}) : n \geq 1\}$ defined in (14.1), (vi) $\Omega_n \rightarrow \Omega_\infty$ for Ω_∞ as in condition (ii), (vii) $C_{1n} \rightarrow C_{1\infty}$ for some matrix $C_{1\infty} \in R^{k\times k}$, and (viii) $B_{1n} \rightarrow B_{1\infty}$ for some matrix $B_{1\infty} \in R^{p_1\times p_1}$.

Lemma 15.1 *Suppose $\widehat{g}_n(\theta_1)$ are moment conditions, $\widehat{D}_{1n}(\theta)$ is defined in* (7.9)*, $\widehat{M}_{1n}(\theta_1)$ is defined in* (7.10) *with $a > 0$, and $p_1 < k$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumption* SS1$_{AR}$*. Then, for the sequence S (or subsequence S_m),*

(a) $AR_{2n} \rightarrow_d AR_{2\infty} := Z_\infty'\Omega_\infty^{-1/2}M_{\overline{\Delta}_{1\infty}^a}\Omega_\infty^{-1/2}Z_\infty \sim \chi^2_{k-p_1}$
for some (possibly) random $k \times p_1$ matrix $\overline{\Delta}_{1\infty}^a$ that is independent of Z_∞, where $\overline{\Delta}_{1\infty}^a$ has full column rank p_1 a.s. and

(b) *for all $\eta \in (0,1)$, $P_{F_n}(\phi_{2n}^{AR}(\theta_{1*n}, \eta) > 0) \rightarrow \eta$.*

Comments: (i). Lemma 15.1 establishes Assumptions B(ii), C(ii), and C(iii) for the second-step AR test. It verifies Assumption C(iii) because the $\chi^2_{k-p_1}$ distribution is absolutely continuous on R when $p_1 < k$.

(ii). The definition of the limit random matrix $\overline{\Delta}_{1\infty}^a$ is complicated and its form, beyond having full column rank a.s., is not important. In consequence, for brevity, $\overline{\Delta}_{1\infty}^a$ is defined in the proof of Lemma 15.1 below, see (15.6), rather than in Lemma 15.1 itself.

(iii). A key result of Lemma 15.1(a) is that $\overline{\Delta}_{1\infty}^a$ has full column rank. This uses the full rank perturbation $an^{-1/2}\zeta_1$ introduced in the definition of $\widehat{M}_{1n}$ in (7.10).

(iv). Under strong and semi-strong identification, the term $an^{-1/2}\zeta_1$ in the definition of $\widehat{M}_{1n}$ has no effect on the asymptotic distribution in Lemma 15.1(a).

(v). The proof of Lemma 15.1 uses Lemmas 10.2 and 10.3 and Corollary 16.2 in the SM to Andrews and Guggenberger (2017) (AG1) to obtain the asymptotic distribution of $\widehat{M}_{1n}$.

The following lemma provides conditions under which Assumption C(v) holds for the second-step C(α)-AR test for a sequence S with $\lim \tau_{1n} > 0$, where $\tau_{1n} := \tau_{1p_1n}$ is the smallest singular value of $\Omega_n^{-1/2}G_{1n}$. Let $\theta_1 = (\theta_{11}, ..., \theta_{1p_1})'$.

Assumption SS2$_{\mathbf{AR}}$. For the null sequence S, $\forall K < \infty$, (i) $\liminf_{n\rightarrow\infty} \tau_{1n} > 0$, (ii) $\widehat{g}_n(\theta_1)$ is twice continuously differentiable on $B(\theta_{1*n}, \varepsilon)$ (for all sample realizations) $\forall n \geq 1$ for some $\varepsilon > 0$, (iii) $\widehat{g}_n = O_p(n^{-1/2})$, (iv) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{G}_{1n}(\theta_1) - G_{1n}(\theta_1)|| \rightarrow_p 0$ for $\{G_{1n}(\cdot) : n \geq$

1} defined in (14.1), (v) $\limsup_{n\to\infty} \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||G_{1n}(\theta_1)|| < \infty$, (vi) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||(\partial^2/\partial\theta_{1s}\partial\theta_1')\widehat{g}_n(\theta_1)|| = O_p(1)$ for $s = 1, ..., p_1$, (vii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Gamma}_{1n}(\theta_1) - \Gamma_{1n}(\theta_1)|| = o_p(1)$ for $\{\Gamma_{1n}(\cdot) : n \geq 1\}$ defined in (14.2), (viii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Gamma_{1n}(\theta_1) - \Gamma_{1n}|| \to 0$, (ix) $||\Gamma_{1n}|| = O(1)$, (x) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| \to_p 0$ for $\{\Omega_n(\cdot) : n \geq 1\}$ defined in (14.1), (xi) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Omega_n(\theta_1) - \Omega_n|| \to 0$, (xii) $\liminf_{n\to\infty} \lambda_{\min}(\Omega_n) > 0$, and (xiii) $\Omega_n = O(1)$.

Lemma 15.2 *Suppose $\widehat{g}_n(\theta_1)$ are moment conditions, $\widehat{D}_{1n}(\theta)$ is defined in (7.9), and $\widehat{M}_{1n}(\theta_1)$ is defined in (7.10) with $a \geq 0$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumption SS2$_{AR}$. Then, under the sequence S (or subsequence S_m), for all constants $K < \infty$,*

(a) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{M}_{1n}(\theta_1) - \widehat{M}_{1n}|| = o_p(1)$,

(b) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} \left\| n^{1/2}\widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{g}_n(\theta_1) - n^{1/2}\widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{g}_n \right\| = o_p(1)$,

(c) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} \left\| n^{1/2}\widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{g}_n(\theta_1) - n^{1/2}\widehat{M}_{1n}\widehat{\Omega}_n^{-1/2}\widehat{g}_n \right\| = o_p(1)$, *and*

(d) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |AR_{2n}(\theta_1) - AR_{2n}| = o_p(1)$.

Comments: (i). Lemma 15.2(d) establishes Assumption C(v) for the second-step C(α)-AR test for a sequence S with $\liminf_{n\to\infty} \tau_{1n} > 0$.

(ii). Lemma 15.2 does not require $a > 0$, but Lemma 15.1 above does.

15.2 Proofs of Lemmas 15.1 and 15.2

Proof of Lemma 15.1. We have $\widehat{\Omega}_n^{-1/2} \to_p \Omega_\infty^{-1/2}$ and $\Omega_\infty^{-1/2}$ is nonsingular by Assumptions SS1$_{AR}$(iii), (v), and (vi).

We write

$$V_{1\infty} = \begin{bmatrix} \Omega_\infty & \Gamma_{1\infty}' \\ \Gamma_{1\infty} & \Omega_{G_1\infty} \end{bmatrix}, \text{ where } \Omega_\infty \in R^{k\times k},\ \Gamma_{1\infty} \in R^{(p_1k)\times k},\ \text{and } \Omega_{G_1\infty} \in R^{(p_1k)\times(p_1k)}. \quad (15.4)$$

By the argument in the proof of Lemma 10.2 in Section 15 of the SM to AG1, we have

$$\begin{aligned} n^{1/2}\begin{pmatrix} \widehat{g}_n \\ vec(\widehat{D}_{1n} - E_{F_n}\widehat{G}_{1n}) \end{pmatrix} &\to_d \begin{pmatrix} Z_\infty \\ Z_{G_1\infty} - \Gamma_{1\infty}\Omega_\infty^{-1}Z_\infty \end{pmatrix} \\ &\sim N\left(0^{(p_1+1)k}, \begin{pmatrix} \Omega_\infty & 0^{k\times(p_1k)} \\ 0^{(p_1k)\times k} & \Omega_{D_1\infty} \end{pmatrix}\right), \text{ where} \\ \Omega_{D_1\infty} &:= \Omega_{G_1\infty} - \Gamma_{1\infty}\Omega_\infty^{-1}\Gamma_{1\infty}', \end{aligned} \quad (15.5)$$

using Assumptions SS1$_{AR}$(ii)–(vi).

We partition $B_{1\infty}$ and $C_{1\infty}$ (defined in Assumption SS1$_{AR}$) and define $\overline{\Delta}_{1\infty}$ and $\overline{\Delta}^a_{1\infty}$ as follows:

$$\begin{aligned}
B_{1\infty} &= [B_{1\infty,q_1} : B_{1\infty,p_1-q_1}],\ C_{1\infty} = [C_{1\infty,q_1} : C_{1\infty,k-q_1}], \\
L^{\diamond}_{p_1-q_1} &:= \begin{bmatrix} 0^{q_1\times(p_1-q_1)} \\ Diag\{\lim n^{1/2}\tau_{1(q_1+1)n}, ..., \lim n^{1/2}\tau_{1p_1n}\} \\ 0^{(k-p_1)\times(p_1-q_1)} \end{bmatrix} \in R^{k\times(p_1-q_1)}, \\
vec(\overline{D}_{1\infty}) &:= Z_{G_{1\infty}} - \Gamma_{1\infty}\Omega_{\infty}^{-1}Z_{\infty} \text{ for } \overline{D}_{1\infty} \in R^{k\times p_1}, \\
\overline{\Delta}_{1\infty} &= [\overline{\Delta}_{1\infty,q_1} : \overline{\Delta}_{1\infty,p_1-q_1}] \in R^{k\times p_1},\ \overline{\Delta}_{1\infty,q_1} := C_{1\infty,q_1}, \\
\overline{\Delta}_{1\infty,p_1-q_1} &:= C_{1\infty}L^{\diamond}_{p_1-q_1} + \Omega_{\infty}^{-1/2}\overline{D}_{1\infty}B_{1\infty,p_1-q_1},\ \text{and} \\
\overline{\Delta}^a_{1\infty} &:= \overline{\Delta}_{1\infty} + a\zeta_1 B_{1\infty}S_{1\infty}, \qquad (15.6)
\end{aligned}$$

where $B_{1\infty,q_1} \in R^{p_1\times q_1}$, $B_{1\infty,p_1-q_1} \in R^{p_1\times(p_1-q_1)}$, $C_{1\infty,q_1} \in R^{k\times q_1}$, $C_{1\infty,k-q_1} \in R^{k\times(k-q_1)}$, $\overline{\Delta}_{1\infty,q_1} \in R^{k\times q_1}$, $\overline{\Delta}_{1\infty,p_1-q_1} \in R^{k\times(p_1-q_1)}$, and $S_{1\infty}$ is defined in (15.3).[14] The limits in $L^{\diamond}_{p_1-q_1}$ exist by Assumption SS1$_{AR}$(i). Note that $\overline{\Delta}_{1\infty,q_1}$ ($:= C_{1\infty,q_1}$) has full column rank q_1 because $C_{1\infty}$ is an orthogonal matrix (since $C_{1n} \to C_{1\infty}$ by Assumption SS1$_{AR}$(vii) and C_{1n} is orthogonal for all n by definition).

Using (15.5), by the proof of Lemma 10.3 in Section 16 of the SM to AG1 with p, $\widehat{D}_n$, $\widehat{W}_n$, W_{F_n}, $\widehat{U}_n$, U_{F_n}, $\overline{D}_h$, $\overline{\Delta}_h$, h_2, h_3, and $h^{\diamond}_{1,p-q}$ in AG1 set equal to p_1, $\widehat{D}_{1n}$, $\widehat{\Omega}_n^{-1/2}$, Ω_n, I_{p_1}, I_{p_1}, $\overline{D}_{1\infty}$, $\overline{\Delta}_{1\infty}$, $B_{1\infty}$, $C_{1\infty}$, and $L^{\diamond}_{p_1-q_1}$, respectively, we have

$$n^{1/2}\Omega_n^{-1/2}\widehat{D}_{1n}T_{1n} \to_d \overline{\Delta}_{1\infty},\ \text{where } T_{1n} := B_{1n}S_{1n}. \qquad (15.7)$$

This result uses Assumptions SS1$_{AR}$(i)–(viii).

We have

$$T_{1n} := B_{1n}S_{1n} \to B_{1\infty}S_{1\infty} \qquad (15.8)$$

using $S_{1n} \to S_{1\infty}$ and Assumption SS1$_{AR}$(viii).

We have $\widehat{\Omega}_n^{-1/2} \to_p \Omega_{1\infty}^{-1/2}$ by Assumptions SS1$_{AR}$(iii), (v), and (vi). This, (15.7), and (15.8) combine to yield

$$n^{1/2}(\widehat{\Omega}_n^{-1/2}\widehat{D}_{1n} + an^{-1/2}\zeta_1)T_{1n} = n^{1/2}\widehat{\Omega}_n^{-1/2}\widehat{D}_{1n}T_{1n} + a\zeta_1 T_{1n} \to_d \overline{\Delta}_{1\infty} + a\zeta_1 B_{1\infty}S_{1\infty} =: \overline{\Delta}^a_{1\infty}. \qquad (15.9)$$

[14] For simplicity, there is some abuse of notation here, e.g., $B_{1\infty,q_1}$ and $B_{1\infty,p_1-q_1}$ denote different matrices even if $p_1 - q_1$ happens to equal q_1.

Using the notation introduced in (15.6), we can write the limit random matrix in (15.9) as

$$\overline{\Delta}_{1\infty}^{a} := \overline{\Delta}_{1\infty} + a\zeta_1 B_{1\infty} S_{1\infty} = [\overline{\Delta}_{1\infty,q_1} : \overline{\Delta}_{1\infty,p_1-q_1} + a\zeta_1 B_{1\infty,p_1-q_1}] \tag{15.10}$$

because $B_{1\infty}S_{1\infty} = [0^{p_1\times q_1} : B_{1\infty,p_1-q_1}]$ by the definition of $S_{1\infty}$ in (15.3). As noted above, $\overline{\Delta}_{1\infty,q_1}$ has full column rank q_1. In addition, $\zeta_1 B_{1\infty,p_1-q_1} \in R^{k\times(p_1-q_1)}$ is a matrix of independent standard normal random variables (because $B_{1\infty}$ is an orthogonal matrix) and $\zeta_1 B_{1\infty,p_1-q_1}$ is independent of $\overline{\Delta}_{1\infty,p_1-q_1}$. By Corollary 16.2 of AG1, these results and $a > 0$ imply that $\overline{\Delta}_{1\infty}^{a}$ has full column rank p_1 a.s.

The matrix $\overline{\Delta}_{1\infty}^{a}$ is independent of Z_∞ because $\overline{\Delta}_{1\infty}^{a}$ is a nonrandom function of $(\overline{D}_{1\infty}, \zeta_1)$, ζ_1 is independent of $(Z_\infty, \overline{D}_{1\infty})$ by definition, and $\overline{D}_{1\infty}$ is independent of Z_∞ since they are jointly normal with zero covariance (because $Evec(\overline{D}_{1\infty})Z_\infty' = E(Z_{G_1\infty} - \Gamma_{1\infty}\Omega_\infty^{-1}Z_\infty)Z_\infty' = 0^{(p_1k)\times k}$) using (15.6) and Assumption SS1$_{AR}$(ii)).

Given a matrix A, the projection matrix P_A is invariant to the multiplication of A by any nonzero constant and the post-multiplication of A by any nonsingular matrix. In consequence, by the continuous mapping theorem,

$$\widehat{M}_{1n} := I_k - P_{\widehat{\Omega}_n^{-1/2}\widehat{D}_{1n} + an^{-1/2}\zeta_1} = I_k - P_{n^{1/2}[\widehat{\Omega}_n^{-1/2}\widehat{D}_{1n} + an^{-1/2}\zeta_1]T_{1n}} \to_d M_{\overline{\Delta}_{1\infty} + a\zeta_1 B_{1\infty}S_{1\infty}} =: M_{\overline{\Delta}_{1\infty}^{a}}, \tag{15.11}$$

where the second equality holds for n large because T_{1n} is nonsingular for n large (because B_{1n} is orthogonal and S_{1n} is nonsingular for n large by its definition in (15.3) and the definition of q_1 in (15.2)) and the convergence uses (15.9) and the fact, established above, that $\overline{\Delta}_{1\infty}^{a}$ has full column rank $p_1 \leq k$ (which implies that the function $J(\overline{\Delta}_{1\infty}^{a}) = (\overline{\Delta}_{1\infty}^{a\prime}\overline{\Delta}_{1\infty}^{a})^{-1}$ is well-defined and continuous a.s. so the continuous mapping theorem is applicable). The convergence in (15.11) holds jointly with $n^{1/2}\widehat{g}_n \to_d Z_\infty$ (using Assumption SS1$_{AR}$(ii)).

The result of part (a) follows from (15.11), $\widehat{\Omega}_n^{-1/2} \to_p \Omega_\infty^{-1/2}$, and Assumption SS1$_{AR}$(ii) using the continuous mapping theorem. We have $Z_\infty'\Omega_\infty^{-1/2}M_{\overline{\Delta}_{1\infty}^{a}}\Omega_\infty^{-1/2}Z_\infty \sim \chi^2_{k-p_1}$ conditional on $\overline{\Delta}_{1\infty}^{a}$ (because, as shown above, $\overline{\Delta}_{1\infty}^{a}$ and Z_∞ are independent and $\overline{\Delta}_{1\infty}^{a}$ has full column rank p_1 a.s. and, by Assumption SS1$_{AR}$(ii), $\Omega_\infty^{-1/2}Z_\infty \sim N(0^k, I_k)$) and, hence, unconditionally as well.

Part (b) follows immediately from part (a) because $\phi_{2n}^{AR}(\theta_{1*n}, \eta) = AR_{2n} - \chi^2_{k-p_1}(1-\eta)$ and $\chi^2_{k-p_1}(1-\eta)$ is the $1-\eta$ quantile of the $\chi^2_{k-p_1}$ distribution. □

Proof of Lemma 15.2. For any $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$, element-by-element mean-value expansions give

$$\begin{aligned}\widehat{g}_n(\theta_1) &= \widehat{g}_n + \frac{\partial}{\partial\theta_1}\widehat{g}_n(\theta_1)(\theta_1 - \theta_{1*n}) + \left(\frac{\partial}{\partial\theta_1'}\widehat{g}_n(\widetilde{\theta}_{1n}) - \frac{\partial}{\partial\theta_1'}\widehat{g}_n(\theta_1)\right)(\theta_1 - \theta_{1*n}) \\ &= \widehat{g}_n + \frac{\partial}{\partial\theta_1'}\widehat{g}_n(\theta_1)(\theta_1 - \theta_{1*n}) + O_p(n^{-1}), \qquad (15.12)\end{aligned}$$

where $\widetilde{\theta}_{1n}$ lies between θ_1 and θ_{1*n} and may differ across the rows of $(\partial/\partial\theta_1')\widehat{g}_n(\widetilde{\theta}_{1n})$ and, hence, satisfies $\widetilde{\theta}_{1n} - \theta_{1*n} = O_p(n^{-1/2})$ (because $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$), the first equality uses Assumption SS2$_{AR}$(ii), and the second equality uses mean-value expansions of $(\partial/\partial\theta_1')\widehat{g}_n(\widetilde{\theta}_{1n})$ and $(\partial/\partial\theta_1')\widehat{g}_n(\theta_1)$ about θ_{1*n} and Assumption SS2$_{AR}$(vi).

For part (a), given the definition of $\widehat{M}_{1n}(\theta_1)$ in (7.10), it suffices to show that

$$\begin{aligned}&\text{(I)} \quad \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{D}_{1n}(\theta_1) - \widehat{D}_{1n}|| = o_p(1), \\ &\text{(II)} \quad \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Omega}_n(\theta_1) - \widehat{\Omega}_n|| = o_p(1), \qquad (15.13)\end{aligned}$$

(III) $\widehat{D}_{1n}$ has singular values that are bounded away from 0 and ∞ wp$\to$1, (IV) $\widehat{\Omega}_n$ has eigenvalues that are bounded away from 0 and ∞ wp$\to$1, and (V) $an^{-1/2}\zeta_1 = o_p(1)$.

Condition (II) holds by Assumptions SS2$_{AR}$(x) and (xi). Condition (IV) holds by Assumptions SS2$_{AR}$(x)–(xiii). Condition (V) holds because a and ζ_1 do not depend on n. Because $\widehat{D}_{1n}(\theta)$ is a simple function of $\widehat{G}_{1n}(\theta)$, $\widehat{\Gamma}_{1n}(\theta)$, $\widehat{\Omega}_n^{-1}(\theta)$, and $\widehat{g}_n(\theta)$, see (7.9), condition (I) holds if

$$\begin{aligned}&\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{G}_{1n}(\theta_1) - \widehat{G}_{1n}|| = o_p(1), \quad \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Gamma}_{1n}(\theta_1) - \widehat{\Gamma}_{1n}|| = o_p(1), \\ &\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{g}_n(\theta_1) - \widehat{g}_n|| = o_p(1), \qquad (15.14)\end{aligned}$$

and conditions (II) and (IV) hold (because $\widehat{G}_{1n}$, $\widehat{\Gamma}_{1n}$, and $\widehat{g}_n$ are $O_p(1)$). The first condition in (15.14) holds by mean-value expansions of the elements of $\widehat{G}_{1n}(\theta_1)$ about θ_{1*n} using Assumptions SS2$_{AR}$(ii) and (vi). The second condition in (15.14) holds by Assumptions SS2$_{AR}$(vii) and (viii). The third condition in (15.14) holds by (15.12) and Assumptions SS2$_{AR}$(iv) and (v). Hence, condition (I) holds.

To establish condition (III), we have

$$\widehat{D}_{1n} = G_{1n} + o_p(1), \qquad (15.15)$$

by the definition of $\widehat{D}_{1n}$ in (7.9) and Assumptions SS2$_{AR}$(iii), (iv), (vii), (ix), (x), and (xii). The singular values of G_{1n} are bounded away from 0 and ∞ by Assumptions SS2$_{AR}$(i) and (v) because τ_{1n} is the smallest singular value of $\Omega_n^{-1/2}G_{1n}$ and the eigenvalues of $\Omega_n^{-1/2}$ are bounded away from 0 and ∞ by Assumptions SS2$_{AR}$(xii) and (xiii). This and (15.15) establish condition (III), which completes the proof of part (a).

Part (b) is established as follows: For all $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$,

$$\begin{aligned}
& n^{1/2}\widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{g}_n(\theta_1) - n^{1/2}\widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{g}_n \\
&= n^{1/2}\widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)\frac{\partial}{\partial\theta_1'}\widehat{g}_n(\theta_1)(\theta_1-\theta_{1*n}) + \widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)O_p(n^{-1/2}) \\
&= n^{1/2}\widehat{M}_{1n}(\theta_1)(\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{1n}(\theta_1) + an^{-1/2}\zeta_1)(\theta_1-\theta_{1*n}) - n^{1/2}\widehat{M}_{1n}(\theta_1)an^{-1/2}\zeta_1(\theta_1-\theta_{1*n}) \\
&\quad + n^{1/2}\widehat{M}_{1n}(\theta_1)\widehat{\Omega}_n^{-1/2}(\theta_1)[\widehat{\Gamma}_{11n}(\theta_1) : \ \dots \ : \widehat{\Gamma}_{1p_1n}(\theta_1)]\left(I_{p_1}\otimes\widehat{\Omega}_n^{-1}(\theta_1)\widehat{g}_n(\theta_1)\right)(\theta_1-\theta_{1*n}) \\
&\quad + O_p(n^{-1/2}) \\
&= o_p(1), \qquad (15.16)
\end{aligned}$$

where the $O_p(n^{-1/2})$ terms holds uniformly over $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$, the first equality uses (15.12), the second equality uses the definition of $\widehat{D}_{1n}(\theta_1)$ in (7.9) and the fact that $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}||\widehat{M}_{1n}(\theta_1)|| = O_p(1)$ because the eigenvalues of $\widehat{M}_{1n}(\theta_1)$ equal zero or one (since $\widehat{M}_{1n}(\theta_1)$ is a projection matrix) and $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}||\widehat{\Omega}_n^{-1/2}(\theta_1)|| = O_p(1)$ by Assumptions SS2$_{AR}$(x)–(xii), and the third equality uses (1) $\widehat{M}_{1n}(\theta_1)[\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{1n}(\theta_1) + an^{-1/2}\zeta_1] = 0^{k\times p_1}$ (because $\widehat{M}_{1n}(\theta_1)$ projects onto the orthogonal complement of the space spanned by $\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{1n}(\theta_1) + an^{-1/2}\zeta_1$, see (7.10)), (2) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}||\widehat{M}_{1n}(\theta_1)|| = O_p(1)$ as above, (3) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}||\widehat{\Omega}_n^{-j}(\theta_1)|| = O_p(1)$ for $j = 1/2, 1$ as above, (4) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}||\widehat{g}_n(\theta_1)|| = o_p(1)$ (by (15.12) and Assumptions SS2$_{AR}$(iii)–(v)), (5) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}||\theta_1 - \theta_{1*n}|| = O(n^{-1/2})$, and (6) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}\widehat{\Gamma}_{1n}(\theta_1) = \sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}[\widehat{\Gamma}_{11n}(\theta_1)' : \cdots : \widehat{\Gamma}_{1p_1n}(\theta_1)']' = O_p(1)$ by Assumptions SS2$_{AR}$(vii)–(ix).

Part (c) holds by parts (a) and (b), $\widehat{g}_n = O_p(n^{-1/2})$ (which holds by Assumption SS2$_{AR}$(iii)), and $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})}||\widehat{\Omega}_n^{-1/2}(\theta_1) - \widehat{\Omega}_n^{-1/2}|| = o_p(1)$ (which is implied by Assumptions SS2$_{AR}$(x)–(xii)).

Part (d) follows from part (c) and $n^{1/2}\widehat{M}_{1n}\widehat{\Omega}_n^{-1/2}\widehat{g}_n = O_p(1)$ (which holds by (2) and (3) above and $\widehat{g}_n = O_p(n^{-1/2})$) given the definition of $AR_{2n}(\theta)$ in (7.10). $\square$

16 Verification of Assumptions for the Second-Step C(α)-LM Test

16.1 Second-Step C(α)-LM Test Results

This section verifies Assumptions B(ii) and C(ii)-C(v) for the second-step C(α)-LM test defined in Section 7.4.2. The results in this section apply only to moment condition models.

We employ the same definitions as in Sections 8.1.1, 14.1, and 15.1. In addition, we define $\tau_{21n}, ..., \tau_{2p_2n}$, q_2, C_{2n}, Υ_{2n}, B_{2n}, S_{2n}, $S_{2\infty}$, and $\overline{\Delta}^a_{2\infty}$ as $\tau_{11n}, ..., \tau_{1p_1n}$, q_1, C_{1n}, Υ_{1n}, B_{1n}, S_{1n}, $S_{1\infty}$, and $\overline{\Delta}^a_{1\infty}$ are defined in Section 15.1, respectively, but with subscripts 2 in place of 1 throughout.

Given the definitions above, $C_{2n}\Upsilon_{2n}B'_{2n}$ is a SVD of $\Omega_n^{-1/2}G_{2n}$ and its singular values are $\tau_{21n}, ..., \tau_{2p_2n}$. We choose the compact SVD of $\Omega_n^{-1/2}G_{2n}$ specified in (8.6) with $\theta = (\theta'_{1*n}, \theta'_{20})'$ to be the compact SVD that is obtained from the SVD $C_{2n}\Upsilon_{2n}B'_{2n}$ by deleting the non-essential rows and columns of C_{2n}, Υ_{2n}, and B_{2n}. Given the definition of q_2, we have $S_{2n} \to S_{2\infty}$. In the case of (local) strong or semi-strong identification, $q_2 = p_2$ and $S_{2\infty} = 0^{k\times p_2}$. In the case of (local) weak identification, $S_{2\infty} \neq 0^{k\times p_2}$.

As defined in (10.2), τ_n is the smallest singular value of $\Omega_n^{-1/2}G_n \in R^{k\times p}$, where $p = p_1 + p_2$.

We let $r_{jn} := r_{jF_n}$ for r_{jF} defined in (8.5) for $j = 1, 2$ and $C_{*n} := C_{*F_n}$ for C_{*F} defined in (8.7).

For the second-step (SS) C(α)-LM test, we use the following assumptions.

Assumption SS1$_{\mathbf{LM}}$. For the null sequence S, (i) $\lim n^{1/2}\tau_{2sn} \in [0, \infty]$ exists $\forall s \leq p_2$, (ii) $n^{1/2}(\widehat{g}'_n, vec(\widehat{G}_{1n} - E_{F_n}\widehat{G}_{1n})', vec(\widehat{G}_{2n} - E_{F_n}\widehat{G}_{2n})')' \to_d (Z'_\infty, Z'_{G_1\infty}, Z'_{G_2\infty})' \sim N(0^{(p+1)k}, V_\infty)$ for some variance matrix $V_\infty \in R^{(p+1)k\times(p+1)k}$ whose first k rows are denoted by $[\Omega_\infty : \Gamma'_{1\infty} : \Gamma'_{2\infty}]$ for $\Omega_\infty \in R^{k\times k}$ and $\Gamma_{j\infty} \in R^{(p_jk)\times k}$ for $j = 1, 2$, (iii) $\widehat{\Gamma}_{2n} \to_p \Gamma_{2\infty}$ for $\Gamma_{2\infty}$ as in condition (ii), (iv) $C_{2n} \to C_{2\infty}$ for some matrix $C_{2\infty} \in R^{k\times k}$, (v) $B_{2n} \to B_{2\infty}$ for some matrix $B_{2\infty} \in R^{p_2\times p_2}$, (vi) $\widehat{G}_n - G_n \to_p 0$ and $G_n \to G_\infty$ for some matrix $G_\infty \in R^{k\times p}$, where $G_n := E_{F_n}\widehat{G}_n$, and (vii) $\widehat{\sigma}^2_{jsn} - \sigma^2_{jsn} \to_p 0$ for $\{\sigma^2_{jsn} : n \geq 1\}$ defined in (14.3) and $\sigma^2_{jsn} \to \sigma^2_{js\infty}$ for some scalars $\sigma^2_{js\infty} > 0$ $\forall s = 1, ..., p_j$, $\forall j = 1, 2$.

Assumption SS2$_{\mathbf{LM}}$. For the null sequence S, $\forall K < \infty$, (i) $\liminf_{n\to\infty} \tau^\Phi_n > K^*_U$ for $K^*_U > 0$ defined in (7.11), (ii) $\widehat{g}_n(\theta_1, \theta_2)$ is differentiable in θ_2 at θ_{20} and $(\partial/\partial\theta'_2)\widehat{g}_n(\theta_1, \theta_{20})$ is differentiable in θ_1 with both holding $\forall\theta_1 \in B(\theta_{1*n}, \varepsilon)$ (for all sample realizations), $\forall n \geq 1$, for some $\varepsilon > 0$, (iii) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})} ||\widehat{G}_{2n}(\theta_1) - G_{2n}(\theta_1)|| \to_p 0$ for $\{G_{2n}(\cdot) : n \geq 1\}$ defined in (14.1), (iv) $\limsup_{n\to\infty} \sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})} ||G_{2n}(\theta_1)|| < \infty$, (v) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})} ||(\partial^2/\partial\theta_{1s}\partial\theta'_2)\widehat{g}_n(\theta_1, \theta_{20})|| = O_p(1)$ for $s = 1, ..., p_1$, (vi) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})} ||\widehat{\Gamma}_{2n}(\theta_1) - \Gamma_{2n}(\theta_1)|| = o_p(1)$ for $\{\Gamma_{2n}(\cdot) : n \geq 1\}$ defined in (14.2), and (vii) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})} ||\Gamma_{2n}(\theta_1) - \Gamma_{2n}|| \to 0$.

Given the quantities Ω_∞, G_∞, and $\sigma^2_{js\infty}$ in Assumption SS1$_{LM}$, we define

$$\begin{aligned} ICS^*_\infty &:= \lambda^{1/2}_{\min}(\Phi'_\infty G'_\infty \Omega^{-1}_\infty G_\infty \Phi_\infty),\ \Phi_\infty := Diag\{\sigma^{-1}_{11\infty}, ..., \sigma^{-1}_{1p_1\infty}, \sigma^{-1}_{21\infty}, ..., \sigma^{-1}_{2p_2\infty}\} \in R^{p\times p}, \\ WI_\infty &:= 1 - s\left(\frac{ICS^*_\infty - K^*_L}{K^*_U - K^*_L}\right),\ \text{and } D^\dagger_\infty := (M_{\overline{\Delta}^a_{1\infty}} + WI_\infty P_{\overline{\Delta}^a_{1\infty}})\overline{\Delta}^a_{2\infty}. \end{aligned} \tag{16.1}$$

As defined, $WI_\infty = 0$ if $ICS^*_\infty \geq K^*_U$, $WI_\infty = 1$ if $ICS^*_\infty \leq K^*_L$, and $WI_\infty \in [0,1]$ otherwise.

The following lemma verifies Assumptions B(ii), C(ii), and C(iii) for the second-step C(α)-LM test.

Lemma 16.1 *Suppose $\widehat{g}_n(\theta_1)$ are moment conditions, $\widehat{D}_{jn}(\theta)$ is defined in* (7.9) *for $j = 1,2$, $\widehat{M}_{1n}(\theta_1)$ is defined in* (7.10) *with $a > 0$ and $p_2 \geq 1$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumptions* SS1$_{AR}$ *and* SS1$_{LM}$*. Then, for the sequence S (or subsequence S_m),*

(a) *$D^\dagger_\infty$ has full column rank p_2 a.s.,*

(b) *$LM_{2n} \to_d LM_{2\infty} := Z'_\infty \Omega^{-1/2}_\infty P_{D^\dagger_\infty} \Omega^{-1/2}_\infty Z_\infty \sim \chi^2_{p_2}$, where $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, D^\dagger_\infty)$ is independent of Z_∞ and $\overline{\Delta}^a_{j\infty}$ has full column rank p_j a.s. for $j = 1,2$, and*

(c) *for all $\eta \in (0,1)$, $\limsup_{n\to\infty} P_{F_n}(\phi^{LM}_{2n}(\theta_{1*n}, \eta) > 0) = \eta$.*

Comments: (i). For the second-step LM test, Lemma 16.1(c) establishes Assumptions B(ii) and C(ii). Lemma 16.1(b) establishes Assumption C(iii) because the $\chi^2_{p_2}$ distribution is absolutely continuous on R when $p_2 \geq 1$. Assumption C(iv) automatically holds for the second-step LM test provided $p_2 \geq 1$ because its nominal level η critical value is the $1-\eta$ quantile of the $\chi^2_{p_2}$ distribution which is nondecreasing in η for $\eta \in (0,1)$.

(ii). The result of Lemma 16.1(a) is key because it allows one to use the continuous mapping theorem to obtain the asymptotic distribution of the LM_{2n} statistic.

(iii). When $\liminf_{n\to\infty} \tau_n > 0$ (i.e., under strong local identification of θ), $\overline{\Delta}^a_{j\infty}$ reduces to $\overline{\Delta}_{j\infty}$ for $j = 1,2$, where $\overline{\Delta}_{1\infty}$ is defined in (15.6) and $\overline{\Delta}_{2\infty}$ is defined analogously, and the terms $an^{-1/2}\zeta_1$ and $an^{-1/2}\zeta_2$ do not affect the asymptotic distribution of LM_{2n}.

(iv). The proof of Lemma 16.1 uses Lemmas 10.2 and 10.3 in the SM to AG1 to obtain the asymptotic distribution of $\widehat{D}_{2n}$ after suitable rescaling and right-hand side (rhs) rotation, but not recentering.

(v). To prove that the result in Comment (iii) to Theorem 8.2 holds (which considers the pure C(α)-LM test (in which case $WI_n(\theta) := 0$), we establish below that Lemma 16.1 holds with $WI_\infty = 0$ provided Assumptions SS1$_{LM}$(vi) and (vii) are replaced by (vi) r_{jn} $(:= r_{jF_n}) = r_{j\infty}$ for all n sufficiently large for some constant $r_{j\infty} \in \{0, ..., p_j\}$ for $j = 1,2$, and (vii) $\lambda_{\min}(C'_{*n}C_{*n}) \geq \delta$

$\forall n \geq 1$ for some $\delta > 0$.

The next lemma provides conditions under which Assumption C(v) holds for the second-step C(α)-LM test for sequences S with $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U^*$.

Lemma 16.2 *Suppose $\widehat{g}_n(\theta_1)$ are moment conditions, $\widehat{D}_{jn}(\theta)$ is defined in (7.9) for $j = 1, 2$ and $\widehat{M}_{1n}(\theta_1)$ is defined in (7.10) with $a \geq 0$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumptions* SS1_{AR}, SS2_{AR}, SS1_{LM}, SS2_{LM}, *and* $\text{SL2}_{LM,QLR1}$ *with* $\text{SL2}_{LM,QLR1}\text{(i)}$ *deleted. Then, under the sequence S (or subsequence S_m), for all constants $K < \infty$,*

(a) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1) - \widehat{\Omega}_n^{-1/2}\widehat{D}_{2n}|| = o_p(1)$ *and*

(b) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |LM_{2n}(\theta_1) - LM_{2n}| = o_p(1)$.

Comments: (i). Lemma 16.2(b) establishes Assumption C(v) for the second-step C(α)-LM test for a sequence S with $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U^*$ (> 0).

(ii). Lemma 16.2 does not require $a > 0$, but Lemma 16.1 above does.

(iii). Lemma 16.2 holds for the pure C(α)-LM test (in which case $WI_n(\theta) := 0$) with the condition $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U^*$ (> 0) in Assumption SS2_{LM}(i) replaced by the condition $\liminf_{n\to\infty} \tau_n > 0$.

16.2 Proofs of Lemmas 16.1 and 16.2

Proof of Lemma 16.1. We write

$$V_\infty = \begin{bmatrix} \Omega_\infty & \Gamma'_{1\infty} & \Gamma'_{2\infty} \\ \Gamma_{1\infty} & \Omega_{G_1\infty} & \Omega'_{G_2G_1\infty} \\ \Gamma_{2\infty} & \Omega_{G_2G_1\infty} & \Omega_{G_2\infty} \end{bmatrix}, \text{ where } \Omega_\infty \in R^{k\times k},\ \Gamma_{j\infty} \in R^{(p_jk)\times k},\ \Omega_{G_j\infty} \in R^{(p_jk)\times(p_jk)}, \tag{16.2}$$

and $\Omega_{G_2G_1\infty} \in R^{(p_2k)\times(p_1k)}$ for $j = 1, 2$. By the argument in the proof of Lemma 10.2 in Section 15 of the SM to AG1, we have

$$n^{1/2}\begin{pmatrix} \widehat{g}_n \\ vec(\widehat{D}_{1n} - E_{F_n}\widehat{G}_{1n}) \\ vec(\widehat{D}_{2n} - E_{F_n}\widehat{G}_{2n}) \end{pmatrix} \tag{16.3}$$

$$\to_d \begin{pmatrix} Z_\infty \\ Z_{G_1\infty} - \Gamma_{1\infty}\Omega_\infty^{-1}Z_\infty \\ Z_{G_2\infty} - \Gamma_{2\infty}\Omega_\infty^{-1}Z_\infty \end{pmatrix} \sim N\left(0^{(p+1)k}, \begin{pmatrix} \Omega_\infty & 0^{k\times(p_1k)} & 0^{k\times(p_2k)} \\ 0^{(p_1k)\times k} & \Omega_{D_1\infty} & \Omega'_{D_2D_1\infty} \\ 0^{(p_2k)\times k} & \Omega_{D_2D_1\infty} & \Omega_{D_2\infty} \end{pmatrix}\right), \text{ where}$$

$$\Omega_{D_j\infty} := \Omega_{G_j\infty} - \Gamma_{j\infty}\Omega_\infty^{-1}\Gamma'_{j\infty} \text{ for } j = 1, 2 \text{ and } \Omega_{D_2D_1\infty} := \Omega_{G_2G_1\infty} - \Gamma_{2\infty}\Omega_\infty^{-1}\Gamma'_{1\infty}$$

using (16.2) and Assumptions $\mathrm{SS1}_{AR}$(iii)–(vi) and $\mathrm{SS1}_{LM}$(ii).

We define $B_{2\infty,q_2}$, $B_{2\infty,p_2-q_2}$, $C_{2\infty,q_2}$, $C_{2\infty,k-q_2}$, $L^{\diamond}_{p_2-q_2}$, $\overline{D}_{2\infty}$, $\overline{\Delta}_{2\infty}$, $\overline{\Delta}_{2\infty,q_2}$, $\overline{\Delta}_{2\infty,p_2-q_2}$, and $\overline{\Delta}^a_{2\infty}$ using the definitions in (15.6) with subscripts 2 in place of 1. The limits in $L^{\diamond}_{p_2-q_2}$ exist by Assumption $\mathrm{SS1}_{LM}$(i). We define T_{2n} as in (15.8) with the subscript 2 in place of 1. As in (15.7)–(15.9) with subscripts 2 in place of 1, we have

$$n^{1/2}(\widehat{\Omega}_n^{-1/2}\widehat{D}_{2n} + an^{-1/2}\zeta_2)T_{2n} = n^{1/2}\widehat{\Omega}_n^{-1/2}\widehat{D}_{2n}T_{2n} + a\zeta_2 T_{2n} \to_d \overline{\Delta}_{2\infty} + a\zeta_2 B_{2\infty}S_{2\infty} =: \overline{\Delta}^a_{2\infty} \tag{16.4}$$

using Assumptions $\mathrm{SS1}_{AR}$(iii), (v), and (vi) and $\mathrm{SS1}_{LM}$(i)–(v). By the same argument as given just below (15.10), the limit random matrix $\overline{\Delta}^a_{2\infty}$ has full column rank p_2 a.s., as stated in part (b).

The convergence results in (15.11) and (16.4) hold jointly (using Assumption $\mathrm{SS1}_{LM}$(ii)). In addition, by the same argument as in the paragraph following (15.10), $\overline{\Delta}^a_{1\infty}$ and $\overline{\Delta}^a_{2\infty}$ are jointly independent of Z_∞, as stated in part (b).

Now, we prove part (a). We have

$$\begin{aligned} ICS^*_n &:= \lambda^{1/2}_{\min}\left(\widehat{\Phi}_n\widehat{G}'_n\widehat{\Omega}_n^{-1}\widehat{G}_n\widehat{\Phi}_n\right) \to_p \lambda^{1/2}_{\min}\left(\Phi_\infty G'_\infty\Omega_\infty^{-1}G_\infty\Phi_\infty\right) =: ICS^*_\infty \text{ and} \\ WI_n &\to_p WI_\infty, \end{aligned} \tag{16.5}$$

where the first and last definitions in the first line are given in (7.6) and (16.1), respectively, the convergence in probability in the first line holds using Assumptions $\mathrm{SS1}_{AR}$(iii), (v), and (vi) (which yield $\widehat{\Omega}_n^{-1} \to_p \Omega_\infty^{-1}$), Assumptions $\mathrm{SS1}_{LM}$(vi) and (vii) (which yield $\widehat{\Phi}_n \to_p \Phi_\infty$ and $\widehat{G}_n \to_p G_\infty$), and Slutsky's Theorem (because the smallest eigenvalue of a matrix is a continuous function of the matrix), and the second line holds by the first line, the definition of $WI_n(\theta)$ in (7.11), and Slutsky's Theorem (because the function $s(\cdot)$ in (7.8) is assumed to be continuous).

When $ICS^*_\infty \le K^*_L$, we have $WI_\infty = 1$,

$$D^\dagger_\infty := (M_{\overline{\Delta}^a_{1\infty}} + WI_\infty P_{\overline{\Delta}^a_{1\infty}})\overline{\Delta}^a_{2\infty} = \overline{\Delta}^a_{2\infty}, \tag{16.6}$$

and $\overline{\Delta}^a_{2\infty}$ has full rank p_2 by the same argument as used to prove Lemma 15.1(a) in (15.10), which uses Corollary 16.2 of AG1, with $\overline{\Delta}^a_{2\infty}$ in place of $\overline{\Delta}^a_{1\infty}$.

When $ICS^*_\infty > 0$, we have $\lambda^{1/2}_{\min}(\Phi_{j\infty}G'_{j\infty}\Omega_\infty^{-1}G_{j\infty}\Phi_{j\infty}) \ge \lambda^{1/2}_{\min}(\Phi_\infty G'_\infty\Omega_\infty^{-1}G_\infty\Phi_\infty) := ICS^*_\infty > 0$ for $j = 1, 2$ using the definition of a minimum eigenvalue. Since $\sigma^2_{js\infty} > 0$ $\forall s = 1, ..., p_j$, $\forall j = 1, 2$ by Assumption $\mathrm{SS1}_{LM}$(vii), this implies that $\lambda^{1/2}_{\min}(G'_{j\infty}\Omega_\infty^{-1}G_{j\infty}) > 0$ for $j = 1, 2$. Using Assumptions $\mathrm{SS1}_{AR}$(vi) and $\mathrm{SS1}_{LM}$(vi), this implies that $\tau_{jn} := \lambda^{1/2}_{\min}(G'_{jn}\Omega_n^{-1}G_{jn}) \to \lambda^{1/2}_{\min}(G'_{j\infty}\Omega_\infty^{-1}G_{j\infty}) > 0$, where $G_\infty = [G_{1\infty} : G_{2\infty}]$ for $G_{j\infty} \in R^{k\times p_j}$, τ_{jn} is defined in (10.2), and $G_{jn} := E_{F_n}\widehat{G}_{jn}$, see (14.1), for $j = 1, 2$. In turn, this gives $q_j = p_j$ for $j = 1, 2$ because, by definition, q_j satisfies $\lim n^{1/2}\tau_{jsn} = \infty$ for $1 \leq s \leq q_j$ (see (15.2) and the second paragraph of this section). Finally, $q_j = p_j$ for $j = 1, 2$ implies that

$$\overline{\Delta}^a_{j\infty} = \overline{\Delta}_{j\infty} \text{ for } j = 1, 2 \tag{16.7}$$

using the definition of $\overline{\Delta}^a_{1\infty}$ in (15.6) and the analogous definition of $\overline{\Delta}^a_{2\infty}$ specified in the second paragraph of this section.

By an analogous argument, when $ICS^*_\infty > 0$, we have $\tau_n = \lambda^{1/2}_{\min}(G'_n\Omega^{-1}_n G_n) \to \lambda^{1/2}_{\min}(G'_\infty\Omega^{-1}_\infty G_\infty) > 0$, where τ_n is defined in (10.2).

When $ICS^*_\infty > K^*_L \geq 0$, we have

$$\begin{aligned} D^\dagger_\infty &:= (M_{\overline{\Delta}^a_{1\infty}} + WI_\infty P_{\overline{\Delta}^a_{1\infty}})\overline{\Delta}^a_{2\infty} = \left(I_k - s_\infty P_{\overline{\Delta}^a_{1\infty}}\right)\overline{\Delta}^a_{2\infty} = \left(I_k - s_\infty P_{\overline{\Delta}_{1\infty}}\right)\overline{\Delta}_{2\infty}, \text{ where} \\ s_\infty &:= s\left(\frac{ICS^*_\infty - K^*_L}{K^*_U - K^*_L}\right) > 0, \end{aligned} \tag{16.8}$$

the last equality on the first line holds by (16.7), and $s_\infty > 0$ because $s(\cdot)$ is a strictly increasing continuous function on $[0, 1]$ with $s(0) = 0$, see (7.8).

Suppose $(I_k - s_\infty P_{\overline{\Delta}_{1\infty}})\overline{\Delta}_{2\infty}$ has rank less than p_2. Then, $\exists \eta \in R^{p_2}$ with $||\eta|| = 1$ such that $\overline{\Delta}_{2\infty}\eta = s_\infty P_{\overline{\Delta}_{1\infty}}\overline{\Delta}_{2\infty}\eta$. Because $s_\infty > 0$, this occurs only if $\overline{\Delta}_{2\infty}\eta \in \text{col}(\overline{\Delta}_{1\infty})$, where $\text{col}(\cdot)$ denotes the column space of a matrix, because the right-hand side of the equation is in $\text{col}(\overline{\Delta}_{1\infty})$. But, $\overline{\Delta}_{2\infty}\eta \in \text{col}(\overline{\Delta}_{1\infty})$ is a contradiction because $[\overline{\Delta}_{1\infty} : \overline{\Delta}_{2\infty}]$ has full column rank a.s., which we now show.

Thus, to prove part (a), it remains to show that $[\overline{\Delta}_{1\infty} : \overline{\Delta}_{2\infty}]$ has full column rank a.s. when $ICS^*_\infty > K^*_L \geq 0$. It suffices to show $[\overline{\Delta}_{1\infty} : \overline{\Delta}_{2\infty}]$ has full column rank a.s. when $\lim \tau_n > 0$ and $q_j = p_j$ for $j = 1, 2$ (because it is shown above that $ICS^*_\infty > 0$ implies both conditions). We have $\overline{\Delta}_{j\infty} = C_{j\infty,p_j}$ (which is nonrandom) when $q_j = p_j$ for $j = 1, 2$ by (15.6) and the second paragraph of this section, where $C_{j\infty} = [C_{j\infty,p_j} : C_{j\infty,k-p_j}] \in R^{k\times k}$.

We have $C_{jn} \to C_{j\infty}$ and $B_{jn} \to B_{j\infty}$ by Assumptions SS1$_{AR}$(vii) and (viii) and SS1$_{LM}$(iv) and (v), where $C_{jn} = [C_{jn,p_j} : C_{jn,k-p_j}]$ and B_{jn} are the $k \times k$ and $p_j \times p_j$ orthogonal matrices in the singular value decompositions $\Omega^{-1/2}_n G_{jn} = C_{jn}\Upsilon_{jn}B'_{jn}$ for $j = 1, 2$, see (15.1). The $k \times p_j$ diagonal matrix Υ_{jn} of singular values of $\Omega^{-1/2}_n G_{jn}$ can be written as $[\Upsilon_{jn,p_j} : 0^{p_j\times(k-p_j)}]'$, where $\Upsilon_{jn,p_j} \in R^{p_j\times p_j}$ is a diagonal matrix with positive diagonal elements for n sufficiently large (since its smallest diagonal element is τ_{jn} and $\tau_{jn} \to \lambda^{1/2}_{\min}(G'_{j\infty}\Omega^{-1}_\infty G_{j\infty}) > 0$ for $j = 1, 2$ as shown above). In consequence, the singular value decomposition $C_{jn}\Upsilon_{jn}B'_{jn}$ equals $C_{jn,p_j}\Upsilon_{jn,p_j}B'_{jn}$, where $C_{jn,p_j} \to$

$C_{j\infty,p_j}$ and $B_{jn} \to B_{j\infty}$ for $j = 1, 2$. Furthermore, $\Omega_n^{-1/2} G_{jn} \to \Omega_\infty^{-1/2} G_{j\infty}$. Hence, $\Upsilon_{jn,p_j} = C'_{jn,p_j}(C_{jn,p_j}\Upsilon_{jn,p_j}B'_{jn})B_{jn} \to C'_{j\infty,p_j}(\Omega_\infty^{-1/2}G_{j\infty})B_{j\infty} := \Upsilon_{j\infty,p_j}$, where $\Upsilon_{j\infty,p_j}$ is a $p_j \times p_j$ diagonal matrix with nonnegative elements because Υ_{jn,p_j} has these properties $n \geq 1$. In consequence,

$$\Omega_\infty^{-1/2} G_{j\infty} = C_{j\infty,p_j}\Upsilon_{j\infty,p_j}B'_{j\infty}. \tag{16.9}$$

Suppose $[\overline{\Delta}_{1\infty} : \overline{\Delta}_{2\infty}] = [C_{1\infty,p_1} : C_{2\infty,p_2}]$ has column rank less than p. Then, there is a vector $\mu \in R^k$ with $||\mu|| = 1$ such that $[C_{1\infty,p_1} : C_{2\infty,p_2}]\mu = 0$. Let $\mu = (\mu'_1, \mu'_2)'$ for $\mu_j \in R^{p_j}$. Let $\xi_j = B_{j\infty}\Upsilon^{-1}_{j\infty,p_j}\mu_j$ for $j = 1, 2$ and $\xi = (\xi'_1, \xi'_2)'$. We have

$$\begin{aligned}
0^k = [C_{1\infty,p_1} : C_{2\infty,p_2}]\mu &= C_{1\infty,p_1}\mu_1 + C_{2\infty,p_2}\mu_2 \\
&= C_{1\infty,p_1}\Upsilon_{1\infty,p_1}B'_{1\infty}B_{1\infty}\Upsilon^{-1}_{1\infty,p_1}\mu_1 + C_{2\infty,p_2}\Upsilon_{2\infty,p_2}B'_{2\infty}B_{2\infty}\Upsilon^{-1}_{2\infty,p_2}\mu_2 \\
&= C_{1\infty,p_1}\Upsilon_{1\infty,p_1}B'_{1\infty}\xi_1 + C_{2\infty,p_2}\Upsilon_{2\infty,p_2}B'_{2\infty}\xi_2 \\
&= \Omega_\infty^{-1/2}G_{1\infty}\xi_1 + \Omega_\infty^{-1/2}G_{2\infty}\xi_2 \\
&= \lim \Omega_n^{-1/2}G_{1n}\xi_1 + \lim \Omega_n^{-1/2}G_{2n}\xi_2 \\
&= \lim \Omega_n^{-1/2}G_n\xi \\
&\neq 0^k,
\end{aligned} \tag{16.10}$$

where the fourth equality holds by the definition of ξ_j, the fifth equality holds by (16.9), the sixth equality holds because $\lim \Omega_n^{-1/2}G_n = \Omega_\infty^{-1/2}G_\infty$ by Assumptions $\mathrm{SS1}_{AR}$ and $\mathrm{SS1}_{LM}$, $G_n = [G_{1n} : G_{2n}]$, and $G_\infty = [G_{1\infty} : G_{2\infty}]$, the last equality holds because $G_n = [G_{1n} : G_{2n}]$, and the inequality holds because $\lim \tau_n > 0$ (shown above), τ_n is the smallest singular value of $\Omega_n^{-1/2}G_n$, and $\xi \neq 0$ (because $B_{j\infty}$ is an orthogonal matrix, $\Upsilon^{-1}_{j\infty,p_j}$ is nonsingular, and $\mu \neq 0$). Equation (16.10) is a contradiction, which completes the proof that $[\overline{\Delta}_{1\infty} : \overline{\Delta}_{2\infty}]$ has full column rank p when $\lim \tau_n > 0$. This completes the proof of part (a).

Next, we prove part (a) for the case of a pure $C(\alpha)$-LM test, in which case $WI_n(\theta) := 0$, $WI_\infty := 0$, and $D^\dagger_\infty = M_{\overline{\Delta}^a_{1\infty}}\overline{\Delta}^a_{2\infty}$, when Assumptions $\mathrm{SS1}_{LM}$(vi) and (vii) are replaced by the two conditions (vi) and (vii) in Comment (v) to Lemma 16.1. If $[\overline{\Delta}^a_{1\infty} : \overline{\Delta}^a_{2\infty}]$ has full column rank p, then the matrix $M_{\overline{\Delta}^a_{1\infty}}\overline{\Delta}^a_{2\infty}$ has rank p_2. This can be proved by showing that if $M_{\overline{\Delta}^a_{1\infty}}\overline{\Delta}^a_{2\infty}$ has rank less than p_2, then $[\overline{\Delta}^a_{1\infty} : \overline{\Delta}^a_{2\infty}]$ has column rank less than p. Let $(\cdot)^+$ denote the Moore-Penrose generalized inverse. Suppose $M_{\overline{\Delta}^a_{1\infty}}\overline{\Delta}^a_{2\infty}$ has rank less than p_2. Then, there exists a nonzero vector

$\varphi \in R^{p_2}$ such that $\overline{\Delta}^a_{2\infty}\varphi = P_{\overline{\Delta}^a_{1\infty}}\overline{\Delta}^a_{2\infty}\varphi$. That is,

$$\begin{aligned}&\overline{\Delta}^a_{2\infty}\varphi = \overline{\Delta}^a_{1\infty}\upsilon, \text{ where } \upsilon := (\overline{\Delta}^{a\prime}_{1\infty}\overline{\Delta}^a_{1\infty})^{+}\overline{\Delta}^{a\prime}_{1\infty}\overline{\Delta}^a_{2\infty}\varphi, \text{ and}\\ &[\overline{\Delta}^a_{1\infty} : \overline{\Delta}^a_{2\infty}]\zeta = 0^k, \text{ where } \zeta := (\upsilon', -\varphi')' \neq 0^k.\end{aligned} \tag{16.11}$$

In this case, $[\overline{\Delta}^a_{1\infty} : \overline{\Delta}^a_{2\infty}]$ has column rank less than p, which establishes the claim in the first sentence of the paragraph.

To prove part (a) in the pure $C(\alpha)$-LM case, it remains to show that $[\overline{\Delta}^a_{1\infty} : \overline{\Delta}^a_{2\infty}]$ has full column rank p. By its definition and (15.10),

$$\overline{\Delta}^a_{j\infty} = [C_{j\infty,q_j} : \overline{\Delta}_{j\infty,p_j-q_j} + a\zeta_j B_{j\infty,p_j-q_j}] \text{ for } j = 1, 2. \tag{16.12}$$

Suppose

$$[C_{1\infty,q_1} : C_{2\infty,q_2}] \text{ has full column rank } q_1 + q_2. \tag{16.13}$$

Then, by Corollary 16.2 of AG1, $[\overline{\Delta}^a_{1\infty} : \overline{\Delta}^a_{2\infty}]$ has full column rank $p_1 + p_2 = p$ a.s. conditional on $\overline{\Delta}_{j\infty,p_j-q_j}$ for $j = 1, 2$ and, hence, unconditionally as well. This holds because, conditional on $\overline{\Delta}_{j\infty,p_j-q_j}$ for $j = 1, 2$,

$$[\overline{\Delta}_{1\infty,p_1-q_1} + a\zeta_1 B_{1\infty,p_1-q_1} : \overline{\Delta}_{2\infty,p_2-q_2} + a\zeta_2 B_{2\infty,p_2-q_2}] \in R^{k\times(p_1-q_1+p_2-q_2)} \tag{16.14}$$

has a multivariate normal distribution with identity variance matrix multiplied by a constant (since $\zeta := [\zeta_1 : \zeta_2]$ has a multivariate normal distribution with identity variance matrix by assumption and $B_{j\infty,p_j-q_j}$ has orthonormal columns for $j = 1, 2$). This is sufficient to verify the condition on the variance matrix of $M_2\Delta_{p-q_*}\xi_2$ in Corollary 16.2 of AG1 (where $M_2\Delta_{p-q_*}\xi_2$ is defined in Corollary 16.2 of AG1).

Thus, to prove part (a) in the pure $C(\alpha)$-LM case, it remains to show (16.13). First, we show $q_j \leq r_{j\infty}$ for $j = 1, 2$. By the definition of q_j (see (15.2)), we have $n^{1/2}\tau_{jsn} \to \infty\ \forall s \leq q_j$ and $\limsup_{n\to\infty} n^{1/2}\tau_{jsn} < \infty\ \forall s > q_j$. Because $r_{jn} := r_{jF_n}$ is the rank of $\Omega^{-1/2}_{F_n} E_{F_n}\widehat{G}_{jn}$ (see (8.5)) and τ_{jsn} is the sth largest singular value of the same matrix, $\tau_{jsn} = 0\ \forall s > r_{jn}$, $\forall n \geq 1$. By condition (vi) in Comment (v) to Lemma 16.1, $r_{jn} = r_{j\infty}$ for some $r_{j\infty} \in \{0, ..., p_j\}$ for all n sufficiently large. The latter two results imply that $\limsup_{n\to\infty} n^{1/2}\tau_{jsn} < \infty\ \forall s > r_{j\infty}$. In consequence, $q_j \leq r_{j\infty}$.

Let C_{jn,q_j} denote the first q_j columns of C_{jn} for $j = 1, 2$. Let $C_{jn,r_{j\infty}}$ denote the first $r_{j\infty}$

columns of C_{jn} for $j = 1, 2$. Now, we have

$$\begin{aligned}
&\lambda_{\min}([C_{1\infty,q_1} : C_{2\infty,q_2}]'[C_{1\infty,q_1} : C_{2\infty,q_2}]) \\
&= \lim \lambda_{\min}([C_{1n,q_1} : C_{2n,q_2}]'[C_{1n,q_1} : C_{2n,q_2}]) \\
&\geq \lim \lambda_{\min}([C_{1n,r_{1\infty}} : C_{2n,r_{1\infty}}]'[C_{1n,r_{1\infty}} : C_{2n,r_{1\infty}}]) \\
&= \lim \lambda_{\min}([C_{*1F_n} : C_{*2F_n}]'[C_{*1F_n} : C_{*2F_n}]) \\
&= \lim \lambda_{\min}(C_{*F_n}' C_{*F_n}) \\
&\geq \delta \\
&> 0, \qquad (16.15)
\end{aligned}$$

where the first equality holds by Assumptions SS1$_{AR}$(vii) and SS1$_{LM}$(iv) (because the smallest eigenvalue of a matrix is a continuous function of the matrix), the first inequality holds because $q_j \leq r_{j\infty}$, the second equality holds by the argument given in the following paragraph, the third equality holds by definition, see (8.7), and the last two inequalities hold by condition (vii) in Comment (v) to Lemma 16.1 and $C_{*n} := C_{*F_n}$.

The second equality of (16.15) holds by the following argument. As stated above (see the third paragraph of this section), we choose the compact SVD of $\Omega_n^{-1/2} G_{jn}$ specified in (8.6) with $\theta = (\theta_{1*n}', \theta_{20}')'$ to be the compact SVD that is obtained from the SVD $C_{jn} \Upsilon_{jn} B_{jn}'$ by deleting the non-essential rows and columns of C_{jn}, Υ_{jn}, and B_{jn} for $j = 1, 2$. This implies that the matrix containing the first r_{jn} columns of C_{jn}, which is denoted by $C_{jn,r_{jn}}$, equals C_{*jF_n} (defined in (8.6)). Since $r_{jn} = r_{j\infty}$ for all n sufficiently large by condition (vi) in Comment (v) to Lemma 16.1, we obtain $C_{jn,r_{j\infty}} = C_{*jF_n}$ for all n sufficiently large for $j = 1, 2$, which establishes the second equality in (16.15). This completes the proof of part (a) in the pure $C(\alpha)$-LM case because (16.15) establishes (16.13).

Next, we complete the proof of part (b) using the result of part (a). By the convergence results in (15.11), (16.4), and (16.5) which hold jointly (using Assumption SS1$_{LM}$(ii)), the continuous mapping theorem gives

$$P_{D_n^\dagger} = P_{(\widehat{M}_{1n} + WI_n \widehat{P}_{1n}) n^{1/2} (\widehat{\Omega}_n^{-1/2} \widehat{D}_{2n} + a n^{-1/2} \zeta_2) T_{2n}} \rightarrow_d P_{(M_{\overline{\Delta}_{1\infty}^a} + WI_\infty P_{\overline{\Delta}_{1\infty}^a}) \overline{\Delta}_{2\infty}^a} = P_{D_\infty^\dagger}, \qquad (16.16)$$

where the equality holds by the definition of $D_n^\dagger(\theta)$ in (7.12) and because a projection matrix P_A is invariant to the multiplication of A by any nonzero constant and the post-multiplication of A by any nonsingular matrix and the continuous mapping theorem applies because of the a.s. full column rank property of $D_\infty^\dagger$ established in part (a) of the lemma. The convergence in (16.16) holds jointly with $n^{1/2}\widehat{g}_n \to_d Z_\infty$ by (16.3), which uses Assumption $\mathrm{SS1}_{LM}$(ii).

The convergence result of part (b) follows from (16.16), $n^{1/2}\widehat{g}_n \to_d Z_\infty$, and $\widehat{\Omega}_n^{-1/2} \to_p \Omega_\infty^{-1/2}$ using the continuous mapping theorem. The limit of the test statistic LM_{2n}, defined in (7.13), is $Z_\infty'\Omega_\infty^{-1/2}P_{D_\infty^\dagger}\Omega_\infty^{-1/2}Z_\infty$. This limit has a $\chi^2_{p_2}$ distribution conditional on $D_\infty^\dagger$ and, hence, is unconditionally $\chi^2_{p_2}$ as well, because (i) $\Omega_\infty^{-1/2}Z_\infty \sim N(0^k, I_k)$, (ii) $\Omega_\infty^{-1/2}Z_\infty$ and $D_\infty^\dagger$ are independent (because $D_\infty^\dagger$ is a deterministic function of $\overline{\Delta}^a_{j\infty}$ for $j = 1, 2$), and (iii) $D_\infty^\dagger$ has full rank p_2 a.s. This completes the proof of part (b).

Part (c) holds because

$$\begin{aligned} & \lim P(\phi_{2n}^{LM}(\theta_{1*n}, \eta) > 0) \\ = & \lim P(LM_{2n} > \chi^2_{p_2}(1-\eta)) \\ = & P(Z_\infty'\Omega_\infty^{-1/2}P_{D_\infty^\dagger}\Omega_\infty^{-1/2}Z_\infty > \chi^2_{p_2}(1-\eta)) \\ = & \eta, \end{aligned} \quad (16.17)$$

where the first equality holds by the definition of $\phi_{2n}^{LM}(\theta_{1*n}, \eta)$ in (7.13) with $\theta = (\theta_{1*n}', \theta_{20}')'$, and the second and third equalities hold by part (b) of the lemma. □

Proof of Lemma 16.2. Part (a) of the Lemma holds by condition (I) in (15.13) in the proof of Lemma 15.2 with the subscripts 1 replaced by subscripts 2 (which holds using Assumptions $\mathrm{SS2}_{LM}$(ii) and (iii)–(vii) in place of Assumptions $\mathrm{SS2}_{AR}$(ii) and (iv)–(viii)), combined with conditions (II) and (IV) in (15.13) (which hold because Assumption $\mathrm{SS2}_{AR}$ is imposed in the present lemma).

Now, we prove part (b). Assumption $\mathrm{SS2}_{LM}$(i) implies that $\liminf_{n\to\infty} \tau_{2n} = \liminf_{n\to\infty} \tau_{2p_2n} > 0$, where τ_{2n} is defined in (10.2). In consequence, $q_2 = p_2$, where q_2 is defined as q_1 is defined in (15.2) with subscripts 1 replaced by subscripts 2. By definition $T_{2n} := B_{2n}S_{2n}$, where B_{2n} is orthogonal and $S_{2n} = Diag\{(n^{1/2}\tau_{21n})^{-1}, ..., (n^{1/2}\tau_{2p_2n})^{-1}\}$ using $q_2 = p_2$, see (15.3) with the leading subscripts 1 replaced by 2. These results give $n^{1/2}T_{2n} = O(1)$. This and part (a) of the lemma give

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||n^{1/2}\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1)T_{2n} - n^{1/2}\widehat{\Omega}_n^{-1/2}\widehat{D}_{2n}T_{2n}|| = o_p(1). \quad (16.18)$$

Next, we have

$$ICS_n^* \to_p ICS_\infty^* > K_U^* \quad (16.19)$$

using the result in (16.5), the fact that $ICS_\infty^* := \lambda_{\min}^{1/2}(\Phi_\infty' G_\infty' \Omega_\infty^{-1} G_\infty \Phi_\infty) = \lim \tau_n^\Phi$ (using Assumptions $\mathrm{SS1}_{AR}$(vi) and $\mathrm{SS1}_{LM}$(vi) and (vii)), and the condition $\liminf_{n\to\infty} \tau_n^\Phi > K_U^*$ (i.e., Assumption

SS2$_{LM}$(i)). Hence, we obtain

$$ICS_n^* > K_U^* \text{ wp} \to 1 \text{ and } WI_n := 1 - s\left(\frac{ICS_n^* - K_L^*}{K_U^* - K_L^*}\right) = 0 \text{ wp} \to 1, \tag{16.20}$$

where the second result uses the first result and the conditions on $s(\cdot)$ in (7.8) (which imply that $s(x) = 1$ for all $x \geq 1$).

We now show that

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |WI_n(\theta_1)| = 0 \text{ wp} \to 1 \tag{16.21}$$

given that $\liminf_{n\to\infty} \tau_n^\Phi > K_U^*$.

By the same argument as used to show (14.5) and (14.9) in the proof of Lemma 14.2 in Section 14, but with $\tau_{n\theta_1}^\Phi$ $(:= \tau_n^\Phi(\theta_1))$ and τ_n^Φ in place of $\tau_{1n\theta_1}^\Phi$ and τ_{1n}^Φ, respectively, we have

$$\liminf_{n\to\infty} \inf_{\theta_1} \tau_{n\theta_1}^\Phi / \tau_n^\Phi = 1 \text{ and } \inf_{\theta_1} ICS_n^{*2}(\theta_1) \geq (\tau_n^\Phi)^2 + o_p(1), \tag{16.22}$$

where $\inf_{\theta_1}$ abbreviates $\inf_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})}$ and likewise with $\sup_{\theta_1}$.

By the same argument as used to show (14.5), but with Φ_n, G_n, $\Phi_{n\theta_1}$, $G_{n\theta_1}$, $\limsup_{n\to\infty} \sup_{\theta_1}$, and $\leq$ in place of Φ_{1n}, G_{1n}, $\Phi_{1n\theta_1}$, $G_{1n\theta_1}$, $\liminf_{n\to\infty} \inf_{\theta_1}$, and $\geq$, respectively, in (14.6) and (14.7), and with λ_n in place of $\lambda_{1n\theta_1}$, where $\lambda_n \in R^p$ is such that $||\lambda_n|| = 1$ and $\lambda_{\min}(\Phi_n G_n' \Omega_n^{-1} G_n \Phi_n) = \lambda_n' \Phi_n G_n' \Omega_n^{-1} G_n \Phi_n \lambda_n$, we obtain $\limsup_{n\to\infty} \sup_{\theta_1} \tau_{n\theta_1}^\Phi / \tau_n^\Phi = 1$. Combining this with the first result in (16.22), we get

$$\lim_{n\to\infty} \sup_{\theta_1} |\tau_{n\theta_1}^\Phi - \tau_n^\Phi| = 0. \tag{16.23}$$

Using (16.23), by the same argument as used to show (14.9), but with $\sup_{\theta_1}$, $+\inf_{\theta_1}$, and $\leq$ in place of $\inf_{\theta_1}$, $-\sup_{\theta_1}$, and $\geq$, respectively, we get $\sup_{\theta_1} ICS_n^{*2}(\theta_1) \leq (\tau_n^\Phi)^2 + o_p(1)$. This and the second result in (16.22) give

$$\sup_{\theta_1} |ICS_n^*(\theta_1) - ICS_n^*| = o_p(1) \tag{16.24}$$

using $ICS_n^* = \tau_n^\Phi + o_p(1)$ by (16.19). In turn, this establishes (16.21) using the same argument as above to show the second result in (16.20).

Now, (16.21) implies that, for $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$, the C($\alpha$)-LM statistic $LM_{2n}(\theta_1)$ can be written as in (7.12) and (7.13) but with $WI_n(\theta_1) = 0$ wp$\to 1$. That is, wp$\to 1$, the C(α)-LM statistic can be written as

$$LM_{2n}(\theta_1) = (n^{1/2}\widetilde{g}_n(\theta_1)'\widehat{M}_{1n}(\theta_1)) \left(n^{1/2}[\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1) + an^{-1/2}\zeta_2]T_{2n}\right)$$
$$\times \left(\left(n^{1/2}[\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1) + an^{-1/2}\zeta_2]T_{2n}\right)' \widehat{M}_{1n}(\theta_1)n^{1/2}[\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1) + an^{-1/2}\zeta_2]T_{2n}\right)^{-1}$$
$$\times \left(n^{1/2}[\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1) + an^{-1/2}\zeta_2]T_{2n}\right)' \widehat{M}_{1n}(\theta_1)n^{1/2}\widetilde{g}_n(\theta_1). \tag{16.25}$$

Each of the multiplicands in (16.25) differs from its counterpart evaluated at θ_{1*n} by $o_p(1)$ uniformly over $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$ $\forall K < \infty$ by Lemma 15.2(a) and (c) and (16.18). We have $||n^{1/2}\widetilde{g}_n(\theta_{1*n})||^2 = n\widehat{g}_n'\widehat{\Omega}_n^{-1}\widehat{g}_n = O_p(1)$ by (13.3). In addition, using (15.11) and (16.4),

$$\widehat{M}_{1n}n^{1/2}[\widehat{\Omega}_n^{-1/2}\widehat{D}_{2n} + an^{-1/2}\zeta_2]T_n \rightarrow_d M_{\overline{\Delta}_{1\infty}^a}\overline{\Delta}_{2\infty}^a \tag{16.26}$$

and the latter has full column rank p_2 by Lemma 16.1(a) under Assumptions SS1$_{AR}$, SS2$_{AR}$, SS1$_{LM}$, and SS2$_{LM}$, since $WI_\infty = 0$ and $D_\infty^\dagger = M_{\overline{\Delta}_{1\infty}^a}\overline{\Delta}_{2\infty}^a$ in the present case by (16.1), (16.5), and (16.20). In consequence, when $\theta_1 = \theta_{1*n}$, the term in the second line of (16.25) that is inverted converges in distribution to a matrix that is nonsingular a.s. (using $\widehat{M}_{1n}^2 = \widehat{M}_{1n}$). This, (16.25), and the results immediately following (16.25) establish the result of part (b). $\square$

17 Verification of Assumptions for the Second-Step C(α)-QLR1 Test

17.1 Second-Step C(α)-QLR1 Test Results

This section verifies Assumptions B(ii) and C(ii)-C(v) for the second-step C(α)-QLR1 test defined in Section 7.4.3. The results in this section apply only to moment condition models.

We employ the same definitions as in Sections 8.1.1, 14.1, 15.1, and 16.1. In particular, the following quantities, which appear in the asymptotic distribution in Lemma 17.1 below, are defined as follows: Z_∞ and Ω_∞ are defined in Assumption SS1$_{LM}$(ii), the (possibly random) $k \times p_1$ matrix $\overline{\Delta}_{1\infty}^a$ is defined in (15.6), and the (possibly random) $k \times p_2$ matrix $\overline{\Delta}_{2\infty}^a$ is defined in (15.6) with subscripts 2 in place of 1 throughout. As defined, $\Omega_\infty^{-1/2}Z_\infty \sim N(0^k, I_k)$.

By definition,

$$
\begin{aligned}
\Omega_n(\theta_1) &:= \Omega_{F_n}(\theta_1) := Var_{F_n}(n^{1/2}\widehat{g}_n(\theta_1)) \in R^{k\times k}, \\
G_{2n}(\theta_1) &:= E_{F_n}\widehat{G}_{2n}(\theta_1) \in R^{k\times p_2}, \\
G_{2si}(\theta_1) &:= \frac{\partial}{\partial\theta_{2s}'}g_i(\theta_1) \in R^k \ \forall s = 1, ..., p_2, \\
\sigma_{2sn}^2(\theta_1) &:= Var_{F_n}(||G_{2si}(\theta_1)||) \in R \ \forall s = 1, ..., p_2, \text{ and} \\
\Phi_{2n}(\theta_1) &:= Diag\{\sigma_{21n}^{-1}(\theta_1), ..., \sigma_{2p_2n}^{-1}(\theta_1)\} \in R^{p_2\times p_2}. \qquad (17.1)
\end{aligned}
$$

We write a SVD of $\Omega_n^{-1/2}G_{2n}\Phi_{2n}$ $(:= \Omega_n^{-1/2}(\theta_{1*n})G_{2n}(\theta_{1*n})\Phi_{2n}(\theta_{1*n}))$ as

$$
\Omega_n^{-1/2}G_{2n}\Phi_{2n} = C_{2n}^{\Phi}\Upsilon_{2n}^{\Phi}B_{2n}^{\Phi\prime}, \qquad (17.2)
$$

where $C_{2n}^{\Phi} \in R^{k\times k}$ and $B_{2n}^{\Phi} \in R^{p_2\times p_2}$ are orthogonal matrices and $\Upsilon_{2n}^{\Phi} \in R^{k\times p_2}$ has the singular values $\tau_{21n}^{\Phi}, ..., \tau_{2p_2n}^{\Phi}$ of $\Omega_n^{-1/2}G_{2n}\Phi_{2n}$ in nonincreasing order on its diagonal and zeros elsewhere. Suppose $\lim n^{1/2}\tau_{2sn}^{\Phi} \in [0, \infty]$ exists for $s = 1, ..., p_2$. (This is Assumption SS1$_{QLR1}$(i) below.) Let q_2^{Φ} $(\in \{0, ..., p_2\})$ be such that

$$
\lim n^{1/2}\tau_{2sn}^{\Phi} = \infty \text{ for } 1 \leq s \leq q_2^{\Phi} \text{ and } \lim n^{1/2}\tau_{2sn}^{\Phi} < \infty \text{ for } q_2^{\Phi} + 1 \leq s \leq p_2. \qquad (17.3)
$$

Note that $q_2^{\Phi} = q_2$ under Assumption SS1$_{QLR1}$ below, where q_2 is defined in (15.2) with subscripts 2 in place of 1.[15] For notational simplicity, let $\tau_{2n}^{\Phi} := \tau_{2p_2n}^{\Phi}$. That is, τ_{2n}^{Φ} is the smallest singular value of $\Omega_n^{-1/2}G_{2n}\Phi_{2n}$.

Define

$$
\begin{aligned}
h_{2\infty,s}^{\Phi} &:= \lim n^{1/2}\tau_{2sn}^{\Phi} < \infty \ \forall s = q_2 + 1, ..., p_2, \\
S_{2n}^{\Phi} &:= Diag\{(n^{1/2}\tau_{21n}^{\Phi})^{-1}, ..., (n^{1/2}\tau_{2q_1n}^{\Phi})^{-1}, 1, ..., 1\} \in R^{p_2\times p_2} \text{ and} \\
S_{2\infty}^{\Phi} &:= Diag\{0, ..., 0, 1, ..., 1\} \in R^{p_2}, \qquad (17.4)
\end{aligned}
$$

where q_2^{Φ} zeros appear in $S_{2\infty}^{\Phi}$. We have $S_{2n}^{\Phi} \to S_{2\infty}^{\Phi}$. In the case of local strong or semi-strong identification of θ_2 given θ_{1*n}, $q_2^{\Phi} = p_2^{\Phi}$ and $S_{2\infty}^{\Phi} = 0^{k\times p_2}$. In the case of local weak identification of θ_2 given θ_{1*n}, $S_{2\infty}^{\Phi} \neq 0^{k\times p_2}$.

For the second-step (SS) C(α)-QLR1 test, we use the following assumptions.

[15] This holds because $\{(\tau_{2sn}^{\Phi})^2 : s = 1, ..., p_2\}$ and $\{(\tau_{2sn})^2 : s = 1, ..., p_2\}$ are the eigenvalues of $\Phi_{2n}G_{2n}'\Omega_n^{-1}G_{2n}\Phi_{2n}$ and $G_{2n}'\Omega_n^{-1}G_{2n}$, respectively, and $\delta_1 \leq \liminf_{n\to\infty}\lambda_{\min}(\Phi_{2n}) \leq \limsup_{n\to\infty}\lambda_{\sup}(\Phi_{2n}) \leq \delta_2$ for some constants $\delta_1 > 0$ and $\delta_2 < \infty$ by Assumption SS1$_{QLR1}$.

Assumption SS1$_{\mathbf{QLR1}}$. For the null sequence S, (i) $\lim n^{1/2}\tau_{2sn}^{\Phi} \in [0,\infty]$ exists $\forall s \leq p_2$, (ii) $C_{2n}^{\Phi} \to C_{2\infty}^{\Phi}$ for some matrix $C_{2\infty}^{\Phi} \in R^{k\times k}$, and (iii) $B_{2n}^{\Phi} \to B_{2\infty}^{\Phi}$ for some matrix $B_{2\infty}^{\Phi} \in R^{p_2\times p_2}$.

Assumption SS2$_{\mathbf{QLR1}}$. For the null sequence S, $\forall K < \infty$, (i) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})} |\widehat{\sigma}_{2sn}^2(\theta_1) - \sigma_{2sn}^2(\theta_1)| \to_p 0$ for $\{\sigma_{2sn}^2(\cdot) : n \geq 1\}$ defined in (17.1) $\forall s = 1,...,p_2$, (ii) $\sup_{\theta_1\in B(\theta_{1*n},K/n^{1/2})} |\sigma_{2sn}^2(\theta_1) - \sigma_{2sn}^2| \to 0$ $\forall s = 1,...,p_2$, and (iii) $\sigma_{2sn}^2 \to \sigma_{2s\infty}^2$ for some constant $\sigma_{2s\infty}^2 \in (0,\infty)$ $\forall s = 1,...,p_2$.

The following quantities appear in the expression for the asymptotic distribution of rk_{2n} that is specified below. Define $Z_{D_2\infty} \in R^{k\times p_2}$ by

$$vec(Z_{D_2\infty}) := Z_{G_2\infty} - \Gamma_{2\infty}\Omega_\infty^{-1}Z_\infty \in R^{p_2k}, \tag{17.5}$$

where Z_∞, $Z_{G_2\infty}$, and $\Gamma_{2\infty}$ are defined in Assumption SS1$_{LM}$. The matrix $Z_{D_2\infty}$ has a normal distribution and is independent of Z_∞ because of the joint normality of $Z_{G_2\infty}$ and Z_∞ and $Cov(vec(Z_{D_2\infty}), Z_\infty) = E(Z_{G_2\infty} - \Gamma_{2\infty}\Omega_\infty^{-1}Z_\infty)Z_\infty' = EZ_{G_2\infty}Z_\infty' - \Gamma_{2\infty} = 0^{(p_2k)\times k}$. Partition the (nonrandom) matrices $B_{2\infty}^{\Phi}$ and $C_{2\infty}^{\Phi}$ as

$$B_{2\infty}^{\Phi} = (B_{2\infty,q_2}^{\Phi}, B_{2\infty,p_2-q_2}^{\Phi}) \text{ and } C_{2\infty}^{\Phi} = (C_{2\infty,q_2}^{\Phi}, C_{2\infty,k-q_2}^{\Phi}), \tag{17.6}$$

where $q_2 = q_2^{\Phi}$, $B_{2\infty,q_2}^{\Phi} \in R^{p_2\times q_2}$, $B_{2\infty,p_2-q_2}^{\Phi} \in R^{p_2\times(p_2-q_2)}$, $C_{2\infty,q_2}^{\Phi} \in R^{k\times q_2}$, and $C_{2\infty,k-q_2}^{\Phi} \in R^{k\times(k-q_2)}$. For simplicity, there is some abuse of notation here, e.g., $B_{2\infty,q_2}^{\Phi}$ and $B_{2\infty,p_2-q_2}^{\Phi}$ denote different matrices even if $p_2 - q_2$ happens to equal q_2.

Next, define the (possibly random) matrix $\overline{\Delta}_{2\infty}^{\Phi}$ as follows:

$$\begin{aligned}
\overline{\Delta}_{2\infty}^{\Phi} &= (\overline{\Delta}_{2\infty,q_2}^{\Phi}, \overline{\Delta}_{2\infty,p_2-q_2}^{\Phi}) \in R^{k\times p_2},\ \overline{\Delta}_{2\infty,q_2}^{\Phi} := C_{2\infty,q_2}^{\Phi} \in R^{k\times q_2}, \\
\overline{\Delta}_{2\infty,p_2-q_2}^{\Phi} &:= C_{2\infty}^{\Phi} h_{2\infty,p_2-q_2}^{\Phi\diamond} + \Omega_\infty^{-1/2} Z_{D_2\infty}\Phi_{2\infty}B_{2\infty,p_2-q_2}^{\Phi} \in R^{k\times(p_2-q_2)}, \\
h_{2\infty,p_2-q_2}^{\Phi\diamond} &:= \begin{bmatrix} 0^{q_2\times(p_2-q_2)} \\ Diag\{h_{2\infty,q_2+1}^{\Phi},...,h_{2\infty,p_2}^{\Phi}\} \\ 0^{(k-p_2)\times(p_2-q_2)} \end{bmatrix} \in R^{k\times(p_2-q_2)},\ \text{and} \\
\Phi_{2\infty} &:= Diag\{\sigma_{21\infty}^{-1},...,\sigma_{2p_2\infty}^{-1}\} \in R^{p_2\times p_2}.
\end{aligned} \tag{17.7}$$

When $\lim n^{1/2}\tau_{2n}^{\Phi} < \infty$, the lemma below shows that the asymptotic distribution of rk_{2n} is given by

$$rk_{2\infty} := \lambda_{\min}(K\overline{\Delta}_{2\infty,p_2-q_2}^{\Phi\prime} C_{2\infty,k-q_2}^{\Phi} C_{2\infty,k-q_2}^{\Phi\prime} \overline{\Delta}_{2\infty,p_2-q_2}^{\Phi}). \tag{17.8}$$

Define

$$
\begin{aligned}
AR^{\dagger}_{2\infty} &:= Z'_{\infty}\Omega_{\infty}^{-1/2}(M_{\overline{\Delta}^a_{1\infty}} + WI_{\infty}P_{\overline{\Delta}^a_{1\infty}})\Omega_{\infty}^{-1/2}Z_{\infty}, \\
LM_{2\infty} &:= Z'_{\infty}\Omega_{\infty}^{-1/2}P_{D^{\dagger}_{\infty}}\Omega_{\infty}^{-1/2}Z_{\infty}, \text{ and} \\
QLR1_{2\infty} &:= \frac{1}{2}\left(AR^{\dagger}_{2\infty} - rk_{2\infty} + \sqrt{(AR^{\dagger}_{2\infty} - rk_{2\infty})^2 + 4LM_{2\infty}\cdot rk_{2\infty}}\right). \qquad (17.9)
\end{aligned}
$$

The random variables $AR^{\dagger}_{2\infty}$ and $LM_{2\infty}$ give the asymptotic distributions of $AR^{\dagger}_{2n}$ and LM_{2n} (as stated in Lemma 16.1 for the LM_{2n} statistic). Note that, when $WI_{\infty} = 1$, we have $M_{\overline{\Delta}^a_{1\infty}} + WI_{\infty}P_{\overline{\Delta}^a_{1\infty}} = I_k$, $AR^{\dagger}_{2\infty} = Z'_{\infty}\Omega_{\infty}^{-1}Z_{\infty} \sim \chi^2_k$. The following lemma shows that $QLR1_{2\infty}$ is the asymptotic distribution of $QLR1_{2n}$ when $\lim n^{1/2}\tau^{\Phi}_{2n} < \infty$.

The following lemma verifies Assumptions B(ii), C(ii), and C(iii) for the second-step C(α)-QLR1 test.

Lemma 17.1 *Suppose $\widehat{g}_n(\theta_1)$ are moment conditions, $\widehat{M}_{1n}(\theta_1)$ is defined in* (7.10) *with $a > 0$, $p_1 < k$, and $p_2 \geq 1$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumptions* SS1$_{AR}$, SS1$_{LM}$, *and* SS1$_{QLR1}$. *Then, for the sequence S (or subsequence S_m),*

(a) $AR^{\dagger}_{2n} \to_d AR^{\dagger}_{2\infty}$,

(b) *when* $\lim n^{1/2}\tau^{\Phi}_{2n} < \infty$, (i) $rk_{2n} \to_d rk_{2\infty}$, (ii) $QLR1_{2n} \to_d QLR1_{2\infty}$, *where* $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, D^{\dagger}_{\infty}, rk_{2\infty})$ *are independent of Z_{∞} and $\overline{\Delta}^a_{j\infty}$ has full column rank p_j a.s. for $j = 1, 2$, and* (iii) $c^{QLR1}(1-\eta, rk_{2n}, WI^{\dagger}_n) \to_d c^{QLR1}(1-\eta, rk_{2\infty}, 1)$ *and the convergence is joint with that in part* (b)(ii),

(c) *when* $\lim n^{1/2}\tau^{\Phi}_{2n} = \infty$, (i) $rk_{2n} \to_p \infty$, (ii) $QLR1_{2n} \to_d LM_{2\infty} \sim \chi^2_{p_2}$, *and* (iii) $c^{QLR1}(1-\eta, rk_{2n}, WI^{\dagger}_n) \to_p \chi^2_{p_2}(1-\eta)$, *and*

(d) *for all* $\eta \in (0,1)$, $\lim P_{F_n}(\phi^{QLR1}_{2n}(\theta_{1*n}, \eta) > 0) = \eta$.

Comments: (i). For the second-step C(α)-QLR1 test, Lemma 17.1(d) establishes Assumptions B(ii) and C(ii). For a sequence S with $\lim n^{1/2}\tau^{\Phi}_{2n} = \infty$, Lemma 17.1(c) establishes Assumption C(iii) (because the asymptotic $\chi^2_{p_2}$ distribution of $QLR1_{2n}$ is absolutely continuous on R when $p_2 \geq 1$ and the probability limit of $c^{QLR1}(1-\eta, rk_{2n}, WI^{\dagger}_n)$ is the constant $\chi^2_{p_2}(1-\eta)$). Assumption C(iv) holds because the conditional critical value $c^{QLR1}(1-\eta, rk_{2n}, WI^{\dagger}_n)$ is nondecreasing in η since $c^{QLR1}(1-\eta, r, w)$ is the $1-\eta$ quantile of $QLR1(r, w)$, see (7.16).

(ii). Under local strong and semi-strong identification of θ_1 given θ_{20}, the terms $an^{-1/2}\zeta_1$ and $an^{-1/2}\zeta_2$, which arise in the definition of $QLR1_{2n}$, do not affect the asymptotic distributions in Lemma 17.1(b) and (c).

(iii). The proof of Lemma 17.1(b)(i) and (c)(i) uses Theorem 10.4 in the SM to AG1.

The next lemma provides conditions under which Assumption C(v) holds for the second-step C(α)-QLR1 test for sequences S with $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U^* > 0$.

Lemma 17.2 *Suppose $\widehat{g}_n(\theta_1)$ are moment conditions, $\widehat{D}_{jn}(\theta)$ is defined in* (7.9) *for $j = 1, 2$, and $\widehat{M}_{1n}(\theta_1)$ is defined in* (7.10) *with $a \geq 0$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumptions* SS1$_{AR}$, SS2$_{AR}$, SS1$_{LM}$, SS2$_{LM}$, *and* SS2$_{QLR1}$. *Then, under the sequence S (or subsequence S_m), for all constants $K < \infty$,*

(a) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |QLR1_{2n}(\theta_1) - QLR1_{2n}| = o_p(1)$ *and*

(b) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |c^{QLR1}(1-\eta, rk_{2n}(\theta_1), WI_n^{\dagger}(\theta_1)) - c^{QLR1}(1-\eta, rk_{2n}, WI_n^{\dagger})| = o_p(1)$ $\forall \eta \in (0, 1)$.

Comments: Lemma 17.2 does not require $a > 0$, but Lemma 17.1 above does.

17.2 Proofs of Lemmas 17.1 and 17.2

Proof of Lemma 17.1. First, we prove part (a). By (16.5), $WI_n \to_p WI_\infty$. By (15.11), $\widehat{M}_{1n} \to_d M_{\overline{\Delta}_{1\infty}^a}$ and $\widehat{P}_{1n} = I_k - \widehat{M}_{1n} \to_d I_k - M_{\overline{\Delta}_{1\infty}^a} = P_{\overline{\Delta}_{1\infty}^a}$. Hence,

$$\widehat{M}_{1n} + WI_n \widehat{P}_{1n} \to_d M_{\overline{\Delta}_{1\infty}^a} + WI_\infty P_{\overline{\Delta}_{1\infty}^a}. \tag{17.10}$$

This, $n^{1/2}\widehat{g}_n \to_d Z_\infty$ and $\widehat{\Omega}_n^{-1/2} \to_p \Omega_\infty^{-1/2}$ give

$$\begin{aligned} AR_{2n}^{\dagger} :=& \; n\widehat{g}_n' \widehat{\Omega}_n^{-1/2} \left(\widehat{M}_{1n} + WI_n \widehat{P}_{1n} \right) \widehat{\Omega}_n^{-1/2} \widehat{g}_n \\ &\to_d Z_\infty' \Omega_\infty^{-1/2} \left(M_{\overline{\Delta}_{1\infty}^a} + WI_\infty P_{\overline{\Delta}_{1\infty}^a} \right) \Omega_\infty^{-1/2} Z_\infty =: AR_{2\infty}^{\dagger}, \end{aligned} \tag{17.11}$$

which establishes part (a).

Parts (b)(i) and (c)(i) of the lemma hold by Theorem 10.4 in the SM to AG1 (using Assumption SS1$_{QLR1}$ to guarantee the existence of $\lim n^{1/2}\tau_{2sn}^{\Phi}$, $C_{2\infty}^{\Phi}$, and $B_{2\infty}^{\Phi}$). To make this clear, the following is the correspondence between the quantities in (17.1)–(17.7) above, which define the asymptotic distribution $rk_{2\infty}$ in part (b)(i), and those in the asymptotic distribution in Lemma

10.2, (10.16), and (10.17) in the SM to AG1:

$$\begin{aligned}
&Z_{D_{2\infty}} \leftrightarrow \overline{D}_h,\ G_{2n} \leftrightarrow E_{F_n}G_i,\ \Omega_n^{-1/2} \leftrightarrow W_{F_n},\ \Phi_{2n} \leftrightarrow U_{F_n},\\
&(\tau^{\Phi}_{21n}, ..., \tau^{\Phi}_{2p_2n}) \leftrightarrow (\tau_{1F_n}, ..., \tau_{p_2F_n}),\ (h^{\Phi}_{2\infty,q_2+1}, ..., h^{\Phi}_{2\infty,p_2}) \leftrightarrow (h_{1,q+1}, ..., h_{1,p})\\
&B^{\Phi}_{2\infty} \leftrightarrow h_2,\ C^{\Phi}_{2\infty} \leftrightarrow h_3,\ B^{\Phi}_{2,q_2,\infty} \leftrightarrow h_{2,q},\ B^{\Phi}_{2,p_2-q_2,\infty} \leftrightarrow h_{2,p-q},\\
&C^{\Phi}_{2\infty,q_2} \leftrightarrow h_{3,q},\ C^{\Phi}_{2\infty,k-q_2} \leftrightarrow h_{3,k-q},\ \overline{\Delta}^{\Phi}_{2\infty} \leftrightarrow \overline{\Delta}_h,\\
&\overline{\Delta}^{\Phi}_{2\infty,q_2} \leftrightarrow \overline{\Delta}_{h,q},\ \overline{\Delta}^{\Phi}_{2\infty,p_2-q_2} \leftrightarrow \overline{\Delta}_{h,p-q},\ h^{\Phi\diamond}_{2\infty,p_2-q_2} \leftrightarrow h^{\diamond}_{1,p-q},\\
&\Omega_\infty^{-1/2} \leftrightarrow h_{71} := W_1(h_7),\ \text{and}\ \Phi_{2\infty} \leftrightarrow h_{81} := U_1(h_8). \qquad (17.12)
\end{aligned}$$

Next, let

$$J_{2n} := n\widetilde{g}_n'\left(\widehat{M}_{1n} + WI_n\widehat{P}_{1n} - P_{D^\dagger_{2n}}\right)\widetilde{g}_n. \qquad (17.13)$$

It follows from (7.13) and (7.14) that

$$AR^\dagger_{2n} = LM_{2n} + J_{2n}. \qquad (17.14)$$

We now prove part (b)(ii). We have

$$\begin{aligned}
QLR1_{2n} :=\ &\frac{1}{2}\left(AR^\dagger_{2n} - rk_{2n} + \sqrt{(AR^\dagger_{2n} - rk_{2n})^2 + 4LM_{2n}\cdot rk_{2n}}\right)\\
\to_d\ &\frac{1}{2}\left(AR^\dagger_{2\infty} - rk_{2\infty} + \sqrt{(AR^\dagger_{2\infty} - rk_{2\infty})^2 + 4LM_{2\infty}\cdot rk_{2\infty}}\right)\\
=:\ &QLR1_{2\infty}, \qquad (17.15)
\end{aligned}$$

where the first and last equalities hold by the definitions of $QLR1_{2n}$ and $QLR1_{2\infty}$ in (7.14) and (17.9), respectively, and the convergence holds by (17.11), $rk_{2n} \to_d rk_{2\infty}$ (by part (b)(i) of the lemma), and $LM_{2n} \to_d LM_{2\infty}$ by Lemma 16.1(b). (The latter three convergence results hold jointly because they all rely on Assumption SS1_{LM}(ii).)

We have $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, D^\dagger_\infty)$ is independent of Z_∞ and $\overline{\Delta}^a_{j\infty}$ has full column rank p_j a.s. for $j = 1, 2$ by Lemma 16.1(b). In addition, $rk_{2\infty}$ is independent of Z_∞ because $rk_{2\infty}$ is a deterministic function of $Z_{D_{2\infty}}$ by (17.7) and (17.8) and $Z_{D_{2\infty}}$ is independent of Z_∞, see the discussion following (17.5).

Next, we prove part (b)(iii). First, we show that $WI^\dagger_n = 1$ wp→ 1. We have $WI_n \to_p WI_\infty$ (by (16.5)), $0 \le ICS^*_\infty = \lim \tau^\Phi_n \le \liminf_{n\to\infty} \tau^\Phi_{2n} = 0$ (by (16.5), the definitions of τ^Φ_n and τ^Φ_{2n} in (10.3), and $\lim n^{1/2}\tau^\Phi_{2n} < \infty$, which is assumed in part (b)), $WI_\infty = 1 - s(0) = 1$ (by (16.1), $ICS^*_\infty = 0$, and the definition of $s(\cdot)$ in (7.8)), and $WI^\dagger_n := 1(WI_n > 0)$. These results combine to establish that $WI^\dagger_n = 1$ wp→ 1. Hence, when proving part (b)(iii), we can suppose $WI^\dagger_n = 1$ a.s.

To prove part (b)(iii), we need to show $c^{QLR1}(1-\eta, rk_{2n}, 1) \to_d c^{QLR1}(1-\eta, rk_{2\infty}, 1)$. This holds by part (b)(i) of the lemma and the continuous mapping theorem provided $c^{QLR1}(1-\eta, r, 1)$ is continuous at all $r \geq 0$.

Now we establish the latter. For notational simplicity, let $c(r) := c^{QLR1}(1-\eta, r, 1)$ and $QLR1(r) := QLR1(r, 1)$. Given any $r_* \geq 0$ and sequence $\{r_n \geq 0 : n \geq 1\}$ such that $r_n \to r_*$ (as $n \to \infty$), it suffices to show that for any subsequence $\{v_n\}$ of $\{n\}$ there exists a subsubsequence $\{m_n\}$ such that $c(r_{m_n}) \to c(r_*)$. We have: (i) $QLR1(r_n) \to QLR1(r_*)$ a.s. (by the definition of $QLR1(r)$ in (7.16)), (ii) given any subsequence $\{v_n\}$ of $\{n\}$, there exists a subsubsequence $\{m_n\}$ such that $c_\infty = \lim c(r_{m_n})$ exists and is finite (because part (i) implies that $\{c(r) : |r - r_*| \leq \varepsilon\}$ lies in a compact set for some $\varepsilon > 0$), and (iii) $P(QLR1(r_*) = c_\infty) = 0$ (because $QLR1(r_*)$ has an absolutely continuous distribution). Results (i)–(iii) imply $1(QLR1(r_{m_m})) \leq c(r_{m_m})) \to 1(QLR1(r_*) \leq c_\infty)$ a.s. Hence, by the dominated convergence theorem,

$$P(QLR1(r_*) \leq c_\infty) = \lim P(QLR1(r_{m_m})) \leq c(r_{m_m})) = 1 - \eta, \tag{17.16}$$

where the last equality holds because $c(r_{m_m})$ is the $1-\eta$ quantile of $QLR1(r_{m_m}))$ for all $n \geq 1$. Equation (17.16) implies that c_∞ is the $1-\eta$ quantile of $QLR1(r_*)$, which is unique (because the distribution function of $QLR1(r_*)$ is continuous and strictly increasing on R_+). That is, $c_\infty = c(r_*)$, which completes the proof that $c^{QLR1}(1-\eta, r, 1)$ is continuous at all $r \geq 0$.

We now prove part (c)(ii). By part (c)(i), $rk_{2n} \to_p \infty$. By (17.14) and some algebra, we have $(AR_{2n}^\dagger - rk_{2n})^2 + 4LM_{2n} \cdot rk_{2n} = (LM_{2n} - J_{2n} + rk_{2n})^2 + 4LM_{2n} \cdot J_{2n}$. Therefore,

$$QLR1_{2n} = \frac{1}{2}\left(LM_{2n} + J_{2n} - rk_{2n} + \sqrt{(LM_{2n} - J_{2n} + rk_{2n})^2 + 4LM_{2n} \cdot J_{2n}}\right). \tag{17.17}$$

Using a mean-value expansion of the square-root expression in (17.17) about $(LM_{2n} - J_{2n} + rk_{2n})^2$, we have

$$\sqrt{(LM_{2n} - J_{2n} + rk_{2n})^2 + 4LM_{2n} \cdot J_{2n}} = LM_{2n} - J_{2n} + rk_{2n} + (2\sqrt{\zeta_n})^{-1} 4LM_{2n} \cdot J_{2n} \tag{17.18}$$

for an intermediate value ζ_n between $(LM_{2n} - J_{2n} + rk_{2n})^2$ and $(LM_{2n} - J_{2n} + rk_{2n})^2 + 4LM_{2n} \cdot J_{2n}$. It follows that

$$QLR1_{2n} = LM_{2n} + o_p(1) \to_d \chi^2_p, \tag{17.19}$$

where the equality holds because $(\sqrt{\zeta_n})^{-1} = o_p(1)$ (since $rk_{2n} \to_p \infty$, $LM_{2n} = O_p(1)$ by Lemma 16.1(b), and $J_{2n} = AR_{2n}^\dagger - LM_{2n} = O_p(1)$ using part (a) of the lemma) and the convergence holds by Lemma 16.1(b).

We now prove part (c)(iii). By an analogous argument to that used to prove (17.19), we obtain

$$QLR1(r,w) \rightarrow \chi^2_{p_2} \text{ a.s. as } r \rightarrow \infty \tag{17.20}$$

for $w = 0$ or 1, where $QLR1(r,w)$ is defined in (7.16). In consequence, $c^{QLR1}(1-\eta, r, w) \rightarrow_p \chi^2_{p_2}(1-\eta)$ as $r \rightarrow \infty$ for $w = 0$ or 1. This, $rk_{2n} \rightarrow_p \infty$ (by part (c)(i)), and $WI_n^\dagger \in \{0,1\}$ imply that $c^{QLR1}(1-\eta, rk_n, WI_n^\dagger) \rightarrow_p \chi^2_{p_2}(1-\eta)$, which establishes part (c)(iii).

Now, we prove part (d). First, consider the case when $\lim n^{1/2}\tau^\Phi_{2n} = \infty$. By parts (c)(ii) and (c)(iii),

$$QLR1_{2n} - c^{QLR1}(1-\eta, rk_{2n}, WI_n) \rightarrow_d LM_{2\infty} - \chi^2_{p_2}(1-\eta). \tag{17.21}$$

In consequence,

$$\lim P_{F_n}(\phi^{QLR1}_{2n}(\theta_{1*n}, \eta) > 0) = P(LM_{2\infty} - \chi^2_{p_2}(1-\eta) > 0) = \eta, \tag{17.22}$$

where the first equality holds using $P(LM_{2\infty} = \chi^2_{p_2}(1-\eta)) = 0$ (because $LM_{2\infty} \sim \chi^2_{p_2}$ by part (c)(ii)) and the second equality holds because $LM_{2\infty} \sim \chi^2_{p_2}$.

Next, we prove part (d) when $\lim n^{1/2}\tau^\Phi_{2n} < \infty$. By parts (b)(ii) and (b)(iii),

$$QLR1_{2n} - c^{QLR1}(1-\eta, rk_n, WI_n) \rightarrow_d QLR1_{2\infty} - c^{QLR1}(1-\eta, r_{2\infty}, 1). \tag{17.23}$$

Thus,

$$\lim P_{F_n}(\phi^{QLR1}_{2n}(\theta_{1*n}, \eta) > 0) = P(QLR1_{2\infty} > c^{QLR1}(1-\eta, r_{2\infty}, 1)) \tag{17.24}$$

provided $P(QLR1_{2\infty} = c^{QLR1}(1-\eta, r_{2\infty}, 1)) = 0$, which holds if $P(QLR1_{2\infty} = c^{QLR1}(1-\eta, r_{2\infty}, 1) | \overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, r_{2\infty}) = 0$ a.s. The latter holds if, conditional on $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_{2\infty})$, $QLR1_{2\infty}$ is absolutely continuous, which we now show.

As shown in the proof of part (b)(iii), $WI_\infty = 1$. This implies that $AR^\dagger_{2\infty} = Z'_\infty \Omega^{-1}_\infty Z_\infty$ by the definition of $AR^\dagger_{2\infty}$ in (17.9). In the present case where $WI_\infty = 1$, define

$$J_{2\infty} := AR^\dagger_{2\infty} - LM_{2\infty} = Z'_\infty \Omega^{-1/2}_\infty (I_k - P_{D^\dagger_{2\infty}}) \Omega^{-1/2}_\infty Z_\infty. \tag{17.25}$$

Conditional on $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_\infty)$, $J_{2\infty} \sim \chi^2_{k-p_2}$ because (i) $I_k - P_{D^\dagger_{2\infty}}$ is a projection matrix with rank $k - p_2$ a.s. (since $D^\dagger_{2\infty}$ has rank p_2 a.s. by Lemma 16.1(a)), (ii) $\Omega^{-1/2}_\infty Z_\infty \sim N(0^k, I_k)$ (by Assumptions SS1$_{AR}$(ii) and (iii)), and (iii) $\Omega^{-1/2}_\infty Z_\infty$ and $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_\infty)$ are independent (by part (b)(ii)). In addition, conditional on $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_{2\infty})$, $LM_{2\infty}$ and $J_{2\infty}$ are independent because $P_{D^\dagger_{2\infty}} \Omega^{-1/2}_\infty Z_\infty$ and $(I_k - P_{D^\dagger_{2\infty}}) \Omega^{-1/2}_\infty Z_\infty$ are jointly normally distributed and uncorrelated conditional on $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_{2\infty})$.

In sum, conditional on $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_\infty)$, $LM_{2\infty} \sim \chi^2_{p_2}$, $J_{2\infty} \sim \chi^2_{k-p_2}$, $LM_{2\infty}$ and $J_{2\infty}$ are independent, and, hence, $AR^\dagger_{2\infty} = LM_{2\infty} + J_{2\infty} \sim \chi^2_{p_2} + \chi^2_{k-p_2}$, where $\chi^2_{p_2}$ and $\chi^2_{k-p_2}$ are independent. Thus, the conditional distribution of $QLR1_{2\infty}$ (defined in (17.9)) given $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_{2\infty})$ with $rk_{2\infty} = r$ is the same as that of

$$QLR1(r, 1) := \frac{1}{2}\left(\chi^2_{p_2} + \chi^2_{k-p_2} - r + \sqrt{(\chi^2_{p_2} + \chi^2_{k-p_2} - r)^2 + 4\chi^2_{p_2} r}\right) \tag{17.26}$$

for all conditioning values of $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty})$, which is absolutely continuous for all $r \geq 0$. Hence, $QLR1_{2\infty}$ is absolutely continuous conditional on $(\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, rk_{2\infty})$ a.s., which completes the proof of (17.24).

Using the result in (17.26), we obtain

$$\begin{aligned} P(QLR1_{2\infty} > c^{QLR1}(1-\eta, r_{2\infty}, 1)) &= E[P(QLR1_{2\infty} > c^{QLR1}(1-\eta, r_{2\infty}, 1)|\overline{\Delta}^a_{1\infty}, \overline{\Delta}^a_{2\infty}, r_{2\infty})] \\ &= E[P(QLR1(r_{2\infty}, 1) > c^{QLR1}(1-\eta, r_{2\infty}, 1)|r_{2\infty})] \\ &= \eta, \end{aligned} \tag{17.27}$$

where the last equality holds because $c^{QLR1}(1-\eta, r, 1)$ is the $1-\eta$ quantile of the distribution of $QLR1(r, 1)$, see (7.16), and $QLR1(r, 1)$ is absolutely continuous for all $r \geq 0$. This and (17.24) establish part (d) when $\lim n^{1/2}\tau^\Phi_{2n} < \infty$. $\square$

Proof of Lemma 17.2. Let $\Pi_n(\theta_1)$ denote $LM_{2n}(\theta_1)$, $\widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1)$, $WI_n(\theta_1)$, or $AR^\dagger_{2n}(\theta_1)$, and let Π_n denote LM_{2n}, $\widehat{\Omega}_n^{-1/2}\widehat{D}_{2n}$, WI_n, or $AR^\dagger_{2n}$. For each definition of $\Pi_n(\theta_1)$ and Π_n, we have: for all $K < \infty$,

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Pi_n(\theta_1) - \Pi_n|| = o_p(1) \tag{17.28}$$

by Lemma 16.2(b) for $\Pi_n(\theta_1) = LM_{2n}(\theta_1)$, by Lemma 16.2(a) for $\Pi_n(\theta_1) = \widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1)$, by (16.20) and (16.21) for $\Pi_n(\theta_1) = WI_n(\theta_1)$, and for $\Pi_n(\theta_1) = AR^\dagger_{2n}(\theta_1)$ by the combination of Lemma 15.2(c), an analogous result to Lemma 15.2(c) with $\widehat{P}_{1n}(\theta_1)$ in place of $\widehat{M}_{1n}(\theta_1)$, and the result for $\Pi_n(\theta_1) = WI_n(\theta_1)$.

First, we prove part (b) of the lemma. By an analogous proof to that of condition (II) in (15.13) with $\widehat{\sigma}^2_{2sn}(\theta_1)$ in place of $\widehat{\Omega}_n(\theta_1)$ (using Assumptions $SS2_{QLR1}$(i) and (ii) in place of Assumptions $SS2_{AR}$(x) and (xi)), we have

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\widehat{\sigma}^2_{2sn}(\theta) - \widehat{\sigma}^2_{2sn}| = o_p(1) \ \forall s = 1, ..., p_2. \tag{17.29}$$

Using this, the definition of $\widehat{\Phi}_{2n}(\theta)$ in (7.15), and Assumptions SS2$_{QLR1}$(i)–(iii) (which imply that $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} \widehat{\sigma}_{2sn}^2(\theta_1) = O_p(1)$ and $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} 1/\widehat{\sigma}_{2sn}(\theta_1) = O_p(1)$), we obtain

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Phi}_{2n}(\theta) - \widehat{\Phi}_{2n}|| = o_p(1). \tag{17.30}$$

Combining (17.30), (17.28) with $\Pi_n(\theta_1) = \widehat{\Omega}_n^{-1/2}(\theta_1)\widehat{D}_{2n}(\theta_1)$, and the definition of $rk_{2n}(\theta_1)$ in (7.15) gives

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |rk_{2n}(\theta_1) - rk_{2n}| = o_p(1). \tag{17.31}$$

Equation (17.31), the result of (17.28) with $\Pi_n(\theta_1) = WI_n(\theta_1)$, and the continuity of $c^{QLR1}(1 - \eta, r, w)$ in $r \geq 0$ for all $\eta \in (0,1)$ and $w = 0$ combine to establish part (b) of the lemma.

The result of part (a) of the lemma follows from (17.31), (17.28) with $\Pi_n(\theta_1) = AR_{2n}^{\dagger}(\theta_1)$ and $\Pi_n(\theta_1) = LM_{2n}(\theta_1)$, and the functional form of $QLR1_{2n}(\theta_1)$ in (7.14). □

18 Amalgamation of High-Level Conditions

In this section, we amalgamate results given in the preceding sections of the SM for the two-step AR/AR, AR/LM, and AR/QLR1 tests.

18.1 Amalgamation Results for the AR/AR Test

The following theorem provides high-level (HL) sufficient conditions for Assumptions B and C to hold for the two-step AR/AR test. This theorem amalgamates the results of Lemmas 12.1, 12.2, 13.1, 14.1, 15.1, and 15.2 for the two-step AR/AR test.

Assumption HL1$_{\mathbf{AR/AR}}$. For the null sequence S, for some $\varepsilon > 0$ and $\forall K < \infty$, (i) $\sup_{\theta_1 \in B(\theta_{1*n}, \varepsilon)} ||\widehat{g}_n(\theta_1) - g_n(\theta_1)|| = o_p(1)$ for $\{g_n(\cdot) : n \geq 1\}$ defined in (14.1), (ii) $g_n = 0^k$ $\forall n \geq 1$, (iii) $\theta_{1*n} \to \theta_{1*\infty}$ for some $\theta_{1*\infty} \in \Theta_1$, (iv) $\widehat{g}_n(\theta_1)$ is twice continuously differentiable on $B(\theta_{1*n}, \varepsilon)$ (for all sample realizations) $\forall n \geq 1$, (v) $n^{1/2}(\widehat{g}_n', vec(\widehat{G}_{1n} - E_{F_n}\widehat{G}_{1n})')' \to_d (Z_\infty', Z_{G_1\infty}')' \sim N(0^{(p_1+1)k}, V_{1\infty})$ for some variance matrix $V_{1\infty} \in R^{(p_1+1)k \times (p_1+1)k}$ whose first k rows are denoted by $[\Omega_\infty : \Gamma_{1\infty}']$ for $\Omega_\infty \in R^{k \times k}$ and $\Gamma_{1\infty} \in R^{(p_1 k) \times k}$, (vi) $\sup_{\theta_1 \in B(\theta_{1*n}, \varepsilon)} ||\widehat{G}_{1n}(\theta_1) - G_{1n}(\theta_1)|| = o_p(1)$ for $\{G_{1n}(\cdot) : n \geq 1\}$ defined in (14.1), (vii) $\sup_{\theta_1 \in B(\theta_{1*n}, \varepsilon)} ||G_{1n}(\theta_1)|| = O(1)$, (viii) $\sup_{\theta_1 \in B(\theta_{1*n}, \varepsilon_n)} ||G_{1n}(\theta_1) - G_{1n}|| = o(1)$ for all sequences of positive constants $\varepsilon_n \to 0$, (ix) $G_{1n} \to G_{1\infty}$ for some matrix $G_{1\infty} \in R^{k \times p_1}$, (x) $E_{F_n}\widehat{\xi}_{1n} = O(1)$, where $\widehat{\xi}_{1n} := \max_{s,u \leq p_1} \sup_{\theta_1 \in B(\theta_{1*\infty}, \varepsilon)} ||(\partial^2/\partial\theta_{1s}\theta_{1u})\widehat{g}_n(\theta_1)||$, (xi) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| \to_p 0$ for $\{\Omega_n(\cdot) : n \geq 1\}$ defined in (14.1), (xii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Omega_n(\theta_1) - \Omega_n|| \to 0$, (xiii) $\liminf_{n \to \infty} \lambda_{\min}(\Omega_n) > 0$, (xiv) $\Omega_n \to \Omega_\infty$ for Ω_∞

as in condition (v), (xv) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Gamma}_{1n}(\theta_1) - \Gamma_{1n}(\theta_1)|| = o_p(1)$ for $\{\Gamma_{1n}(\cdot) : n \geq 1\}$ defined in (14.2), (xvi) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Gamma_{1n}(\theta_1) - \Gamma_{1n}|| \to 0$, (xvii) $\Gamma_{1n} \to \Gamma_{1\infty}$ for $\Gamma_{1\infty}$ as in condition (v), (xviii) $\lim n^{1/2}\tau_{1sn} \in [0, \infty]$ exists $\forall s \leq p_1$, (xix) $C_{1n} \to C_{1\infty}$ for some matrix $C_{1\infty} \in R^{k \times k}$, (xx) $B_{1n} \to B_{1\infty}$ for some matrix $B_{1\infty} \in R^{p_1 \times p_1}$, (xxi) $c_n \to 0$ for $\{c_n : n \geq 1\}$ in (7.3), (xxii) $nc_n \to \infty$, (xxiii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\widehat{\sigma}^2_{1sn}(\theta_1) - \sigma^2_{1sn}(\theta_1)| \to_p 0$ for $\{\sigma^2_{1sn}(\cdot) : n \geq 1\}$ $\forall s = 1, ..., p_1$ defined in (14.3), (xxiv) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\sigma^2_{1sn}(\theta_1) - \sigma^2_{1sn}| \to 0$ $\forall s = 1, ..., p_1$, and (xxv) $\liminf_{n \to \infty} \sigma^2_{1sn} > 0$ $\forall s = 1, ..., p_1$.

Assumption W. For the null sequence S, (i) $\widehat{W}_{1n}$ is symmetric and psd, and (ii) $\widehat{W}_{1n} \to_p W_{1\infty}$ for some nonrandom nonsingular matrix $W_{1\infty} \in R^{k \times k}$.

Theorem 18.1 *Suppose $\widehat{g}_n(\theta_1)$ are moment conditions, $\widehat{D}_{1n}(\theta)$ is defined in* (7.9)*, $\widehat{M}_{1n}(\theta_1)$ is defined in* (7.10) *with $a > 0$, $\widehat{Q}_n(\theta)$ is the* GMM *criterion function defined in* (7.2)*, CS_{1n} is the first-step* AR CS *CS^{AR}_{1n}, $\phi_{2n}(\theta_1, \eta)$ is the second-step* C(α)-AR *test $\phi^{AR}_{2n}(\theta_1, \eta)$, $\widehat{\alpha}_{2n}(\theta_1)$ is defined in* (7.4)–(7.8)*, and $p_1 < k$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumption* HL1$_{AR/AR}$.

(a) *Suppose, in addition, the sequence S (or subsequence S_m) is such that $\lim \tau^{\Phi}_{1n} < K_L$ (where $K_L < \infty$ appears in the definition of $\widehat{\alpha}_{2n}(\theta_1)$ in* (7.8)*). Then, Assumption* B *holds for the sequence S (or subsequence S_m).*

(b) *Suppose, in addition (to the conditions stated before part (a)), the sequence S (or subsequence S_m) is such that $\liminf_{n \to \infty} \tau_{1n} > 0$ and Assumption* W *holds. Then, Assumption* C *holds for the sequence S (or subsequence S_m).*

The following lemma provides high-level sufficient conditions for Assumption OE to hold for a sequence S for the two-step AR/AR test. This lemma amalgamates the results of Lemmas 12.3, 12.4, 13.2, and 14.2.

Assumption HL2$_{\mathbf{AR/AR}}$. For the null sequence S, (i) $\liminf_{n \to \infty} \tau^{\Phi}_{1n} > K_U$ (for $K_U > 0$ as in the definition of $\widehat{\alpha}_{2n}(\theta_1)$ in (7.8)), (ii) $\sup_{\theta_1 \in \Theta_1} n^{1/2}||\widehat{g}_n(\theta_1) - g_n(\theta_1)|| = O_p(1)$ for $\{g_n(\cdot) : n \geq 1\}$ defined in (14.1), (iii) $\lim_{n \to \infty} \inf_{\theta_1 \notin B(\theta_{1*n}, K_n/n^{1/2})} n^{1/2}||g_n(\theta_1)|| = \infty$ for all sequences $K_n \to \infty$, (iv) $\liminf_{n \to \infty} \inf_{\theta_1 \notin B(\theta_{1*n}, \varepsilon)} ||g_n(\theta_1)|| > 0$ $\forall \varepsilon > 0$, (v) $\sup_{\theta_1 \in \Theta_1} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| = o_p(1)$ for $\{\Omega_n(\cdot) : n \geq 1\}$ defined in (14.1), (vi) $\sup_{\theta_1 \in \Theta_1} ||\Omega_n(\theta_1)|| = O(1)$, and (vii) $\liminf_{n \to \infty} \inf_{\theta_1 \in \Theta_1} \lambda_{\min}(\Omega_n(\theta_1)) > 0$.

Lemma 18.2 *Suppose $\widehat{g}_n(\theta_1)$, $\widehat{D}_{1n}(\theta)$, $\widehat{M}_{1n}(\theta_1)$, CS_{1n}, $\phi_{2n}(\theta_1, \eta)$, and $\widehat{\alpha}_{2n}(\theta_1)$ are as in Theorem* 18.1*, $a > 0$, and $p_1 < k$. Let S be a null sequence (or S_m a null subsequence) that satisfies Assumptions* HL1$_{AR/AR}$, HL2$_{AR/AR}$, *and* W. *Then, Assumption* OE *holds for the sequence S (or subsequence S_m).*

18.2 Amalgamation Results for the AR/LM and AR/QLR1 Tests

The following theorem provides high-level sufficient conditions for Assumptions B and C to hold for the two-step AR/LM and AR/QLR1 tests. For the two-step AR/LM test, this theorem amalgamates the results of Lemmas 12.1, 12.2, 13.1, 14.1, 16.1, and 16.2. For the AR/QLR1 test, it amalgamates the results of Lemmas 12.1, 12.2, 13.1, 14.1, 17.1, and 17.2.

Assumption HL1$_{\mathbf{AR/LM}}$. For the null sequence S, $\forall K < \infty$, (i) $\widehat{g}_n(\theta_1, \theta_2)$ is differentiable in θ_2 at θ_{20} and $(\partial/\partial\theta_2')\widehat{g}_n(\theta_1, \theta_2)$ is differentiable in θ_1 with both holding $\forall \theta_1 \in B(\theta_{1*n}, \varepsilon)$ (for all sample realizations), $\forall n \geq 1$, for some $\varepsilon > 0$, (ii) $n^{1/2}(\widehat{g}_n', vec(\widehat{G}_{1n} - E_{F_n}\widehat{G}_{1n})', vec(\widehat{G}_{2n} - E_{F_n}\widehat{G}_{2n})')'$ $\rightarrow_d (Z_\infty', Z_{G1\infty}', Z_{G2\infty}')' \sim N(0^{(p+1)k}, V_\infty)$ for some variance matrix $V_\infty \in R^{(p+1)k \times (p+1)k}$ whose first k rows are denoted by $[\Omega_\infty : \Gamma_{1\infty}' : \Gamma_{2\infty}']$ for $\Omega_\infty \in R^{k \times k}$ and $\Gamma_{j\infty} \in R^{(p_j k) \times k}$ for $j = 1, 2$, (iii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{G}_{2n}(\theta_1) - G_{2n}(\theta_1)|| \rightarrow_p 0$ for $\{G_{2n}(\cdot) : n \geq 1\}$ defined in (14.1), (iv) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||G_{2n}(\theta_1) - G_{2n}|| \rightarrow 0$, (v) $G_{2n} \rightarrow G_{2\infty}$ for some matrix $G_{2\infty} \in R^{k \times p_2}$, (vi) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||(\partial^2/\partial\theta_{1s}\partial\theta_2')\widehat{g}_n(\theta_1, \theta_{20})|| = O_p(1)$ for $s = 1, ..., p_1$, (vii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})}$ $||\widehat{\Gamma}_{2n}(\theta_1) - \Gamma_{2n}(\theta_1)|| = o_p(1)$ for $\{\Gamma_{2n}(\cdot) : n \geq 1\}$ defined in (14.2), (viii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})}$ $||\Gamma_{2n}(\theta_1) - \Gamma_{2n}|| \rightarrow 0$, (ix) $\Gamma_{2n} \rightarrow \Gamma_{2\infty}$ for $\Gamma_{2\infty}$ as in condition (ii), (x) $\lim n^{1/2}\tau_{2sn} \in [0, \infty]$ exists $\forall s \leq p_2$ (where τ_{2sn} is defined in the paragraph containing (10.2), (xi) $C_{2n} \rightarrow C_{2\infty}$ for some matrix $C_{2\infty} \in R^{k \times k}$, (xii) $B_{2n} \rightarrow B_{2\infty}$ for some matrix $B_{2\infty} \in R^{p_2 \times p_2}$, (xiii) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\widehat{\sigma}_{2sn}^2(\theta_1) - \sigma_{2sn}^2(\theta_1)| \rightarrow_p 0$ for $\{\sigma_{2sn}^2(\cdot) : n \geq 1\}$ $\forall s = 1, ..., p_2$ defined in (17.1), (xiv) $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\sigma_{2sn}^2(\theta_1) - \sigma_{2sn}^2| \rightarrow 0$ $\forall s = 1, ..., p_2$, and (xv) $\sigma_{jsn}^2 \rightarrow \sigma_{js\infty}^2$ for some constant $\sigma_{js\infty}^2 \in (0, \infty)$ $\forall s = 1, ..., p_2, \forall j = 1, 2$.

Assumption HL1$_{\mathbf{AR/QLR1}}$. For the null sequence S, (i) $\lim n^{1/2}\tau_{2sn}^\Phi \in [0, \infty]$ exists $\forall s \leq p_2$ (where $\tau_{2sn}^\Phi := \tau_{2sn}^\Phi(\theta_{1*n}, \theta_{20})$ is defined in the paragraph containing (17.2), (ii) $C_{2n}^\Phi \rightarrow C_{2\infty}^\Phi$ for some matrix $C_{2\infty}^\Phi \in R^{k \times k}$, and (iii) $B_{2n}^\Phi \rightarrow B_{2\infty}^\Phi$ for some matrix $B_{2\infty}^\Phi \in R^{p_2 \times p_2}$.

Lemma 18.3 *Suppose the conditions in Theorem* 18.1 *hold except that* $\phi_{2n}(\theta_1, \eta)$ *is the second-step* C(α)-LM *test* $\phi_{2n}^{LM}(\theta_1, \eta)$ *or the second-step* C(α)-QLR1 *test* $\phi_{2n}^{QLR1}(\theta_1, \eta)$, $\widehat{\alpha}_{2n}(\theta_1)$ *is defined accordingly in* (7.6)–(7.8)*, and* $p_1 < k$ *is replaced by* $p_2 \geq 1$ *for the* C(α)-LM *test and by* $p_2 \geq 1$ *and* $p \leq k$ *for the* C(α)-QLR1 *test. Let* S *be a null sequence* (*or* S_m *a null subsequence*) *that satisfies Assumptions* HL1$_{AR/AR}$ *and* HL1$_{AR/LM}$ *and, for the second-step* C(α)-QLR1 *test, Assumption* HL1$_{AR/QLR1}$ *as well.*

(a) *Suppose, in addition, the sequence* S (*or subsequence* S_m) *is such that* $\lim \tau_n^\Phi < K_L$ (*where* $K_L < \infty$ *appears in the definition of* $\widehat{\alpha}_{2n}(\theta_1)$ *in* (7.8)). *Then, Assumption* B *holds for the sequence* S (*or subsequence* S_m).

(b) *Suppose, in addition (to the conditions stated before part (a)), the sequence S (or subsequence S_m) is such that* $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U^*$ *and Assumption* W *holds. Then, Assumption* C *holds for the sequence S (or subsequence S_m).*

Comment: When $LM_{2n}(\theta)$ is the pure C(α)-LM statistic, i.e., $WI_n(\theta) := 0$, Lemma 18.3(a) holds provided conditions (vi) and (vii) in Comment (v) to Lemma 16.1 hold, and Lemma 18.3(b) holds with the weaker condition $\liminf_{n\to\infty} \tau_n > 0$ in place of $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U^*$. The same is true when the $QLR1_{2n}(\theta)$ is the pure C(α)-QLR1 statistic and $WI_n(\theta) := 0$ in the QLR1 critical value function.

The following lemma provides high-level sufficient conditions for Assumption OE to hold for a sequence S for the two-step AR/LM and AR/QLR1 tests. This lemma amalgamates the results of Lemmas 12.3, 12.4, 13.2, and 14.2 for these tests.

Assumption HL2$_{\mathbf{AR/LM,QLR1}}$. Assumption HL2$_{AR/AR}$ holds with τ_n^{Φ} in place of τ_{1n}^{Φ} in part (i).

Lemma 18.4 *Suppose* $\widehat{g}_n(\theta_1)$, $\widehat{D}_{1n}(\theta)$, $\widehat{M}_{1n}(\theta_1)$, CS_{1n}, *and* $\{c_n : n \geq 1\}$ *are as in Theorem* 18.1, $\phi_{2n}(\theta_1, \eta)$ *is the second-step* C(α)-LM $\phi_{2n}^{LM}(\theta_1, \eta)$ *or* C(α)-QLR1 *test* $\phi_{2n}^{QLR1}(\theta_1, \eta)$, $\widehat{\alpha}_{2n}(\theta_1)$ *is defined accordingly in* (7.6)–(7.8), $a > 0$, *and* $p_2 \geq 1$. *Let S be a null sequence (or S_m a null subsequence) that satisfies Assumptions* HL1$_{AR/AR}$, HL1$_{AR/LM}$, HL2$_{AR/LM,QLR1}$, *and* W. *Then, Assumption* OE *holds for the sequence S (or subsequence S_m).*

Comment: Lemma 18.4 differs from Lemma 18.2 because the second-step data-dependent significance level differs between the second-step C(α)-AR test, which is considered in the latter lemma, and the second-step C(α)-LM and C(α)-QLR1 tests, which are considered in the former lemma.

18.3 Proofs of Theorem 18.1 and Lemmas 18.2, 18.3, and 18.4

Proof of Theorem 18.1. Assumption B(i) holds by Lemma 13.1, which employs Assumption FS1$_{AR}$, because Assumption HL1$_{AR/AR}$(xi) $\Rightarrow$ Assumption FS1$_{AR}$(i); HL1$_{AR/AR}$(xiv) $\Rightarrow$ FS1$_{AR}$(ii); HL1$_{AR/AR}$(v) $\Rightarrow$ FS1$_{AR}$(iii); and HL1$_{AR/AR}$(xiii) $\Rightarrow$ FS1$_{AR}$(iv).

Assumptions B(ii), C(ii), and C(iii) hold by Lemma 15.1 (and Comment (i) following it), which employs Assumption SS1$_{AR}$, because $a > 0$; $p_1 < k$; Assumption HL1$_{AR/AR}$(xviii) $\Rightarrow$ Assumption SS1$_{AR}$(i); HL1$_{AR/AR}$(v) $\Rightarrow$ SS1$_{AR}$(ii); HL1$_{AR/AR}$(xiii) & HL1$_{AR/AR}$(xiv) $\Rightarrow$ SS1$_{AR}$(iii); HL1$_{AR/AR}$(xv) & HL1$_{AR/AR}$(xvii) $\Rightarrow$ SS1$_{AR}$(iv); HL1$_{AR/AR}$(xi) $\Rightarrow$ SS1$_{AR}$(v); HL1$_{AR/AR}$(xiv) $\Rightarrow$ SS1$_{AR}$(vi); HL1$_{AR/AR}$(xix) $\Rightarrow$ SS1$_{AR}$(vii), and HL1$_{AR/AR}$(xx) $\Rightarrow$ SS1$_{AR}$(viii).

Assumption B(iii) holds under the conditions of Theorem 18.1(a) by Lemma 14.1, which employs Assumption SL1$_{AR}$, because the assumption $\lim \tau_{1n}^{\Phi} < K_L$ of Theorem 18.1(a) $\Rightarrow$ Assumption SL1$_{AR}$(i); Assumption HL1$_{AR/AR}$(vi) $\Rightarrow$ Assumption SL1$_{AR}$(ii); HL1$_{AR/AR}$(vii) $\Rightarrow$ SL1$_{AR}$(iii);

$\text{HL1}_{AR/AR}$(xi) $\Rightarrow$ SL1_{AR}(iv); $\text{HL1}_{AR/AR}$(xiii) $\Rightarrow$ SL1_{AR}(v); $\text{HL1}_{AR/AR}$(xiv) $\Rightarrow$ SL1_{AR}(vi); $\text{HL1}_{AR/AR}$(xxiii) $\Rightarrow$ SL1_{AR}(vii); and $\text{HL1}_{AR/AR}$(xxv) $\Rightarrow$ SL1_{AR}(viii).

Assumption C(i) holds under the conditions of Theorem 18.1(b) by Lemma 12.1, which employs Assumptions ES1 and ES2, because the result of Lemma 12.2 $\Rightarrow$ Assumption ES1 and Lemma 12.2 applies here because Assumption FOC is verified below; Assumption $\text{HL1}_{AR/AR}$(iv) $\Rightarrow$ ES2(i); Assumption $\text{HL1}_{AR/AR}$(v) $\Rightarrow$ ES2(ii); $\text{HL1}_{AR/AR}$(vi) & $\text{HL1}_{AR/AR}$(vii) $\Rightarrow$ ES2(iii); $\text{HL1}_{AR/AR}$(xxii) $\Rightarrow$ ES2(iv), and Assumption W $\Rightarrow$ ES2(v).

For the verification of Assumption C(i) under the conditions of Theorem 18.1(b), it remains to show that Assumption FOC, which is employed in Lemma 12.2, holds. We have: the assumption $\liminf_{n\to\infty} \tau_{1n} > 0$ of Theorem 18.1(b) $\Rightarrow$ Assumption FOC(i); Assumption $\text{HL1}_{AR/AR}$(i) $\Rightarrow$ FOC(ii); $\text{HL1}_{AR/AR}$(ii) $\Rightarrow$ FOC(iii); $\text{HL1}_{AR/AR}$(iii) $\Rightarrow$ FOC(iv); $\text{HL1}_{AR/AR}$(iv) $\Rightarrow$ FOC(v); $\text{HL1}_{AR/AR}$(v) $\Rightarrow$ FOC(vi); $\text{HL1}_{AR/AR}$(vi) $\Rightarrow$ FOC(viii); $\text{HL1}_{AR/AR}$(vii) $\Rightarrow$ FOC(ix); $\text{HL1}_{AR/AR}$(viii) $\Rightarrow$ FOC(x); $\text{HL1}_{AR/AR}$(ix) $\Rightarrow$ FOC(xi); $\text{HL1}_{AR/AR}$(xiii) $\Rightarrow$ FOC(xiii); Markov's inequality and $\text{HL1}_{AR/AR}$(x) $\Rightarrow$ FOC(xiv); and Assumption W $\Rightarrow$ FOC(xvi). In addition, because we are considering the moment condition model here, by the paragraph following Assumption FOC, Assumptions FOC(vii), (xii), and (xv) are implied by Assumptions FOC(v) and (xiv), which have just been verified, and Assumption $\text{HL1}_{AR/AR}$(x).

As noted in Section 15 above, Assumption C(iv) holds automatically for the second-step AR test provided $p_1 < k$ (which is assumed here) because its nominal level η critical value is the $1-\eta$ quantile of the $\chi^2_{k-p_1}$ distribution which is nondecreasing in η for $\eta \in (0,1)$ when $p_1 < k$.

Assumption C(v) holds under the conditions of Theorem 18.1(b) by Lemma 15.2(d), which employs Assumption SS2_{AR}, because the assumption $\liminf_{n\to\infty} \tau_{1n} > 0$ of Theorem 18.1(b) $\Rightarrow$ Assumption SS2_{AR}(i); Assumption $\text{HL1}_{AR/AR}$(iv) $\Rightarrow$ Assumption SS2_{AR}(ii); $\text{HL1}_{AR/AR}$(v) $\Rightarrow$ SS2_{AR}(iii); $\text{HL1}_{AR/AR}$(vi) $\Rightarrow$ SS2_{AR}(iv); $\text{HL1}_{AR/AR}$(vii) $\Rightarrow$ SS2_{AR}(v); Markov's inequality and $\text{HL1}_{AR/AR}$(x) $\Rightarrow$ SS2_{AR}(vi); $\text{HL1}_{AR/AR}$(xv) $\Rightarrow$ SS2_{AR}(vii); $\text{HL1}_{AR/AR}$(xvi) $\Rightarrow$ SS2_{AR}(viii); $\text{HL1}_{AR/AR}$(xvii) $\Rightarrow$ SS2_{AR}(ix); $\text{HL1}_{AR/AR}$(xi) $\Rightarrow$ SS2_{AR}(x); $\text{HL1}_{AR/AR}$(xii) $\Rightarrow$ SS2_{AR}(xi); $\text{HL1}_{AR/AR}$(xiii) $\Rightarrow$ SS2_{AR}(xii); and $\text{HL1}_{AR/AR}$(xiv) $\Rightarrow$ SS2_{AR}(xiii). $\square$

Proof of Lemma 18.2. As required by Assumption OE, Assumption C holds for the sequence S by Theorem 18.1(b), using Assumptions $\text{HL1}_{AR/AR}$ and $\text{HL2}_{AR/AR}$(i). Note that the condition $\liminf_{n\to\infty} \tau_{1n} > 0$ of Theorem 18.1(b) is implied by Assumption $\text{HL2}_{AR/AR}$(i) (i.e., $\liminf_{n\to\infty} \tau^{\Phi}_{1n} > K_U$ for $K_U > 0$) plus Assumption HL1_{AR}(xxv), which implies that $\limsup_{n\to\infty} \lambda_{\max}(\Phi_{1n}) < \infty$, where $\Phi_{1n} := \Phi_{1n}(\theta_{1*n}, \theta_{20})$ is defined in (8.3) and (14.3).

By Lemma 12.3, the results of Lemmas 12.4 and 13.2 imply that Assumption OE(i) holds. Hence, we need to verify the assumptions used in Lemmas 12.4 and 13.2 to verify Assumption

OE(i).

First, we verify Assumptions ES3 and ES4, which are imposed in Lemma 12.4. Assumptions $\mathrm{HL2}_{AR/AR}$(i) & $\mathrm{HL1}_{AR/AR}$(xxv) $\Rightarrow$ Assumption ES3(i); $\mathrm{HL1}_{AR/AR}$(iv) $\Rightarrow$ ES3(iii); $\mathrm{HL1}_{AR/AR}$(v) $\Rightarrow$ ES3(iv); $\mathrm{HL1}_{AR/AR}$(vi) $\Rightarrow$ ES3(v); $\mathrm{HL1}_{AR/AR}$(viii) $\Rightarrow$ ES3(vi); $\mathrm{HL1}_{AR/AR}$(ix) $\Rightarrow$ ES3(vii); $\mathrm{HL1}_{AR/AR}$(xi) $\Rightarrow$ ES3(viii); $\mathrm{HL1}_{AR/AR}$(xiii) $\Rightarrow$ ES3(ix); $\mathrm{HL1}_{AR/AR}$(xxi) $\Rightarrow$ ES3(x); and W $\Rightarrow$ ES3(xi). Assumption ES3(ii) holds by Lemma 12.1 under Assumptions ES1 and ES2; it is shown in the proof of Theorem 18.1 that Assumption $\mathrm{HL1}_{AR/AR}$ $\Rightarrow$ Assumption ES2; Lemma 12.2 verifies Assumption ES1 using Assumption FOC; and Assumption FOC is verified in Theorem 18.1 (and the present lemma imposes Assumption $\mathrm{HL1}_{AR/AR}$, which is employed in Theorem 18.1). Assumption $\mathrm{HL2}_{AR/AR}$(ii) $\Rightarrow$ Assumption ES4(i) and $\mathrm{HL2}_{AR/AR}$(iv) $\Rightarrow$ ES4(ii). This completes the verification of Assumptions ES3 and ES4.

Second, we verify Assumption $\mathrm{FS2}_{AR}$, which is used in Lemma 13.2. Assumption $\mathrm{HL2}_{AR/AR}$(ii) $\Rightarrow$ Assumption $\mathrm{FS2}_{AR}$(i); $\mathrm{HL2}_{AR/AR}$(iii) $\Rightarrow$ $\mathrm{FS2}_{AR}$(ii); $\mathrm{HL2}_{AR/AR}$(v) $\Rightarrow$ $\mathrm{FS2}_{AR}$(iii); $\mathrm{HL2}_{AR/AR}$(vi) $\Rightarrow$ $\mathrm{FS2}_{AR}$(iv); and $\mathrm{HL2}_{AR/AR}$(vii) $\Rightarrow$ $\mathrm{FS2}_{AR}$(v). This completes the verification of Assumption OE(i).

Lastly, Assumption OE(ii) holds under Assumption $\mathrm{SL2}_{AR}$ by Lemma 14.2. Hence, we need to verify Assumption $\mathrm{SL2}_{AR}$. Assumption $\mathrm{HL2}_{AR/AR}$(i) $\Rightarrow$ Assumption $\mathrm{SL2}_{AR}$(i); $\mathrm{HL1}_{AR/AR}$(vi) $\Rightarrow$ $\mathrm{SL2}_{AR}$(ii); $\mathrm{HL1}_{AR/AR}$(viii) $\Rightarrow$ $\mathrm{SL2}_{AR}$(iii); $\mathrm{HL1}_{AR/AR}$(ix) $\Rightarrow$ $\mathrm{SL2}_{AR}$(iv); $\mathrm{HL1}_{AR/AR}$(xi) $\Rightarrow$ $\mathrm{SL2}_{AR}$(v); $\mathrm{HL1}_{AR/AR}$(xii) $\Rightarrow$ $\mathrm{SL2}_{AR}$(vi); $\mathrm{HL1}_{AR/AR}$(xiii) $\Rightarrow$ $\mathrm{SL2}_{AR}$(vii); $\mathrm{HL1}_{AR/AR}$(xxiii) $\Rightarrow$ $\mathrm{SL2}_{AR}$(viii); $\mathrm{HL1}_{AR/AR}$(xxiv) $\Rightarrow$ $\mathrm{SL2}_{AR}$(ix); and $\mathrm{HL1}_{AR/AR}$(xxv) $\Rightarrow$ $\mathrm{SL2}_{AR}$(x). This completes the proof of the lemma. $\square$

Proof of Lemma 18.3. For the AR/LM and AR/QLR1 tests, Assumptions B(i) and C(i) hold by the same arguments as given in the proof of Theorem 18.1 (using the fact that $\liminf_{n\to\infty} \tau_n^{\Phi} > K_U^*$ (> 0), which is assumed in Lemma 18.3(b), implies that $\liminf_{n\to\infty} \tau_{1n} > 0$, which is assumed in Theorem 18.1(b), when verifying Assumption C(i)).

For the AR/LM and AR/QLR1 tests, Assumption B(iii) holds under the conditions of Lemma 18.3(a) by Lemma 14.1, which employs Assumption $\mathrm{SL1}_{LM,QLR1}$ (for the second-step C(α)-LM and C(α)-QLR1 tests), because the assumption $\lim \tau_n^{\Phi} < K_L$ of Lemma 18.3(a) $\Rightarrow$ Assumption $\mathrm{SL1}_{LM,QLR1}$(i); Assumption $\mathrm{HL1}_{AR/AR}$ $\Rightarrow$ Assumptions $\mathrm{SL1}_{AR}$(ii)–(viii), as shown in the proof of Theorem 18.1, and the latter conditions constitute Assumption $\mathrm{SL1}_{LM,QLR1}$(ii); and $\mathrm{HL1}_{AR/LM}$(xiii)–(xv) $\Rightarrow$ $\mathrm{SL1}_{LM,QLR1}$(iii) and (iv).

For the second-step C(α)-LM test, Assumptions B(ii), C(ii), and C(iii) hold by Lemma 16.1 and Comment (i) following it, which employs Assumptions $\mathrm{SS1}_{AR}$ and $\mathrm{SS1}_{LM}$, because Assumption $\mathrm{HL1}_{AR/AR}$ implies Assumption $\mathrm{SS1}_{AR}$ (as shown in the proof of Theorem 18.1), and Assumption $\text{HL1}_{AR/LM}$ implies Assumption SS1_{LM}. The latter holds because Assumption $\text{HL1}_{AR/LM}$(x) $\Rightarrow$ Assumption SS1_{LM}(i); $\text{HL1}_{AR/LM}$(ii) $\Rightarrow$ SS1_{LM}(ii); $\text{HL1}_{AR/LM}$(vii) & (ix) with $\theta_1 = \theta_{1*n}$ $\Rightarrow$ SS1_{LM}(iii); $\text{HL1}_{AR/LM}$(xi) $\Rightarrow$ SS1_{LM}(iv); $\text{HL1}_{AR/LM}$(xii) $\Rightarrow$ SS1_{LM}(v); $\text{HL1}_{AR/AR}$(vi) & (ix) and $\text{HL1}_{AR/LM}$(iii) & (v) $\Rightarrow$ SS1_{LM}(vi); and $\text{HL1}_{AR/AR}$(xxiii) and $\text{HL1}_{AR/LM}$(xiii) & (xv) $\Rightarrow$ SS1_{LM}(vii).

For the second-step C(α)-QLR1 test, Assumptions B(ii), C(ii), and C(iii) hold by Lemma 17.1 and Comment (i) following it because Lemma 17.1 relies on Assumptions SS1_{AR} and SS1_{LM}, which have just been verified, as well as on Assumption SS1_{QLR1}, which holds because Assumption $\text{HL1}_{AR/QLR1}$(i)–(iii) $\Rightarrow$ Assumption SS1_{QLR1}.

For the second-step C(α)-LM test, Assumption C(iv) holds provided $p_2 \geq 1$ (which is assumed here) because its nominal level η critical value is the $1-\eta$ quantile of the $\chi^2_{p_2}$ distribution which is nondecreasing in η for $\eta \in (0,1)$ when $p_2 \geq 1$.

For the second-step C(α)-QLR1 test, Assumption C(iv) holds because its conditional critical value $c^{QLR1}(1-\eta, rk_{2n}(\theta_1), WI_n(\theta_1))$ is nondecreasing in η since $c^{QLR1}(1-\eta, r, w)$ is the $1-\eta$ quantile of $QLR1(r,w)$, see (7.16).

For the second-step C(α)-LM test, Assumption C(v) holds under the conditions of Lemma 18.3 by Lemma 16.2(b), which employs Assumptions SS1_{AR}, SS2_{AR}, SS1_{LM}, and SS2_{LM}, because Assumption HL1_{AR} $\Rightarrow$ Assumption SS1_{AR}, as shown above; $\text{HL1}_{AR/LM}$ $\Rightarrow$ SS1_{LM}, as shown above; the condition $\liminf_{n\to\infty} \tau_n^\Phi > K_U^*$ in Lemma 18.3(b), $\text{HL1}_{AR/AR}$(xxiii), and $\text{HL1}_{AR/LM}$(xv) with $j = 1 \Rightarrow \liminf_{n\to\infty} \tau_{1n}^\Phi > K_U^* \Rightarrow$ Assumption SS2_{AR}(i); Assumption HL1_{AR} $\Rightarrow$ all parts of Assumption SS2_{AR} except its part (i) (as shown in the proof of Theorem 18.1); the condition $\liminf_{n\to\infty} \tau_n > 0$ in Lemma 18.3(b), $\text{HL1}_{AR/AR}$(xxiii), and $\text{HL1}_{AR/LM}$(xiii) & (xv) $\Rightarrow$ Assumption SS2_{LM}(i); $\text{HL1}_{AR/LM}$(i) $\Rightarrow$ SS2_{LM}(ii); and $\text{HL1}_{AR/LM}$(iii)–(viii) $\Rightarrow$ SS2_{LM}(iii)–(vii).

For the second-step C(α)-QLR1 test, Assumption C(v) holds under the conditions of Lemma 18.3(b) by Lemma 17.2 because Lemma 17.2 relies on Assumptions SS2_{AR} and SS2_{LM}, which have just been verified above, and Assumption SS2_{QLR1}, which holds by Assumption $\text{HL1}_{AR/LM}$(xiii)–(xv). $\square$

Proof of Lemma 18.4. As required by Assumption OE, Assumption C holds for the sequence S by Lemma 18.3(b), using Assumptions $\text{HL1}_{AR/AR}$, $\text{HL1}_{AR/LM}$, $\text{HL2}_{AR/LM,QLR1}$, and, for the C(α)-QLR1 test, $\text{HL2}_{AR/AR}$(i) as well. Note that the condition $\liminf_{n\to\infty} \tau_n^\Phi > K_U^*$ of Lemma 18.3(b) is implied by Assumption $\text{HL2}_{AR/AR}$(i) (i.e., $\liminf_{n\to\infty} \tau_n^\Phi > K_U$) and $K_U^* < K_L \leq K_U$ (which holds by the definition of the constant K_U^* following (7.11)).

The verification of Assumption OE(i) is the same as in the proof of Lemma 18.2.

Assumption OE(ii) holds under Assumption $\text{SL2}_{LM,QLR1}$ by Lemma 14.2. Hence, we need to verify Assumption $\text{SL2}_{LM,QLR1}$. Assumption $\text{HL2}_{AR/LM,QLR1}$ $\Rightarrow$ Assumption $\text{SL2}_{LM,QLR1}$(i); $\text{HL1}_{AR/AR}$ $\Rightarrow$ $\text{SL2}_{LM,QLR1}$(ii) (which consists of SL2_{AR}(v)–(x)), as shown in the proof of Lemma 18.2 above; $\text{HL1}_{AR/LM}$(iii)–(v) $\Rightarrow$ $\text{SL2}_{LM,QLR1}$(iii)–(v); and $\text{HL1}_{AR/LM}$(xiii)–(xv) $\Rightarrow$ $\text{SL2}_{LM,QLR1}$(vi)–(viii). □

19 Proof of Theorem 8.1

This section proves Theorem 8.1 using the results in Section 18, which, in turn, uses the results in Sections 12-14.

19.1 Proof of Theorem 8.1

To prove Theorem 8.1 we find it useful to reparametrize (θ_1, F) with parameter space $\mathcal{F}_{AR/AR}$, defined in (8.8), to a parameter λ with parameter space $\Lambda_{AR/AR}$. The parameter λ is chosen such that for some subvector of λ convergence of a drifting subsequence of the subvector allows one to verify Assumption $\text{HL1}_{AR/AR}$, which is employed in Theorem 18.1, and Assumption $\text{HL2}_{AR/AR}$, which is employed in Lemma 18.2.

Let $\{h_n(\lambda) : n \geq 1\}$ be a sequence of functions on a space Λ. The parameter λ and function $h_n(\lambda)$ are of the following form:
(i) $\lambda = (\lambda_1, ..., \lambda_d, \lambda_{d+1})'$, where $\lambda_j \in R$ $\forall j \leq d$ and λ_{d+1} belongs to some infinite-dimensional pseudo-metric space,[16] and
(ii) $h_n(\lambda) = (h_{n,1}(\lambda), ..., h_{n,J}(\lambda))'$ and

$$h_{n,j}(\lambda) = \begin{cases} n^{1/2}\lambda_j & \text{for } j = 1, ..., J_R \\ \lambda_j & \text{for } j = J_R + 1, ..., J, \end{cases} \quad \text{for some } J_R \leq d. \tag{19.1}$$

Define

$$\begin{aligned} H = \{ & h \in (R \cup \{\pm\infty\})^J : h_{m_n}(\lambda_{m_n}) \to h \text{ for some subsequence } \{m_n\} \\ & \text{of } \{n\} \text{ and some sequence } \{\lambda_{m_n} \in \Lambda : n \geq 1\}\}. \end{aligned} \tag{19.2}$$

The result in the following lemma is established in the proof of Theorem 2.2 in Andrews, Cheng, and Guggenberger (2011). For completeness, we provide a proof below.

[16] For notational simplicity, we stack d real-valued quantities and one infinite-dimensional quantity into the vector λ.

Lemma 19.1 *For any sequence* $\{\lambda_n \in \Lambda : n \geq 1\}$ *and any subsequence* $\{w_n\}$ *of* $\{n\}$ *there exists a subsequence* $\{m_n\}$ *of* $\{w_n\}$ *such that* $h_{m_n}(\lambda_{m_n}) \to h$ *for some* $h \in H$.

Comment: Lemma 19.1 is useful in establishing the correct asymptotic size of any two-step test, not just the two-step AR/AR test.

Now, we specify λ and Λ that are used with the two-step AR/AR test.

Let $\Omega_F(\theta) := Var_F(g_i(\theta))$. We write a SVD of $\Omega_F^{-1/2}(\theta_1)E_F G_{1i}(\theta_1)$ as

$$\Omega_F^{-1/2}(\theta_1)E_F G_{1i}(\theta_1) = C_{1F}(\theta_1)\Upsilon_{1F}(\theta_1)B_{1F}(\theta_1)', \tag{19.3}$$

where $C_{1F}(\theta_1) \in R^{k\times k}$ and $B_{1F}(\theta_1) \in R^{p_1\times p_1}$ are orthogonal matrices and $\Upsilon_{1F}(\theta_1) \in R^{k\times p_1}$ has the singular values $\tau_{11F}(\theta_1), ..., \tau_{1p_1F}(\theta_1)$ of $\Omega_F^{-1/2}(\theta_1)E_F G_{1i}(\theta_1)$ in nonincreasing order on its diagonal and zeros elsewhere.

Let $\tau^{\Phi}_{1p_1F}(\theta_1)$ denote the smallest singular value of $\Omega_F^{-1/2}(\theta_1)E_F G_{1i}(\theta_1)\Phi_{1F}(\theta_1)$, where $\Phi_{1F}(\theta_1)$ is defined in (8.3).

We define the elements of λ to be[17]

$$\begin{aligned}
\lambda_{1,\theta_1,F} &:= (\tau_{11F}(\theta_1), ..., \tau_{1p_1F}(\theta_1))' \in R^{p_1},\\
\lambda_{2,\theta_1,F} &:= B_{1F}(\theta_1) \in R^{p_1\times p_1},\\
\lambda_{3,\theta_1,F} &:= C_{1F}(\theta_1) \in R^{k\times k},\\
\lambda_{4,\theta_1,F} &:= E_F G_{1i}(\theta_1) \in R^{k\times p_1},\\
\lambda_{5,\theta_1,F} &:= E_F \begin{pmatrix} g_i(\theta_1) \\ vec(G_{1i}(\theta_1) - E_F G_{1i}(\theta_1)) \end{pmatrix} \begin{pmatrix} g_i(\theta_1) \\ vec(G_{1i}(\theta_1) - E_F G_{1i}(\theta_1)) \end{pmatrix}' \in R^{(p_1+1)k\times(p_1+1)k},\\
\lambda_{6,\theta_1,F} &:= \theta_1,\\
\lambda_{7,\theta_1,F} &:= (\tau_{1p_1F}(\theta_1), \tau^{\Phi}_{1p_1F}(\theta_1))'\\
\lambda_{8,\theta_1,F} &:= F, \text{ and}\\
\lambda = \lambda_{\theta_1,F} &:= (\lambda_{1,\theta_1,F}, ..., \lambda_{8,\theta_1,F}).
\end{aligned} \tag{19.4}$$

We let $\lambda_{5,g,\theta_1,F}$ denote the upper left $k \times k$ submatrix of $\lambda_{5,\theta_1,F.}$ Thus, $\lambda_{5,g,\theta_1 F} = E_F g_i(\theta_1)g_i(\theta_1)' = \Omega_F(\theta_1)$ for $(\theta_1, F) \in \mathcal{F}_{AR/AR}$.

We consider the parameter space $\Lambda_{AR/AR}$ for λ that corresponds to $\mathcal{F}_{AR/AR}$. The parameter

[17] For simplicity, when writing $\lambda = (\lambda_{1,F}, ..., \lambda_{8,F})$, we allow the elements to be scalars, vectors, matrices, and distributions and likewise in similar expressions.

space $\Lambda_{AR/AR}$ and the function $h_n(\lambda)$ are defined by

$$\Lambda_{AR/AR} := \{\lambda : \lambda = (\lambda_{1,\theta_1,F}, ..., \lambda_{8,\theta_1,F}) \text{ for some } (\theta_1, F) \in \mathcal{F}_{AR/AR}\} \text{ and}$$
$$h_n(\lambda) := (n^{1/2}\lambda_{1,\theta_1,F}, \lambda_{2,\theta_1 F}, \lambda_{3,\theta_1,F}, \lambda_{4,\theta_1,F}, \lambda_{5,\theta_1,F}, \lambda_{6,\theta_1,F}, \lambda_{7,\theta_1,F}). \tag{19.5}$$

By the definition of $\mathcal{F}_{AR/AR}$, $\Lambda_{AR/AR}$ indexes distributions that satisfy the null hypothesis $H_0 : \theta_2 = \theta_{20}$. Redundant elements in $(\lambda_{1,\theta_1,F}, ..., \lambda_{8,\theta_1,F})$, such as the redundant off-diagonal elements of the symmetric matrix $\lambda_{5,\theta_1,F}$, are not needed, but do not cause any problem. The dimension J of $h_n(\lambda)$ equals the number of elements in $(\lambda_{1,F}, ..., \lambda_{7,F})$.

We define λ and $h_n(\lambda)$ as in (19.4) and (19.5) because, as shown below, verifying Assumption HL1$_{AR/AR}$, which is employed in Theorem 18.1, and Assumptions HL2$_{AR/AR}$, which is employed in Lemma 18.2, for subsequences requires convergence of the corresponding subsequences of $n^{1/2}\lambda_{1,\theta_{1n},F_n}$ and $\lambda_{j,\theta_{1n},F_n}$ for $j = 2, ..., 7$.

For notational convenience,

$$\{\lambda_{n,h} : n \geq 1\} \text{ denotes a sequence } \{\lambda_n \in \Lambda_{AR/AR} : n \geq 1\} \text{ for which } h_n(\lambda_n) \to h \in H \tag{19.6}$$

for H defined in (19.2) with Λ equal to $\Lambda_{AR/AR}$.[18] By the definitions of $\Lambda_{AR/AR}$ and $\mathcal{F}_{AR/AR}$, $\{\lambda_{n,h} : n \geq 1\}$ is a sequence of distributions that satisfies the null hypothesis $H_0 : \theta_2 = \theta_{20}$. Below, "all sequences $\{\lambda_{w_n,h} : n \geq 1\}$" means "all sequences $\{\lambda_{w_n,h} : n \geq 1\}$ for any $h \in H$," where H is defined with Λ equal to $\Lambda_{AR/AR}$, and likewise with n in place of w_n. To maintain the notation employed above that θ_{1*n} denotes the true value of θ_1, we let $\{(\theta_{1*n}, F_n) : n \geq 1\}$ denote the sequence of (θ_1, F) values in $\mathcal{F}_{AR/AR}$ that corresponds to $\{\lambda_{n,h} : n \geq 1\}$.

We decompose h (defined by (19.2), (19.4), and (19.5)) analogously to the decomposition of λ: $h = (h_1, ..., h_7)$, where $\lambda_{j,\theta_1,F}$ and h_j have the same dimensions for $j = 1, ..., 7$. We further decompose the vector h_1 as $h_1 = (h_{1,1}, ..., h_{1,\min\{k,p_1\}})'$, where elements of h_1 could equal ∞. In addition, we let $h_{5,g}$ denote the upper left $k \times k$ submatrix of h_5. In consequence, under a sequence $\{\lambda_{n,h} : n \geq 1\}$, we have

$$\begin{aligned} n^{1/2}\tau_{1sF_n}(\theta_{1*n}) &\to h_{1,s} \geq 0 \ \forall s \leq p_1, \\ \lambda_{j,\theta_{1*n},F_n} &\to h_j \ \forall j = 2, ..., 7, \text{ and} \\ \lambda_{5,g,\theta_{1*n},F_n} &= \Omega_{F_n}(\theta_{1*n}) = Var_{F_n}(g_i(\theta_{1*n})) \to h_{5,g}. \end{aligned} \tag{19.7}$$

[18] Analogously, for any subsequence $\{w_n : n \geq 1\}$, $\{\lambda_{w_n,h} : n \geq 1\}$ denotes a sequence $\{\lambda_{w_n} \in \Lambda_{AR/AR} : n \geq 1\}$ for which $h_{w_n}(\lambda_{w_n}) \to h \in H$.

By the conditions in $\mathcal{F}_{AR/AR}$, $h_{5,g}$ is pd.

The following lemma verifies Assumptions HL1$_{AR/AR}$, which is employed in Theorem 18.1 and Lemma 18.2, for all subsequences $\{\lambda_{w_n,h} : n \geq 1\}$.

Lemma 19.2 *Suppose* $\widehat{g}_n(\theta_1)$ *are the moment functions defined in* (3.3), $g_i(\theta)$ *satisfies the differentiability condition in Theorem* 8.1, $\{c_n : n \geq 1\}$ *are as in Theorem* 8.1, $a > 0$, $p_1 < k$, Θ *is open,* Θ_{1*} *is bounded, and* $B(\Theta_{1*}, \varepsilon) \subset \Theta_1$ *for some* $\varepsilon > 0$. *Let the null parameter space be* $\mathcal{F}_{AR/AR}$. *Then, for all subsequences* $\{\lambda_{w_n,h} : n \geq 1\}$, *Assumption* HL1$_{AR/AR}$ *holds and* $\lim \tau_{1n}$ *and* $\lim \tau_{1n}^{\Phi}$ *exist.*

The following lemma verifies Assumption HL2$_{AR/AR}$.

Lemma 19.3 *Suppose* $\widehat{g}_n(\theta_1)$ *are the moment functions defined in* (3.3), $g_i(\theta)$ *satisfies the differentiability condition in Theorem* 8.1, *and the null parameter space is* $\mathcal{F}_{AR/AR}$. *Let* S *be a null sequence* (*or* S_m *a null subsequence*) *for which Assumption* SI *holds. Then, Assumption* HL2$_{AR/AR}$ *holds for the sequence* S (*or the subsequence* S_m).

Now, we prove Theorem 8.1 using Theorems 5.1 and 18.1 and Lemmas 18.2 and 19.1-19.3.

Proof of Theorem 8.1. The result of Theorem 8.1(a) follows from the high-level result Theorem 5.1(a) provided Assumption CAL holds. Assumption CAL requires that for any null sequence S and any subsequence $\{w_n\}$ of $\{n\}$, there exists a subsubsequence $\{m_n\}$ such that S_m satisfies Assumption B or C. Theorem 18.1 provides high-level conditions under which Assumption B or C holds for a subsequence S_m. The condition required for Theorem 18.1 is Assumption HL1$_{AR/AR}$.

By Lemma 19.1, for any null sequence S or, equivalently, any sequence $\{\lambda_n \in \Lambda : n \geq 1\}$, and any subsequence $\{w_n\}$ of $\{n\}$, there exists a subsubsequence $\{m_n\}$ such that $h_{m_n}(\lambda_{m_n}) \to h$ for some $h \in H$. By Lemma 19.2, for the subsequence $\{m_n\}$ (that satisfies $h_{m_n}(\lambda_{m_n}) \to h$ for some $h \in H$), Assumption HL1$_{AR/AR}$ holds and $\lim \tau_{1n}$ and $\lim \tau_{1n}^{\Phi}$ exist. Given this, by Theorem 18.1(a) and (b), the subsequence S_m satisfies Assumption B when $\lim \tau_{1m_n}^{\Phi} < K_L$ and it satisfies Assumption C when $\liminf_{n\to\infty} \tau_{1m_n} > 0$ and Assumption W holds, which is assumed.

By definition, τ_{1n}^{Φ} is the smallest singular value of $\Omega_n^{-1/2} G_{1n} \Phi_{1n}$, see (8.3), and $\Phi_{1n} := Diag\{\sigma_{11n}^{-1}, ..., \sigma_{1p_1n}^{-1}\}$, where $\sigma_{1sn}^2 := Var_{F_n}(||G_{1si}||)$, see (14.3). Given these definitions and the condition $Var_F(||G_{1si}(\theta_1)||) \geq \delta$ for $(\theta_1, F) \in \mathcal{F}_{AR/AR}$, see (8.8), we have: $\lim \tau_{1m_n}^{\Phi} \geq K_L$ implies $\liminf_{n\to\infty} \tau_{1m_n} > 0$. Hence, every subsequence S_m with $\lim \tau_{1m_n}^{\Phi} < K_L$ satisfies Assumption B and every subsequence S_m with $\lim \tau_{1m_n}^{\Phi} \geq K_L$ satisfies Assumption C. This completes the verification of Assumption CAL and the proof of Theorem 8.1(a).

Now we prove Theorem 8.1(b). The result of Theorem 8.1(b) that $AsyNRP = \alpha$ holds for a sequence S if for any subsequence $\{w_n\}$ of $\{n\}$, there exists a subsubsequence $\{m_n\}$ such that $\lim P_{\theta_{*m_n},F_{m_n}}(\phi^{SV}_{2m_n}(\alpha) > 0)) = \alpha$ for the corresponding subsubsequence S_m, where $\theta_{*m_n} = (\theta'_{1*m_n}, \theta'_{20})'$. Take the subsubsequence $\{m_n\}$ as above to be such that $h_{m_n}(\lambda_{m_n}) \to h$ for some $h \in H$. Then, by Lemma 19.2, for the subsubsequence S_m, Assumption HL1$_{AR/AR}$ holds.

The $\lim P_{\theta_{*m_n},F_{m_n}}(\phi^{SV}_{2m_n}(\alpha) > 0)) = \alpha$ result for the subsubsequence S_m follows from the high-level result Theorem 5.1(c) provided Assumption OE holds for the subsubsequence S_m. Assumption OE holds for the subsubsequence S_m by Lemma 18.2. The conditions required for Lemma 18.2 are Assumptions HL1$_{AR/AR}$, HL2$_{AR/AR}$, and W. Assumption HL1$_{AR/AR}$ is verified in a previous paragraph. In addition, Assumption W is assumed to hold. Hence, to establish Theorem 8.1(b), it remains to verify Assumption HL2$_{AR/AR}$. Using Assumption SI, which is imposed in Theorem 8.1(b), these conditions hold by Lemma 19.3. This completes the proof of Theorem 8.1(b).

Theorem 8.1(c) follows immediately from Theorem 8.1(a) and (b).

Theorem 8.1(d) and (e) hold by Theorem 5.1(d) and (e), respectively, because a sequence S that satisfies Assumption SI is shown above to satisfy Assumption OE.

To establish Theorem 8.1(f), we use the high-level CS results given in Theorem 5.1(f), rather than the high-level test results given in Theorem 5.1(a)–(e). This requires verifying the CS versions of Assumptions B, C, CAL, and OE, rather than the test versions. The only difference between the CS and test versions is that θ_{2*n} appears throughout in place of θ_{20}. The verification of the CS versions these conditions is the same as given above but with some adjustments.

First, we adjust the definition of λ that appears in (19.4) and (19.5). Specifically, we define λ as in (19.4), but with θ in place of θ_1 throughout. We retain the definition of $h_n(\lambda)$ given in (19.5), but with θ in place of θ_1 in λ.

Second, we adjust the parameter space $\Lambda_{AR/AR}$, which appears in (19.5), to the following parameter space:

$$\Lambda_{\Theta,AR/AR} := \{\lambda : \lambda = (\lambda_{1,\theta,F}, ..., \lambda_{9,\theta,F}) \text{ for some } (\theta, F) \in \mathcal{F}_{\Theta,AR/AR}\}, \tag{19.8}$$

where $\mathcal{F}_{\Theta,AR/AR}$ is defined in (8.9) using the test parameter space $\mathcal{F}_{AR/AR}(\theta_2)$ for $\theta_2 \in \Theta_{2*}$. Note that the moment conditions in $\mathcal{F}_{AR/AR}(\theta_2)$ hold uniformly over $\theta_2 \in \Theta_{2*}$ by the definitions of $\mathcal{F}_{\Theta,AR/AR}$. For example, $E_F||g_i(\theta_1)||^{2+\gamma} = E_F||g_i(\theta_1, \theta_{20})||^{2+\gamma} \leq M$ (by the definition of $\mathcal{F}_{AR,AR}$ in (8.8)) $\forall \theta_{20} \in \Theta_{2*}$ (by the definition of $\mathcal{F}_{\Theta,AR/AR}$) implies that $\sup_{\theta_2 \in \Theta_{2*}} E_F||g_i(\theta_1, \theta_2)||^{2+\gamma} \leq M$. This is used in the adjusted proofs everywhere the moment conditions are employed in the unadjusted proofs.

Third, we use the assumption that Θ_* is bounded and $B(\Theta_*, \varepsilon) \subset \Theta$ for some $\varepsilon > 0$ to ensure that $\lambda_{6,\theta_n,F_n} = \theta_n$ has a limit in $\Theta \subset R^p$, call it θ_∞, for all sequences $\{\lambda_{n,h} : n \geq 1\}$ (rather than a limit whose elements might equal $\pm\infty$). The assumption that $B(\Theta_*, \varepsilon) \subset \Theta$ guarantees that the mean-value expansions that appear in (12.3), (12.6)–(12.9), (12.19), (15.12), (15.14), (19.12), (19.13), (19.18), (19.22, (19.37), and (19.41) also hold when θ_{20} is replaced by θ_{2*n}.

Given the adjustments above, the results in Lemmas 12.1–12.4, 13.1, 13.2, 15.1, 15.2, 14.1, 14.2, 18.1, and 18.2 hold with the null sequence S replaced by a sequence S which has θ_{20} replaced by θ_{2*n}.

Furthermore, Lemmas 19.2 and 19.3 hold with the adjustments to λ and $\Lambda_{AR/AR}$ stated immediately above. In consequence, the verification of Assumptions B, C, CAL, and OE given above goes through when θ_{2*n} appears throughout in place of θ_{20}. This completes the proof of Theorem 8.1(f). $\square$

19.2 Proofs of Lemmas 19.1-19.3

Proof of Lemma 19.1. Let $\{w_n\}$ be some subsequence of $\{n\}$. Let $h_{w_n,j}(\lambda_{w_n})$ denote the jth component of $h_{w_n}(\lambda_{w_n})$ for $j = 1, ..., J$. Let $m_{1,n} = w_n \; \forall n \geq 1$. For $j = 1$, either (1) $\limsup_{n\to\infty} |h_{m_{j,n},j}(\lambda_{m_{j,n}})| < \infty$ or (2) $\limsup_{n\to\infty} |h_{m_{j,n},j}(\lambda_{m_{j,n}})| = \infty$. If (1) holds, then for some subsequence $\{m_{j+1,n}\}$ of $\{m_{j,n}\}$,

$$h_{m_{j+1,n},j}(\lambda_{m_{j+1,n}}) \to h_j \text{ for some } h_j \in R. \tag{19.9}$$

If (2) holds, then for some subsequence $\{m_{j+1,n}\}$ of $\{m_{j,n}\}$,

$$h_{m_{j+1,n},j}(\lambda_{m_{j+1,n}}) \to h_j, \text{ where } h_j = \infty \text{ or } -\infty. \tag{19.10}$$

Applying the same argument successively for $j = 2, ..., J$ yields a subsequence $\{m_n\} = \{m_{J+1,n}\}$ of $\{w_n\}$ for which $h_{m_n,j}(\lambda_{m_n^*}) \to h_j \; \forall j \leq J$, which establishes the result of the Lemma. $\square$

Proof of Lemma 19.2. For notational simplicity, we prove the result for a sequence $\{\lambda_{n,h} : n \geq 1\}$. The same arguments go through with n replaced by w_n to obtain the subsequence results that are stated in the lemma.

We do not verify Assumptions HL1$_{AR/AR}$(i), HL1$_{AR/AR}$(ii), ... in numerical order because some of these conditions are used in the verification of others. For brevity, we abbreviate Assumptions HL1$_{AR/AR}$(i), HL1$_{AR/AR}$(ii), ... by Assumptions (i), (ii), ...

Assumption (iv) requires that $\widehat{g}_n(\theta_1)$ is twice continuously differentiable on $B(\theta_{1*n}, \varepsilon) \; \forall n \geq 1$ for some $\varepsilon > 0$. Assumption (iv) holds because the present lemma imposes the differentiability condition in Theorem 8.1 (which states that $g_i(\theta_1)$ is twice continuously differentiable in θ_1 on Θ_1 for all sample realizations), and $B(\theta_{1*n}, \varepsilon) \subset \Theta_1 \; \forall n \geq 1$ for some $\varepsilon > 0$ because $\theta_{1*n} \in \Theta_{1*}$ by the definition of $\mathcal{F}$ and $B(\Theta_{1*}, \varepsilon) \subset \Theta_1$ by an assumption of the lemma.

Assumption (i) requires that $\sup_{\theta_1 \in B(\theta_{1*n}, \varepsilon)} ||\widehat{g}_n(\theta_1) - g_n(\theta_1)|| = o_p(1)$ for some nonrandom functions $\{g_n(\cdot) : n \geq 1\}$ and some $\varepsilon > 0$. We verify Assumption (i) with $g_n(\theta_1) = E_{F_n} \widehat{g}_n(\theta_1)$. This condition is a uniform WLLN. Because $\theta_{1*n} \to \theta_{1*\infty} \in \Theta_1$ by (19.31) below, it suffices to establish this result with $B(\theta_{1*\infty}, \varepsilon)$ in place of $B(\theta_{1*n}, \varepsilon)$. Since $B(\theta_{1*\infty}, \varepsilon)$ is a bounded set, sufficient conditions for this uniform WLLN's are

$$\begin{aligned} &\text{(a) } \widehat{g}_n(\theta_1) - g_n(\theta_1) = o_p(1) \ \forall \theta_1 \in B(\theta_{1*\infty}, \varepsilon) \text{ and} \\ &\text{(b) } \sup_{\theta_a \in B(\theta_{1*\infty}, \varepsilon)} \sup_{\theta_1 \in B(\theta_a, \varepsilon_n)} ||\widehat{g}_n(\theta_1) - g_n(\theta_1) - (\widehat{g}_n(\theta_a) - g_n(\theta_a))|| = o_p(1) \end{aligned} \tag{19.11}$$

for all sequences of constants $\{\varepsilon_n > 0 : n \geq 1\}$ for which $\varepsilon_n \to 0$, e.g., see Theorem 1(a) of Andrews (1991a). Conditions (a) and (b) are pointwise WLLN's and stochastic equicontinuity, respectively.

Condition (b) of (19.11) is established as follows. In the verification of condition (b) we assume $k = 1$ for notational simplicity and without loss of generality (wlog) (because the verification can be done separately for each element of $\widehat{g}_n(\cdot) - g_n(\cdot)$). Consider any $\theta_a \in B(\theta_{1*\infty}, \varepsilon)$ and $\theta_1 \in B(\theta_a, \varepsilon_n)$. Element-by-element two-term Taylor expansions of $g_i(\theta_1)$ about θ_a give

$$g_i(\theta_1) = g_i(\theta_a) + G_{1i}(\theta_a)(\theta_1 - \theta_a) + \sum_{j=1}^{p_1} (\theta_{1j} - \theta_{aj}) \frac{\partial}{\partial \theta_{1j}} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_a), \tag{19.12}$$

where $\theta_a = (\theta_{a1}, ..., \theta_{ap_1})'$ and $\widetilde{\theta}_{1i}$ lies between θ_1 and θ_a. Element-by-element mean-value expansions of $G_{1i}(\theta_a)$ about θ_{1*n} give

$$G_{1i}(\theta_a) = G_{1i} + \sum_{j=1}^{p_1} (\theta_{aj} - \theta_{1*nj}) \frac{\partial}{\partial \theta_{1j}} G_{1i}(\overline{\theta}_{1i}), \tag{19.13}$$

where $\theta_{1*n} = (\theta_{1*n1}, ..., \theta_{1*np_1})'$ and $\overline{\theta}_{1i}$ lies between θ_a and θ_{1*n} and may differ across the columns of $(\partial / \partial \theta_{1j}) G_{1i}(\overline{\theta}_{1i})$. Equation (19.13) uses the assumption of Theorem 8.1, which is imposed in this lemma, that $g_i(\theta_1)$ is twice continuously differentiable in θ_1 on Θ_1 and $G_{1i}(\theta_a) := (\partial / \partial \theta_1') g_i(\theta_a)$.

Substituting (19.13) into (19.12) and taking expectations gives

$$\begin{aligned} &\sup_{\theta_a \in B(\theta_{1*\infty}, \varepsilon)} \sup_{\theta_1 \in B(\theta_a, \varepsilon_n)} ||g_n(\theta_1) - g_n(\theta_a)|| \\ &\leq ||G_{1n}|| \cdot ||\theta_1 - \theta_a|| + E_{F_n} \xi_{1i} ||\theta_1 - \theta_a||^2 + E_{F_n} \xi_{1i} ||\theta_a - \theta_{1*n}|| \cdot ||\theta_1 - \theta_a|| \\ &= o(1), \end{aligned} \tag{19.14}$$

where $g_n := E_{F_n} g_i$, $G_{1n} := E_{F_n} G_{1i}$, the inequality uses $||(\partial / \partial \theta_{1j}) G_{1i}(\widetilde{\theta}_{1i})|| \leq p_1^{1/2} \xi_{1i}$ (when $k = 1$)

and $||(\partial/\partial\theta_{1j})G_{1i}(\overline{\theta}_{1i})|| \leq p_1^{1/2}\xi_{1i}$ for ξ_{1i} defined in (8.2), and the equality uses $||\theta_1 - \theta_a|| \leq \varepsilon_n \to 0$, $||\theta_a - \theta_{1*n}|| \leq 2\varepsilon$ for n sufficiently large, and the conditions in $\mathcal{F}_{AR/AR}$ that $E_{F_n}\xi_{1i}^2 \leq M$ and $E_{F_n}||vec(G_{1i})||^{2+\gamma} := E_{F_n}||vec(G_{1i}(\theta_{1*n}))||^{2+\gamma} \leq M$ for $(\theta_{1*n}, F_n) \in \mathcal{F}_{AR/AR}$.

Similarly, substituting (19.13) into (19.12) and taking averages over $i = 1, ..., n$ gives

$$\begin{aligned}
&\sup_{\theta_a \in B(\theta_{1*\infty},\varepsilon)} \sup_{\theta_1 \in B(\theta_a,\varepsilon_n)} ||\widehat{g}_n(\theta_1) - \widehat{g}_n(\theta_a)|| \\
&\leq ||\widehat{G}_{1n}|| \cdot ||\theta_1 - \theta_a|| + p_1 n^{-1}\sum_{i=1}^n \xi_{1i}||\theta_1 - \theta_a||^2 + p_1 n^{-1}\sum_{i=1}^n \xi_{1i}||\theta_a - \theta_{1*n}|| \cdot ||\theta_1 - \theta_a|| \\
&= o_p(1), \qquad (19.15)
\end{aligned}$$

where the equality uses $||\theta_1 - \theta_a|| \leq \varepsilon_n \to 0$, $||\theta_a - \theta_{1*n}|| \leq 2\varepsilon$ for n large, and $||\widehat{G}_{1n}|| = O_p(1)$ and $n^{-1}\sum_{i=1}^n \xi_{1i} = O_p(1)$, which hold by Markov's inequality using the same moment conditions as used in (19.14).

Equations (19.14) and (19.15) combine to verify condition (b) in (19.11).

Condition (a) in (19.11) holds by the WLLN's for independent L^2-bounded random variables. Again, for notational simplicity, we assume that $k = 1$. The L^2-boundedness condition holds by replacing θ_a by θ_{1*n} in (19.12), taking the inner product of the resulting expression with itself, and then taking expectations. This yields: $\forall\theta_1 \in B(\theta_{1*\infty}, \varepsilon)$,

$$\begin{aligned}
&E_{F_n} g_i(\theta_1)' g_i(\theta_1) = E_{F_n}\eta_n'\eta_n = O(1), \text{ where} \\
&\eta_n := g_i + G_{1i} \times (\theta_1 - \theta_{1*n}) + \sum_{j=1}^{p_1}(\theta_{1j} - \theta_{1*nj})\frac{\partial}{\partial\theta_{1j}}G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n}) \qquad (19.16)
\end{aligned}$$

and the second equality holds using $||\theta_1 - \theta_{1*n}|| \leq 2\varepsilon$ for n large and using the moment conditions listed after (19.14). This completes the verification of Assumption (i).

Assumption (ii) requires $g_n = 0^k \ \forall n \geq 1$, where $g_n = g_n(\theta_{1*n})$. By the verification of Assumption (i), $g_n(\theta_{1*n}) = E_{F_n}\widehat{g}_n(\theta_{1*n})$. Hence, $g_n = 0^k \ \forall n \geq 1$ holds by the condition in $\mathcal{F}_{AR/AR}$ that $E_{F_n} g_i(\theta_{1*n}) = 0^k \ \forall(\theta_{1*n}, F_n) \in \mathcal{F}_{AR/AR}$.

Assumption (vi) requires $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon)} ||\widehat{G}_{1n}(\theta_1) - G_{1n}(\theta_1)|| = o_p(1)$ for some nonrandom functions $\{G_{1n}(\cdot) : n \geq 1\}$. We verify Assumption (vi) with $G_{1n}(\theta_1) = E_{F_n}\widehat{G}_{1n}(\theta_1)$ and we assume $k = 1$ for notational simplicity. Assumption (vi) is a uniform WLLN. Its verification is similar to, but simpler than, the verification of Assumption (i). To verify stochastic equicontinuity, one uses (19.13) with θ_a and θ_{1*n} replaced by θ_1 and θ_a, respectively. Then, the analogues of (19.14) and (19.15) are

$$\sup_{\theta_a \in B(\theta_{1*\infty},\varepsilon)} \sup_{\theta_1 \in B(\theta_a,\varepsilon_n)} ||G_{1n}(\theta_1) - G_{1n}(\theta_a)|| \le p_1 E_{F_n}\xi_{1i}||\theta_1 - \theta_a|| = o(1) \text{ and}$$
$$\sup_{\theta_a \in B(\theta_{1*\infty},\varepsilon)} \sup_{\theta_1 \in B(\theta_a,\varepsilon_n)} ||\widehat{G}_{1n}(\theta_1) - \widehat{G}_{1n}(\theta_a)|| \le p_1 n^{-1}\sum_{i=1}^{n} \xi_{1i}||\theta_1 - \theta_a|| = o_p(1), \quad (19.17)$$

where the two equalities use $||\theta_1 - \theta_a|| \le \varepsilon_n \to 0$, $E_{F_n}\xi_{1i}^2 \le M$, and $n^{-1}\sum_{i=1}^n \xi_{1i} = O_p(1)$ as above. This verifies stochastic equicontinuity.

To verify the pointwise WLLN's, i.e., $\widehat{G}_{1n}(\theta_1) - G_{1n}(\theta_1) = o_p(1)$ $\forall\theta_1 \in B(\theta_{1*\infty},\varepsilon)$, which is analogous to condition (a) in (19.11), we use the WLLN's for independent L^2-bounded random variables. The L^2-boundedness condition holds by an analogous argument to that given in (19.16). This completes the verification of Assumption (vi).

Assumption (vii) requires $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon)} ||G_{1n}(\theta_1)|| = O(1)$. By the verification of Assumption (vi) above, $G_{1n}(\theta_1) = E_{F_n}G_{1i}(\theta_1)$. Hence, Assumption (vii) holds by the moment condition on $G_{1i}(\theta_1)$ in $\mathcal{F}_{AR/AR}$.

Assumption (viii) requires $\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon_n)} ||G_{1n}(\theta_1) - G_{1n}|| = o(1)$ for all sequences of positive constants $\varepsilon_n \to 0$. For notational simplicity and wlog, we suppose $k = p_1 = 1$. For $\theta_1 \in B(\theta_{1*n},\varepsilon_n)$, element-by-element mean-value expansions of $G_{1i}(\theta_1)$ about θ_{1*n} give

$$G_{1i}(\theta_1) = G_{1i} + \frac{\partial}{\partial\theta_1}G_{1i}(\overline{\theta}_{1i})(\theta_1 - \theta_{1*n}), \quad (19.18)$$

where $\overline{\theta}_{1i}$ lies between θ_1 and θ_{1*n}. Taking expectations in (19.18) gives

$$\sup_{\theta_1 \in B(\theta_{1*n},\varepsilon_n)} |G_{1n}(\theta_1) - G_{1n}| \le E_{F_n}\xi_{1i} \sup_{\theta_1 \in B(\theta_{1*n},\varepsilon_n)} |\theta_1 - \theta_{1*n}| = o(1), \quad (19.19)$$

where the inequality uses $|(\partial/\partial\theta_1)G_{1i}(\overline{\theta}_{1i})| \le \xi_{1i}$ and the equality uses $\varepsilon_n \to 0$ and $E_{F_n}\xi_{1i} \le M$ for $\theta_1 \in B(\theta_{1*n},\varepsilon_n)$ and $(\theta_{1*n}, F_n) \in \mathcal{F}_{AR/AR}$. This verifies Assumption (viii).

Assumption (x) requires $E_{F_n}\widehat{\xi}_{1n} = O(1)$, where $\widehat{\xi}_{1n} := \max_{s,u \le p_1} \sup_{\theta_1 \in B(\theta_{1*\infty},\varepsilon)} ||(\partial^2/\partial\theta_{1s}\theta_{1u})\widehat{g}_n(\theta_1)||$. By the definition of ξ_{1i} in (8.2), $\widehat{\xi}_{1n} \le n^{-1}\sum_{i=1}^n \xi_{1i}$ for $\varepsilon > 0$ sufficiently small that $B(\theta_{1*\infty},\varepsilon) \subset \Theta_1$. Hence, Assumption (x) holds by the moment condition $E_F\xi_{1i}^2 \le M$ $\forall(\theta_1, F) \in \mathcal{F}_{AR/AR}$.

Assumption (xi) requires $\sup_{\theta_1 \in B(\theta_{1*n},K/n^{1/2})} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| \to_p 0$ for some nonrandom functions $\{\Omega_n(\cdot) : n \ge 1\}$. We verify Assumption (xi) with $\Omega_n(\theta_1) = E_{F_n}g_i(\theta_1)g_i(\theta_1)' - E_{F_n}g_i(\theta_1)$

$\times E_{F_n} g_i(\theta_1)'$. We have

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| \tag{19.20}$$
$$\leq \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Omega}_n(\theta_1) - \widehat{\Omega}_n|| + \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Omega_n(\theta_1) - \Omega_n|| + ||\widehat{\Omega}_n - \Omega_n||$$

using the triangle inequality. Next, we have

$$||\widehat{\Omega}_n - \Omega_n|| = \left\| n^{-1} \sum_{i=1}^n g_i g_i' - \widehat{g}_n \widehat{g}_n' - (E_{F_n} g_i g_i' - E_{F_n} \widehat{g}_n E_{F_n} \widehat{g}_n') \right\| \tag{19.21}$$
$$\leq \left\| n^{-1} \sum_{i=1}^n g_i g_i' - E_{F_n} g_i g_i' \right\| + (||\widehat{g}_n|| + ||g_n||) \left\| n^{-1} \sum_{i=1}^n g_i - E_{F_n} g_i \right\| = o_p(1),$$

where the first equality holds by the triangle inequality and standard manipulations and the second equality holds by the WLLN for independent $L^{1+\gamma/2}$-bounded random variables for $\gamma > 0$ as in $\mathcal{F}_{AR/AR}$ using the moment conditions in $\mathcal{F}_{AR/AR}$ and $||\widehat{g}_n||+||g_n|| = O_p(1)$ using Markov's inequality and the moment conditions in $\mathcal{F}_{AR/AR}$.

To verify Assumption (xi), it remains to show that the first and second summands on the rhs of (19.20) are $o_p(1)$. Assumption (xii) requires $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Omega_n(\theta_1) - \Omega_n|| \to 0$. Hence, verifying Assumption (xii) shows that the second summand on the rhs of (19.20) is $o_p(1)$.

Now we verify Assumption (xii). For notational simplicity, we assume $k = p_1 = 1$ when verifying Assumption (xii). (The results for $k, p_1 \geq 1$ hold by analogous arguments.) For $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$, element-by-element two-term Taylor expansions of $g_i(\theta_1)$ about θ_{1*n} give

$$g_i(\theta_1) = g_i + G_{1i} \times (\theta_1 - \theta_{1*n}) + (\theta_1 - \theta_{1*n}) \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n}), \tag{19.22}$$

where $\widetilde{\theta}_{1i}$ lies between θ_1 and θ_{1*n}. Taking expectations in (19.22) gives

$$g_n(\theta_1) = g_n + G_{1n} \times (\theta_1 - \theta_{1*n}) + (\theta_1 - \theta_{1*n}) E_{F_n} \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n}) \text{ and}$$
$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |g_n(\theta_1) - g_n| \leq |G_{1n}| \times |\theta_1 - \theta_{1*n}| + E_{F_n} \xi_{1i} |\theta_1 - \theta_{1*n}|^2 = o(1), \tag{19.23}$$

where $g_n := E_{F_n} g_i$, $G_{1n} := E_{F_n} G_{1i}$, and the inequality uses the conditions in $\mathcal{F}_{AR/AR}$ that $E_{F_n} \xi_{1i}^2 \leq M$, $E_{F_n} ||vec(G_{1i})||^{2+\gamma} := E_{F_n} ||vec(G_{1i}(\theta_{1*n}))||^{2+\gamma} \leq M$ for $(\theta_{1*n}, F_n) \in \mathcal{F}_{AR/AR}$.

Using (19.22) and taking expectations, we have: uniformly over $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$,

$$
\begin{aligned}
E_{F_n} g_i(\theta_1) g_i(\theta_1) =\ & E_{F_n} g_i g_i + E_{F_n} G_{1i}^2 \times (\theta_1 - \theta_{1*n})^2 + E_{F_n} \left(\frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i}) \right)^2 (\theta_1 - \theta_{1*n})^4 \\
& + 2E_{F_n} g_i G_{1i} \times (\theta_1 - \theta_{1*n}) + 2E_{F_n} g_i \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n})^2 \\
& + 2E_{F_n} G_{1i} \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n})^3 \\
=\ & E_{F_n} g_i g_i + o(1), \qquad (19.24)
\end{aligned}
$$

where the second equality holds using $|\theta_1 - \theta_{1*n}| \le K/n^{1/2}$, the moment conditions in $\mathcal{F}_{AR/AR}$ referred to above, the inequality $|(\partial/\partial\theta_{1j})G_{1i}(\widetilde{\theta}_{1i})| \le \xi_{1i}$, and the Cauchy-Bunyakovsky-Schwarz inequality. Equations (19.23) and (19.24) yield

$$
\begin{aligned}
& \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\Omega_n(\theta_1) - \Omega_n| \\
= & \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |E_{F_n} g_i(\theta_1) g_i(\theta_1) - E_{F_n} g_i g_i - E_{F_n} g_i(\theta_1) E_{F_n} g_i(\theta_1) + E_{F_n} g_i E_{F_n} g_i| \\
= & \ o(1), \qquad (19.25)
\end{aligned}
$$

where the second equality uses the triangle inequality and $E_{F_n} g_i(\theta_1) = O(1)$, which holds by the moment conditions in $\mathcal{F}_{AR/AR}$. This completes the verification of Assumption (xii).

Next, we show that the first summand on the rhs of (19.20) is $o_p(1)$, which is needed to complete the verification of Assumption (xi). For notational simplicity, we assume $k = p_1 = 1$ in this paragraph. Using (19.22), we have: uniformly over $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$,

$$
\begin{aligned}
& n^{-1} \sum_{i=1}^n g_i(\theta_1) g_i(\theta_1) \\
= & \ n^{-1} \sum_{i=1}^n g_i g_i + n^{-1} \sum_{i=1}^n G_{1i}^2 \times (\theta_1 - \theta_{1*n})^2 + n^{-1} \sum_{i=1}^n \left(\frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i}) \right)^2 (\theta_1 - \theta_{1*n})^4 \\
& + 2n^{-1} \sum_{i=1}^n g_i G_{1i} \times (\theta_1 - \theta_{1*n}) + 2n^{-1} \sum_{i=1}^n g_i \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n})^2 \\
& + 2n^{-1} \sum_{i=1}^n G_{1i} \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n})^3 \\
= & \ n^{-1} \sum_{i=1}^n g_i g_i + o_p(1), \qquad (19.26)
\end{aligned}
$$

where the second equality holds using $||\theta_1 - \theta_{1*n}|| \le K/n^{1/2}$, the inequality $||(\partial/\partial\theta_{1j})G_{1i}(\widetilde{\theta}_{1i})|| \le \xi_{1i}$, the WLLN for independent $L^{1+\gamma/2}$-bounded random variables for $\gamma > 0$ as in $\mathcal{F}_{AR/AR}$, the moment conditions in $\mathcal{F}_{AR/AR}$, and the Cauchy-Bunyakovsky-Schwarz inequality. By similar, but simpler calculations, uniformly over $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$,

$$n^{-1}\sum_{i=1}^{n} g_i(\theta_1) = n^{-1}\sum_{i=1}^{n} g_i + o_p(1). \tag{19.27}$$

Equations (19.26) and (19.27) imply that the first summand on the rhs of (19.20) is $o_p(1)$. This completes the verification of Assumption (xi).

Assumption (xiii) holds by the condition in $\mathcal{F}_{AR/AR}$ that $\lambda_{\min}(\Omega_n) = \lambda_{\min}(E_{F_n} g_i(\theta_{1*n}) g_i(\theta_{1*n})')$ $\geq \delta\ \forall(\theta_{1*n}, F_n) \in \mathcal{F}_{AR/AR}$.

Assumption (xv) requires $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\widehat{\Gamma}_{1n}(\theta_1) - \Gamma_{1n}(\theta_1)|| = o_p(1)$ for some nonrandom functions $\{\Gamma_{1n}(\cdot) : n \geq 1\}$, where $\widehat{\Gamma}_{1n}(\theta)$ is defined in (14.2). We verify Assumption (xv) with $\Gamma_{1n}(\theta_1)$ defined as in (14.2), i.e., $\Gamma_{1n}(\theta_1) := E_{F_n} vec(G_{1i}(\theta_1) - E_{F_n} G_{1i}(\theta_1)) g_i(\theta_1)'$ using the identical distribution assumption in $\mathcal{F}_{AR/AR}$. The verification is quite similar to that of Assumption (xi) with $vec(G_{1i}(\theta_1)) g_i(\theta_1)'$ and $vec(G_{1i}(\theta_1))$ in place of $g_i(\theta_1) g_i(\theta_1)'$ and $g_i(\theta_1)$, respectively. In consequence, we do not provide all of the details. Analogues of (19.20), (19.21), (19.26), and (19.27) hold by analogous arguments, so it suffices to show that an analogue of the second summand on the rhs of (19.20) holds. Assumption (xvi) requires $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Gamma_{1n}(\theta_1) - \Gamma_{1n}|| \to 0$. Hence, verifying Assumption (xvi) shows that the analogue of the second summand on the rhs of (19.20) is $o_p(1)$. We verify Assumption (xvi) below. This completes the verification of Assumption (xv).

Assumption (xvi) requires $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Gamma_{1n}(\theta_1) - \Gamma_{1n}|| \to 0$, where $\Gamma_{1n}(\theta_1)$ is defined in the previous paragraph. For notational simplicity, we assume $k = p_1 = 1$ in the verification of Assumption (xvi). Using (19.18) with $\varepsilon_n = K/n^{1/2}$ and (19.22) and taking expectations, we obtain

$$\begin{aligned}
& E_{F_n} G_{1i}(\theta_1) g_i(\theta_1) - E_{F_n} G_{1i} g_i \\
= {} & E_{F_n} G_{1i}^2(\theta_1 - \theta_{1*n}) + E_{F_n} G_{1i} \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n})^2 + E_{F_n} \frac{\partial}{\partial \theta_1} G_{1i}(\overline{\theta}_{1i}) g_i(\theta_1 - \theta_{1*n}) \\
& + E_{F_n} \frac{\partial}{\partial \theta_1} G_{1i}(\overline{\theta}_{1i}) G_{1i}(\theta_1 - \theta_{1*n})^2 + E_{F_n} \frac{\partial}{\partial \theta_1} G_{1i}(\overline{\theta}_{1i}) \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n})^3 \\
= {} & o(1),
\end{aligned} \tag{19.28}$$

where the second equality holds uniformly over $\theta_1 \in B(\theta_{1*n}, K/n^{1/2})$ using $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})}$ $|\theta_1 - \theta_{1*n}| = o(1)$, $|(\partial/\partial\theta_1) G_{1i}(\overline{\theta}_{1i})| \leq \xi_{1i}$, $|(\partial/\partial\theta_1) G_{1i}(\widetilde{\theta}_{1i})| \leq \xi_{1i}$, $E_{F_n} g_i^2 + E_{F_n} G_{1i}^2 + E_{F_n} \xi_{1i}^2$ $= O(1)$ by the moment conditions in $\mathcal{F}_{AR/AR}$, and the Cauchy-Bunyakovsky-Schwarz inequality.

Next, we have

$$\begin{aligned}
&\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\Gamma_{1n}(\theta_1) - \Gamma_{1n}| \\
&= \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |E_{F_n} G_{1i}(\theta_1) g_i(\theta_1) - E_{F_n} G_{1i} g_i - E_{F_n} G_{1i}(\theta_1) E_{F_n} g_i(\theta_1) + E_{F_n} G_{1i} E_{F_n} g_i| \\
&= o(1), \qquad (19.29)
\end{aligned}$$

where the second equality holds by (19.19) with $\varepsilon_n = K/n^{1/2}$, (19.23), (19.28), and the moment conditions in $\mathcal{F}_{AR/AR}$, which imply that $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})}(|E_{F_n} vec(G_{1i}(\theta_1))| + |E_{F_n} g_i(\theta_1)|) = O(1)$ (when $k = p_1 = 1$). This completes the verification of Assumption (xvi).

Assumption (xxiii) requires $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\widehat{\sigma}^2_{1sn}(\theta_1) - \sigma^2_{1sn}(\theta_1)| \to_p 0$ for some nonrandom functions $\{\sigma^2_{1sn}(\cdot) : n \geq 1\}$ $\forall s = 1, ..., p_1$, where $\widehat{\sigma}^2_{1sn}(\theta_1)$ is defined in (7.4). We verify Assumption (xxiii) with $\sigma^2_{1sn}(\theta_1) := E_{F_n}||G_{1si}(\theta_1)||^2 - (E_{F_n}||G_{1si}(\theta_1)||)^2$. Provided Assumption (xxiv) holds, Assumption (xxiii) holds by the same argument as for Assumption (xv) with $||G_{1si}(\theta_1)||^2$ and $||G_{1si}(\theta_1)||$ in place of $vec(G_{1i}(\theta_1)) g_i(\theta_1)'$ and $vec(G_{1i}(\theta_1))$, respectively (which in turn relies on the verification of Assumption (xi)).

Now we verify Assumption (xxiv), which requires that $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |\sigma^2_{1sn}(\theta_1) - \sigma^2_{1sn}| \to 0$ $\forall s = 1, ..., p_1$. The latter is implied by $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |E_{F_n}||G_{1si}(\theta_1)||^2 - E_{F_n}||G_{1si}||^2| \to 0$ and $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} |E_{F_n}||G_{1si}(\theta_1)|| - (E_{F_n}||G_{1si}||)| \to 0$. For notational simplicity, we suppose $k = p_1 = 1$. Using (19.18) with $\varepsilon_n = K/n^{1/2}$ and taking expectations, we have

$$\begin{aligned}
&\sup_{\theta_1} |E_{F_n}|G_{1i}(\theta_1)|^2 - E_{F_n}|G_{1i}|^2| \\
&= \sup_{\theta_1} \left| E_{F_n} \left| G_{1i} + \frac{\partial}{\partial \theta_1} G_{1i}(\overline{\theta}_{1i})(\theta_1 - \theta_{1*n}) \right|^2 - E_{F_n}|G_{1i}|^2 \right| \\
&\leq \sup_{\theta_1} E_{F_n} \left| \frac{\partial}{\partial \theta_1} G_{1i}(\overline{\theta}_{1i})(\theta_1 - \theta_{1*n}) \right|^2 \\
&\quad + 2 \sup_{\theta_1} E_{F_n} \left| \frac{\partial}{\partial \theta_1} G_{1i}(\overline{\theta}_{1i})(\theta_1 - \theta_{1*n}) G_{1i} \right| \\
&\leq E_{F_n} \xi^2_{1i} K^2/n + 2 E_{F_n}(\xi_{1i}|G_{1i}|) K/n^{1/2} \\
&= o_p(1), \qquad (19.30)
\end{aligned}$$

where $\sup_{\theta_1}$ denotes $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})}$, the first equality uses (19.18), the first inequality holds by the triangle inequality, the second inequality uses $||(\partial/\partial\theta_1) G_{1i}(\overline{\theta}_{1i})|| \leq \xi_{1i}$, and the last equality uses $E_{F_n} \xi^2_{1i} \leq M$ and $E_F |G_{1i}|^{2+\gamma} \leq M$ (when $k = p_1 = 1$) for $(\theta_{1*n}, F_n) \in \mathcal{F}_{AR/AR}$ and the Cauchy-Bunyakovsky-Schwarz inequality. Establishing the analogous result to that in (19.30) with

$|G_{1si}(\theta_1)|^2$ and $|G_{1si}|^2$ replaced by $|G_{1si}(\theta_1)|$ and $|G_{1si}|$ is quite similar.

Assumption (xxv) requires that $\liminf_{n\to\infty} \sigma^2_{1sn} > 0 \; \forall s = 1, ..., p_1$, where $\sigma^2_{1sn} := E_{F_n}||G_{1si}||^2 - (E_{F_n}||G_{1si}||)^2$. This holds by the condition in $\mathcal{F}_{AR/AR}$ that $Var_F(||G_{1si}||) \geq \delta > 0 \; \forall s = 1, ..., p_1$.

By (19.6), $\{\lambda_{n,h} : n \geq 1\}$ denotes a sequence $\{\lambda_n \in \Lambda_{AR/AR} : n \geq 1\}$ for which $h_n(\lambda_n) \to h \in H$. By (19.7), we have

$$
\begin{aligned}
n^{1/2}\lambda_{1,\theta_{1*n},F_n} &:= n^{1/2}(\tau_{11F_n}(\theta_{1*n}), ..., \tau_{1p_1F_n}(\theta_{1*n}))' \to h_1, \\
\lambda_{2,\theta_{1*n},F_n} &:= B_{1F_n}(\theta_{1*n}) \to h_2 =: B_{1\infty}, \\
\lambda_{3,\theta_{1*n},F_n} &:= C_{1F_n}(\theta_{1*n}) \to h_3 =: C_{1\infty}, \\
\lambda_{4,\theta_{1*n},F_n} &:= E_{F_n} G_{1i}(\theta_{1*n}) \to h_4 =: G_{1\infty}, \\
\lambda_{5,\theta_{1*n},F_n} &:= E_{F_n} \begin{pmatrix} g_i(\theta_{1*n}) \\ vec(G_{1i}(\theta_{1*n}) - E_F G_{1i}(\theta_{1*n})) \end{pmatrix} \begin{pmatrix} g_i(\theta_{1*n}) \\ vec(G_{1i}(\theta_{1*n}) - E_F G_{1i}(\theta_{1*n})) \end{pmatrix}' \\
&\to \;\; h_5 =: V_{1\infty} := \begin{pmatrix} \Omega_\infty & \Gamma'_{1\infty} \\ \Gamma_{1\infty} & h_{5,G_1G_1} \end{pmatrix}, \qquad (19.31) \\
\lambda_{6,\theta_{1*n},F_n} &:= \theta_{1*n} \to h_6 =: \theta_{1*\infty}, \text{ and} \\
\lambda_{7,\theta_{1*n},F_n} &:= (\tau_{1p_1F_n}(\theta_{1*n}), \tau^{\Phi}_{1p_1F_n}(\theta_{1*n}))' := (\tau_{1n}, \tau^{\Phi}_{1n})' \to h_7 =: (\lim \tau_{1n}, \lim \tau^{\Phi}_{1n})',
\end{aligned}
$$

where $\Omega_\infty \in R^{k\times k}$, $\Gamma_{1\infty} \in R^{(p_1k)\times k}$, $h_{5,G_1G_1} \in R^{(p_1k)\times(p_1k)}$, and the second equality in the second last line holds by the notation introduced in (10.1) and (10.2). The convergence results in (19.31) verify Assumptions (iii), (ix), (xiv), (xvii), (xviii), (xix), and (xx). Note that $h_6 =: \theta_{1*\infty}$ lies in Θ_1 as required by Assumption (iii) because Θ_{1*} is bounded and $B(\Theta_{1*}, \varepsilon) \subset \Theta_1$ for some $\varepsilon > 0$ by the assumptions of the present lemma. In addition, the last convergence result in (19.31) guarantees that $\lim \tau_{1n}$ and $\lim \tau^{\Phi}_{1n}$, which appear in Theorem 18.1(b), exist.

Assumption (v) holds by the univariate CLT for triangular arrays of rowwise independent $L^{2+\gamma}$-bounded random variables (where $L^{2+\gamma}$-boundedness holds by the moment conditions in $\mathcal{F}_{AR/AR}$), the convergence condition $Var_{F_n}(n^{1/2}b'(\widehat{g}'_n, vec(\widehat{G}_{1n} - E_{F_n}\widehat{G}_{1n}))') \to b'V_{1\infty}b \; \forall b \in R^{(p_1+1)k}$ with $||b|| > 0$ (which holds by the convergence results for $\lambda_{5,\theta_{1*n},F_n}$ in (19.31)), and the Cramér-Wold device.

Assumptions (xxi) and (xxii) hold by the assumptions of the lemma on $\{c_n : n \geq 1\}$, which are the same as in Theorem 8.1, that $c_n \to 0$ and $nc_n \to \infty$.

This completes the verification of Assumption HL1$_{AR/AR}$ and of the existence of $\lim \tau_{1n}$ and $\lim \tau^{\Phi}_{1n}$. □

The proof of Lemma 19.3 uses the following lemma when verifying Assumption HL2$_{AR/AR}$(ii).

Lemma 19.4 *Let $\mathcal{B}$ be a pseudometric space with pseudometric ρ. Let $\{\nu_n(\cdot) : n \geq 1\}$ be a sequence of real-valued stochastic processes on $\mathcal{B}$. Suppose* (i) *$\mathcal{B}$ is totally bounded,* (ii) *$\{\nu_n(\cdot) : n \geq 1\}$ is stochastically equicontinuous under ρ, i.e., $\forall \varepsilon, \eta > 0$ $\exists \delta > 0$ such that* $\limsup_{n\to\infty} P(\sup_{\beta_1,\beta_2 \in \mathcal{B}: \rho(\beta_1,\beta_2)<\delta} |\nu_n(\beta_1) - \nu_n(\beta_2)| > \eta) < \varepsilon$, *and* (iii) $\nu_n(\beta) = O_p(1)$ $\forall \beta \in \mathcal{B}$. *Then,* $\sup_{\beta \in \mathcal{B}} |\nu_n(\beta)| = O_p(1)$.

Comments: (i). The result of Lemma 19.4 also holds if the stochastic equicontinuity condition is weaken by replacing "$\forall \eta > 0$" to "for some $\eta > 0$."

(ii). The result of Lemma 19.4 could be obtained by establishing the weak convergence of $\{\nu_n(\cdot) : n \geq 1\}$ to some limit process and applying the continuous mapping theorem. But, condition (iii) of Lemma 19.4 is noticeably weaker than the weak convergence of all finite-dimensional distributions of $\nu_n(\cdot)$, which would be needed to establish weak convergence. Condition (iii) can be verified straightforwardly using Markov's inequality when $\nu_n(\beta)$ is a sample average for $\beta \in \mathcal{B}$.

Proof of Lemma 19.4. Let $B(\beta, \delta)$ denote a closed ball in $\mathcal{B}$ centered at β with radius δ under the pseudometric ρ. Because $\mathcal{B}$ is totally bounded, there exists a finite number of balls in $\mathcal{B}$, say J_δ balls, that cover $\mathcal{B}$. Let the centers of these balls be $\{\beta_{j\delta} \in \mathcal{B} : j = 1, ..., J_\delta\}$. We have

$$\sup_{\beta\in\mathcal{B}} |\nu_n(\beta)| = \max_{j \leq J_\delta} \sup_{\beta \in B(\beta_{j\delta},\delta)} |\nu_n(\beta)| \leq \max_{j\leq J_\delta} |\nu_n(\beta_{j\delta})| + \xi_{\delta n} \leq \sum_{j=1}^{J_\delta} |\nu_n(\beta_{j\delta})| + \xi_{\delta n}, \text{ where}$$
$$\xi_{\delta n} := \max_{j \leq J_\delta} \sup_{\beta \in B(\beta_{j\delta},\delta)} |\nu_n(\beta) - \nu_n(\beta_{j\delta})| \tag{19.32}$$

and the first inequality holds by the triangle inequality.

Given any $\varepsilon > 0$ and some $\eta > 0$, e.g., $\eta = 1$ suffices, take $\delta > 0$ such that $\limsup_{n\to\infty} P(\xi_{\delta n} > \eta) < \varepsilon/2$. Such a value δ exists by the stochastic equicontinuity condition (ii). For $0 < K < \infty$, we have

$$\begin{aligned}
\limsup_{n\to\infty} P\left(\sup_{\beta\in\mathcal{B}} |\nu_n(\beta)| > K\right) &\leq \limsup_{n\to\infty} P\left(\sum_{j=1}^{J_\delta} |\nu_n(\beta_{j\delta})| + \xi_{\delta n} > K\right) \\
&\leq \limsup_{n\to\infty} \left(P\left(\sum_{j=1}^{J_\delta} |\nu_n(\beta_{j\delta})| + \xi_{\delta n} > K, \xi_{\delta n} \leq \eta\right) + P(\xi_{\delta n} > \eta)\right) \\
&\leq \sum_{j=1}^{J_\delta} \limsup_{n\to\infty} P\left(|\nu_n(\beta_{j\delta})| > \frac{K-\eta}{J_\delta}\right) + \varepsilon/2 \\
&< \varepsilon,
\end{aligned} \tag{19.33}$$

where the first inequality holds by (19.32), the second and third inequalities hold by standard manipulations, and the last inequality holds for K sufficiently large using the assumption that $\nu_n(\beta_{j\delta}) = O_p(1)\ \forall j \leq J_\delta$ by condition (iii). $\square$

Proof of Lemma 19.3. We have: Assumption SI(ii) $\Rightarrow$ Assumption HL2$_{AR/AR}$(i), SI(i) $\Rightarrow$ HL2$_{AR/AR}$(iv), SI(iii) $\Rightarrow$ HL2$_{AR/AR}$(vi), and SI(vi) $\Rightarrow$ HL2$_{AR/AR}$(vii).

Next, we verify Assumption HL2$_{AR/AR}$(iii). It suffices to show Assumption HL2$_{AR/AR}$(iii) holds for sequences $K_n \to \infty$ such that $K_n/n^{1/2} \to 0$ because $\inf_{\theta_1 \notin B(\theta_{1*n}, K/n^{1/2})} ||n^{1/2} g_n(\theta_1)||$ is nonincreasing in K. Hence, we assume that $K_n/n^{1/2} \to 0$.

For $n \geq 1$, let $\theta_{1n}^\dagger \in B(\theta_{1*n}, K_n/n^{1/2})$ satisfy

$$n^{1/2}||g_n(\theta_{1n}^\dagger)|| = \inf_{\theta_1 \notin B(\theta_{1*n}, K_n/n^{1/2})} n^{1/2}||g_n(\theta_1)|| + \varepsilon_n, \tag{19.34}$$

where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Such values $\{\theta_{1n}^\dagger : n \geq 1\}$ always exist. Assumption HL2$_{AR/AR}$(iii) holds iff $n^{1/2}||g_n(\theta_{1n}^\dagger)|| \to \infty$.

Define

$$d_n^\dagger := ||\theta_{1n}^\dagger - \theta_{1*n}|| \text{ and } s_n^\dagger := n^{1/2}||g_n(\theta_{1n}^\dagger)||. \tag{19.35}$$

We want to show that $s_n^\dagger \to \infty$. This holds if every subsequence $\{m_n : n \geq 1\}$ of $\{n\}$ has a subsubsequence $\{v_n : n \geq 1\}$ such that $s_{v_n}^\dagger \to \infty$. Given an arbitrary subsubsequence $\{v_n\}$ either $\liminf_{n\to\infty} d_{v_n}^\dagger > 0$ or $\liminf_{n\to\infty} d_{v_n}^\dagger = 0$.

First, suppose $\liminf_{n\to\infty} d_{v_n}^\dagger > 0$. Let $\varepsilon \in (0, \liminf_{n\to\infty} d_{v_n}^\dagger)$. We have

$$s_{v_n}^\dagger = v_n^{1/2}||g_{v_n}(\theta_{1v_n}^\dagger)|| \geq \inf_{\theta_1 \notin B(\theta_{1*v_n}, \varepsilon)} v_n^{1/2}||g_{v_n}(\theta_{1v_n}^\dagger)|| \to \infty, \tag{19.36}$$

as desired, where the inequality holds using the definitions of $d_n^\dagger$ and ε and the convergence holds by Assumption SI(i) and $v_n^{1/2} \to \infty$.

Second, suppose $\liminf_{n\to\infty} d_{v_n}^\dagger = 0$. Then, there exists a subsequence $\{r_n\}$ of $\{v_n\}$ for which $\lim_{n\to\infty} d_{r_n}^\dagger = 0$ and, in this case, we that show $s_{r_n}^\dagger \to \infty$, which completes the proof of Assumption HL2$_{AR/AR}$(iii). For notational simplicity, we replace r_n by n and assume $\lim d_n^\dagger = 0$. By definition, $\theta_{1n}^\dagger \notin B(\theta_{1*n}, K_n/n^{1/2})$. Hence, $K_n/n^{1/2} \leq ||\theta_{1n}^\dagger - \theta_{1*n}|| = d_n^\dagger \to 0$. Element-by-element mean-value expansions about θ_{1*n} give

$$g_n(\theta_{1n}^\dagger) = G_{1n}(\widetilde{\theta}_{1n})(\theta_{1n}^\dagger - \theta_{1*n}) = (G_{1n} + o(1))(\theta_{1n}^\dagger - \theta_{1*n}), \tag{19.37}$$

where $\widetilde{\theta}_{1n}$ lies between $\theta_{1n}^\dagger$ and θ_{1*n} and may differ across the rows of $G_{1n}(\widetilde{\theta}_{1n})$, the first equality uses the assumption that $g_i(\theta)$ satisfies the differentiability condition in Theorem 8.1 and the condition in

$\mathcal{F}_{AR/AR}$ that $E_{F_n} g_i(\theta_{1*n}) = 0^k$, and the second equality uses $||\widetilde{\theta}_{1n} - \theta_{1*n}|| \leq ||\theta_{1n}^\dagger - \theta_{1*n}|| = d_n^\dagger \to 0$ and $\sup_{\theta_1 \in B(\theta_{1*n}, \varepsilon_n)} ||G_{1n}(\theta_1) = G_{1n}|| = o_p(1)$ for all sequences of positive constants $\varepsilon_n \to 0$, which is Assumption HL1$_{AR/AR}$(viii), and is verified in (19.18) and (19.19).

We have

$$\begin{aligned} n||g_n(\theta_{1n}^\dagger)||^2 &= n||(G_{1n} + o(1))(\theta_{1n}^\dagger - \theta_{1*n})||^2 \\ &\geq n \inf_{\lambda \in R^{p_1}: ||\lambda||=1} ||(G_{1n} + o(1))\lambda||^2 \times ||\theta_{1n}^\dagger - \theta_{1*n}||^2 \\ &\geq n \left(\inf_{\lambda \in R^{p_1}: ||\lambda||=1} ||G_{1n}\lambda||^2 + o(1) \right) (K_n/n^{1/2})^2 \\ &\geq \left(\inf_{\lambda \in R^{p_1}: ||\lambda||=1} ||\Omega_n^{-1/2} G_{1n}\lambda||^2 / \lambda_{\max}^2(\Omega_n^{-1/2}) + o(1) \right) K_n^2 \\ &= \left(\tau_{1n}^2 \lambda_{\min}(\Omega_n) + o(1) \right) K_n^2 \\ &\to \infty, \end{aligned} \tag{19.38}$$

where the first equality uses (19.37), the second inequality holds because $G_{1n} = O(1)$ (by the moment condition $E_F||vec(G_{1i}(\theta_1))||^{2+\gamma} \leq M$ in $\mathcal{F}_{AR/AR}$) and because $\theta_{1n}^\dagger \notin B(\theta_{1*n}, K_n/n^{1/2})$, the last equality uses the definition of τ_{1n}, and the convergence to ∞ uses the assumption that $K_n \to \infty$, $\liminf_{n\to\infty} \lambda_{\min}(\Omega_n) > 0$ (which holds by the condition $\lambda_{\min}(\Omega_F(\theta_1)) \geq \delta$ in $\mathcal{F}_{AR/AR}$), and the fact that $\liminf_{n\to\infty} \tau_{1n} > 0$, which we now show. We obtain $\liminf_{n\to\infty} \tau_{1n} > 0$ using Assumption SI(ii) (i.e., $\liminf_{n\to\infty} \tau_{1n}^\Phi > K_U$), the definition of τ_{1n}^Φ in (10.3), the definition of Φ_{1n} in (14.3), and the conditions $Var_F(||G_{1si}(\theta_1)||) \geq \delta$ for all $s = 1, ..., p_1$ in $\mathcal{F}_{AR/AR}$. This completes the verification of Assumption HL2$_{AR/AR}$(iii).

Now, we show that Assumptions SI(iv) and (v) imply Assumption HL2$_{AR/AR}$(ii). Because Assumption HL2$_{AR/AR}$(ii) can be verified element-by-element, we assume without loss of generality that $k = 1$. We use Lemma 19.4 with $\beta = \theta_1$, $\mathcal{B} = \Theta_1$, and

$$\nu_n(\theta_1) := n^{1/2}(\widehat{g}_n(\theta_1) - E_{F_n}\widehat{g}_n(\theta_1)). \tag{19.39}$$

Condition (iii) of Lemma 19.4, i.e., $\nu_n(\theta_1) = O_p(1)\ \forall \theta_1 \in \Theta_1$, holds because for any $\varepsilon > 0$,

$$P_{F_n}(\nu_n(\theta_1) > K) \leq E_{F_n}\nu_n^2(\theta_1)/K^2 = E_{F_n}(g_i(\theta_1) - E_{F_n} g_i(\theta_1))^2/K^2 < \varepsilon \tag{19.40}$$

for all $n \geq 1$, where the first inequality holds by Markov's inequality, the equality holds because $\{g_i(\theta_1) : i \leq n\}$ are i.i.d. under F_n for each $n \geq 1$, and the last inequality holds for K sufficiently large using Assumption SI(iii).

We verify conditions (i) and (ii) of Lemma 19.4, i.e., Θ_1 is totally bounded under ρ and $\{\nu_n(\cdot) : n \geq 1\}$ is stochastically equicontinuous under ρ, using Theorem 4 in Andrews (1994, p. 2277). Here, ρ is defined by $\rho(\theta_a, \theta_b) := \limsup_{n\to\infty}(E_{F_n}(g_i(\theta_a) - g_i(\theta_b))^2)^{1/2}$ for $\theta_a, \theta_b \in \Theta_1$. Theorem 4 requires Assumptions B-D of Andrews (1994) to hold. Assumption C on p. 2269 of Andrews (1994) holds by the independence assumption in $\mathcal{F}_{AR/AR}$. Assumption B on p. 2268 (an envelope condition) holds by Assumption SI(iii), which states that $\limsup_{n\to\infty} E_{F_n} \sup_{\theta_1\in\Theta_1} ||g_i(\theta_1)||^r < \infty$ for some $r > 2$.

We verify Assumption D (Ossiander's L^p entropy condition) in Andrews (1994) using Theorem 5 in Andrews (1994, p. 2281) with $p = 2$. To apply Theorem 5 it suffices that the functions $\{g_i(\theta_1) : \theta_1 \in \Theta_1\}$ are a type II class of functions (i.e., Lipschitz functions, see p. 2270 of Andrews (1994)) with Θ_1 bounded and $\limsup_{n\to\infty} E_{F_n} B_i^r < \infty$ for some $r > 2$, where B_i is a random Lipschitz "constant." This holds because, by the mean-value theorem, using the differentiability of $g_i(\theta_1)$, the convexity of Θ_1 imposed in Assumption SI(v), and the moment condition in Assumption SI(iv), we have

$$||g_i(\theta_a) - g_i(\theta_b))|| \leq B_{1i}||\theta_a - \theta_b|| \ \forall \theta_a, \theta_b \in \Theta_1 \text{ for } B_{1i} := \sup_{\theta_1\in\Theta_1} ||\frac{\partial}{\partial\theta_1} g_i(\theta_1)||, \tag{19.41}$$

where $\limsup_{n\to\infty} E_{F_n} B_{1i}^r < \infty$. And, Θ_1 is bounded by Assumption SI(v). This completes the verification of Assumption HL2$_{AR/AR}$(ii).

Next, we verify Assumption HL2$_{AR/AR}$(v), i.e., $\sup_{\theta_1\in\Theta_1} ||\widehat{\Omega}_n(\theta_1) - \Omega_n(\theta_1)|| = o_p(1)$, where $\widehat{\Omega}_n(\theta_1)$ is defined in (3.6), and $\Omega_n(\theta_1) = E_{F_n} g_i(\theta_1) g_i(\theta_1)' - E_{F_n} g_i(\theta_1) E_{F_n} g_i(\theta_1)'$. We do so by obtaining uniform WLLN's over Θ_1 for averages over $i \leq n$ of $g_i(\theta_1)$ and $g_i(\theta_1)g_i(\theta_1)'$. For the average over $i \leq n$ of $g_i(\theta_1)$, a uniform WLLN's holds (i.e., $\sup_{\theta_1\in\Theta_1} ||\widehat{g}_n(\theta_1) - E_{F_n}\widehat{g}_n(\theta_1)|| \to_p 0$) by Assumption HL2$_{AR/AR}$(ii), which is verified above.

To obtain a uniform WLLN's for the average over $i \leq n$ of $g_i(\theta_1)g_i(\theta_1)'$, we use the following generic uniform WLLN's. Let $\{s_i(\theta_1) : i \leq n, n \geq 1\}$ be some vector-valued random functions on Θ_1, where $s_i(\theta_1) := s(W_i, \theta_1)$. Let $\widehat{s}_n(\theta_1) := n^{-1}\sum_{i=1}^n s_i(\theta_1)$. Sufficient conditions for a uniform WLLN's for these random functions under $\{F_n : n \geq 1\}$ (i.e., $\sup_{\theta_1\in\Theta_1} ||\widehat{s}_n(\theta_1) - E_{F_n}\widehat{s}_n(\theta_1)|| \to_p 0$) are

(a) $\widehat{s}_n(\theta_1) - E_{F_n}\widehat{s}_n(\theta_1) \to_p 0 \ \forall \theta_1 \in \Theta_1$,

(b) $||s_i(\theta_a) - s_i(\theta_b)|| \leq B_{si}||\theta_a - \theta_b||, \ \forall \theta_a, \theta_b \in \Theta_1,$ where $\limsup_{n\to\infty} n^{-1}\sum_{i=1}^n E_{F_n} B_{si} < \infty$, and

(c) Θ_1 is bounded, (19.42)

e.g., see Theorem 1(a) and Lemma 2(a) of Andrews (1991a).

Now, consider $s_i(\theta_1) = g_i(\theta_1)g_i(\theta_1)'$. A pointwise WLLN's for $n^{-1}\sum_{i=1}^n g_i(\theta_1)g_i(\theta_1)'$ holds for each fixed $\theta_1 \in \Theta_1$ under the i.i.d. condition in $\mathcal{F}_{AR/AR}$ for each fixed F and Assumption SI(iii). Hence, condition (a) of (19.42) holds. Condition (c) of (19.42) holds immediately by Assumption SI(v).

Using (19.41), we obtain

$$\begin{aligned} & ||g_i(\theta_a)g_i(\theta_a)' - g_i(\theta_b)g_i(\theta_b)'|| \\ & \leq 2 \sup_{\theta_1 \in \Theta_1} ||g_i(\theta_1)|| \cdot ||g_i(\theta_a) - g_i(\theta_b)|| \\ & \leq 2 \sup_{\theta_1 \in \Theta_1} ||g_i(\theta_1)||B_i||\theta_a - \theta_b||, \ \forall \theta_a, \theta_b \in \Theta_1, \end{aligned} \tag{19.43}$$

where for matrix arguments $||\cdot||$ denotes the Frobenious norm and the first inequality uses the triangle inequality. Combining (19.43) with $\limsup_{n\to\infty} E_{F_n} \sup_{\theta_1\in\Theta_1} ||g_i(\theta_1)||^2 < \infty$ and $\limsup_{n\to\infty} E_{F_n} B_i^2 < \infty$, which hold by Assumptions SI(iii) and (iv), and using the Cauchy-Bunyakovsky-Schwarz inequality verifies condition (b) of (19.42). This completes the verification of a uniform WLLN's for the average over $i \leq n$ of $g_i(\theta_1)g_i(\theta_1)'$, which completes the verification of Assumption HL2$_{AR/AR}$(v). $\square$

20 Proof of Theorem 8.2

The proof of Theorem 8.2 is similar to that of Theorem 8.1 given in Section 19. But, it uses an adjusted definition of $\lambda_{\theta_1,F}$ in (19.4) and a parameter space $\Lambda_{AR/LM,QLR1}$ (defined below) in place of $\Lambda_{AR/AR}$.

As above, $\Omega_F(\theta) := Var_F(g_i(\theta))$. We write a SVD of $\Omega_F^{-1/2}(\theta_1)E_F G_{2i}(\theta_1)$ as

$$\Omega_F^{-1/2}(\theta_1)E_F G_{2i}(\theta_1) = C_{2F}(\theta_1)\Upsilon_{2F}(\theta_1)B_{2F}(\theta_1)', \tag{20.1}$$

where $C_{2F}(\theta_1) \in R^{k\times k}$ and $B_{2F}(\theta_1) \in R^{p_2\times p_2}$ are orthogonal matrices and $\Upsilon_{2F}(\theta_1) \in R^{k\times p_2}$ has the singular values $\tau_{21F}(\theta_1), ..., \tau_{2p_2F}(\theta_1)$ of $\Omega_F^{-1/2}(\theta_1)E_F G_{2i}(\theta_1)$ in nonincreasing order on its diagonal and zeros elsewhere.

We write a SVD of $\Omega_F^{-1/2}(\theta_1)E_F G_{2i}(\theta_1)\Phi_{2F}(\theta_1)$ as

$$\Omega_F^{-1/2}(\theta_1)E_F G_{2i}(\theta_1)\Phi_{2F}(\theta_1) = C_{2F}^{\Phi}(\theta_1)\Upsilon_{2F}^{\Phi}(\theta_1)B_{2F}^{\Phi}(\theta_1)', \tag{20.2}$$

where $C_{2F}^{\Phi}(\theta_1) \in R^{k\times k}$ and $B_{2F}^{\Phi}(\theta_1) \in R^{p_2\times p_2}$ are orthogonal matrices and $\Upsilon_{2F}^{\Phi}(\theta_1) \in R^{k\times p_2}$ has the singular values $\tau^{\Phi}_{21F}(\theta_1), ..., \tau^{\Phi}_{2p_2F}(\theta_1)$ of $\Omega_F^{-1/2}(\theta_1)E_F G_{2i}(\theta_1)\Phi_{2F}(\theta_1)$ in nonincreasing order on its diagonal and zeros elsewhere.

Let $\tau_{pF}(\theta_1)$ denote the smallest singular value of $\Omega_F^{-1/2}(\theta_1)E_F G_i(\theta_1)$ and let $\tau^{\Phi}_{pF}(\theta_1)$ denote the smallest singular value of $\Omega_F^{-1/2}(\theta_1)E_F G_i(\theta_1)\Phi_F(\theta_1)$, where $\Phi_F(\theta_1)$ is defined in (8.4).

The adjusted definition of $\lambda_{\theta_1,F}$ is as follows. First, $(\tau_{21F}(\theta_1), ..., \tau_{2p_2F}(\theta_1))' \in R^{p_2}$ and $(\tau^{\Phi}_{21F}(\theta_1), ..., \tau^{\Phi}_{2p_2F}(\theta_1))' \in R^{p_2}$ are added onto $\lambda_{1,\theta_1,F}$ so that $\lambda_{1,\theta_1,F} \in R^{p_1+2p_2}$. Second, $G_i(\theta_1)$ appears in place of $G_{1i}(\theta_1)$ in $\lambda_{4,\theta_1,F}$ and $\lambda_{5,\theta_1,F}$. Third, the following six elements are added onto $\lambda_{\theta_1,F}$: $\lambda_{9,\theta_1,F} := B_{2F}(\theta_1) \in R^{p_2 \times p_2}$, $\lambda_{10,\theta_1,F} := C_{2F}(\theta_1) \in R^{k \times k}$, $\lambda_{11,\theta_1,F} := (\sigma^2_{11F}(\theta_1), ..., \sigma^2_{1p_1F}(\theta_1), \sigma^2_{21F}(\theta_1), ..., \sigma^2_{2p_2F}(\theta_1))' \in R^p$, where $\sigma^2_{jsF}(\theta_1) := Var_F(||G_{jsi}(\theta_1)||)$ $\forall s = 1, ..., p_j$, $\forall j = 1, 2$, $\lambda_{12,\theta_1,F} := B^{\Phi}_{2F}(\theta_1) \in R^{p_2 \times p_2}$, $\lambda_{13,\theta_1,F} := C^{\Phi}_{2F}(\theta_1) \in R^{k \times k}$, and $\lambda_{14,\theta_1,F} := (\tau_{pF}(\theta_1), \tau^{\Phi}_{pF}(\theta_1))'$. Fourth, h_5 in (19.31) is defined by

$$h_5 := V_\infty := \begin{pmatrix} \Omega_\infty & \Gamma'_\infty \\ \Gamma_\infty & h_{5,GG} \end{pmatrix}, \tag{20.3}$$

where $\Omega_\infty \in R^{k \times k}$, $\Gamma_\infty \in R^{(pk) \times k}$, $h_{5,GG} \in R^{(pk) \times (pk)}$. Fifth, $h_{n,j}(\lambda)$ in (19.5) is defined to equal λ_j for $j = 9, ..., 14$.

The parameter space $\Lambda_{AR/LM,QLR1}$ (for λ) that we use here is defined analogously to $\Lambda_{AR/AR}$ in (19.5), but is based on the adjusted definition of λ and the parameter space $\mathcal{F}_{AR/LM,QLR1}$, rather than $\mathcal{F}_{AR/AR}$. We use the same function $h_n(\lambda)$ here as defined as in (19.5), but based on the adjusted definition of λ.

The proof of Theorem 8.2 uses the following lemma. This lemma verifies Assumptions $\text{HL1}_{AR/LM}$ and $\text{HL1}_{AR/QLR1}$, which are employed in Lemmas 18.3 and 18.4, for all subsequences $\{\lambda_{w_n,h} : n \geq 1\}$. The subsequences $\{\lambda_{w_n,h} : n \geq 1\}$ considered in this lemma are based on the adjusted definitions of $\lambda_{\theta_1,F}$ and $h_{n,j}(\lambda)$ given immediately above.

Lemma 20.1 *Suppose $\widehat{g}_n(\theta_1)$ are the moment functions defined in* (3.3), *$g_i(\theta)$ satisfies the differentiability conditions in Theorem* 8.2, $c_n \to 0$, $nc_n \to \infty$, $a > 0$, *and* $p_2 \geq 1$. *Let the null parameter space be $\mathcal{F}_{AR/LM,QLR1}$. Then, for all subsequences $\{\lambda_{w_n,h} : n \geq 1\}$, Assumptions* $\text{HL1}_{AR/LM}$ *and* $\text{HL1}_{AR/QLR1}$ *hold and* $\lim \tau_{m_n}$ *and* $\lim \tau^{\Phi}_{m_n}$ *exist.*

Comment: When $LM_{2n}(\theta)$ is the pure C(α)-LM statistic (i.e., $WI_n(\theta) := 0$), the parameter space $\mathcal{F}_{AR/LM,QLR1}$ is restricted as in (8.12), and the definition of $\lambda_{\theta_1,F}$ is augmented by $\lambda_{15,\theta_1,F} := (r_{1n}, r_{2n})'$, Lemma 20.1 holds and, in addition, conditions (vi) and (vii) in Comment (v) to Lemma 16.1 hold. The same is true when $QLR1_{2n}(\theta)$ is the pure C(α)-QLR1 statistic and $WI_n(\theta) := 0$ in the QLR1 critical value function. (These results are proved following the proof of Lemma 20.1

below.) These results imply that the results of Lemma 18.3(a) and (b) hold (using the Comment to Lemma 18.3).

Proof of Theorem 8.2. The proof of Theorem 8.2 is the same as that of Theorem 8.1 given in Section 19, but (i) with the definitions of $\lambda_{\theta_1,F}$ and $h_{n,j}(\lambda)$ adjusted as in the paragraph containing (20.3), (ii) using Lemmas 18.3 and Lemma 18.4 in place of Theorem 18.1 and Lemma 18.2, (iii) with $\mathcal{F}_{AR/LM,QLR1}$ and $\mathcal{F}_{\Theta,AR/LM,QLR1}$ in place of $\mathcal{F}_{AR/AR}$ and $\mathcal{F}_{\Theta,AR/AR}$, and (iv) using Lemma 20.1 in addition to Lemma 19.2. Lemma 20.1 shows that $\lim \tau_{m_n}$ and $\lim \tau^{\Phi}_{m_n}$ exist, which is used when Lemma 18.3 is employed and in showing that a subsequence S_m satisfies Assumption B when $\lim \tau^{\Phi}_{m_n} < K_L$ and Assumption C when $\liminf_{n\to\infty} \tau^{\Phi}_{m_n} > K^*_U$, as in the proof of Theorem 8.1 (with $\liminf_{n\to\infty} \tau^{\Phi}_{m_n} > K^*_U$ in place of $\liminf_{n\to\infty} \tau_{1m_n} > 0$).

The proof of part (a) uses Lemma 18.3, which employs Assumptions HL1$_{AR/AR}$ and HL1$_{AR/LM}$, and for the second-step C(α)-QLR1 test, Assumption HL1$_{QLR1}$ as well. These assumptions are verified for the parameter space $\mathcal{F}_{AR/LM,QLR1}$ by Lemmas 19.2 (using $\mathcal{F}_{AR/LM,QLR1} \subset \mathcal{F}_{AR/AR}$) and 20.1.

The proof of part (b) uses Lemma 18.4, which employs Assumptions HL1$_{AR/AR}$, HL1$_{AR/LM}$, HL2$_{AR/LM,QLR1}$, and W. Assumptions HL1$_{AR/AR}$ and HL1$_{AR/LM}$ are verified in the proof of part (a) of the theorem. Assumption HL2$_{AR/LM,QLR1}$ is the same as Assumption HL2$_{AR/AR}$ except for part (i). Hence, using $\mathcal{F}_{AR/LM,QLR1} \subset \mathcal{F}_{AR/AR}$, Lemma 19.3 verifies all of Assumption HL2$_{AR/LM,QLR1}$ except part (i). Part (i) of Assumption HL2$_{AR/LM,QLR1}$ is implied by Assumption SI2, which is imposed in part (b) of the theorem. Assumption W is imposed in the theorem, so it holds by assumption.

Parts (c)–(f) of the theorem hold by the same arguments as given in the proof of these parts of Theorem 8.1. □

Proof of Lemma 20.1. For notational simplicity, we consider a sequence $\{\lambda_{n,h} : n \geq 1\}$, rather than a subsequence $\{\lambda_{w_n,h} : n \geq 1\}$.

Assumption HL1$_{AR/LM}$(i) holds by the differentiability conditions that are imposed in Theorem 8.2, but not in Theorem 8.1.

Assumption HL1$_{AR/LM}$(ii) holds by the CLT using the moment conditions in $\mathcal{F}_{AR/LM,QLR1}$ (including the condition $E_F||vec(G_{2i}(\theta_1))'||^{2+\gamma} \leq M$, which does not appear in $\mathcal{F}_{AR/AR}$) by the same argument as in the verification of Assumption HL1$_{AR/AR}$(v) given in the paragraph following (19.31) in the proof of Lemma 19.2. Note that the variance matrix $V_{1\infty}$ in Assumption HL1$_{AR/AR}$(v) is the upper left $(p_1+1)k \times (p_1+1)k$ sub-matrix of V_∞, where V_∞ is the limit of $\lambda_{5,\theta_{1*n},F_n}$ in (20.3).

The verification of Assumptions HL1$_{AR/LM}$(iii)–(v) is the same as for Assumptions HL1$_{AR/AR}$

(vi), (viii), and (ix) in the proof of Lemma 19.2 (see (19.17), the two paragraphs that follow (19.17), and (19.31)) with subscripts 2 in place of 1 and using $E_F\xi_{2i}^2 \leq M$ in place of $E_F\xi_{1i}^2 \leq M$.

Assumption HL1$_{AR/LM}$(vi) holds by the condition $E_F\xi_{12i} \leq M$ in $\mathcal{F}_{AR/LM,QLR1}$ and Markov's inequality.

The verification of Assumptions HL1$_{AR/LM}$(vii)–(ix) is the same as for Assumptions HL1$_{AR/AR}$(xv)–(xvii) in the proof of Lemma 19.2 (see (19.28), the paragraph preceding (19.28), (19.29), and (19.31)) with subscripts 2 in place of 1, using the condition $E_F\xi_{2i}^2 \leq M$ in place of $E\xi_{1i}^2 \leq M$, and using $G_i(\theta_1)$ in place of $G_{1i}(\theta_1)$ in the definitions of $\lambda_{4,\theta_1,F}$ and $\lambda_{5,\theta_1,F}$ as specified in the paragraph that contains (20.3).

The verification of Assumptions HL1$_{AR/LM}$(x)–(xii) is the same as for Assumptions HL1$_{AR/AR}$(xviii)–(xx) in the proof of Lemma 19.2 (see (19.31)) using the adjusted definition of $\lambda_{1,\theta_1,F}$ to include $(\tau_{21F}(\theta_1), ..., \tau_{2p_2F}(\theta_1))'$ and the addition of $\lambda_{9,\theta_1,F}$ and $\lambda_{10,\theta_1,F}$ to $\lambda_{\theta_1,F}$, as specified in the paragraph that contains (20.3).

The verification of Assumptions HL1$_{AR/LM}$(xiii) and (xiv) is the same as for Assumptions HL1$_{AR/AR}$(xxiii) and (xxiv) in the proof of Lemma 19.2 with subscripts 2 in place of 1 (see (19.30)) using the $E_F||vec(G_{2i}(\theta_1))||^{2+\gamma} \leq M$ and $E_F\xi_{2i}^2 \leq M$ conditions in the definition of $\mathcal{F}_{AR/LM,QLR1}$.

By (19.6), $\{\lambda_{n,h} : n \geq 1\}$ denotes a sequence $\{\lambda_n \in \Lambda_{AR/LM,QLR1} : n \geq 1\}$ for which $h_n(\lambda_n) \to h \in H$. For the sequence $\{\lambda_{n,h} : n \geq 1\}$, the convergence result of Assumption HL1$_{AR/LM}$(xv) (i.e., $\sigma_{jsn}^2 \to \sigma_{js\infty}^2$ $\forall s = 1, ..., p_j$, $\forall j = 1, 2$) holds by the addition of $\lambda_{11,\theta_1,F}$ to $\lambda_{\theta_1,F}$, as specified in the paragraph that contains (20.3) because $\sigma_{jsn}^2 := Var_{F_n}(||G_{jsi}||)$ $\forall s = 1, ..., p_j$, $\forall j = 1, 2$, see (17.1). The result of Assumption HL1$_{AR/LM}$(xv) that $\sigma_{js\infty}^2 \in (0, \infty)$ $\forall s = 1, ..., p_j$, $\forall j = 1, 2$ holds by the $Var_F(||G_{jsi}||) \geq \delta$ and $E_F||vec(G_{ji}(\theta_1))||^{2+\gamma} \leq M$ conditions for $j = 1, 2$ in the definitions of $\mathcal{F}_{AR/AR}$ and $\mathcal{F}_{AR/LM,QLR1}$. This completes the verification of Assumption HL1$_{AR/LM}$.

By (19.5) and the adjusted definition of $\lambda_{1,\theta_{1*n},F_n}$, for the sequence $\{\lambda_{n,h} : n \geq 1\}$, $n^{1/2}\tau_{2sn}^\Phi$ converges to some value in $[0, \infty]$ $\forall s \leq p_2$. Hence, Assumption HL1$_{AR/QLR1}$(i) holds. The convergence results of Assumptions HL1$_{AR/QLR1}$(ii) and (iii) (i.e., $C_{2n}^\Phi \to C_{2\infty}^\Phi$ and $B_{2n}^\Phi \to B_{2\infty}^\Phi$) hold by the addition of $\lambda_{12,\theta_1,F}$ and $\lambda_{13,\theta_1,F}$ to $\lambda_{\theta_1,F}$, as specified in the paragraph that contains (20.3). This completes the verification of Assumption HL1$_{AR/QLR1}$.

The limits $\lim \tau_n$ and $\lim \tau_n^\Phi$ exist by the addition of $\lambda_{14,\theta_1,F}$ to $\lambda_{\theta_1,F}$. $\square$

Now we prove the Comment to Lemma 20.1, which requires that we verify conditions (vi) and (vii) stated in Comment (v) to Lemma 16.1. Condition (vi) (i.e., $r_{jn} = r_{j\infty}$ for n sufficiently large) holds by the addition of $\lambda_{15,\theta_1,F}$ to $\lambda_{\theta_1,F}$, which implies that $\lim r_{jn}$ exists, and the fact that r_{jn} can only take on a finite number of values. Condition (vii) holds by (8.12) (i.e., $\lambda_{\min}(C_{*F}(\theta_1)'C_{*F}(\theta_1)) \geq \delta$).

The Comments to Lemmas 18.3 and 20.1 imply that Lemma 18.3 holds for the pure C(α)-LM and pure C(α)-QLR1 tests provided $\mathcal{F}_{AR/LM,QLR1}$ is restricted as in (8.12) and $\lambda_{15,\theta_1,F}$ is added to $\lambda_{\theta_1,F}$.

Lastly, the proof of Comment (iii) to Theorem 8.2 is the same as the proof of Theorem 8.2 given above using the results of Lemma 18.3, which hold when $\mathcal{F}_{AR/LM,QLR1}$ is restricted as in (8.12) and the test is the pure C(α)-LM or pure C(α)-QLR1 test (for which $WI_n(\theta) := 0$).

21 Proof of Theorem 11.1

The proof of Theorem 11.1 is analogous to that of Theorems 8.1 and 8.2. In the time series case, we define λ, $\{\lambda_{n,h} : n \geq 1\}$, and Λ as in (19.5) and (19.6) and the discussion around (19.8) for the AR/AR test and CS, respectively, and as in the discussion around (20.3) for the AR/LM and AR/QLR1 tests and CS's. But, we define $\lambda_{5,\theta_1,F}$ and $\lambda_{5,\theta,F}$ differently from the i.i.d. case. We define

$$\lambda_{5,\theta_1,F} := V_F(\theta_1) \text{ and } \lambda_{5,\theta,F} := V_F(\theta), \tag{21.1}$$

where $V_F(\theta)$ is defined in (11.3), rather than as in (19.4). In consequence, $\lambda_{5,\theta_{1n},F_n} \to h_5$ implies that $V_{F_n}(\theta_{1n}) \to h_5$ and the condition $V_{F_n}(\theta_{1*n}) \to V_\infty$ in Assumption V holds with $\theta_{1*n} = \theta_{1n}$ and $V_\infty = h_5$ (and analogously for CS's and Assumption V-CS). We let $\Lambda_{TS,AR/AR}$ denote the time series version of $\Lambda_{AR/AR}$. It is defined as in (19.5), but with $\mathcal{F}_{TS,AR/AR}$ in place of $\mathcal{F}_{AR/AR}$ and with the changes described above.

The proof of Theorem 11.1 uses the CLT given in the following lemma. This lemma employs Corollary 1 in de Jong (1997) and is analogous to Lemma 20.1 in Section 20 in the SM to AG1.

Lemma 21.1 *Let* $f_i(\theta) := (g_i(\theta)', vec(G_i(\theta))')'$. *We have: for tests,* $w_n^{-1/2} \sum_{i=1}^{w_n} (f_i(\theta_{1*n}) - E_{F_n} f_i(\theta_{1*n})) \to_d N(0^{(p+1)k}, h_5)$ *under all subsequences* $\{w_n\}$ *and all sequences* $\{\lambda_{w_n,h} \in \Lambda_{TS,AR/AR} : n \geq 1\}$, *and for* CS's *the same result holds with* θ_{*n} *in place of* θ_{1*n}.

We use the following stochastic equicontinuity result, which is a special case of Hansen (1996) Theorem 3, in the proof of Theorem 11.1. The strong mixing numbers of a triangular array of random vectors are defined in the usual way, e.g., see Hansen (1996).

Lemma 21.2 *Suppose* (i) $\{W_{ni} : i \leq n, n \geq 1\}$ *are row-wise identically distributed, strong mixing random vectors taking values in a set* $\mathcal{W}$, (ii) $\mathcal{B}$ *is a bounded subset of* R^{d_β}, (iii) $\mathcal{F}_{Lip}$ *is a set of real-valued functions* $s(w, \beta)$ *on* $\mathcal{W} \times \mathcal{B}$ *that satisfy* $|s(w, \beta_1) - s(w, \beta_2)| \leq B(w)||\beta_1 - \beta_2||$ *for some Lipschitz function* $B(w)$ *on* $\mathcal{W}$, (iv) $\limsup_{n\to\infty} E|s(W_{ni}, \beta)|^r < \infty \ \forall \beta \in \mathcal{B}$ *for some*

$r > 2$, (v) $\limsup_{n\to\infty} E|B(W_{ni})|^r < \infty$, *and* (vi) *the strong mixing numbers of* $\{W_{ni} : i \leq n, n \geq 1\}$ *satisfy* $\sum_{m=1}^{\infty} \alpha_m^{1/q-1/r} < \infty$ *for some* $q > d_\beta$ *and* $2 \leq q < r$. *Then,* $\forall \varepsilon, \eta > 0$ $\exists \delta > 0$ *such that* $\limsup_{n\to\infty} P(\sup_{\beta_1,\beta_2 \in \mathcal{B}:||\beta_1-\beta_2||<\delta} |\nu_n(\beta_1) - \nu_n(\beta_2)| > \eta) < \varepsilon$, *where* $\nu_n(\beta) := n^{-1/2} \sum_{i=1}^{n}(s(W_{ni}, \beta) - Es(W_{ni}, \beta))$.

Comment: The same constant r appears in conditions (iv)–(vi).

Proof of Theorem 11.1. The proof is the same as the proofs of Theorems 8.1 and 8.2 and Lemmas 19.2, 19.3, and 20.1, given in Sections 19 and 20, with some modifications. The modifications affect the proofs of Lemmas 19.2, 19.3, and 20.1. No modifications are needed elsewhere. We describe the modifications for tests. The modifications for CS's are analogous with θ_{*n} in place of θ_{1*n}.

The first modification is the change in the definition of $\lambda_{5,\theta_1,F}$ described in (21.1). Equation (21.1) and the $\lambda_{\min}(\cdot)$ condition in $\mathcal{F}_{TS,AR/AR}$ imply that Assumptions $\text{HL1}_{AR/AR}$(xiv) and (xvii) (stated in Section 18) hold.

The second modification is the use of a WLLN for triangular arrays of strong mixing random vectors, rather than i.i.d. random vectors, when verifying Assumptions $\text{HL1}_{AR/AR}$(i) (in condition (a) in (19.11)), $\text{HL1}_{AR/AR}$(vi) (in the paragraph following (19.17)), and $\text{HL1}_{AR/AR}$(xxiii) (in the paragraph following (19.29)), and when verifying Assumptions $\text{HL1}_{AR/LM}$(iii) and (xiii) in the proof of Lemma 20.1. For the WLLN, we use Example 4 of Andrews (1988), which shows that for a strong mixing row-wise-stationary triangular array $\{W_{ni} : i \leq n, n \geq 1\}$ we have $n^{-1} \sum_{i=1}^{n}(\xi(W_{ni}) - E_{F_n}\xi(W_{ni})) \to_p 0$ for any real-valued function $\xi(\cdot)$ (that may depend on n) for which $\sup_{n\geq 1} E_{F_n}||\xi(W_{ni})||^{1+\delta} < \infty$ for some $\delta > 0$.

The third modification is the use of a CLT for triangular arrays of strong mixing random vectors, rather than i.i.d. random vectors, when verifying Assumption $\text{HL1}_{AR/AR}$(v) at the end of the proof of Lemma 19.2 and when verifying Assumption $\text{HL1}_{AR/LM}$(ii) in the proof of Lemma 20.1. For the CLT, we use Lemma 21.1.

The fourth modification is to use Assumption V to verify Assumptions $\text{HL1}_{AR/AR}$(xi) and (xv) in the proof of Lemma 19.2 and Assumption $\text{HL1}_{AR/LM}$(vii) in the proof of Lemma 20.1 with $\Omega_n(\theta)$, $\Gamma_{1n}(\theta)$, and $\Gamma_{2n}(\theta)$ defined by the submatrices of $V_{F_n}(\theta)$, defined in (11.3) and partitioned as in (11.4), e.g., $\Gamma_{1n}(\theta) := (\Gamma_{11F_n}(\theta)', ..., \Gamma_{1p_1F_n}(\theta)')'$.

The fifth modification is the verification of Assumptions $\text{HL1}_{AR/AR}$(xii) and (xvi) and $\text{HL1}_{AR/LM}$(viii). Assumption $\text{HL1}_{AR/AR}$(xii) requires $\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Omega_n(\theta_1) - \Omega_n|| \to 0$, where $\Omega_n(\theta_1) := \sum_{m=-\infty}^{\infty}(E_F g_i(\theta) g_{i-m}'(\theta) - E_F g_i(\theta) E_F g_i(\theta)')$ is defined in (11.1). For notational simplicity, we suppose that $k = 1$, so that $\Omega_n(\theta_1)$ is a scalar. To verify Assumption $\text{HL1}_{AR/AR}$(xii), we use the two-term Taylor expansion in (19.22) and write

$$\begin{aligned} g_i(\theta_1) &= g_i + \delta_{ni}(\theta_1), \text{ where} \\ \delta_{ni}(\theta_1) &:= G_{1i} \times (\theta_1 - \theta_{1*n}) + (\theta_1 - \theta_{1*n}) \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i})(\theta_1 - \theta_{1*n}) \end{aligned} \tag{21.2}$$

and $\widetilde{\theta}_{1i}$ lies between θ_1 and θ_{1*n}.

By the standard strong mixing covariance inequality of Davydov (1968), for two function $s_1(\cdot)$ and $s_1(\cdot)$ on $\mathcal{W}$ and some $\gamma > 0$ and $C_1 < \infty$,

$$\begin{aligned} ||Cov_F(s_1(W_i), s_2(W_{i-m}))|| &\leq C_1 ||s_1(W_i)||_{F,2+\gamma} \, ||s_2(W_i)||_{F,2+\gamma} \, \alpha_F^{\gamma/(2+\gamma)}(m) \\ &\leq C_1 C^{\gamma/(2+\gamma)} ||s_1(W_i)||_{F,2+\gamma} \, ||s_2(W_i)||_{F,2+\gamma} \, m^{-d\gamma/(2+\gamma)}, \text{ where} \\ ||s(W_i)||_{F,2+\gamma} &:= (E_F ||s(W_i)||^{2+\gamma})^{1/(2+\gamma)}, \end{aligned} \tag{21.3}$$

$d\gamma/(2+\gamma) > 1$, and the second inequality uses the condition on the strong mixing numbers in the definition of $\mathcal{F}_{TS,AR/AR}$ in (11.2).

Using (21.2), we have

$$\begin{aligned} \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\delta_{ni}(\theta_1)||_{F_n,2+\gamma} &\leq ||G_{1i}||_{F_n,2+\gamma} \, K/n^{1/2} + \left\| \frac{\partial}{\partial \theta_1} G_{1i}(\widetilde{\theta}_{1i}) \right\|_{F_n,2+\gamma} K^2/n \\ &= O(n^{-1/2}), \end{aligned} \tag{21.4}$$

where the equality holds using the definition of ξ_{1i} in (8.2) and the moment conditions $E_F \xi_{1i}^{2+\gamma} \leq M$ and $E_F ||vec(G_{1i}(\theta_1))||^{2+\gamma} \leq M \; \forall (\theta_1, F) \in \mathcal{F}_{TS,AR/AR}$.

Now we bound the mth term in the doubly infinite sum over $m = -\infty, ..., \infty$ that defines $\Omega_n(\theta_1) - \Omega_n$. We have

$$\begin{aligned} A_{nm} &:= \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||E_{F_n} g_i(\theta_1) g_{i-m}(\theta_1) - (E_{F_n} g_i(\theta_1))^2 - E_{F_n} g_i g_{i-m} + (E_{F_n} g_i)^2|| \\ &= \sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||E_{F_n} g_i \delta_{ni-m}(\theta_1) - E_{F_n} g_i E_{F_n} \delta_{ni}(\theta_1) + E_{F_n} \delta_{ni}(\theta_1) g_{i-m} - E_{F_n} \delta_{ni}(\theta_1) E_{F_n} g_i \\ &\quad + E_{F_n} \delta_{ni}(\theta_1) \delta_{ni-m}(\theta_1) - (E_{F_n} \delta_{ni}(\theta_1))^2|| \\ &\leq C_1 C^{\gamma/(2+\gamma)} (2||g_i||_{F_n,2+\gamma} \, ||\delta_{ni}(\theta_1)||_{F_n,2+\gamma} + ||\delta_{ni}(\theta_1)||^2_{F_n,2+\gamma}) m^{-d\gamma/(2+\gamma)} \\ &= O(n^{-1/2}) m^{-d\gamma/(2+\gamma)}, \end{aligned} \tag{21.5}$$

where the $O(n^{-1/2})$ term does not depend on m, the first equality holds by (21.2), the inequality holds by (21.3) applied three times, and the last equality holds by (21.4) and $||g_i||_{F_n,2+\gamma} \leq M$.

We have

$$\sup_{\theta_1 \in B(\theta_{1*n}, K/n^{1/2})} ||\Omega_n(\theta_1) - \Omega_n|| \leq \sum_{m=-\infty}^{\infty} A_{nm} = O(n^{-1/2}) \sum_{m=-\infty}^{\infty} m^{-d\gamma/(2+\gamma)} = O(n^{-1/2}), \quad (21.6)$$

where the first inequality holds by the definition of $\Omega_n(\theta_1)$ in (11.1), the first equality holds by (21.5), and the last equality uses the condition on d in $\mathcal{F}_{TS,AR/AR}$ that $d > (2+\gamma)/\gamma$. This completes the verification of Assumption $\text{HL1}_{AR/AR}$(xii). The verifications of Assumptions $\text{HL1}_{AR/AR}$(xvi) and $\text{HL1}_{AR/LM}$(viii) are similar and hence, for brevity, are not given.

The sixth modification is in the verification of Assumption $\text{HL2}_{AR/AR}$(ii) in (19.39)–(19.41) in the proof of Lemma 19.3. We verify condition (ii) of Lemma 19.4, i.e., $\{\nu_n(\cdot) : n \geq 1\}$ is stochastically equicontinuous for $\nu_n(\theta_1)$ defined in (19.39), using Lemma 21.2 with $s(w, \beta) = g(w, \theta_1, \theta_{20})$, $\beta = \theta_1$, $d_\beta = p_1$, and $\mathcal{B} = \Theta_1$, rather than using Theorem 4 in Andrews (1994, p. 2277). (We use the latter result in the row-wise i.i.d. case because it yields weaker conditions than are obtained by applying Lemma 21.2 in the i.i.d. case.) Conditions (i) and (ii) of Lemma 21.2 hold by the strong mixing condition in $\mathcal{F}_{TS,AR/AR}$ and Assumption SI(v), respectively. Condition (iii) of Lemma 21.2 holds with $B(W_{ni}) = \sup_{\theta_1 \in \Theta_1} ||G_{1i}(\theta_1)||$ by a mean-value expansion using Assumption SI(v), as in (19.41). Conditions (iv), (v), and (vi) of Lemma 21.2 hold by Assumptions SI(iii), SI(iv), and SI-TS(i) (because the conditions in Assumption SI-TS(i) imply that $q > p_1$ and $2 \leq q < r$), respectively.

In addition, when verifying Assumption $\text{HL2}_{AR/AR}$(ii), we verify condition (iii) of Lemma 19.4, i.e., $\nu_n(\theta_1) = O_p(1)\ \forall \theta_1 \in \Theta_1$ for $\nu_n(\theta_1)$ defined in (19.39), using Markov's inequality and the strong mixing covariance inequality in (21.3), rather than Markov's inequality combined with the expression for the variance of an average of i.i.d. random variables, as in (19.40). It suffices to show that $Var_{F_n}(n^{1/2}\widehat{g}_n(\theta_1)) = O_p(1)\ \forall \theta_1 \in \Theta_1$. By change of variables, we have

$$\begin{aligned} Var_{F_n}\left(n^{-1/2}\sum_{i=1}^{n} g_i(\theta_1)\right) &= \sum_{m=-n+1}^{n-1}\left(1 - \frac{|m|}{n}\right) Cov_{F_n}(g_i(\theta_1), g_{i-m}(\theta_1)) \\ &\leq O(1) \sum_{m=-\infty}^{\infty} \alpha_{F_n}^{(r-2)/r}(m) = O(1), \end{aligned} \quad (21.7)$$

where the inequality holds using the first line of (21.3) with r in place of $2+\gamma$, $s_1(W_i) = s_2(W_i) = g_i(\theta_1)$, and $\limsup_{n\to\infty} E_{F_n} \sup_{\theta_1 \in \Theta_1} ||g_i(\theta_1)||^r < \infty$ (by Assumption SI(iii)), and the second equality holds by the conditions on the strong mixing numbers in Assumption SI-TI(i) by the following argument. Given the mixing number condition in Assumption SI-TI(i), it suffices to show that $1/q - 1/r \leq (r-2)/r$, or equivalently, $1/q \leq 1 - 1/r$, or $q \geq r/(r-1)$. We have

$q \geq (q+\delta_1)/(1+\delta_1) \geq (q+\delta_1)/(q+\delta_1-1) = r/(r-1)$, where the first inequality holds because $q \geq 1$ and $\delta_1 > 0$ and the second inequality holds because $q-1 \geq 1$.

The seventh modification is to use Assumption SI-TS(ii) to verify Assumption $\text{HL2}_{AR/AR}$(v).

This completes the proof of Theorem 11.1. $\square$

Proof of Lemma 21.1. The proof is essentially the same as that of Lemma 20.1 in Section 20 in the SM to AG1. For the CS case, it relies on the moment conditions $E_{F_n}||f_i(\theta_{*n})||^{2+\gamma} \leq M < \infty$ for some $M < \infty$, $\forall(\theta_{*n}, F_n) \in \mathcal{F}_{\Theta,TS,AR/AR}$ (or, equivalently, $\forall\lambda_n \in \Lambda_{\Theta,TS,AR/AR}$), and on the strong mixing numbers satisfying $\alpha_{F_n}(m) \leq Cm^{-d}$ for some $d > (2+\gamma)/\gamma$ and some $C < \infty$, $\forall(\theta_{*n}, F_n) \in \mathcal{F}_{\Theta,TS,AR/AR}$, where for notational simplicity we consider the sequence $\{n\}$, rather than a subsequence $\{w_n : n \geq 1\}$. $\square$

Proof of Lemma 21.2. The result of Lemma 21.2 is a special case of Theorem 3 in Hansen (1996) where Hansen's Lipschitz exponent λ equals 1, the average over n in his equations (12) and (13) disappear because of the assumption of row-wise identical distributions (and, hence, the square and square root in his (12) and (13) cancel), his parameter dimension "a" is d_β in our notation, his metric ρ_r is the L^r metric given our assumption of row-wise identical distributions, and the L^r metric on $\mathcal{B}$ can be replaced by the Euclidean metric on $\mathcal{B}$ using the Lipschitz condition (iii) and the moment condition (v) in the statement of the Lemma. $\square$

22 Additional Simulation Results

This section provides additional simulation results to those given in Section 9. The details concerning the models, tests, and simulation scenarios considered are given in Section 9.

22.1 Heteroskedastic Linear IV Model

Table SM-I provides NRP's of the AR/QLR1 test in the heteroskedastic linear IV model of Section 9.1 for sample sizes $n = 50$ and 500 for $k = 4$. Table SM-II does likewise for $n = 100$ and 250 for $k = 8$. The NRP results in Table SM-I are similar to those in Table I for $n = 100$ and 250. Even for $n = 50$, the maximum NRP is .050. On the other hand, the results in Table SM-II for $k = 8$ show some over-rejection of the null with the maximum NRP probability being .064 for $n = 100$ and .056 for $n = 250$.

TABLE SM-I. Null Rejection Probabilities of the Nominal .05 AR/QLR1 Test for $n = 50$ and 500 and $k = 4$, and Base Case Tuning Parameters in the Heteroskedastic Linear Instrumental Variables Model

		$n = 50$					$n = 500$				
	$\|\|\pi_2\|\|$:	40	20	12	4	0	40	20	12	4	0
	40	.043	.043	.043	.046	.050	.049	.049	.049	.047	.046
	20	.040	.040	.040	.042	.048	.049	.049	.048	.046	.045
$\|\|\pi_1\|\|$	12	.037	.037	.038	.040	.044	.048	.048	.047	.044	.043
	4	.018	.018	.019	.022	.033	.037	.034	.030	.030	.037
	0	.000	.000	.000	.000	.001	.000	.000	.001	.001	.001

TABLE SM-II. Null Rejection Probabilities of the Nominal .05 AR/QLR1 Test for $n = 100$ and 250 and $k = 8$, and Base Case Tuning Parameters in the Heteroskedastic Linear Instrumental Variables Model

		$n = 100$					$n = 250$				
	$\|\|\pi_2\|\|$:	40	20	12	4	0	40	20	12	4	0
	40	.043	.045	.049	.061	.064	.045	.045	.046	.054	.056
	20	.040	.042	.046	.059	.063	.044	.044	.044	.053	.055
$\|\|\pi_1\|\|$	12	.038	.039	.042	.057	.061	.042	.041	.040	.051	.054
	4	.014	.014	.017	.036	.047	.019	.019	.020	.038	.044
	0	.000	.000	.000	.001	.001	.000	.001	.001	.001	.001

22.2 Nonlinear IV Model: Inference on the Structural Function

Table SM-III shows little sensitivity of NRP's of the AR/QLR1 test to α_1, but some sensitivity of power to α_1 when $||\pi|| = 4$ and θ_2 is positive. Table SM-III shows no sensitivity of the NRP's and power of the AR/QLR1 test to K_L, K_L^*, and a. The table shows no sensitivity of NRP's to K_{rk}, but some sensitivity of power to K_{rk}. The smallest value of K_{rk}, .25, yields noticeably lower power than larger values.

TABLE SM-III. Sensitivity of NRP and Power of the Nominal .05 AR/QLR1 Test to the Tuning Parameters α_1, K_L, K_{rk}, K_L^*, and a for $||\pi|| = 50$ and 4 and for Five Values of θ_2 for Inference on the Structural Function at $y_1 = 2$ in the Nonlinear Instrumental Variables Model

Tuning Parameter		θ_2 :	$\|\|\pi\|\| = 50$ .00	-.130	-.094	.105	.155	$\|\|\pi\|\| = 4$ .00	-1.15	-.88	2.7	8.8
	.0010		.043	.796	.497	.507	.808	.038	.798	.496	.519	.808
	.0025		.044	.799	.502	.507	.807	.038	.799	.496	.514	.805
α_1	**.0050**		.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
	.0100		.042	.798	.502	.494	.794	.035	.797	.487	.486	.785
	.0150		.041	.795	.498	.483	.784	.032	.794	.475	.465	.766
	.01		.045	.804	.510	.513	.813	.039	.803	.503	.524	.800
K_L	**.05**		.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
	.10		.040	.791	.491	.493	.796	.035	.792	.486	.504	.800
	.25		.046	.742	.445	.430	.735	.041	.769	.451	.465	.789
	.50		.045	.776	.477	.468	.770	.037	.791	.474	.482	.792
K_{rk}	**1.0**		.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
	2.0		.045	.811	.518	.524	.821	.037	.802	.500	.538	.810
	4.0		.045	.812	.521	.530	.824	.038	.800	.500	.565	.826
	.001		.044	.801	.504	.502	.803	.037	.799	.495	.504	.803
K_L^*	**.005**		.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
	.010		.044	.801	.504	.502	.803	.037	.799	.495	.504	.707
	.00		.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
a	$\mathbf{10^{-6}}$		.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
	.01		.043	.801	.504	.502	.803	.036	.799	.495	.504	.798

TABLE SM-IV. Sensitivity of NRP and Power of the Nominal .05 AR/QLR1 Test to the Sample Size, n, and Number of Instruments, k, for $||\pi|| = 50$ and 4 and for Five Values of θ_2 for Inference on the Structural Function at $y_1 = 2$ in the Nonlinear Instrumental Variables Model

		$\|\|\pi\|\| = 50$					$\|\|\pi\|\| = 4$				
	θ_2 :	.00	-.130	-.094	.105	.155	.00	-1.15	-.88	2.7	8.8
	50	.026	.271	.211	.191	.229	.018	.125	.071	.097	.256
	100	.036	.596	.383	.348	.545	.025	.350	.186	.228	.536
n	250	.040	.790	.496	.479	.764	.034	.670	.381	.411	.763
	500	.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
	1000	.048	.775	.481	.489	.787	.042	.868	.570	.572	.755
	4	.044	.801	.504	.502	.803	.037	.799	.495	.504	.800
k	8	.044	.648	.362	.338	.620	.037	.742	.415	.348	.664
	12	.049	.506	.268	.237	.462	.037	.658	.348	.278	.572

22.3 Nonlinear IV Model: Inference on the Derivative of the Structural Function

Table SM-IV provides NRP's for the nominal .05 AR/QLR1 test for hypotheses concerning the derivative of the structural function at $y_1 = 2$. The table shows that the NRP's vary between .007 and .052 over these cases. The lowest NRP's occur for $||\pi|| = 0$. In the base case scenario, $n = 500$ and $k = 4$, the NRP's are in [.034, .047] for $||\pi|| \geq 4$.

TABLE SM-V. Null Rejection Probabilities of the Nominal .05 AR/QLR1 Test for Base Case Tuning Parameters for Inference on the Derivative of the Structural Function at $y_1 = 2$ in the Nonlinear Instrumental Variables

k	n	Errors	$\|\|\pi\|\|$:	100	75	50	35	20	14	8	4	0
4	50	Homoskedastic		.033	.031	.028	.024	.024	.026	.025	.019	.002
4	100	Homoskedastic		.039	.039	.039	.039	.039	.037	.033	.026	.007
4	250	Homoskedastic		.042	.043	.044	.043	.040	.037	.033	.031	.017
4	**500**	**Homoskedastic**		.045	.046	.047	.043	.039	.038	.037	.034	.016
8	100	Homoskedastic		.051	.052	.051	.050	.050	.051	.048	.036	.007
8	250	Homoskedastic		.045	.046	.047	.047	.046	.044	.040	.033	.023
4	250	Heteroskedastic		.032	.031	.030	.029	.026	.023	.017	.011	.008

The results of Table SM-VI are very similar to those of Table SM-III. The results of Table SM-VII are broadly similar to those of Table SM-IV.

TABLE SM-VI. Sensitivity of NRP and Power of the Nominal .05 AR/QLR1 Test to the Tuning Parameters α_1, K_L, K_{rk}, K_L^*, and a for $||\pi|| = 50$ and 4 and for Five Values of θ_2 for Inference on the Derivative of the Structural Function in the Nonlinear Instrumental Variables Model

Tuning			$\|\|\pi\|\| = 50$					$\|\|\pi\|\| = 4$				
Parameter		θ_2 :	.00	-.085	-.061	.070	.104	.00	-.80	-.60	1.6	4.5
	.0010		.046	.796	.495	.497	.792	.035	.805	.505	.520	.811
	.0025		.046	.796	.495	.503	.797	.035	.806	.505	.515	.809
α_1	**.0050**		.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
	.0100		.046	.797	.495	.506	.804	.032	.799	.495	.485	.787
	.0150		.046	.797	.495	.506	.804	.030	.792	.482	.463	.770
	.01		.046	.797	.496	.505	.801	.036	.810	.508	.523	.813
K_L	**.05**		.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
	.10		.044	.789	.486	.489	.788	.032	.799	.494	.505	.802
	.25		.048	.734	.425	.430	.736	.038	.778	.458	.470	.788
	.50		.050	.770	.466	.469	.769	.035	.796	.482	.485	.793
K_{rk}	**1.0**		.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
	2.0		.047	.809	.508	.521	.816	.036	.811	.509	.530	.816
	4.0		.048	.810	.511	.526	.819	.039	.813	.516	.552	.839
	.001		.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
K_L^*	**.005**		.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
	.010		.046	.797	.495	.504	.800	.034	.800	.498	.505	.796
	.00		.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
a	$\mathbf{10^{-6}}$		.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
	.01		.046	.797	.495	.504	.800	.034	.805	.503	.504	.802

TABLE SM-VII. Sensitivity of NRP and Power of the Nominal .05 AR/QLR1 Test to the Sample Size, n, and Number of Instruments, k, for $||\pi|| = 50$ and 4 and for Five Values of θ_2 for Inference on the Derivative of the Structural Function in the Nonlinear Instrumental Variables Model

		$\|\|\pi\|\| = 50$					$\|\|\pi\|\| = 4$				
	θ_2 :	.00	-.085	-.061	.070	.104	.00	-.80	-.60	1.6	4.5
	50	.030	.371	.246	.213	.327	.018	.053	.035	.064	.126
	100	.040	.697	.470	.433	.660	.024	.156	.090	.177	.404
n	250	.043	.834	.544	.520	.805	.031	.492	.278	.380	.723
	500	.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
	1000	.048	.692	.403	.431	.726	.039	.959	.727	.590	.834
	4	.046	.797	.495	.504	.800	.034	.805	.503	.505	.802
k	8	.046	.603	.338	.322	.591	.030	.776	.437	.356	.669
	12	.053	.458	.244	.232	.439	.042	.693	.360	.284	.565

23 Additional Second-Step C(α) Tests

23.1 C(α)-QLR2 Test

Here, we define a C(α) version of the test in Andrews and Guggenberger (2015) (AG), which we refer to as the C(α)-QLR2 test. The second-step C(α)-QLR2 test statistic is

$$\begin{aligned} QLR2_{2n}(\theta) &:= AR_{2n}(\theta) - \lambda_{\min}(n\widehat{Q}_{2n}(\theta)), \text{ where} \\ \widehat{Q}_{2n}(\theta) &:= \left(\widetilde{g}_n(\theta), \widehat{D}^*_{2n}(\theta)\right)' \widehat{M}_{1n}(\theta) \left(\widetilde{g}_n(\theta), \widehat{D}^*_{2n}(\theta)\right) \in R^{(p_2+1)\times(p_2+1)}, \\ \widehat{D}^*_{2n}(\theta) &:= \widehat{\Omega}_n^{-1/2}(\theta)\widehat{D}_{2n}(\theta)\widehat{L}_{2n}^{1/2}(\theta) \in R^{k\times p_2}, \\ \widehat{L}_{2n}(\theta) &:= (\theta_2, I_{p_2})(\widehat{\Sigma}^{\varepsilon}_{2n}(\theta))^{-1}(\theta_2, I_{p_2})' \in R^{p_2\times p_2}, \end{aligned} \tag{23.1}$$

$\widetilde{g}_n(\theta)$ is defined in (7.10), and $\widehat{\Sigma}^{\varepsilon}_{2n}(\theta)$ is defined below.[19]

The C(α)-QLR2 test uses a conditional critical value that depends on the $k \times p_2$ matrix $n^{1/2}\widehat{D}^*_{2n}(\theta_1)$ and the $k\times k$ projection matrix $\widehat{M}_{1n}(\theta_1)$. For nonrandom $D_2 \in R^{k\times p_2}$ and nonrandom symmetric psd $M \in R^{k\times k}$, let

$$QLR2_{k,p_2}(D_2, M) := Z'MZ - \lambda_{\min}((Z, D_2)'M(Z, D_2)), \text{ where } Z \sim N(0^k, I_k). \tag{23.2}$$

[19]Unlike the random perturbation of $\widehat{\Omega}_n^{-1/2}(\theta)\widehat{D}_{1n}(\theta)$ by $an^{-1/2}\zeta_1$ in (7.10) and the random perturbation of $\widehat{\Omega}_n^{-1/2}(\theta)\widehat{D}_{2n}(\theta)$ by $an^{-1/2}\zeta_2$ in (7.13), no random perturbation of $\widehat{D}^*_{2n}(\theta)$ is needed in the definition of $QLR2_{2n}(\theta)$.

Define $c^{QLR2}_{k,p_2}(D_2, M, 1-\alpha)$ to be the $1-\alpha$ quantile of the distribution of $QLR2_{k,p_2}(D_2, M)$. For given D_2 and M, $c^{QLR2}_{k,p_2}(D_2, M, 1-\alpha)$ can be computed by simulation very quickly and easily.

For given $\theta_1 \in \Theta_1$, the nominal level η second-step C(α)-QLR2 test rejects $H_0 : \theta_2 = \theta_{20}$ when

$$\phi^{QLR2}_{2n}(\theta_1, \eta) := QLR2_{2n}(\theta_1, \theta_{20}) - c^{QLR2}_{k,p_2}(n^{1/2}\widehat{D}^*_{2n}(\theta_1, \theta_{20}), \widehat{M}_{1n}(\theta_1, \theta_{20}), 1-\eta) > 0. \quad (23.3)$$

When $p_2 = 1$, the $\lambda_{\min}(n\widehat{Q}_{2n}(\theta))$ term that appears in (23.1) can be solved in closed form. In this case, the $QLR2_{2n}(\theta)$ statistic can be written as

$$\begin{aligned}
QLR2_{2n}(\theta) &:= \frac{1}{2}\left(AR_{2n}(\theta) - rk^*_{2n}(\theta) + \sqrt{(AR_{2n}(\theta) - rk^*_{2n}(\theta))^2 + 4LM^*_{2n}(\theta) \cdot rk^*_{2n}(\theta)}\right), \text{ where} \\
LM^*_{2n}(\theta) &:= n\widetilde{g}_n(\theta)' P_{\widehat{D}^*_{2n}(\theta)} \widetilde{g}_n(\theta), \\
rk^*_{2n}(\theta) &:= n\widehat{D}^*_{2n}(\theta)' \widehat{M}_{1n}(\theta) \widehat{D}^*_{2n}(\theta),
\end{aligned} \quad (23.4)$$

and $AR_{2n}(\theta)$ is defined in (7.10). When $p_2 = 1$, the C(α)-QLR2 critical value, for a nominal level η test, is as in (7.17) with $rk_{2n}(\theta) = rk^*_{2n}(\theta)$ and $WI^\dagger_n(\theta) = 0$: $c^{QLR1}(1-\eta, rk^*_{2n}(\theta), 0)$.

Now, we define $\widehat{\Sigma}^\varepsilon_{2n}(\theta)$. Define

$$\begin{aligned}
\widehat{R}_{2n}(\theta) &:= \left(B_2(\theta)' \otimes I_k\right) \widehat{V}_{2n}(\theta) \left(B_2(\theta) \otimes I_k\right) \in R^{(p_2+1)k \times (p_2+1)k}, \text{ where} \\
\widehat{V}_{2n}(\theta) &:= n^{-1} \sum_{i=1}^n \left(f_{2i}(\theta) - \widehat{f}_{2n}(\theta)\right)\left(f_{2i}(\theta) - \widehat{f}_{2n}(\theta)\right)' \in R^{(p_2+1)k \times (p_2+1)k}, \quad (23.5) \\
f_{2i}(\theta) &:= \begin{pmatrix} g_i(\theta) \\ vec(G_{2i}(\theta)) \end{pmatrix}, \ \widehat{f}_{2n}(\theta) := \begin{pmatrix} \widehat{g}_n(\theta) \\ vec(\widehat{G}_{2n}(\theta)) \end{pmatrix}, \text{ and } B_2(\theta) := \begin{pmatrix} 1 & 0'_{p_2} \\ -\theta_2 & -I_{p_2} \end{pmatrix}.
\end{aligned}$$

Let $\widehat{R}_{2j\ell n}(\theta)$ denote the (j, ℓ) $k \times k$ submatrix of $\widehat{R}_{2n}(\theta)$ for $j, \ell \leq p_2 + 1$.[20]

We define $\widehat{\Sigma}_{2n}(\theta) \in R^{(p_2+1)\times(p_2+1)}$ to be the symmetric pd matrix whose (j, ℓ) element is

$$\widehat{\Sigma}_{2j\ell n}(\theta) = tr(\widehat{R}_{2j\ell n}(\theta)' \widehat{\Omega}^{-1}_n(\theta))/k \quad (23.6)$$

for $j, \ell \leq p_2 + 1$. AG use an eigenvalue-adjusted version of $\widehat{\Sigma}_{2n}(\theta)$, denoted $\widehat{\Sigma}^\varepsilon_{2n}(\theta)$.

The eigenvalue adjustment is defined as follows. Let $H \in R^{d_H \times d_H}$ be any non-zero positive semi-definite (psd) matrix with spectral decomposition $A_H \Lambda_H A'_H$, where $\Lambda_H = Diag\{\lambda_{H1}, ..., \lambda_{Hd_H}\}$ is the diagonal matrix of eigenvalues of H with nonnegative nonincreasing diagonal elements and A_H is a corresponding orthogonal matrix of eigenvectors of H. For $\varepsilon > 0$, the eigenvalue-adjusted

[20] That is, $\widehat{R}_{2j\ell n}(\theta)$ contains the elements of $\widehat{R}_{2n}(\theta)$ indexed by rows $(j-1)k+1$ to jk and columns $(\ell-1)k$ to ℓk.

matrix H^ε is

$$H^\varepsilon := A_H \Lambda_H^\varepsilon A_H', \text{ where } \Lambda_H^\varepsilon := Diag\{\max\{\lambda_{H1}, \lambda_{\max}(H)\varepsilon\}, ..., \max\{\lambda_{Hd_H}, \lambda_{\max}(H)\varepsilon\}\}, \quad (23.7)$$

where $\lambda_{\max}(H)$ denotes the maximum eigenvalue of H. Note that $H^\varepsilon = H$ whenever the condition number of H is less than or equal to $1/\varepsilon$ (for $\varepsilon \leq 1$).[21]

The matrix $\widehat{\Sigma}_{2n}^\varepsilon(\theta)$ is defined in (23.7) with $H = \widehat{\Sigma}_{2n}(\theta)$. Based on the finite-sample simulations, AG recommend using $\varepsilon = .05$.

23.2 C(α)-QLR3 Test

In this section we introduce a C(α) version of the CQLR test in I. Andrews and Mikusheva (2016) (AM), which we refer to as the C(α)-QLR3 test. For given $\theta_1 \in \Theta_1$, the second-step C(α)-QLR3 test statistic is

$$AR_{2n}(\theta_1, \theta_{20}) - \inf_{\theta_2 \in \Theta_2} AR_{2n}(\theta_1, \theta_2). \quad (23.8)$$

Next, we define AM's data-dependent critical value. For $\theta := (\theta_1, \theta_2) \in \Theta$ and $\theta_0 := (\theta_1', \theta_{20}')' \in \Theta$, let

$$\begin{aligned} h_{2n}(\theta) &:= n^{1/2}\widehat{g}_n(\theta) - \widehat{\Omega}_n(\theta, \theta_0)\widehat{\Omega}_n^{-1}(\theta_0, \theta_0)n^{1/2}\widehat{g}_n(\theta_0) \text{ and} \\ g_{2n}^*(\theta) &:= h_{2n}(\theta) + \widehat{\Omega}_n(\theta, \theta_0)\widehat{\Omega}_n^{-1}(\theta_0, \theta_0)\xi_2^*, \end{aligned} \quad (23.9)$$

where $\xi_2^* \sim N(0, \widehat{\Omega}_n(\theta_0, \theta_0))$ given $\widehat{\Omega}_n(\cdot, \cdot)$ and $\widehat{\Omega}_n(\theta) := \widehat{\Omega}_n(\theta, \theta)$. We view $h_{2n}(\theta_1, \theta_2)$ as a stochastic process indexed by θ_2 with θ_1 fixed. It is designed to be asymptotically independent of $n^{1/2}\widehat{g}_n(\theta_1, \theta_{20})$ when $\theta_0 = (\theta_1', \theta_{20}')'$ is the true value. Define

$$\begin{aligned} QLR3_{2n}^*(\theta_1) &:= AR_{2n}^*(\theta_1, \theta_{20}) - \inf_{\theta_2 \in \Theta_2} AR_{2n}^*(\theta_1, \theta_2), \text{ where} \\ AR_{2n}^*(\theta) &:= g_{2n}^*(\theta)'\widehat{\Omega}_n^{-1/2}(\theta)\widehat{M}_{1n}(\theta)\widehat{\Omega}_n^{-1/2}(\theta)g_{2n}^*(\theta). \end{aligned} \quad (23.10)$$

Let $cv_{2n}^{QLR3}(h_{2n}, \theta_1, \eta)$ denote the $1-\eta$ quantile of the conditional distribution of $QLR3_{2n}^*(\theta_1)$ given $h_{2n} := h_{2n}(\cdot)$ and $\widehat{\Omega}_n(\cdot, \cdot)$.

For given $\theta_1 \in \Theta_1$, the θ_1-orthogonalized nominal η second-step CQLR3 test rejects $H_0 : \theta_2 =$

[21] AG shows that the eigenvalue-adjustment procedure possesses the following desirable properties: (i) H^ε is uniquely defined, (ii) $\lambda_{\min}(H^\varepsilon) \geq \lambda_{\max}(H)\varepsilon$, (iii) $\lambda_{\max}(H^\varepsilon)/\lambda_{\min}(H^\varepsilon) \leq \max\{1/\varepsilon, 1\}$, (iv) for all $c > 0$, $(cH)^\varepsilon = cH^\varepsilon$, and (v) $H_n^\varepsilon \to H^\varepsilon$ for any sequence of psd matrices $\{H_n : n \geq 1\}$ with $H_n \to H$.

θ_{20} when

$$\phi_{2n}^{QLR3}(\theta_1, \eta) := AR_{2n}(\theta_1, \theta_{20}) - \inf_{\theta_2 \in \Theta_2} AR_{2n}(\theta_1, \theta_2) - cv_{2n}^{QLR3}(h_{2n}, \theta_1, \eta) > 0. \tag{23.11}$$

The second-step CQL3 test is applicable in moment condition models, but not minimum distance models.

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